Time-Integrators for Nonsmooth Dynamics - Siconos Software tutorial

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Objectives & Motivations

1. Nonsmooth dynamical systems in the large :
   - What is a nonsmooth dynamical system?
     Focus on mechanical systems with contact and friction
   - Why using the nonsmooth modeling framework?
     Why not regularizing?
   - Archetypal example : the bouncing ball (a.k.a. the ping pong ball)

2. Tools and methods for nonsmooth dynamical systems
   - Formulation of nonsmooth Lagrangian systems
   - Numerical time integration: Event-driven and time-stepping schemes
   - One–step problem :
     - from a set of nonlinear equations (Newton's method) to a set of equations and inequalities (Optimization) link with
     - Optimization Theory. Linear Complementarity Problem (LCP), Variational Inequality (VI)

3. Introduction to Siconos software
   - Functionalities.
   - Modeling and simulation principles
   - Examples with a tutorial on Jupyter notebook

What is a NonSmooth Dynamical System (NSDS) ?

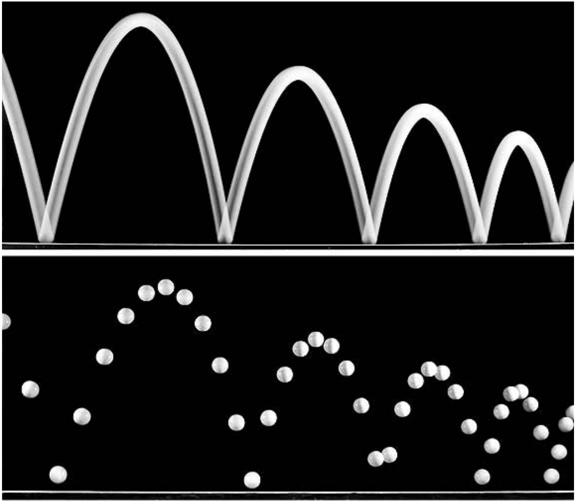
Nonsmooth

- Graphs with kinks, jumps, peaks

Dynamics

- The branch of mechanics that is concerned with the effects of forces on the motion of objects.
- Systems that evolves with time.

What is a Non Smooth Dynamical System (NSDS) ?



What is a NonSmooth Dynamical System (NSDS) ?

A NSDS is a dynamical system characterized by two correlated features:

- a nonsmooth evolution with the respect to time:
  - jumps in the state and/or in its derivatives w.r.t. time
  - generalized solutions (distributions)
- a set of non smooth laws (Generalized equations, inclusions) constraining the state x

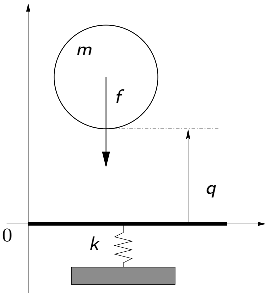
It is a modeling assumption based on two separate time–scales in the evolution of a dynamical system:

1. a small time scale where abrupt changes are located (*e.g.* impacting times)
2. a large time scale where the evolution is slower (*e.g.* free flight motion)

Remarks

It may be the result of a idealization or a passage to the limit.
Similar the continuum Mechanics modeling assumption.

A famous nonsmooth dynamical system: the bouncing ball



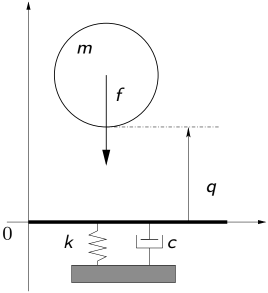
Compliant model

$$m\ddot{q}(t) = f(t)+\lambda = f(t)+\begin{cases} -kq(t) & \text{if} \quad q(t) < 0 \\ 0 & \text{if} \quad q(t) \geq 0 \end{cases} \tag{1}$$

Complementarity formulation

$$\lambda = \begin{cases} -kq & \text{if} \quad q < 0 \\ 0 & \text{if} \quad q \geq 0 \end{cases} \iff 0 \leq \lambda \perp \lambda + kq \geq 0 \tag{2}$$

A famous nonsmooth dynamical system: the bouncing ball



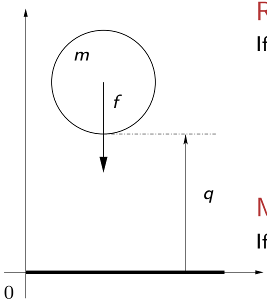
Compliant model with linear damping

$$\begin{cases} m\ddot{q}(t) = f(t) + \lambda \\ \min(0, c\dot{q} + kq) = \lambda \\ \text{if } q \leq 0 \text{ then } \lambda > 0 \\ \text{if } q > 0 \text{ then } \lambda = 0 \end{cases} \tag{3}$$

Complementarity formulation

Possible, but more complicated in order to maintain positive forces.

A famous nonsmooth dynamical system: the bouncing ball



Rigid limit

If we let $k \to +\infty$ (rigid contact with restitution) we get

$$\begin{cases} m\ddot{q}(t) = f(t) + \lambda(t) \\ 0 \leq q(t) \perp \lambda(t) \geq 0 \end{cases} \tag{4}$$

Mandatory impact law (for discrete systems)

If $\dot{q}(t^-) < 0$ and $q(t) = 0$

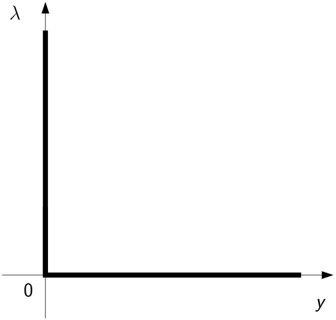
$$\dot{q}(t^+) = -e\dot{q}(t^-) \tag{5}$$

Therefore we pass from a piecewise linear system to a complementarity system

What do we gain doing so (compliance replaced by rigidity)?

Complementarity condition

Signorini's condition in contact mechanics



$$0 \leq y \perp \lambda \geq 0 \quad (6)$$
$$\Updownarrow$$
$$-y \in N_{\boldsymbol{R}_+}(\lambda) \quad (7)$$
$$\Updownarrow$$
$$-\lambda \in N_{\boldsymbol{R}_+}(y) \quad (8)$$
$$\Updownarrow$$
$$\lambda^T(y' - y) \geq 0, \text{ for all } y' \in \boldsymbol{R}_+ \quad (9)$$
$$\Updownarrow$$
$$y^T(\lambda' - \lambda) \geq 0, \text{ for all } \lambda' \in \boldsymbol{R}_+ \quad (10)$$

A well-know concept in Optimization

- Numerous theoretical tools (variational inequalities, complementarity problems, proximal point techniques)
- Numerous numerical tools (pivoting techniques, projected over–relaxation (Gauss–Seidel), semi-smooth Newton methods, interior point methods, ...)

Compliant vs. rigid model

Compliant model

- ⊕ Possibly more realistic models.
  - are we able to accurately know the behavior at contact (relation force/indentation) ?
  - Hertz's contact model for spheres (limited validity !) dissipation ?
- ⊖ Complex contact phenomena.
  - space and time scales are difficult to handle
  - numerous inner variables
- ⊖ Numerical implementation ostensibly more simpler, but numerous issues
  - stiff model, high frequency dynamics (most of the time non physical), stability of integrators, small time–steps, . . .
  - high sensitivity to contact parameters
  - limited smoothness : issues in order and adaptive time–step strategy

Rigid model

- ⊖ Limited description of the contact behavior
- ⊕ Modeling of threshold effects
- ⊕ Simple set of parameters with limited sensitivity
- ⊕ Stable and robust numerical implementation
  - no spurious high frequency dynamics.

Numerical simulation: Stiff problems versus complementarity

Euler discretization of the compliant system (finite k)

$$\left\{ \begin{array}{l} \frac{\dot{q}_{i+1}-\dot{q}_i}{h} = kq_{i+1} \\ \frac{q_{i+1}-q_i}{h} = \dot{q}_i \end{array} \right. \Leftrightarrow \begin{pmatrix} \dot{q}_{i+1} \\ q_{i+1} \end{pmatrix} = \begin{pmatrix} kh^2+1 & kh \\ h & 1 \end{pmatrix} \begin{pmatrix} \dot{q}_i \\ q_i \end{pmatrix} \tag{11}$$

This problem is **stiff** because the eigenvalues γ_1 and γ_2 of $\begin{pmatrix} kh^2+1 & kh \\ h & 1 \end{pmatrix}$ satisfy $\frac{\gamma_1}{\gamma_2} \rightarrow +\infty$ when $k \rightarrow +\infty$.

stiff integrators

Numerical simulation: Stiff problems versus complementarity

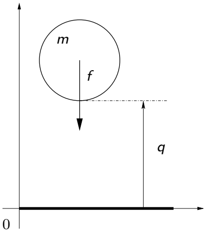
Euler discretization (Moreau's scheme) of the complementarity system (infinite k)

$$\begin{cases} \dot{q}_{i+1} - \dot{q}_i = hf_{i+1} + \lambda_{i+1} \\ 0 \leq \dot{q}_{i+1} + e\dot{q}_i \perp \lambda_{i+1} \geq 0 \\ \\ q_{i+1} = q_i + h\dot{q}_i \end{cases} \tag{12}$$

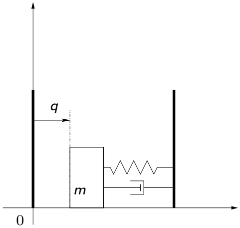
which is nothing else but solving a simple Linear complementarity systems (LCP) (or a quadratic program QP) at each step!!!

Academic examples

The bouncing Ball and the linear impacting oscillator



(a) Bouncing ball example



(b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

NonSmooth Multibody Systems (NSMBS)

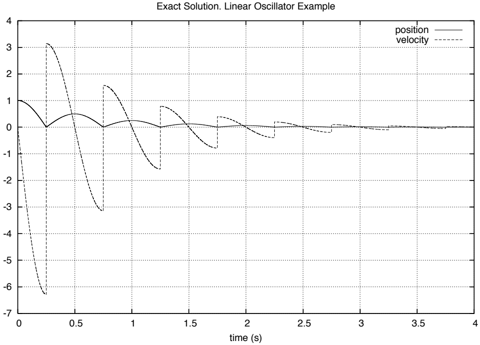


Figure: Analytical solution. Linear Oscillator

NonSmooth Multibody Systems (NSMBS)

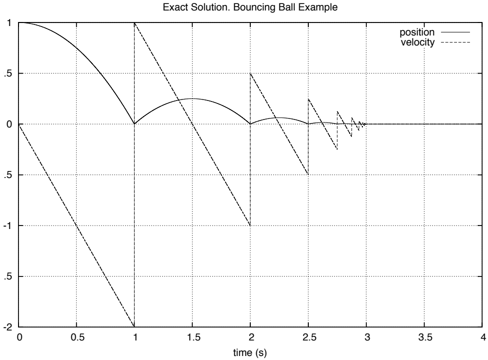


Figure: Analytical solution. Bouncing ball example

NonSmooth Multibody Systems

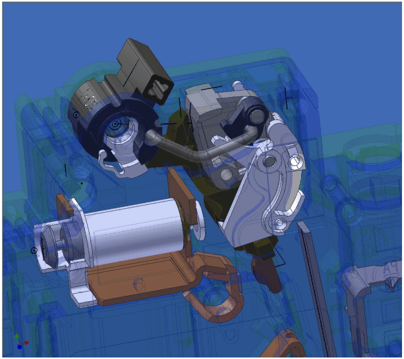
Scleronomous holonomic perfect unilateral constraints

$$\begin{cases} M(q(t))\dot{v} = F(t, q(t), v(t)) + G(q(t))\,\lambda(t), \text{ a.e} \\ \dot{q}(t) = v(t), \\ g(t) = g(q(t)), \quad \dot{g}(t) = G^T(q(t))v(t), \\ 0 \leq g(t) \perp \lambda(t) \geq 0, \\ \dot{g}^+(t) = -e\dot{g}^-(t), \end{cases} \tag{13}$$

where $G(q) = \nabla g(q)$ and e is the coefficient of restitution.

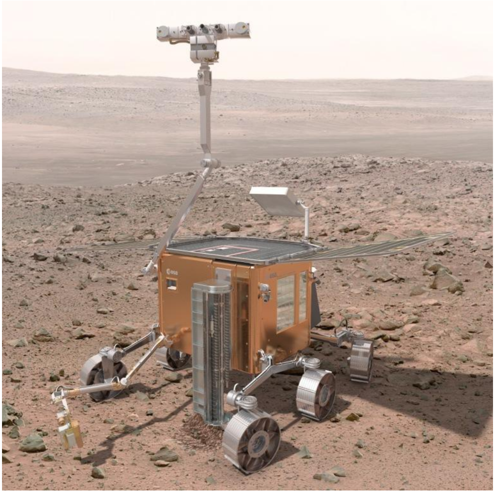
Mechanical systems with contact, impact and friction

Simulation of Circuit breakers (INRIA/Schneider Electric)



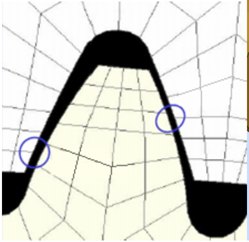
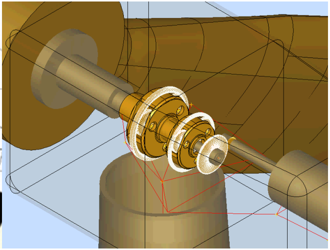
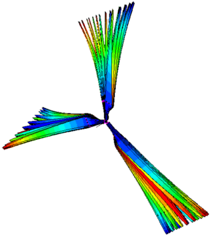
Mechanical systems with contact, impact and friction

Simulation of the ExoMars Rover (INRIA/Trasys Space/ESA)



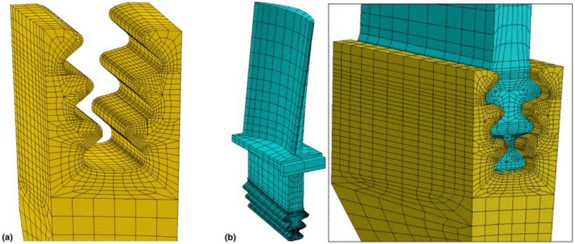
Mechanical systems with contact, impact and friction

Simulation of wind turbines (DYNAWIND project) collaboration with O. Brüls (Université de Liège)

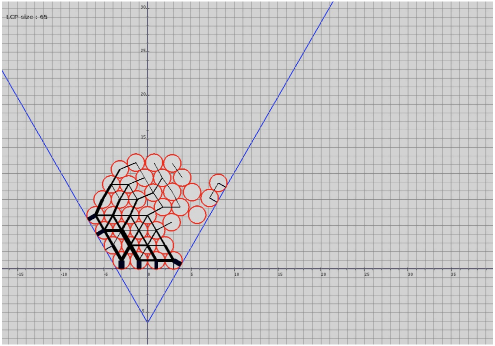
Mechanical systems with contact, impact and friction

Simulation of blades



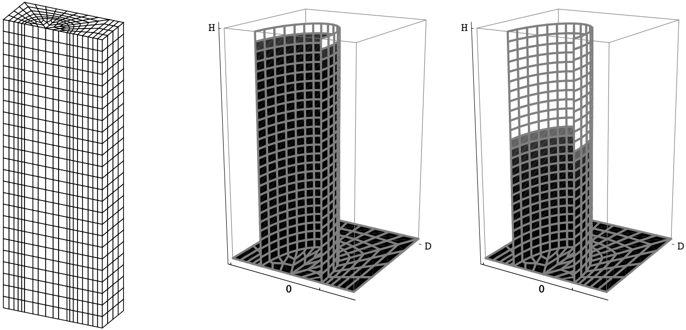
Mechanical systems with contact, impact and friction

Stack of beads with perturbation



Mechanical systems with contact, impact and friction

FEM models with contact, friction cohesion, etc...



Joint work with Y. Monerie, IRSN.

Mechanical systems with contact, impact and friction

Divided Materials and Masonry

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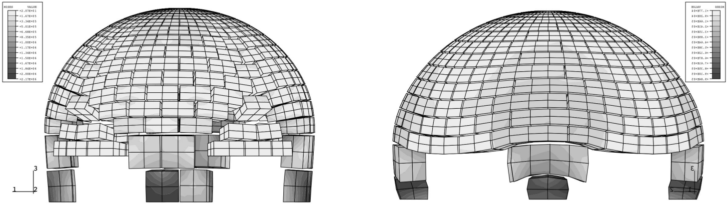
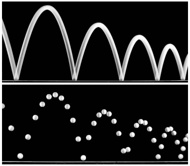


Siconos Software[3]

An opensource platform for the modeling, the simulation and the control of nonsmooth dynamical systems.

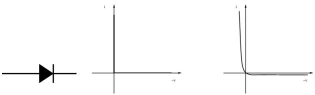
Vincent Acary, Maurice Brémond, Olivier Huber, Franck Pérignon, Stephen Sinclair.



Dynamical systems with . . .

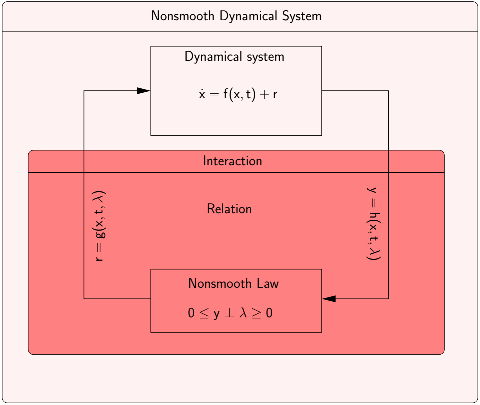
a nonsmooth evolution with respect to time:
solutions not differentiable everywhere,
discontinuities
(jumps in the state or its derivatives . . .),

nonsmooth laws, constraining the state.



In practice, what should you do in Siconos?

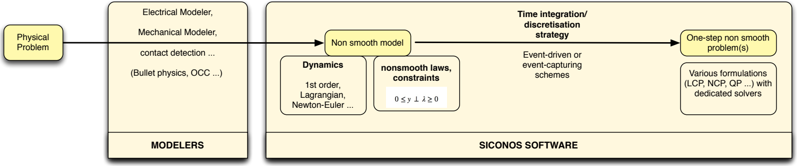
- Describe you nonsmooth model :
  - dynamics $\rightarrow$ first or second order ODE
  - nonsmooth laws
  - constraints, $\rightarrow$ some algebraic equations



- Choose time-integration method, formulation and solvers
- Solve, post-process and so on

See Bouncing-Ball example thereafter.

Siconos pipeline



Interlude: Software Engineering in Siconos

Project started 12 years ago.

- written in C/C++ (300000 lines) and wrapped into Python with swig,
- Siconos documentation and examples: http://siconos.gforge.inria.fr
- Hosted on github, Apache2 license: https://github.com/siconos/siconos
- Build automation process (cmake)
- Continuous integration : check *http://cdash-bipop.inrialpes.fr/index.php?project=Siconos*
- Quite easy to install on Unix-like systems (Linux, Macosx)
- Siconos is a library $\rightarrow$ easy to call from other softwares
- Packages on demand for standard OS (ubuntu, debian ...)

Siconos usage

- build and install (check doc) $\rightarrow$ libraries, headers and py files
- write a C++ program or a python script and run

```
siconos yourscript.cpp[py]
```

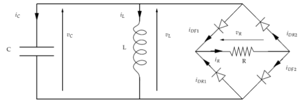
- check examples for inspiration

Warning : siconos == computational core. Some tools for post-processing, contact detection . . . are provided for convenience only.

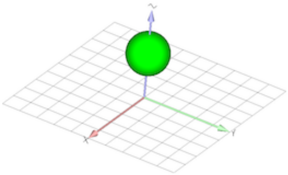
Siconos tutorials

Two notebooks are available as tutorials for Siconos basics functionalities and usage:

- A 4 diodes bridge wave rectifier.



- A ball bouncing on the ground.



https://github.com/fperignon/siconos-tutorials, hit 'launch binder' at the back of the page.

Principle of nonsmooth event tracking methods (Event-driven schemes)

Time-decomposition of the dynamics in

- *modes*, time-intervals in which the dynamics is smooth,
- discrete events, times where the dynamics is nonsmooth.

Comments

On the numerical point of view, we need

- detect events with for instance root-finding procedure.
  - Dichotomy and interval arithmetic
  - Newton procedure for C^2 function and polynomials
- solve the non smooth dynamics at events with a reinitialization rule of the state,
- integrate the smooth dynamics between two events with any ODE solvers.

Principle of nonsmooth event capturing methods (Time–stepping schemes)

1. A unique formulation of the dynamics is considered. For instance, a dynamics in terms of measures.
$$\begin{cases} -mdv + fdt = di \\ \dot{q} = v^+ \\ 0 \leq di \perp v^+ \geq 0 \text{ if } q \leq 0 \end{cases} \tag{14}$$
2. The time-integration is based on a consistent approximation of the equations in terms of measures. For instance,
$$\int_{]t_k,t_{k+1}]} dv = \int_{]t_k,t_{k+1}]} dv = (v^+(t_{k+1}) - v^+(t_k)) \approx (v_{k+1} - v_k) \tag{15}$$
3. Consistent approximation of measure inclusion.

$$0 \leq di \perp v^+ \geq 0 \text{ if } q \leq 0 \quad \rightarrow \quad \begin{cases} p_{k+1} \approx \displaystyle\int_{]t_k,t_{k+1}]} di \\ 0 \leq p_{k+1} \perp v_{k+1} \geq 0 \quad \text{if } \tilde{q}_k \leq 0 \end{cases} \tag{16}$$

Nonsmooth Lagrangian Dynamics

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that
$$v^+ = \dot{q}^+ \tag{17}$$
- q is related to this velocity by
$$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\,dt \tag{18}$$
- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that
$$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{19}$$

NonSmooth Multibody Systems

Scleronomous holonomic perfect unilateral constraints

$$\begin{cases} M(q(t))\dot{v} = F(t, q(t), v(t)) + G(q(t))\,\lambda(t), \text{ a.e} \\ \dot{q}(t) = v(t), \\ g(t) = g(q(t)), \quad \dot{g}(t) = G^T(q(t))v(t), \\ 0 \leq g(t) \perp \lambda(t) \geq 0, \\ \dot{g}^+(t) = -e\dot{g}^-(t), \end{cases} \tag{20}$$

where $G(q) = \nabla g(q)$ and e is the coefficient of restitution.

Nonsmooth Lagrangian Dynamics

Definition (Nonsmooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{21}$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- The nonsmooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References

[11, 12, 8, 9]

Nonsmooth Lagrangian Dynamics

Measures Decomposition (for dummies)

$$\left\{\begin{array}{rlll} dv = & \gamma\, dt + & (v^+ - v^-)\, d\nu + & dv_s \\ di = & f\, dt + & p\, d\nu + & di_s \end{array}\right. \tag{22}$$

where

- $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- f is the Lebesgue measurable force,
- $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v, i.e. where $(v^+ - v^-) \neq 0$,i.e. $d\nu = \sum_i \delta_{t_i}$
- p is the purely atomic impact percussions such that $pd\nu = \sum_i p_i \delta_{t_i}$
- dv_S and di_S are singular measures with the respect to $dt + d\eta$.

Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the nonsmooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{23}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{24}$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t, q, v)dt = fdt \tag{25}$$

or

$$M(q)\gamma^+ + F(t, q, v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{26}$$

State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

Two main implementations

- Moreau–Jean time–stepping scheme
- Schatzman–Paoli time–stepping scheme

Moreau–Jean's Time stepping scheme [9, 7]

Principle

$$\begin{cases} M(q_{k+\theta})(v_{k+1} - v_k) - hF_{k+\theta} = p_{k+1} = G(q_{k+\theta})P_{k+1}, & (27a) \\ q_{k+1} = q_k + hv_{k+\theta}, & (27b) \\ U_{k+1} = G^T(q_{k+\theta})\, v_{k+1} & (27c) \\ \begin{array}{ll} 0 \leq U^\alpha_{k+1} + eU^\alpha_k \perp P^\alpha_{k+1} \geq 0 & \text{if} \quad \bar{g}^\alpha_{k,\gamma} \leq 0 \\ P^\alpha_{k+1} = 0 & \text{otherwise} \end{array} . & (27d) \end{cases}$$

with

- $\theta \in [0, 1]$
- $x_{k+\theta} = (1 - \theta)x_{k+1} + \theta x_k$
- $F_{k+\theta} = F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta})$
- $\bar{g}_{k,\gamma} = g_k + \gamma h U_k, , \gamma \geq 0$ is a prediction of the constraints.

Schatzman–Paoli's Time stepping scheme [10]

Principle

$$\begin{cases} M(q_{k+1})(q_{k+1} - 2q_k + q_{k-1}) - h^2 F_{k+\theta} = p_{k+1}, & (28a) \\ v_{k+1} = \dfrac{q_{k+1} - q_{k-1}}{2h}, & (28b) \\ -p_{k+1} \in N_K\left(\dfrac{q_{k+1} + e q_{k-1}}{1+e}\right), & (28c) \end{cases}$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geq 0\}$

$$0 \leq g\left(\frac{q_{k+1} + e q_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1} + e q_{k-1}}{1+e}\right) P_{k+1} \geq 0 \qquad (29)$$

Comparison

Shared mathematical properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Mechanical properties

- Position vs. velocity constraints
- Respect of the impact law in one step (Moreau) vs. Two-steps(Schatzman)
- Linearized constraints rather than nonlinear.

But ...

But

Both schemes are quite inaccurate and "dissipate" a lot of energy of vibrations. This is a consequence of the first order approximation of the smooth forces term F

Recent improvements

- Nonsmooth generalized α schemes [6, 5]
- Time discontinuous Galerkin methods [13, 14]
- Stabilized index-2 formulation [2, 1]
- Geometric integrators.

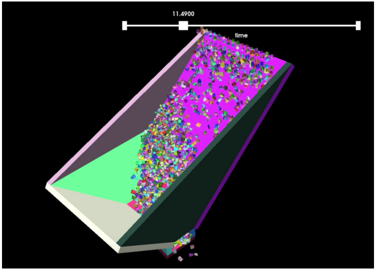
Siconos tutorial - part 2

See Python notebook ...

Many other (and more interesting . . .) examples

in siconos/examples directory

- Electronics
- Mechanics/Multibody
- Biology
- Control

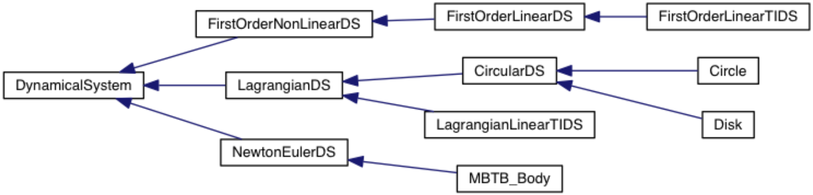


26 000 contacts

Siconos Short Review

Dynamical systems in siconos

The class used to describe a set of ordinary differential equations.



First order

$$M\dot{x}(t) = f(x, t, z) + r \quad (30)$$
$$x(t_0) = x_0$$

Lagrangian systems

$$M\ddot{q}(t) = f(t, \dot{q}, q, z) + p \quad (31)$$
$$q(t_0) = q_0, \ \dot{q}(t_0) = v_0$$

- user-defined plug-in functions

+ interface with Bullet Physics (*http://bulletphysics.org*)

Relations

Algebraic equations used to describe constraints (unilateral or equality)

- First order

$$\begin{aligned} output &= y = & h(X, t, \lambda, Z) \\ input &= R = & g(X, t, \lambda, Z) \end{aligned}$$

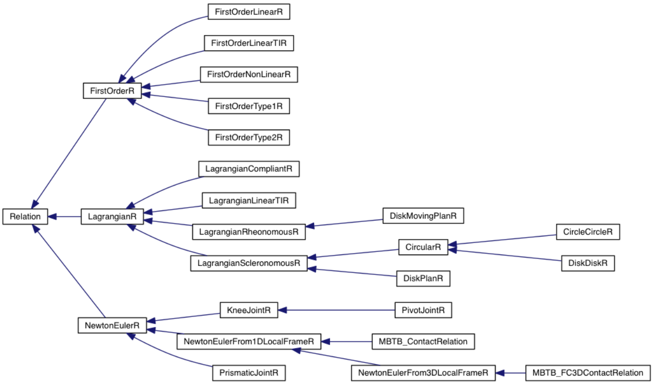
- Lagrangian

$$\begin{aligned} y &= & h(Q, Z) \\ \dot{y} &= & G_0(Q, Z)\dot{Q} \\ P &= & G_0^t(Q, Z)\lambda \end{aligned}$$

with $G_0(Q, Z) = \nabla_Q h(Q, Z)$

- user-defined plug-in functions

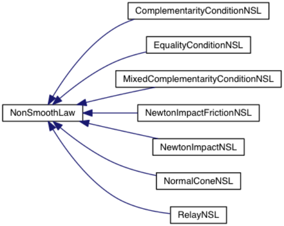
Relations


Relation
FirstOrderR
FirstOrderLinearR
FirstOrderLinearTIR
FirstOrderNonLinearR
FirstOrderType1R
FirstOrderType2R
LagrangianR
LagrangianCompliantR
LagrangianLinearTIR
LagrangianRheonomousR
DiskMovingPlanR
LagrangianScleronomousR
CircularR
CircleCircleR
DiskDiskR
DiskPlanR
NewtonEulerR
KneeJointR
PivotJointR
NewtonEulerFrom1DLocalFrameR
MBTB_ContactRelation
NewtonEulerFrom3DLocalFrameR
MBTB_FC3DContactRelation
PrismaticJointR

Nonsmooth laws

- Complementarity Conditions

$$0 \leq y \perp \lambda \geq 0$$



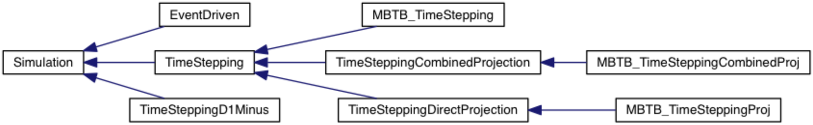
- Newton impact

$$\begin{aligned} &\textit{if } y(t) = 0 \\ &0 \leq \dot{y}(t^+) \quad + \quad e\dot{y}(t^-) \perp \lambda \geq 0 \end{aligned}$$

- Newton impact-friction
- Relay

$$\begin{array}{lll} \dot{y} & = 0 & , d \leq \lambda \leq c \\ \dot{y} & \geq 0 & , \lambda = c \\ \dot{y} & \leq 0 & , \lambda = d \end{array}$$

Simulation classes



Based on the two above-mentioned strategies

- Event-Capturing
- Event-Tracking

One-step Nonsmooth problems

- Linear Complementarity Problems

$$\begin{aligned} w &= q + Mz \\ 0 \leq w &\perp z \geq 0 \end{aligned}$$

- Mixed Linear Complementarity Problems

$$\begin{aligned} 0 &= Au + Cv + a \\ z &= Du + Bv + b \\ 0 \leq w &\perp z \geq 0 \end{aligned}$$

- Relay Problems
- Equalities
- Affine Variationnal Inequalities

One-step NonSmooth problems: Signorini's condition and Coulomb's friction

- 2D/3D friction contact problem

$$\begin{cases} \hat{u} = Wr + q + \left[\left[\begin{array}{c} \mu^\alpha \| u^\alpha_{\mathrm{T}} \| \\ 0 \\ 0 \end{array} \right]^T, \alpha = 1 \ldots n_c \right]^T \\ K^\star_\mu \ni \hat{u} \perp r \in K_\mu \end{cases}$$

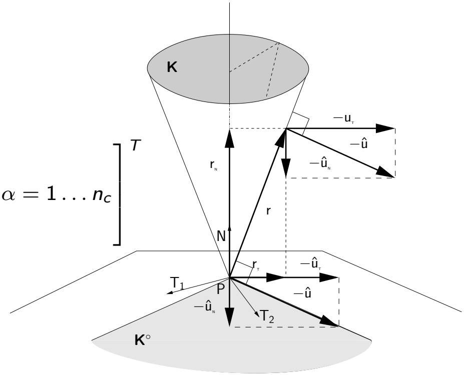
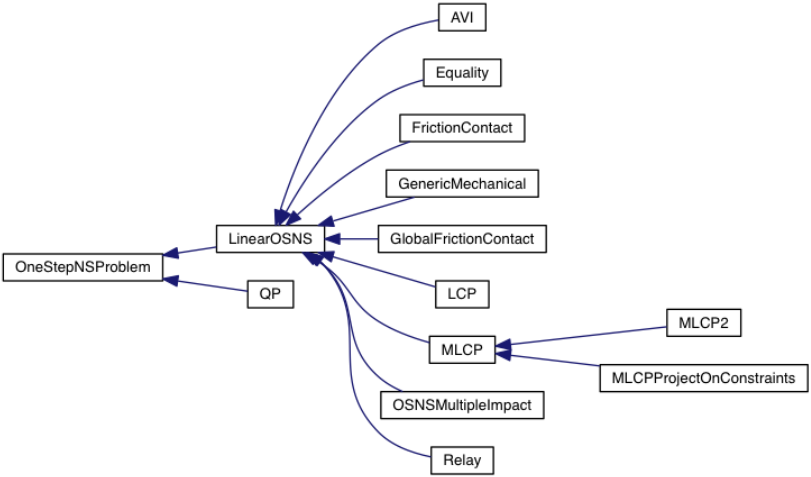


Figure: Coulomb's friction and the modified velocity $\hat{u}$. The sliding case.

More details concerning friction-contact formulation in [4].

One-step NonSmooth problems



Nonsmooth Solvers

- LCP solvers
- lexicographic Lemke
- QP Solver
- NSQP Solver
- CPG Solver
- PGS Solver
- RPGS Solver
- PSOR Solver
- NewtonMin Solver
- NewtonFB Solver
- Newton min + FB Solver
- Path (Ferris) Solver
- Enumerative Solver
- Latin Solver
- Latin_ Solver
- Block solver (Gauss Seidel)
- Friction-Contact solvers
- 2D solvers
- CPG
- 3D solvers
- Non-Smooth Gauss Seidel

Thank you for your attention!

Try it , ... and do not hesiate to contribute :

- *http://siconos.gforge.inria.fr*
- *https://github.com/siconos/siconos*

[1] V. Acary. Projected event-capturing time-stepping schemes for nonsmooth mechanical systems with unilateral contact and Coulomb's friction. Computer Methods in Applied Mechanics and Engineering, 256:224 – 250, 2013. ISSN 0045-7825. doi: 10.1016/j.cma.2012.12.012. URL http://www.sciencedirect.com/science/article/pii/S0045782512003829.

[2] V. Acary. Energy conservation and dissipation properties of time-integration methods
for the nonsmooth elastodynamics with contact. Zeitschrift für Angewandte Mathematik und Mechanik, submitted, 2014.

[3] V. Acary and F. Pérignon. An introduction to SICONOS. Technical Report TR-0340, INRIA, http://hal.inria.fr/inria-00162911/en/, 2007.

[4] Vincent Acary, Maurice Brémond, Tomasz Koziara, and Franck Pérignon. FCLIB: a collection of discrete 3D Frictional Contact problems. Technical Report RT-0444, February 2014. URL https://hal.inria.fr/hal-00945820.

[5] Olivier Brüls, Vincent Acary, and Alberto Cardona. Simultaneous enforcement of constraints at position and velocity levels in the nonsmooth generalized-α scheme. Computer Methods in Applied Mechanics and Engineering, 281: 131–161, November 2014. doi: 10.1016/j.cma.2014.07.025. URL http://hal.inria.fr/hal-01059823.

[6] Q. Z. Chen, V. Acary, G. Virlez, and O. Brüls. A nonsmooth generalized-α scheme for flexible multibody systems with unilateral constraints. International Journal for Numerical Methods in Engineering, 96(8):487–511, 2013. ISSN 1097-0207. doi: 10.1002/nme.4563. URL http://dx.doi.org/10.1002/nme.4563.

[7] M. Jean. The non smooth contact dynamics method. Computer Methods in Applied Mechanics and Engineering, 177:235–257, 1999. Special issue on computational modeling of contact and friction, J.A.C. Martins and A. Klarbring, editors.

[8] J.J. Moreau. Liaisons unilatérales sans frottement et chocs inélastiques. cras, 296 série II:1473–1476, 1983.

[9] J.J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. In J.J. Moreau and P.D. Panagiotopoulos, editors, Nonsmooth mechanics and applications, number 302 in CISM, Courses and lectures, pages 1–82. CISM 302, Spinger Verlag, Wien- New York, 1988. Formulation mathematiques tire du livre Contacts mechanics.

[10] L. Paoli and M. Schatzman. A numerical scheme for impact problems I: The one-dimensional case. SIAM Journal of Numerical Analysis, 40(2):702–733, 2002.

[11] M. Schatzman. Sur une classe de probl√é¬®mes hyperboliques non linéaires. Comptes Rendus de l'Académie des Sciences Série A, 277:671–674, 1973.

[12] M. Schatzman. A class of nonlinear differential equations of second order in time. nla, 2(3):355–373, 1978.

[13] T. Schindler and V. Acary. Timestepping schemes for nonsmooth dynamics based on discontinuous galerkin methods: Definition and outlook. Mathematics and Computers in Simulation, 95:180–199, 2013. ISSN 0378-4754. doi: 10.1016/j.matcom.2012.04.012. URL http://www.sciencedirect.com/science/article/pii/S0378475412001231.

[14] T. Schindler, S. Rezaei, J. Kursawe, and V. Acary. Half-explicit timestepping schemes on velocity level based on time-discontinuous galerkin methods. Computer methods in Applied Mechanics in Engineering, 290(15):250–276, 2015.