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BUCKLING AND NON-LINEAR VIBRATIONS OF A PIEZOELECTRIC STRATIFIED PLATE – APPLICATION TO A MEMS BIOSENSOR

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1 INTRODUCTION

The large amplitude non-linear vibratory behavior of a stratified circular plate with a piezoelectric patch is addressed in this study. The goal is to build an efficient model of a resonating MEMS (Micro Electro Mechanical System) bio-sensor whose purpose is to detect in real-time the presence of a given molecule (the so called target molecule) in a liquid medium.

The bio-sensor under study is constituted of a matrix of micromachined stratified circular membranes [1] (see Fig. 1). Each membrane can be individually actuated on its fundamental vibration mode by a piezoelectric thin patch. A biorecognition molecule, capable of immobilizing the target molecule, is glued on the top face of the membrane. When the membrane is in contact with a liquid medium that contains the target molecule, the latter is trapped, thus adding mass and lowering the fundamental resonance frequency. Detecting and measuring this frequency shift lead to estimate the concentration of the target molecule in the aqueous solution. At present, whereas the manufacturing of the biosensor is well overcomed, its fine dynamical behavior still needs to be modeled and simulated [1] in order to design such a biosensor.

Two main issues have to be modeled. First, the manufacturing processes of the membranes create prestresses, the intensity and sign of which differ from one layer to another. A consequence is that the shape of the membrane at rest is not plane and that buckling has to be taken into account (see Fig. 2, left). Secondly, non-linear large amplitude vibrations have been experimentally observed in normal use of the sensor, in the form of Duffing-like resonance curves (see Fig. 2, right). Similar phenomena are also encountered in beam piezoelectric MEMS resonators [3].

The membrane is modeled as a stratified circular plate, clamped at its boundary, with one piezoelectric layer and the other composed of linear-elastic-homogeneous-isotropic materials. Von-Kármán-like equations with a kirchhoff-love kinematics across all layers [4], are used. Those PDEs are expanded onto the normal mode basis of the plate *without* prestress. A set of second order differential equations, coupled by linear, quadratic and cubic terms, is obtained. The static solutions lead to predict the geometry of the buckled equilibrium position. Then, the modes of the membrane (frequencies and shapes) in post-buckling vibrations can be obtained. Finally, the dynamic non-linear behaviour of the system, in the vicinity of the fundamental mode of the *prestressed* membrane, is obtained with the framework of non-linear normal modes [5, 6]. In this article, only the general problem formulation is exposed. More results will be proposed at the colloquium.

2 GOVERNING EQUATIONS

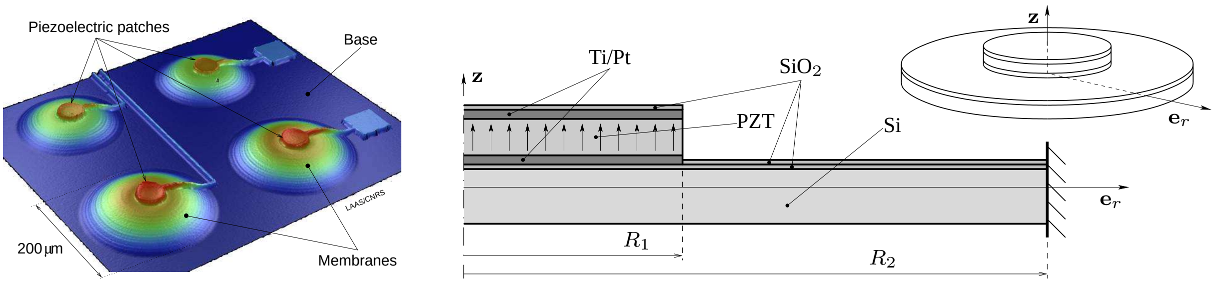

Figure 1. A 2×2 resonant membranes matrix biosensor. Cross section of a layered membrane.

A circular stratified plate with a circular piezoelectric (PZT) film at centre is considered (Fig. 1). The silicon (Si), silicon oxyde (SiO_2) and electrode (Ti/Pt) layers are constituted of isotropic homogeneous materials. The electrodes, plugged to an external voltage generator, create in the PZT layer an electric field in the transverse-$\mathbf{z}$ direction. This enables both to polarize the PZT layer (in the $\mathbf{z}$ direction) and to deform it in the radial-$\mathbf{e}_r$ direction to drive the system in vibration. The PZT layer material is homogeneous and transversely-isotropic in the longitudinal plane of the plate. The classical laminated plate theory [4] is used, with von-Kármán-like non-linear strain-displacement relationships. The electric field, directed in the transverse-$\mathbf{z}$ direction, is assumed uniform through the thickness of the plate. An axisymmetric geometry is considered and a polar (r, θ, z) coordinate system is used. The plate is composed of two parts: a centre circular region, for $r \in [0, R_1]$ and an annular outer region for $r \in [R_1, R_2]$ (Fig. 1). The plate is clamped at its circular edge, for $r = R_2$.

We consider here only axisymmetric deformations of the plate. The strains in a point of coordinates (r, θ, z) write $\boldsymbol{\varepsilon} = \{\varepsilon_r \, \varepsilon_\theta\}^{\mathrm{t}} = \boldsymbol{\epsilon} + z\boldsymbol{\kappa}$, with $\boldsymbol{\epsilon} = \{\epsilon_r \, \epsilon_\theta\}^{\mathrm{t}}$, $\boldsymbol{\kappa} = \{\kappa_r \, \kappa_\theta\}^{\mathrm{t}}$ and

$$\epsilon_r = u_{r,r} + 1/2 w_{,r}^2, \quad \epsilon_\theta = u_r/r, \quad \kappa_r = -w_{,rr}, \quad \kappa_\theta = -w_{,r}/r. \tag{1}$$

In the above equations, u_r and w are respectively radial and transverse displacements and $\circ_{,r}$ is the first derivative of $\circ$ with respect to r. The constitutive relation for the linear piezoelectric material writes:

$$\boldsymbol{\sigma} = \boldsymbol{Q}\left(\boldsymbol{\epsilon} + z\boldsymbol{\kappa}\right) - \sigma_{0P}\mathbf{1}, \quad \text{with} \quad \sigma_{0P} = \sigma_0 + \frac{e_{31}}{h_p}V(t), \tag{2}$$

where $\boldsymbol{\sigma} = \{\sigma_r \, \sigma_\theta\}^{\mathrm{t}}$ are the stresses, $\boldsymbol{Q}$ is the plate stiffness operator and $\mathbf{1} = \{1\,1\}^{\mathrm{t}}$. σ_0 represents the manufacturing prestresses, $V(t)$ is the voltage applied between the two Ti/Pt electrodes, e_{31} is the piezoelectric constant and h_p is the PZT layer thickness. By integrating Eq. (2) over the thickness, the following constitutive relations are obtained, with respect to the membrane forces $\boldsymbol{N} = \{N_r \, N_\theta\}^{\mathrm{t}}$, the moments $\boldsymbol{M} = \{M_r \, M_\theta\}^{\mathrm{t}}$ and the stiffness matrices $\boldsymbol{A}$ (extensional), $\boldsymbol{B}$ (bending-extensional) and $\boldsymbol{D}$ (bending) [4]:

$$\boldsymbol{N} = \boldsymbol{A}\boldsymbol{\epsilon} + \boldsymbol{B}\boldsymbol{\kappa} - N_{0P}\mathbf{1}, \quad \boldsymbol{M} = \boldsymbol{B}\boldsymbol{\epsilon} + \boldsymbol{D}\boldsymbol{\kappa} - M_{0P}\mathbf{1}, \tag{3}$$

where N_{0P} and M_{0P} are tensions and moments created by both the prestresses and the piezoelectric converse effect, stemming from the σ_{0P} term in Eq. (2). By inverting Eq. (3), one obtains [2]:

$$\boldsymbol{\epsilon} = \boldsymbol{A}^{*}\boldsymbol{N} + \boldsymbol{B}^{*}\boldsymbol{\kappa} + A^{+}N_{0P}\mathbf{1}, \quad \boldsymbol{M} = -\boldsymbol{B}^{*\,\mathrm{t}}\boldsymbol{N} + \boldsymbol{D}^{*}\boldsymbol{\kappa} - \tilde{M}_{0P}\mathbf{1}, \tag{4}$$

with $\boldsymbol{A}^* = \boldsymbol{A}^{-1}$, $\boldsymbol{B}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}$, $\boldsymbol{D} = \boldsymbol{D} + \boldsymbol{B}\boldsymbol{B}^*$, $A^+ = A^*_{11} + A^*_{12}$, $\tilde{M}_{0P} = B^+ N_{0P} + M_{0P}$ and $B^+ = B^*_{11} + B^*_{12}$. Substituting Eqs. (4) in the equilibrium equations of the plate and using dimensionless variables leads to obtaining the following equations of motions, in term of transverse displacement w and Airy stress function F:

$$L_D(w) + \varepsilon L_B(F) + \Delta M_{0P} + \varepsilon \Delta(B^+ N_{0P}) + \bar{m}\ddot{w} = \varepsilon L(w, F) + p, \tag{5}$$

$$L_A(F) - L_B(w) + \Delta(A^+ N_{0P}) = -1/2 L(w, w). \tag{6}$$

L_A, L_B and L_D are differential operators that include the discontinuity of matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ in $r = R_1$. At any point out of this discontinuity, the operators write $L_A(\circ) = A^*_{11}\Delta\Delta\circ$, $L_B(\circ) = B^*_{12}\Delta\Delta\circ$ and $L_D(\circ) = D^*_{11}\Delta\Delta\circ$ with Δ the laplacian operator. $L(\circ, \circ)$ is the classical bilinear operator found in von Kármán-like equations of motion [7]. $\bar{m}$ is the mass per unit surface of the plate, $\ddot{w}$ is the second derivative of w with respect to time and p is an external pressure. F is related to $\boldsymbol{N}$ by $N_r = F_{,r}/r$ and $N_\theta = F_{,rr}$.

The clamped boundary conditions impose that w and F be finite in $r = 0$ and that at the edge, in $r = R_2$: $w = 0$, $w_{,r} = 0$, $\epsilon_\theta = 0$ and $\epsilon_r - (r\epsilon_\theta)_{,r} = 0$.

3 MODAL EXPANSION

The transverse unknown displacement w is expanded onto the eigenmodes (Φ_s, ω_s) of the short-circuited plate with zero prestresses ($V(t) \equiv 0$, $\sigma_0 = 0$). Those modes are solutions of the linear part of equations (5,6), with $M_{0P} = N_{0P} = 0$. With:

$$w(r, t) = \sum_{s=1}^{+\infty} \Phi_s(r)\, q_s(t), \tag{7}$$

by expanding F onto ad-hoc functions and using the orthogonality properties of the (Φ_s, ω_s), one can show that equations (5,6) are equivalent to the following set, verified by unknown modal coordinates $q_s(t)$, $\forall s \in \mathbb{N}^*$:

$$\begin{aligned} \ddot{q}_s(t) + 2\xi_s\omega_s\dot{q}_s(t) + \omega_s^2 q_s(t) &= k_0^s + k_P^s V(t) + \sum_{q=1}^{+\infty} \left[\alpha^s_{0q} + \alpha^s_{Pq} V(t)\right] q_q(t) \\ &+ \sum_{p=1}^{+\infty}\sum_{q=1}^{+\infty} \beta^s_{pq}\, q_p(t)\, q_q(t) + \sum_{p=1}^{+\infty}\sum_{q=1}^{+\infty}\sum_{r=1}^{+\infty} \Gamma^s_{pqr}\, q_p(t)\, q_q(t)\, q_r(t) + Q_s(t). \end{aligned} \tag{8}$$

A modal damping term has been added in the above equations (ξ_s is the damping factor of the s-th. mode). All coefficients appearing in the above equations (k_0^s, k_P^s, α^s_{0q}, α^s_{Pq}, β^s_{pq}, Γ^s_{pqr} and Q_s) are functions of the (Φ_s, ω_s), with known analytical expressions, not reported here for a sake of brevity.

The initial partial differential equations (5,6) have been replaced by the equivalent discretized problem (8) of coupled non-linear differential equations. The coupling terms stem from different sources. Geometrical non-linearities create the cubic terms (Γ^s_{pqr}), in the same manner as for a homogeneous plate [7]. The layered structure of the plate, if it is non-symmetrical in the transverse-$\mathbf{z}$ direction, adds quadratic non-linearities (β^s_{pqr}). The voltage applied to the piezoelectric layer can be written as $V(t) = \hat{V} + \tilde{V}(t)$, with a constant part $\hat{V}$, that creates the polarization, superimposed to an oscillating part $\tilde{V}(t)$. Both $\hat{V}$ and the prestresses (N_{0P}, M_{0P}) add a constant forcing (k_0^s, $k_P^s\hat{V}$) as well as linear terms (α^s_{0q}, $\alpha^s_{Pq}\hat{V}$). The constant terms are responsible of the non-plane static position of the plate. The linear terms modifies the static position, can create buckling and shifts the natural frequencies ω_s of the plate. The dynamic piezoelectric driving stemming from voltage $\tilde{V}(t)$ creates a direct forcing ($k_P^s\tilde{V}(t)$) as well as a parametric excitation ($\alpha^s_{Pq}\tilde{V}(t)$) of the plate. Q_s stems from the external pressure p.

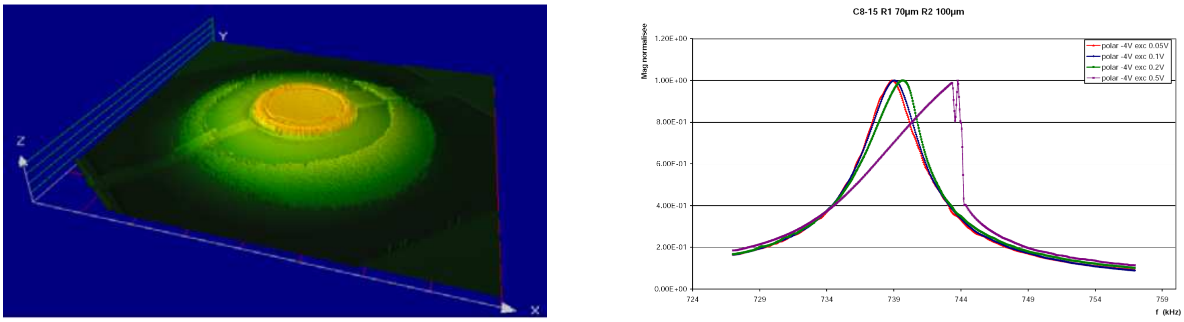

Figure 2. Static non-planar geometry of a membrane and dynamic non-linear behaviour. Experimental results.

4 RESOLUTION

A solution to the problem defined by equations (7,8) is obtained by writing:

$$w(r,t) = \hat{w}(r) + \tilde{w}(r,t) = \sum_{s=1}^{+\infty} \Phi_s(r) \left[\hat{q}_s + \tilde{q}_s(t)\right], \tag{9}$$

where $\hat{w}(r)$ is the static position of the plate and $\tilde{w}(r,t)$ is related to its oscillations with respect to its position at rest $\hat{w}(r)$. By substituting Eq. (9) in (8) with $\tilde{w} = 0$, a set of algebraic non-linear equations is obtained, whose solution (the $\hat{q}_s$) gives the non-planar static behaviour of the membrane depicted on Fig. 2(left). Then, with $\tilde{w} \neq 0$, a set of equations similar to (8), that includes $\hat{q}_s$ and $\tilde{q}_s$, is obtained. After a proper diagonalisation of its linear part and by retaining enough modes Φ_s in the calculation, the natural frequencies of the plate as a function of the prestresses are obtained. Finally, the resolution of the full non-linear set, in the particular case of a resonating piezoelectric excitation in the vicinity of the fundamental frequency of the plate, will be conducted with the formalism of the non-linear normal modes [6], in order to obtain the curves of 2(right).

Modes (Φ_s, ω_s) of the short-circuited plate with no prestresses, with the geometrical and material discontinuity in $r = R_1$, have been calculated in a semi-analytical manner (with Bessel functions), by connecting two continuous annular plates in $r = R_1$. The computations of all coefficients appearing in Eqs. (8) is under progress, as well as the estimation of $\hat{w}$ and $\tilde{w}$. Those results will be presented at the colloquium, along with convergences studies that shows the accuracy of the solution with respect to the number of modes (Φ_s, ω_s) retained in Eq. (9).

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