Investigation of Quasi-Periodic Solutions in Nonlinear Oscillators Featuring Internal Resonance

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Abstract. Quasi-Periodic solutions can arise in assemblies of nonlinear oscillators as a consequence of Neimark-Sacker (NS) bifurcations. In this work we investigate analytically and numerically the system of two coupled oscillators in two different settings featuring 1:2 and 1:3 internal resonance, respectively. More specifically, in the former case, the locus of NS points is obtained analytically and its variation with respect to the system parameters is highlighted. In the latter case, on the contrary, the NS boundary curve is investigated numerically. In both cases the results allow predicting the appearance of quasi-periodic solutions.

Keywords: Quasi-periodic regime, frequency combs, internal resonances, stability, nonlinear vibrations, multiple scales method, Neimark-Sacker bifurcations

1 Motivation

Nonlinear dynamical phenomena in coupled oscillators are connected with the emergence of complex features that have no counterpart in linear theory. In this context, one can cite for example jump phenomena, hysteretic behaviours, quasi-periodic solutions and chaotic vibrations [1–3]. In recent years, a number of investigations highlighting the occurrence of so-called *Frequency Combs* (FCs) in the observed dynamical solutions have been reported in the literature on Micro Electro Mechanical Systems (MEMS). These dynamical behaviours are characterized by the appearance of numerous frequency peaks in the Fourier transform of the vibration data, organized with repeating patterns and constant interspace between the spectral peaks (see Fig. 1). For instance, FCs have important applications in optics and can be used for measurement purposes, see *e.g.* [4, 5].

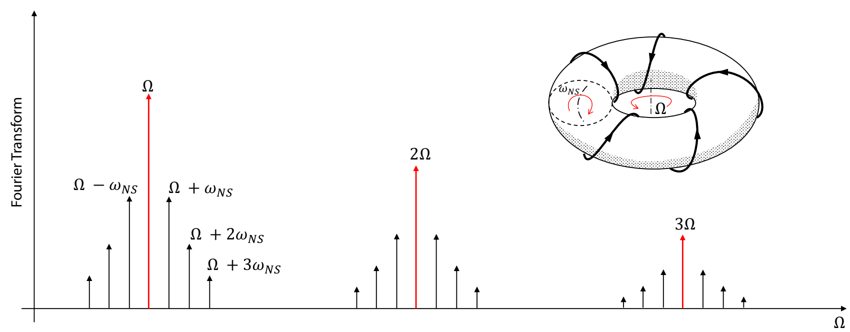

Fig. 1. Schematic representation of a Frequency Comb induced by a Neimarck-Sacker bifurcation. The bifurcation introduces in the system response a new incommensurate frequency ω_{NS} such that the Fourier transform of the system shows peaks at the forced frequency Ω multiples and at each combination $m\Omega \pm n\omega_{NS}$ with m and n integers. This corresponds to a motion of the orbits on a torus in the phase space (qualitative representation in the top right corner).

In MEMS devices, FCs have been experimentally reported *e.g* in [6,7] where a resonator with strong nonlinear coupling between bending and torsional modes is addressed. Other examples are provided in [8], where a tunable FC is considered, or in [9] where a 1:2 internal resonance in a MEMS arch resonator is investigated. Another option to achieve FC is through contact phenomena inducing period-doubling bifurcations, as in [10].

Through the analysis of the experiments available in the literature, a direct relationship between a FC and a Quasi-Periodic (QP) regime can be argued. When a FC is established, the response is characterized by an amplitude modulation in time and an incommensurate frequency ω_{NS} appears along with the driving frequency Ω. Due to nonlinearities, ω_{NS} and Ω combine according to the rule $m\Omega \pm n\omega_{NS}$ with m and n integers and each combination corresponds to a peak in the Fourier transform, see Fig. 1.

This work focuses on nonlinear oscillators featuring 1:2 and 1:3 internal resonances (IR) and on the appearance of QP solutions as the consequence of Neimark-Sacker bifurcations. In this case the emergence of QP regimes and FCs can be easily predicted if the locus of Neimark-Sacker bifurcation points is available [11,12]. Numerical evaluation of the boundary curves of specific bifurcation points, as parameters are varied, is currently an active research topic in the literature. For example, numerical continuation methods can be adapted with few additional constraints to follow the locus of a specific bifurcation point in the parameter space, see *e.g.* [13] for recent examples in nonlinear oscillations.

Even though NS bifurcation points have been identified in these systems since a long time, see *e.g.* [14, 15] and references therein, new results are reported in this document. An analytical expression of the Neimark-Sacker boundary curve is provided for the 1:2 IR case and a numerical continuation method is developed for the systematic computation of the NS boundary curve. The latter method is also applied to the case of a 1:3 IR.

The paper is organized as follows. Section 2 presents the reference nonlinear systems considered in the investigation while Section 3 derives the analytical expression of the Neimark-Sacker boundary for the case of the 1:2 IR. Then in Section 4 the comparison between the analytical and numerical solutions in the systems considered is discussed. Finally, in Section 5 we summarize the outcomes of the work.

2 Reference Systems

This study considers a system of two coupled nonlinear oscillators featuring 1:2 and 1:3 IR. The former one is described through its second order normal form [1] while the latter considers a simplified system. The equations of motion of the former system read:

$$\ddot{q}_1 + \omega_1^2 q_1 + 2\mu_1 \dot{q}_1 + \alpha_{12}^{(1)} q_1 q_2 = F\cos\left(\Omega t\right), \tag{1a}$$

$$\ddot{q}_2 + \omega_2^2 q_2 + 2\mu_2 \dot{q}_2 + \alpha_{11}^{(2)} q_1^2 = 0, \tag{1b}$$

in which $q_i, (i = 1, 2)$ is the displacement of the i-th oscillator and ω_i are the two eigenfrequencies, hereafter assumed to fulfill the relation $\omega_2 \approx 2\omega_1$. Two quadratic coefficients $\alpha_{12}^{(1)}$ and $\alpha_{11}^{(2)}$ are introduced. In the mechanical context with conservative loads, one has $\alpha_{11}^{(2)} = \alpha_{12}^{(1)}/2$. The linear damping coefficients are μ_i, $i = 1, 2$, and an external harmonic forcing with angular frequency Ω and amplitude F is applied to the first oscillator.

For the 1:3 IR, the focus in this contribution is put on a system which is simplified as compared to the complete normal form for a 1:3 internal resonance, following the example selected and analyzed in [6, 7]. The equations of motion read:

$$\ddot{q}_1 + \omega_1^2 q_1 + \mu_1 \dot{q}_1 + \beta_{111}^{(1)} q_1^3 + \beta_{112}^{(1)} q_1^2 q_2 = F\cos\left(\Omega t\right), \tag{2a}$$

$$\ddot{q}_2 + \omega_2^2 q_2 + \mu_2 \dot{q}_2 + \beta_{111}^{(2)} q_1^3 = 0. \tag{2b}$$

In this system, $\beta_{ik}^{(j)}$ are the nonlinear coupling coefficients. Since we are interested in the 1:3 IR, the eigenfrequencies are such that $\omega_2 \approx 3\omega_1$. The monomial terms that are not herein considered as compared to the normal form of the system, are the following: $q_1 q_2^2$ on the first oscillator; q_2^3 and $q_2 q_1^2$ on the second oscillator. Also in this case, if we consider a mechanical context with conservative loads one has $\beta_{111}^{(2)} = \beta_{112}^{(1)}/3$.

3 Neimark-Sacker Analytical Boundary for 1:2 Resonance

This section investigates the frequency response functions (FRF) of the oscillators described by Eq. (1). The appearance of QP solutions was documented *e.g.*

in [14, 16], although the expression of the Neimark-Sacker boundary curve was not detailed. Following the Multiple Scales (MS) method [17, 18], we introduce two different time scales $T_0 = t$ and $T_1 = \varepsilon t$, with ε a small bookkeeping parameter. A linear expansion of the system response as $q_i = q_{i0} + \varepsilon q_{i1}$ is assumed, while the damping is modelled as $\mu_i = \varepsilon \xi_i$, the forcing term as $F = \varepsilon f$ and finally the nonlinear coefficients as $\alpha_{12}^{(1)} = \varepsilon \bar{\alpha}_{12}^{(1)}$, $\alpha_{11}^{(2)} = \varepsilon \bar{\alpha}_{11}^{(2)}$. Under these assumptions we rewrite (1) as :

$$\ddot{q}_1 + \omega_1^2 q_1 = \varepsilon[-2\xi_1 \dot{q}_1 - \bar{\alpha}_{12}^{(1)} q_1 q_2 + f \cos(\Omega t)], \tag{3a}$$

$$\ddot{q}_2 + \omega_2^2 q_2 = \varepsilon[-2\xi_2 \dot{q}_2 - \bar{\alpha}_{11}^{(2)} q_1^2]. \tag{3b}$$

The forcing frequency Ω is close to the first eigenfrequency ω_1 ($\omega_2 \approx 2\omega_1$), and detuning parameters σ_1 and σ_2 are introduced to quantify the mismatches as $\omega_2 = 2\omega_1 + \varepsilon\sigma_1$ and $\Omega = \omega_1 + \varepsilon\sigma_2$. Following a classical path, the solution of the first order system can be expressed in the form $q_{i0} = A_i(T_1)\exp(\mathbb{i}\omega_i T_0) +$ *c.c.*, with $A_i(T_1) = a_i(T_1)/2\exp(\mathbb{i}\theta_i(T_1))$. When q_{i0} is inserted in the second order system, secular terms arise. Forcing them to vanish (solvability conditions), a set of four differential equation is obtained. The resulting system is made autonomous by introducing the angular variables:

$$\gamma_1 = \theta_1 - \sigma_2 T_1, \quad \gamma_2 = 2\theta_1 - \theta_2 - \sigma_1 T_1, \tag{4}$$

The dynamics of the first order solution a_i and γ_i with respect to slow time scale T_1 is governed by:

$$a_1' = \frac{a_1\left(\bar{\alpha}_{12}^{(1)} a_2 \sin(\gamma_2) - 4\xi_1\omega_1\right) - 2f\sin(\gamma_1)}{4\omega_1}, \tag{5a}$$

$$\gamma_1' = \frac{\bar{\alpha}_{12}^{(1)} a_1 a_2 \cos(\gamma_2) - 2f\cos(\gamma_1)}{4a_1\omega_1} - \sigma_2, \tag{5b}$$

$$a_2' = -\frac{\bar{\alpha}_{11}^{(2)} a_1^2 \sin(\gamma_2) + 4\xi_2 a_2\omega_2}{4\omega_2}, \tag{5c}$$

$$\gamma_2' = -\frac{f\cos(\gamma_1)}{a_1\omega_1} - \sigma_1 + \frac{1}{2}\cos(\gamma_2)\left(\frac{\bar{\alpha}_{12}^{(1)} a_2}{\omega_1} - \frac{\bar{\alpha}_{11}^{(2)} a_1^2}{2a_2\omega_2}\right). \tag{5d}$$

The fixed points solutions, associated to forced oscillations of constant amplitudes, can be expressed as function of a_1 and a_2 only. The amplitude equations, employing the physical parameters of Eq. (2), finally read:

$$\begin{aligned} &a_2^3 + \frac{8\omega_1\Gamma}{\alpha_{12}^{(1)}}((\Omega-\omega_1)(\omega_2-2\Omega)+\mu_1\mu_2)a_2^2 + \frac{16\omega_1^2}{\alpha_{12}^{(1)2}}((\Omega-\omega_1)^2+\mu_1^2)a_2 \\ &-\frac{\Gamma\alpha_{11}^{(2)}F^2}{\alpha_{12}^{(1)2}\omega_2} = 0, \quad a_1 = \sqrt{\frac{4\omega_2 a_2}{\alpha_{11}^{(2)}\Gamma}}, \end{aligned} \tag{6}$$

where $\Gamma = \sqrt{\frac{1}{\mu_2^2 + (\omega_2 - 2\Omega)^2}}$.

We now focus on the definition of the Neimark-Sacker (NS) boundary curve. The stability of the solutions depends on the eigenvalues λ of the Jacobian matrix of system (5). The NS bifurcation requires that a pair of complex conjugate eigenvalues crosses the imaginary axis. [19]. Imposing this constraint in the characteristic polynomial of Eqs. (5), one can express the NS boundary as a polynomial in a_2:

$$b_1 a_2^4 + b_2 a_2^3 + b_3 a_2^2 + b_4 a_2 + b_5 = 0. \tag{7}$$

where the expression of each coefficient is reported in appendix A. For fixed values of the system parameters μ_1, μ_2, $\alpha_{12}^{(1)}$, ω_2, ω_1 and spanning the values of Ω one gets the boundary curve for the NS bifurcation as a function $a_2(\Omega)$.

4 Neimark-Sacker Boundary Curve: Numerical Results

In this section the NS boundaries associated with the system of eqs. (1) and (2) are investigated numerically and, in the former case, compared to the analytical prediction developed in the previous section.

The numerical continuation procedure has been implemented in the software MANLAB [20, 21]. The procedure is based on the Standard Augmented system method here briefly introduced, using the general dynamical system notations used in [22]. Let $\mathbf{x}(t)$ be a n-dimensional vector and $\mathbf{f}$ the nonlinear smooth mapping defining the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \boldsymbol{\alpha}), \tag{8}$$

where $\boldsymbol{\alpha}$ is the parameter vector of the system. Let us assume that, for a given value of $\boldsymbol{\alpha}$, a period-1 limit cycle emerges from a NS bifurcation, such that $\mathbf{x}(0) = \mathbf{x}(1)$ (phase condition).

The goal is to follow the locus of NS bifurcation points as $\boldsymbol{\alpha}$ is varied. To this purpose we parametrize the critical multipliers $\lambda_{1,2}$ by means of a scalar variable θ as: $\lambda_{1,2} = e^{\pm i\theta}$. Let us introduce a complex eigenfunction $\mathbf{w}(t)$ of system (8). The system that allows the continuation of NS boundary consists of Eqs. (8) along with his phase condition [22]:

$$\dot{\mathbf{w}} - \mathbf{f}_{\mathbf{x}}(\mathbf{x}(t), \boldsymbol{\alpha})\mathbf{w}(t) = \mathbf{0}, \tag{9a}$$

$$\mathbf{w}(1) - e^{i\theta}\mathbf{w}(0) = \mathbf{0}, \tag{9b}$$

$$\int_0^1 \mathbf{w}^T(t)\mathbf{w}_0(t)\mathrm{d}t \; - 1 \; = 0. \tag{9c}$$

where $\mathbf{w}_0(t)$ is a complex-valued reference eigenfunction at a nearby point on the solution branch and $\mathbf{f}_{\mathbf{x}}$ is the Jacobian of Eq. (8). Eqs. (9) is solved with MANLAB, that implements the Harmonic Balance Method for the computation of periodic solutions and applies Hill's method to evaluate the stability of the system (see [23]). This approach was demonstrated to be extremely efficient and

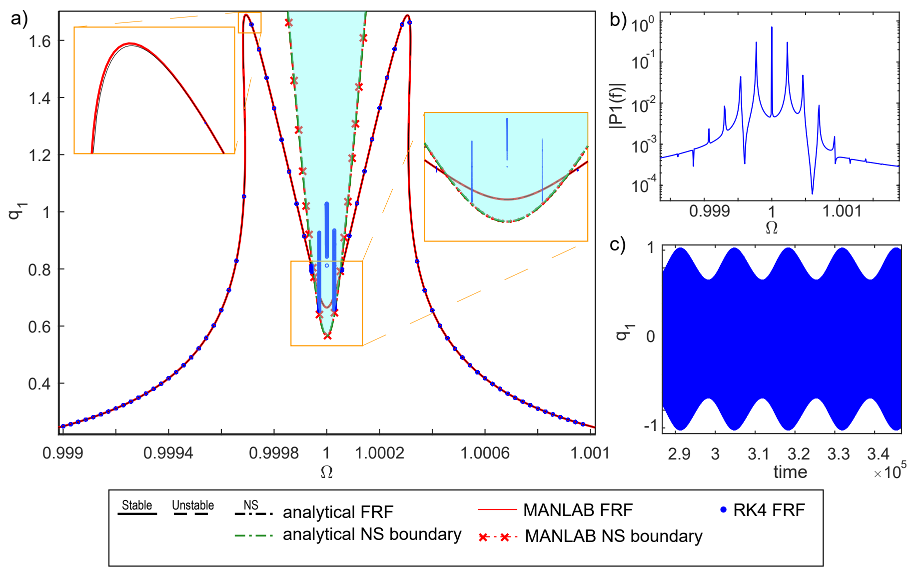

Fig. 2. q_1 as function of the excitation frequency Ω for system (1) with 1:2 IR. Selected parameters are: $\alpha_{12}^{(1)} = 2 \cdot 10^{-3}$, $\alpha_{11}^{(2)} = 1 \cdot 10^{-3}$, $\mu_1 = \mu_2 = 1 \cdot 10^{-4}$, $F = 5 \cdot 10^{-4}$ and $\sigma_1 = 0$. In Fig. a) black lines represent the FRF of the multiple scale system from Eq. (6) (a continuous line denotes a stable branch, dashed unstable, dash dotted marks the Quasi-Periodic regime). The red continuous line represents the FRF of the system Eq. (1) computed though numerical continuation in MANLAB (10 harmonics). The red dot dashed line is the NS boundary obtained with numerical continuation methods (10 harmonics). The dark green dashed line is the analytical NS boundary. The light blue filled region highlights where a QP regime is expected. The blue circle markers represent the RK4 direct time-marching solution (Poincaré sections). A cloud of points arise from the QP regime. Two enlarged view close to the peak and to the NS region allow appreciating the small differences between the solutions. In Fig. b) and c) results for the value $\Omega = 1$ are reported. Fig. b) is the Fourier transform of the time history represented in Fig. c). The Frequency comb and the amplitude modulation are clearly visible

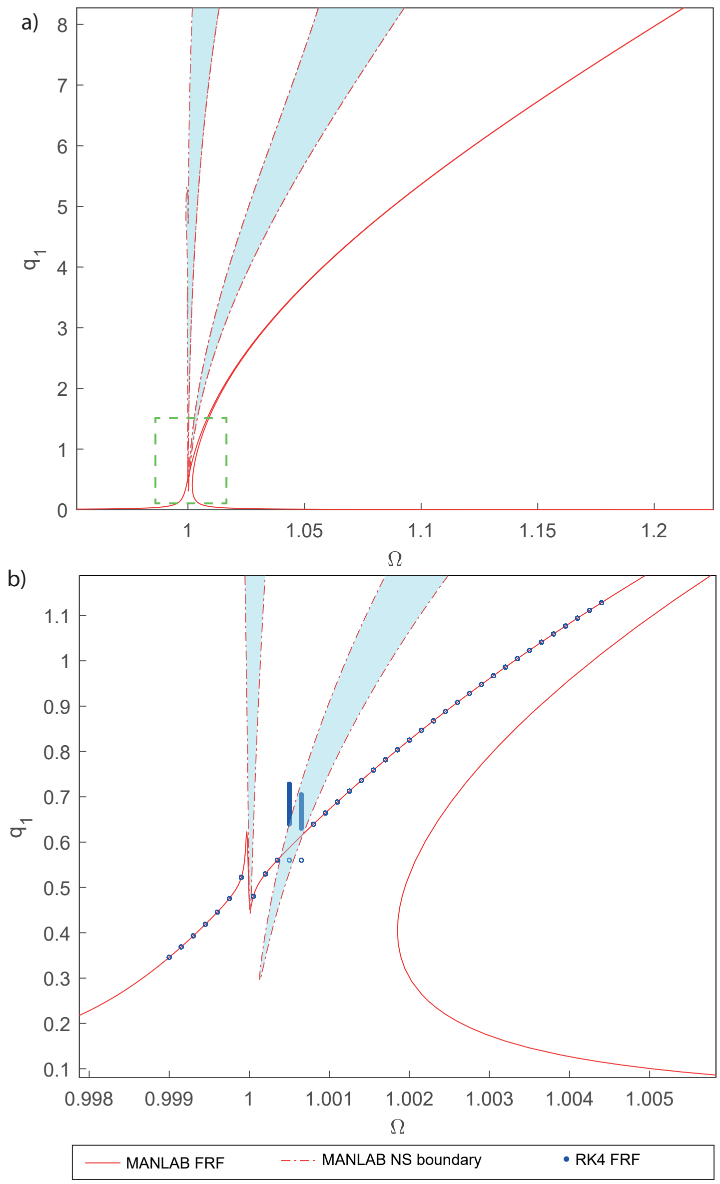

Fig. 3. q_1 as function of the excitation frequency Ω for system (2) obtained with the continuation method (10 harmonics). Selected parameters: $\beta_{111}^{(1)} = 1 \cdot 10^{-2}$, $\beta_{112}^{(1)} = 2 \cdot 10^{-2}$, $\beta_{111}^{(2)} = 6.66 \cdot 10^{-3}$, $\mu_1 = \mu_2 = 1 \cdot 10^{-4}$, $F = 1 \cdot 10^{-3}$ and $\omega_2 = 3\omega_1$. (a) complete FRF, (b) enlarged view close to the NS bifurcations. The red lines represent the FRF of the system. The red dash dotted lines are the NS boundaries achieved with numerical continuation methods. The light blue colouring highlights the regions in the parameter space enveloped in the NS boundary. The blue circle markers represent the RK4 direct time-marching solution (Poincaré sections). A cloud of points arises from the QP regime.

reliable for the analysis of nonlinear systems. To this aim the augmented system (9) must be converted in the frequency domain using a Floquet-Hill formulation.

Fig. 2 shows the results obtained in the case of the 1:2 IR, where the analytical prediction given by Eq. (6) and Eq. (7) are compared to the numerical one obtained by continuation of Eq. (1) with MANLAB and direct time-marching Runge-Kutta 4 (RK4) method. The three approaches provide the same result in the periodic regions and only small differences can be observed close to the resonance peaks between the analytical and continuation approaches. The NS boundary encloses a connected domain centred on the resonance frequency ω_1. Here the time-marching approach reveals the Quasi-Periodic regime and the FC. A nearly perfect match between analytical and numerical NS boundaries is found, thus validating the overall procedure.

For the 1:3 IR case only the numerical solution of Eqs. (2) is reported in Fig. 3. One can observe a more complex pattern with two different tongues of instability regions. More specifically, the NS boundary crosses the frequency response function in two different portions thus creating two narrow regions with QP solutions, one centred on ω_1, the other being shifted to higher frequencies. The local existence of two different regions is a noteworthy difference with respect to the 1:2 IR case. Moreover, it is worth stressing that capturing the narrow and close instability tongues is quite difficult with standard continuation methods. In the proposed example we also show results obtained with a RK4 direct integration close to the unstable regions. The two results have a perfect agreement in the periodic regions and, as expected, the RK4 integration detects the QP regime within the NS boundary.

5 Conclusion

In the present paper we have obtained an analytical expression of the Neimark-Sacker boundary for a system of two coupled oscillators with 1:2 internal resonance. The analytical formula has been validated with an ad-hoc developed numerical continuation of bifurcation points. The numerical method has also been used to compute the NS boundaries for two oscillators featuring 1:3 internal resonance showing that two distinct instability regions exist. These regions have the shape of narrow tongues and are difficult to detect only with direct integration schemes and other standard numerical tools.

A Coefficient of NS Boundary Polynomial

$$b_1 = \frac{\alpha_{12}^{(1)4}\mu_1\mu_2}{64\omega_1^4} \tag{A.1a}$$

$$b_2 = -\frac{\alpha_{12}^{(1)3}\left(\mu_1+\mu_2\right)^2\sqrt{\mu_2^2+\left(\omega_2-2\Omega\right)^2}}{8\omega_1^3} \tag{A.1b}$$

$$b_3 = -\frac{\alpha_{12}^{(1)2}}{2\omega_1^2}\mu_1\mu_2\cdot\left(\mu_1^2+\mu_2^2+2\mu_1\mu_2-2\omega_1\Omega+4\omega_2\Omega+\omega_1^2-\omega_2^2-3\Omega^2\right) \tag{A.1c}$$

$$b_4 = \frac{2\alpha_{12}^{(1)}}{\omega_1}\left(\mu_1+\mu_2\right)^2\sqrt{\mu_2^2+\left(\omega_2-2\Omega\right)^2}\cdot\left(\mu_1^2+2\mu_2\mu_1+\mu_2^2+\left(\omega_1+\omega_2-3\Omega\right)^2\right) \tag{A.1d}$$

$$b_5 = 4\Bigg(\bigg(\mu_1\left(\mu_2^2+\left(\omega_2-2\Omega\right)^2\right)+\mu_2\left(\Omega-\omega_1\right)^2+\mu_2\mu_1^2\bigg)\cdot\bigg(\mu_1\left(4\mu_2^2+\left(\Omega-\omega_1\right)^2\right) + \mu_2\left(\mu_2^2+\left(\omega_2-2\Omega\right)^2\right)+\mu_1^3+4\mu_2\mu_1^2\bigg)-\left(\mu_1+\mu_2\right)^2\left(\mu_1^2+\left(\Omega-\omega_1\right)^2\right) \left(\mu_2^2+\left(\omega_2-2\Omega\right)^2\right)\Bigg) \tag{A.1e}$$

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