# Periodic solutions with Sticking phase of a vibro-impact system

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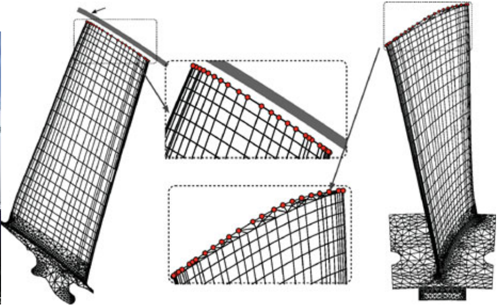
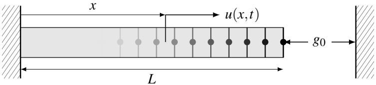
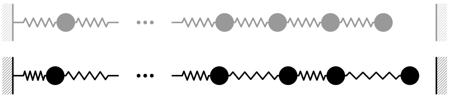
# Motivation : Mechanical engineering



- A turbomachinery compressor blade in contact with a rigid casing. Denis Laxalde ; Mathias Legrand (2012)
- A simplified continuous model : a thin rod in the unilateral contact with a foundation. G. Lebeau -M. Schatzman (1984)



# A discrete model : $N$ degree-of-freedom system



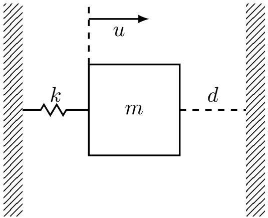

$$
\begin{cases}
\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r}, & (1a) \\
\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 & (1b) \\
u_N(t) \leq d,\ d > 0 & (1c) \\
R(t) \leq 0,\ (u_N(t) - d)\,R(t) = 0, \forall t \geq 0 & (1d)
\end{cases}
$$

where $\mathbf{u}(t) = [u_1(t), \cdots, u_N(t)]^\top$, $\mathbf{r}(t) = [0, \cdots, 0, R(t)]^\top$.
When $u_N(t) = d$, the reflection law is

$$\dot{u}_N^+(t) = -e\dot{u}_N^-(t) \text{ with } e = 1.$$

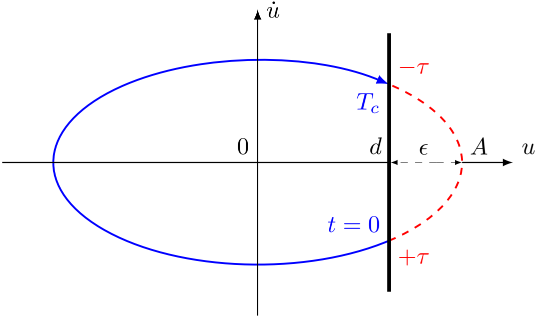
and conserved energy $\mathbf{v}^\top\mathbf{M}\mathbf{v} + \mathbf{u}^\top\mathbf{K}\mathbf{u} = \mathbf{E}(t) = \mathbf{E}(0), \qquad \mathbf{v} = \dot{\mathbf{u}} \quad (= \dot{\mathbf{u}}^\pm)$

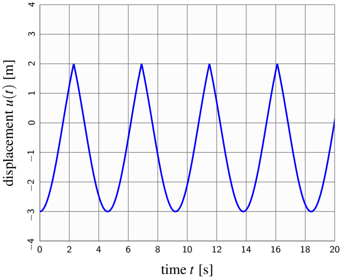
# One degree-of-freedom system



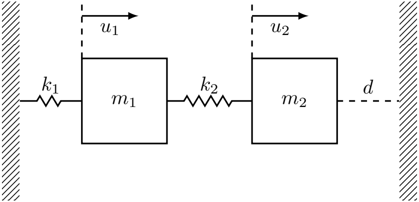

$$
\begin{cases}
m\ddot{u} + ku = R(t) & (2a) \\
u(t) \leq d & (2b) \\
R(t) \leq 0, (u(t) - d)R(t) = 0 & (2c) \\
u(t) = d \Rightarrow \dot{u}^{+}(t) = -\dot{u}^{-}(t) & (2d)
\end{cases}
$$

# One degree-of-freedom system



# Two degrees-of-freedom system




$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \begin{bmatrix} 0 \\ R(t) \end{bmatrix} & \text{(3a)} \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 & \text{(3b)} \\ {\color{red} u_2(t) \leq d}, \ d > 0 & \text{(3c)} \\ R(t) \leq 0, \quad (u_2(t) - d)\, R(t) = 0, \forall t & \text{(3d)} \\ {\color{blue} \dot{\mathbf{u}}^\top \mathbf{M}\dot{\mathbf{u}} + \mathbf{u}^\top \mathbf{K}\mathbf{u} = \mathbf{E}(t) = \mathbf{E}(0)}, & \text{(3e)} \end{cases}$$

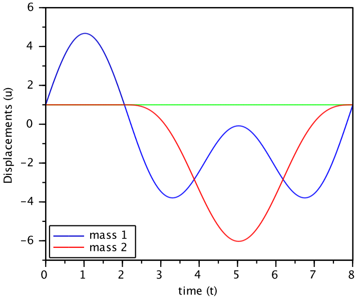
# Sticking phase

## Definition

When $u_2(0) = d$, there is a sticking phase if there exists $\mathcal{T} > 0$ such that

$$u_2(t) = d, \text{ for all } t \in [0, \mathcal{T}]$$

**Goal** : The periodic solutions with one Sticking Phase per Period (1SPP).



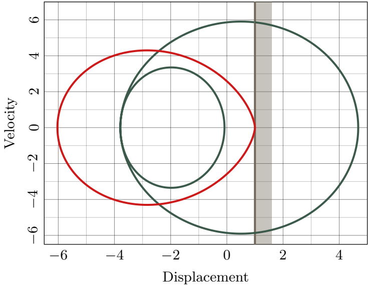
# The occurrence of sticking phase

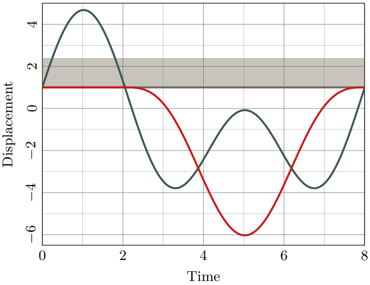
## Proposition (for a proof : Heng, Junca, 2013)

*There exists a sticking phase if and only if :*

1. $u_2(0) = d, \quad \dot{u}_2^-(0) = 0, \quad u_1(0) > d$, *or*
2. $u_2(0) = d, \quad \dot{u}_2^-(0) = 0, \quad u_1(0) = d, \quad \dot{u}_1(0) > 0.$

- Initial data : $u_2(0) = d, \quad \dot{u}_2^-(0) = 0, \quad u_1(0) = d, \quad \dot{u}_1(0) = v.$
- Data at $\mathcal{T}$ : $u_2(\mathcal{T}) = d, \quad \dot{u}_2^-(\mathcal{T}) = 0, \quad u_1(\mathcal{T}) = d, \quad \dot{u}_1(\mathcal{T}) = -v.$



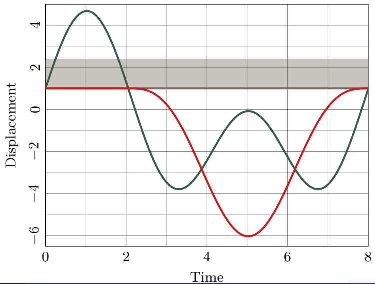



# Looking for periodic solution with 1 SPP

## Definition (1SPP)

A periodic function $\mathbf{u}(t)$ is called a 1SPP : a $T$-periodic solution with one sticking phase per period of the system (3) if there exists $0 < \mathcal{T} < T$ such that (up to a time translation)

1. $u_2 = d$ on $[0, \mathcal{T}]$,
2. $u_2 < d$ on $]\mathcal{T}, T[$
3. $\mathbf{u}(T) = \mathbf{u}(0)$ and $\dot{\mathbf{u}}^-(T) = \dot{\mathbf{u}}^-(0)$.



# Possible initial data and period of solution with 1SPP

Assume that $\mathbf{u}(t)$ is the periodic solution with one sticking phase per period (1SPP) of the system (3) then :

1. the duration of the free flight $s = T - \mathcal{T} > 0$ is necessarily a root of

$$h(s) = \sum_{j=1}^{2} \alpha_j \cot\left(\frac{\omega_j s}{2}\right) = 0, \tag{4}$$

2. $\mathbf{u}$ is the solution associated with the initial data

$$[u_1(0), u_2(0), \dot{u}_1(0), \dot{u}_2(0)] = [d, d, v(s), 0], \tag{5}$$

$$v(s) = \frac{d}{w_1(s)}. \tag{6}$$

3. the period $T$ of $\mathbf{u}$ is a function of $s$ :

$$T(s) = s + \mathcal{T}(v(s)) = s + \frac{2}{\omega} \arctan\left(\xi \frac{d}{w_1(s)}\right). \tag{7}$$

$$\boxed{\{s > 0,\ h(s) = 0\}}$$

$$\downarrow$$

$$\boxed{v(s) = \frac{d}{w_1(s)}}$$

$$\Big\downarrow \text{ Check if } v > 0$$

$$\boxed{\mathcal{T}(v(s)) = \frac{2}{\omega} \arctan\left(\xi v(s)\right)}$$

$$\downarrow$$

$$\boxed{T(s) = s + \mathcal{T}(v(s))}$$

$$\downarrow$$

$$\boxed{\mathbf{U}_0 = [u_{10}, u_{20}, \dot{u}_{10}, \dot{u}_{20}]^\top = [d, d, v(s), 0]^\top}$$

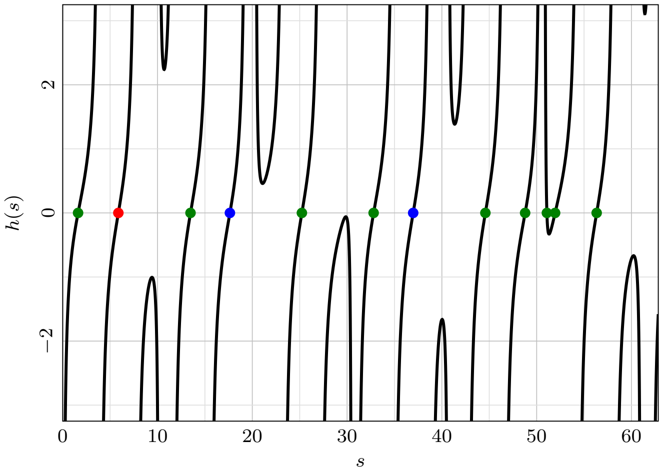
$$\downarrow$$

$$\boxed{\mathbf{U}(t, \mathbf{U}_0, T)}$$

$$\Big\downarrow \text{ Check if } u_2 < d \text{ on } ]\mathcal{T}, T[$$

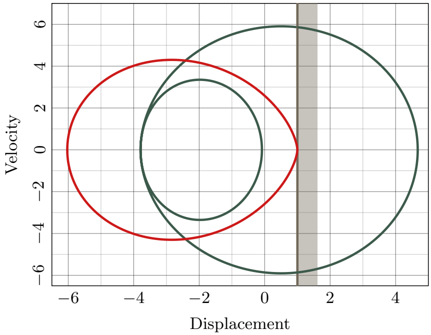
**Admissible 1SPP**

# Simulation of the set of 1SPP

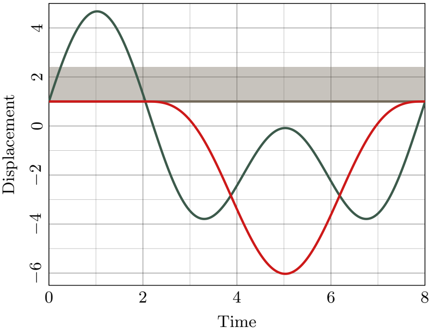



$Z = \{s > 0,\ h(s) = 0\} =$ set of possible free flight time
$Z^+ = \{s \in Z,\ w_1(s) > 0\} =$ set of all the $s$ s.t. $v(s) > 0$,
$Z^{ad} =$ set of $s$ s.t. the solution with 1SPP is admissible

- Orbits and displacements of two masses when
  $m_1 = m_2 = 1, k_1 = k_2 = 1, d = 1.$
  $s \approx 5.95$, $\mathcal{T} \approx 2.05$, $v \approx 5.86$, $T = \mathcal{T} + s \approx 8.00$.

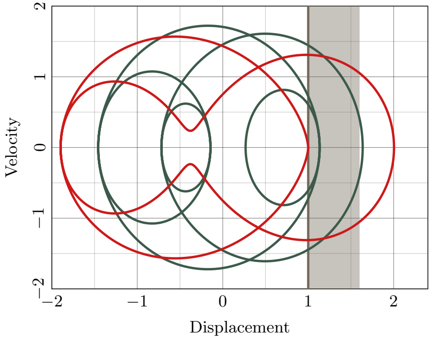



Orbit $(\mathbf{u}, \dot{\mathbf{u}})$

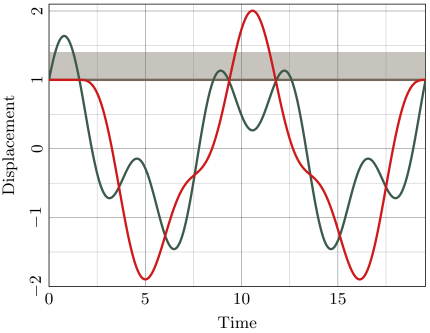



Displacements $\mathbf{u}$ w.r.t time $t$

- Orbits and displacements of two masses when $s \approx 17.97$.

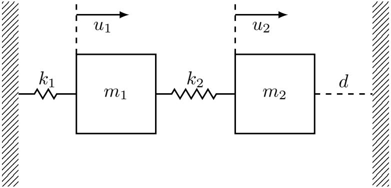



Orbit $(\mathbf{u}, \dot{\mathbf{u}})$




Displacements $\mathbf{u}$ w.r.t time $t$

# The case $d \leq 0$



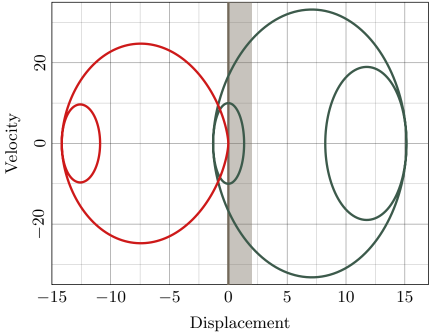

- Sticking phase with finite duration :
  - $d = 0$ : Need the condition that $\dfrac{\omega_1}{\omega_2} \in \mathbb{Q}$.
  - $d < 0$ : The set of admissible 1SPP is found from

$$Z^- = \{s \in Z,\ w_1(s) < 0\}$$

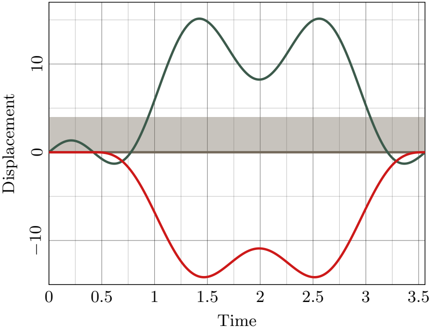
- Sticking phases with infinite duration appear as a special case when

$$\dot{u}_1(0) = 0.$$

# 1SPP with finite sticking phase for $d = 0$ and $v > 0$ arbitrary




Orbit $(\mathbf{u}, \dot{\mathbf{u}})$




Displacements $\mathbf{u}$ w.r.t time $t$

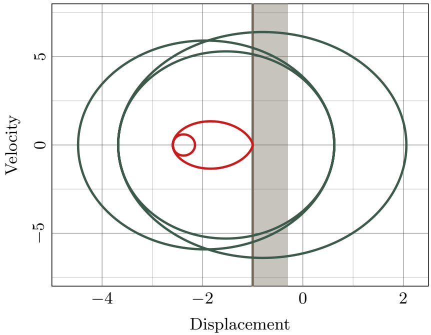
When the ratio

$$\frac{\omega_1}{\omega_2} = \frac{1}{5}$$

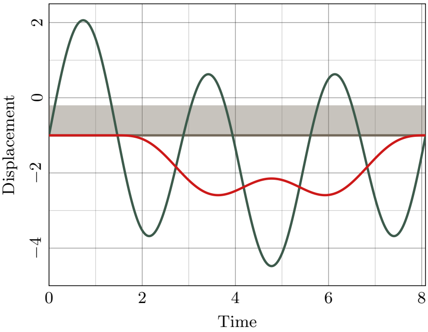
# 1 SPP with finite sticking phase for $d < 0$ and $v > 0$

$$m_1 = 1.0; m_2 = 6.0; k_1 = 1.0; k_2 = 4.0; d = -1.0$$

$$s \approx 6.61;\ v \approx 6.38;\ \mathcal{T} \approx 1.46;\ T \approx 8.08$$




Orbit $(\mathbf{u}, \dot{\mathbf{u}})$




Displacements $\mathbf{u}$ w.r.t time $t$

# $N$ degree-of-freedom system ? ($N \geq 3$)

- Based on

$$m_N \ddot{u}_N = \mathbf{F}(t) + R(t), \quad \mathbf{F}(t) = -\sum_{j=1}^{N} k_{Nj} u_j(t)$$

- At the beginning of sticking phase :

$$\mathbf{F}(0) = 0,\ \mathbf{F}(t) > 0,\ t \gtrsim 0$$

- During sticking time :
  ($N - 1$) degree-of-freedom system.
  The symmetry does not hold.
- The end of sticking phase $\mathcal{T} > 0$ is the first time such that

$$\mathbf{F}(\mathcal{T}) = 0, \quad \mathbf{F}(t) < 0,\ t \gtrsim \mathcal{T}$$

$\mathcal{T}$ is not explicit.

# An example : 3 degree-of-freedom system

- The beginning of the sticking phase : 3 possibilities
  - $u_3(0) = d$ , $\dot{u}_3(0) = 0$, $u_2(0) = d$, $\dot{u}_2(0) > 0$, or
  - $u_3(0) = d$ , $\dot{u}_3(0) = 0$, $u_2(0) = d$, $\dot{u}_2(0) = 0$, $u_1(0) > d$, or
  - $u_3(0) = d$ , $\dot{u}_3(0) = 0$, $u_2(0) = d$, $\dot{u}_2(0) = 0$, $u_1(0) = d$, $\dot{u}_1(0) > 0$.
- The end of the sticking phase :
  - $u_3(\mathcal{T}) = d$ , $\dot{u}_3(0) = 0$, $u_2(\mathcal{T}) = d$, $\dot{u}_2(\mathcal{T}) < 0$, or
  - $u_3(\mathcal{T}) = d$ , $\dot{u}_3(\mathcal{T}) = 0$, $u_2(\mathcal{T}) = d$, $\dot{u}_2(\mathcal{T}) = 0$, $u_1(\mathcal{T}) < d$, or
  - $u_3(\mathcal{T}) = d$ , $\dot{u}_3(\mathcal{T}) = 0$, $u_2(\mathcal{T}) = d$, $\dot{u}_2(\mathcal{T}) = 0$, $u_1(\mathcal{T}) = d$, $\dot{u}_1(\mathcal{T}) < 0$.
- The periodic condition : $\mathbf{U}(T) = \mathbf{U}(0)$

$\rightarrow$ at most 3 parameters and 5 equations.

$\rightarrow$ Overdetermined nonlinear system.

# 1SPP for $N$ degree-of-freedom system ($N > 2$)

- There exists the symmetric 1SPP. [Anders THORIN]

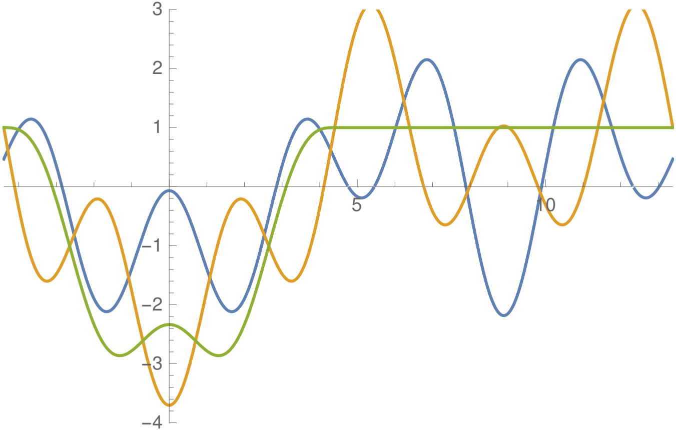



FIGURE : Displacement of masses for a 1SPP of 3 degree-of-freedom system

# Perspectives

- Understand more the dynamics of the free system through the first return map :
  - The return to the wall : yes or no
  - The stability
  - Internal resonance

  ....
- Vibro-impact system with force :
  Non-linear resonance ?
- CONTINUOUS MODEL :
  - reduction of the problem with some symmetries
  - a simple formulation (1D instead 2D)

Thank you

for

your attention !

# References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