NONLINEAR BEHAVIOR OF GONGS THROUGH THE DYNAMICS OF SIMPLE RODS SYSTEMS

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Abstract

In this study, the nonlinear equations of vibration of small systems composed of rods subjected to large deflection are derived and discussed. Through simple arguments, a parallel is drawn between the dynamics exhibited by those rods systems, and the behaviour of nonlinear percussion instruments, such as gongs. First, the physical sources of nonlinearity are identified and discussed. Secondly, the dependence of the frequency of oscillations on the amplitude of vibration is explicited. The nature of this dependence, which can be either of softening or of hardening type, is related to geometrical characteristics, such as thickness and curvature of the structure. Finally, experiments on a Chinese tam-tam are performed in order to confirm the previously discussed points .

1 INTRODUCTION

Nonlinear percussion intruments such as cymbals and gongs can exhibit very complicated behavior, and are thus still not completely understood, in spite of recent studies on the subject (see [1] for a review). For gongs, two different families can be roughly distinguished, with respect to the intensity of the nonlinear effects in the vibration. The first family exhibits only weak nonlinearities in usual playing. Chinese opera gongs are of this kind, and the main musical effect in this case is the contrary pitch glide that they can exhibit in their sound, depending on the balance between the thickness and the curvature of their shell-like geometry [2]. The second family, where one can find the Chinese tam-tam, shows strong nonlinear effects and high dimensional chaotic vibrations, which are responsible of the bright shimmering sound [3, 4].

The purpose of the present work is to investigate the properties of a small system composed of rods subjected to large deflection, in order to understand the principles of nonlinear behaviour observed in complex structures like gongs, plates and shells. Since the system has only one degree of freedom (DOF.), it is able to predict the behaviour of only one mode of vibration of a gong. Thus, only the weakly non-linear response of gongs, without modal coupling, is adressed here.

The simplicity of rods systems enables a physical interpretation of the non-linearities found in the equations, which can be extended to the case of a gong. The contrary pitch glides in Chinese opera gongs are explained either by the hardening or by the softening behavior of the system, whick are linked to geometrical properties. Finally, experiments on a Chinese tam-tam are presented. The different hardening/softening behavior for axisymmetric modes are illustrated and discussed in view of their corresponding modal shapes. This latter part of the work is directly connected to the experiments described in [x] (citer chaignethomastouzeISMA2001).

2 PHYSICAL INTERPRETATION OF THE GEOMETRICAL NON-LINEARITIES

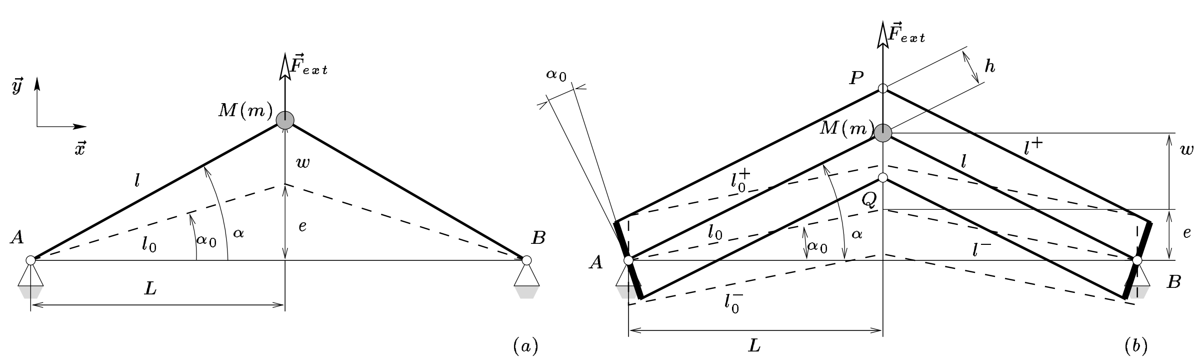

Figure 1: (a) – Sketch of the 2-rods system. Undeformed configuration (angle α_0) in dotted lines. Current configuration (angle α) in solid lines. (b) – Sketch of the 6-rods system.

The system shown in Figure 1(a) is studied. It is composed of a mass m, connected to a frame by two rods, of current length l and angle α. The undeformed configuration is defined by length l_0 and angle α_0. The links between the frames and the rods (A and B), and between the rods and the mass (M), are all perfect joints of revolution, so that the rods are subjected to axial stress only. Due to the symmetry of the system, the motion of the mass is vertical. The symbol w denotes the deflection of mass m from the undeformed position. It is assumed that the gravity forces are negligible with respect to the elastic forces.

2.1 Restoring force

The Green-Lagrange (GL) normal strain in the rods is:

$$e_n = \frac{1}{2}\frac{l^2 - l_0^2}{l_0^2} = \frac{e}{l_0}\frac{w}{l_0} + \frac{1}{2}\left(\frac{w}{l_0}\right)^2 = \sin\alpha_0 \frac{w}{l_0} + \frac{1}{2}\left(\frac{w}{l_0}\right)^2 \tag{1}$$

where $e/l_0 = \sin\alpha_0$.

To take the large deflections of the system into account, the symmetric Piola-Kirchoff (PK) stress tensor is used, and denoted by $\underline{\underline{\pi}}$. It is a measure of the stress in the rods with respect to the *undeformed* configuration [5]. The material of the rods is assumed to be linear, homogeneous and isotropic, with Young's modulus E, so that the constitutive relation leads to a PK normal force in the rods defined by:

$$N_0 = A\pi_n = EA.e_n, \tag{2}$$

where A is the area of a cross-section of the rods and π_n the normal PK stress.

The real (Cauchy) normal force in rods is denoted by N. Its direction corresponds to the current axis of the rod. This force is related to the PK normal force N_0 by [5] the equation:

$$N = \frac{l}{l_0}N_0 \quad \text{so that} \quad N = \frac{l}{l_0}EA.e_n = \frac{1}{2}EA\left[\left(\frac{l}{l_0}\right)^3 - \frac{l}{l_0}\right] \tag{3}$$

The vertical force F_w exerted by the two rods on the mass m is the projection on the $\vec{y}$-axis of the tension N of the rods, so that:

$$F_w = 2N\sin\alpha = 2\underbrace{\left(\frac{w}{l_0} + \sin\alpha_0\right)}_{\frac{l}{l_0}.\sin\alpha}\ \underbrace{EA\left(\sin\alpha_0\frac{w}{l_0} + \frac{1}{2}\left(\frac{w}{l_0}\right)^2\right)}_{\frac{l_0}{l}N} \tag{4a,b}$$

$$= 2EA\left\{\sin^2\alpha_0\frac{w}{l_0} + \frac{3}{2}\sin\alpha_0\left(\frac{w}{l_0}\right)^2 + \frac{1}{2}\left(\frac{w}{l_0}\right)^3\right\} \tag{4c}$$

2.2 Nonlinearities in the flat system: general geometrical nonlinearities

The system is assumed to be flat at rest, which means $\alpha_0 = 0$. In this case, F_w reduces to:

$$F_w = EA\left(\frac{w}{l_0}\right)^3 \tag{5}$$

The restoring force is nonlinear with respect to the vertical displacement of the mass. This geometrical nonlinearity is due to the fact that the *current* configuration cannot be set equal to the *undeformed* configuration, as it is generally assumed in the linear theory. Therefore, the nonlinearity results from two main causes [6]:

- The finite elongation of the rods (Eq. (1)), which leads to an axial force N which is nonlinear with respect to the elongation of the rod (Eq. (3));
- The projection on the vertical $\vec{y}$-axis, with the projection angle α depending on w (Eq. (4a)).

An analogy can be made between the flat rods system and other continuous systems, such as strings and membranes, which are subjected to nonlinear stretching for large magnitude of the transverse displacement.

2.3 Influence of the curvature

If the undeformed configuration presents a non-zero angle α_0, two terms are added in the expression of F_w. Equation (4b) helps to identify the origin of the different terms of F_w in Eq. (4c).

- The *linear term* is the projection of the axial forces in the rods on the vertical axis, with angle α *constant* and equal to the angle at rest α_0. This yields a linear stiffness $2EA\sin^2\alpha_0$ term.
- The *quadratic term* comes from two sources. (i) Following equation (4b), the current stiffness F_w/w of the system depends upon the product of $\sin\alpha$ (the first bracketed expression of Eq. (4b)) and $\sin\alpha_0$ (in the second bracketed expression). Since α depends on w, the stiffness increases (resp. decreases) with α. (ii) The rods are subjected to a finite elongation, whose projection on α (equal to α_0) yields the second part of the quadratic term.
- The *cubic term* results from a combination between two nonlinear effects: the finite elongation of the rods, and the projection on the varying angle α.

A parallel can be made between the latter rods system and an arch (resp. a shell), with its bending stiffness neglected, as the transverse external forces are in equilibrum with the internal membrane forces only. In this case, a finite transverse deflection leads to a stretching of the midline (resp. midsurface) of the arch (resp. shell). The angle α_0 of the rods system is comparable to the curvature of the arch/shell, which adds quadratic terms in the expression of the restoring force [7].

2.4 Influence of the thickness

To study the effect of the bending stiffness of the arches/shells, the system shown in Figure 1(b) is considered. Four supplementary rods are added to the first system. This set can be decomposed into two subsystems of three rods, clamped at one end and connected with revolute joints at the other. All three rods in each subsystem are assumed to remain parallel with each other, and are subjected to axial stresses only. The obtained structure represents a crude approximation of the thickness of an arch/shell, each rod being viewed as a particular fiber. After little algebra, the expression of the restoring force is found to be:

$$\begin{aligned} F_w = 2EA\Bigg\{ &\frac{w}{l_0}\underbrace{\left[3\sin^2\alpha_0 + 2\frac{h^2}{l_0^2}(1+\tan^2\alpha_0)\right]}_{\gamma_1} + \left(\frac{w}{l_0}\right)^2 \underbrace{\sin\alpha_0\left[\frac{9}{2} + 9\frac{h^2}{l_0^2}(1+\tan^2\alpha_0)\right]}_{\gamma_2} \\ &+ \left(\frac{w}{l_0}\right)^3 \underbrace{\left[\frac{3}{2} + 6\frac{h^2}{l_0^2}(1+\tan^2\alpha_0) + 2\frac{h^2}{l_0^2}(1+\tan^2\alpha_0)^2\right]}_{\gamma_3} \Bigg\} \end{aligned} \tag{6}$$

The addition of a "thickness" $2h$ is responsible of the linear stiffness, $4EA(h/l_0)^2$ of the flat system (with $\alpha_0 = 0$). It also adds various non-linear terms, which are due to similar sources: (i) finite elongation of *all* the fibers of the structure, and (ii) projection on the vertical axis, of the varying angle α.

A parallel can be made, here, between this latter rod system and a initially flat bar (or plate) with ($\alpha_0 = 0$), or an curved arch (or a shell) with ($\alpha_0 \neq 0$).

In conclusion, Eq. ((6)) shows that the restoring force in an arch or a shell results from two effects: (i) the *membrane force* (depending on the curvature, represented in our model by α_0 or e) and (ii) the *bending forces* (depending on the thickness, corresponding to h in the system). Their balance creates opposite non-linear effects which are now discussed.

3 NONLINEAR HARDENING OR SOFTENING EFFECTS

A general property of nonlinear oscillators is that, for large amplitudes of vibrations, the frequency of oscillations depends on the amplitude [8]. If the frequency increases with the amplitude of oscillations, the system is said to have a *hardening* behaviour; in the opposite case, it is said to be *softening*. In the case of certain gongs used in Chinese operas, a downward or upward pitch glide in the sound is observed, depending upon the balance between curvature and thickness effects [2]. The motion of the mass of the rods systems is now investigated, in order to shed new light on the behavior of these gongs.

3.1 Equation of motion

Applying the linear balance momentum to the material point M (Fig. 1(b)) leads to:

$$\vec{F}_{ext} + \vec{F}_w = m\ddot{w} \quad \text{and then to} \quad \ddot{w} + \omega_0^2 w + \beta w^2 + \Gamma w^3 = F_{ext}/m \tag{7}$$

where $\ddot{w}$ is the second derivative of w with respect to time and $\vec{F}_{ext} = F_{ext}\vec{y}$ denotes the external force applied to the mass. The linear natural frequency of the system, ω_0, and coefficients β and Γ are defined by:

$$\omega_0^2 = \frac{2EA}{ml_0}\,\gamma_1\,, \quad \beta = \frac{2EA}{ml_0^2}\,\gamma_2\,, \quad \Gamma = \frac{2EA}{ml_0^3}\,\gamma_3\,, \tag{8}$$

with γ_1, γ_2 and γ_3 defined by Eq. ((6)).

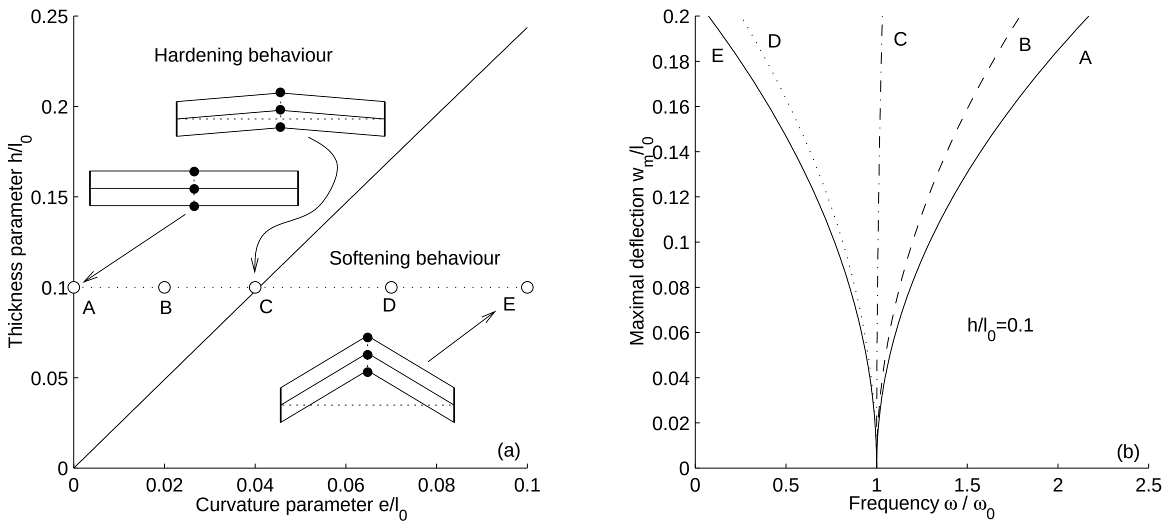

Figure 2: (a) – Behavior of the rods system as a function of h/l_0 and e/l_0. (b) – Theoretical backbone curves ($\tilde{\omega}$ as a function of w_m) of the five undeformed configurations of the system denoted by 'o' in (a), for a constant thickness parameter $h/l_0 = 0.1$ and an increasing curvature parameter e/l_0.

3.2 Multiple-scale solution

Eq. ((7)) is a conservative Duffing equation with a quadratic term added. The free response (with a zero F_{ext}) of this equation can be obtained in the weakly non-linear range (if the amplitude of oscillation, hereafter denoted as w_m, is small) by a perturbation method. Following the method of multiple scales [8], a first-order solution of Eq. ((7)) is:

$$w(t) = w_m\,\cos(\tilde{\omega}t + \varphi) \quad \text{with} \quad \tilde{\omega} = \omega_0\left[1 + \frac{9\Gamma\omega_0^2 - 10\beta^2}{24\omega_0^4}\,w_m^2\right]. \tag{9}$$

Thus, depending on the sign of $9\Gamma\omega_0^2 - 10\beta^2$, the frequency of oscillations increases as a function of w_m (hardening behaviour) or decreases (softening behaviour).

The effects of relative thickness (h/l_0) and curvature e/l_0 are now discussed. After little algebra, it is found that $9\Gamma\omega_0^2 - 10\beta^2$ is positive if:

$$\sin^2\alpha_0 \leq \frac{3\xi + 12\xi^2 + 4\xi^3}{18 + 72\xi + 84\xi^2} \quad \text{with} \quad \xi = \frac{h^2}{l_0^2}(1 + \tan\alpha_0). \tag{10}$$

The system is then assumed to be "thin", which means that $h/l_0 \ll 1$. Thus, $9\Gamma\omega_0^2 - 10\beta^2$ is positive if:

$$\frac{h}{l_0} \geq \sqrt{6}\frac{e}{l_0}\sqrt{1 - \left(\frac{e}{l_0}\right)^2}. \tag{11}$$

Eq. ((11)) shows that the surface h/l_0-e/l_0 is divided in two regions which are shown in Figure 2(a). Five backbone curves are plotted from equation (9) in Figure 2(b), for a constant thickness $h/l_0 = 0.1$ and five different values of e/l_0. Figure 2 shows that if the thickness h is larger than 2.5 times the angle parameter e, the behavior of the system is of the hardening type (upper region) whereas, in the opposite case, it is of the softening type (lower region). This result is found qualitatively similar to spherical shells, where h must be larger to 0.5 times e to obtain a hardening behaviour for the first axisymmetric mode of vibration [9]. When a flat structure (e.g a plate) is considered, for which e vanishes, the behaviour is always of the hardening type [10, 7, 11].

In conclusion, this study shows that the softening/hardening behaviour of our rod-system depends on the balance between its "thickness" h and its "curvature" e. In a shell, the curvature, which is responsible for membrane stiffness, yields a softening behaviour. On the opposite, the thickness, responsible for a bending stiffness, yields an hardening behaviour.

3.3 Experiments on a chinese tam-tam

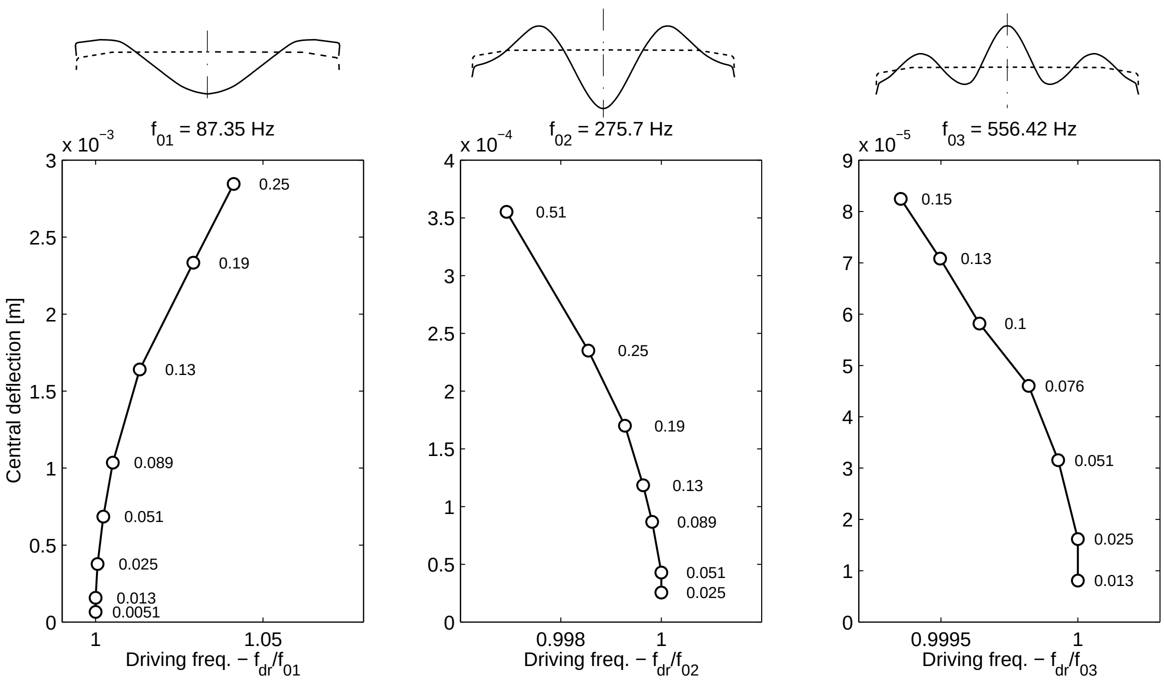

Figure 3: Experimental backbone curves of the lowest three axisymmetric modes of a Chinese tam-tam, and cross-sections of the corresponding modal shapes (the undeformed tam-tam is shown dashed). The measured amplitude of the corresponding harmonic driving force (in N) is specified for each experimental point.

Since our rod system has only one DOF., it can describe only one mode of vibration of a complex structure. To illustrate the theory presented in the previous paragraph, experiments have been made in order to identify the behaviour of the axisymmetric modes of a Chinese tam-tam, whose cross-section is shown dashed in the upper sketches of Figure 3. Backbone curves for the lowest three axisymmetric modes of are presented in Figure 3. The experimental method, involving harmonic central excitation of the gong, is the same than the one used for circular plates in [11]. A photograph of the set-up is shown in [12].

Figure 3 shows that the modes of the gong have different behaviour: some of them exhibits hardening behaviour, whereas others are softening. A numerical modal analysis of our gong (with the finite element method, implemented in CASTEM 2000[13]), validated by the experiments, shows that the axisymmetric mode of lower frequency $f_{01} = 87.35$ Hz involves only the central circular section of the upper surface, and is very similar to the first axisymmetric mode of a clamped circular plate (see the upper sketches of Figure 3 and [12]). This is a result of the stiffening effect of the conical shell and of the ring surrounding the structure (A similar behaviour is noticed in [2]). A corresponding rod system should have a thickness h larger than the curvature parameter e, and thus a hardening behaviour. On the contrary, the axisymmetric modes of larger frequencies ($f_{02} = 275.70$ Hz, $f_{03} = 556.42$ Hz) involve in their oscillations all the area of the upper circular shell, flat at the center, and conical near the rim. Their softening behaviour is then probably caused by the curvature introduced by the conical part. A rod system with a similar behaviour should have a thickness h smaller than its curvature parameter e, and would be of the softening type.

If second order effects are taken into account in the multiple scales developments of equation (9), it can be shown that a given system of the softening type for small w_m, becomes of the hardening type for larger w_m. This particular behaviour has been already pointed out theoretically in [10, 7, 2] for spherical shells. Nevertheless, the curves of figure 3 validate the first order model, as they have been measured for a large range of deflections of the gong: the maximal amplitude shown on figure 3(a) is 3 mm, which is rather large for our gong of thickness 1 mm at the center.

4 CONCLUSION

The simple rods systems studied in this work enables us to explain the behaviour of a single mode vibration of a gong. The main cause of the nonlinearities results from nonlinear stretching of the midplane. Two main geometrical characteristics of the structure have an influence on the behaviour. First, the thickness is responsible for the bending stiffness of the structure, and yields hardening behaviour. Secondly, the curvature introduces membrane stiffness, leading to a softening behaviour. The balance between these two effects fixes the exact behaviour of the system. In particular, it explains why curved Chinese gongs exhibits upward pitch glides, whereas gongs with a flatter upper surface show a downward pitch glide. It confirms the results of Fletcher [2].

The same arguments applied to the modal shapes of the modes of a Chinese tam-tam explain why some of the modes have a hardening behaviour, whereas others are of the softening type. This qualitative interpretation should be confirmed by a quantitative numerical study on shells of non-trivial cross-section, like our Chinese tam-tam.

The simple model proposed here can be extended to take into account more degrees of freedom. This should be useful to explain quantitatively the coupling and the exchanges of energy between the modes, a phenomenon which is particularly relevant in order to understand the transition to chaotic vibrations [3, 4, 12].

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