Wave turbulence: the example of vibrating plates.

Can one hear a Kolmogorov spectrum?

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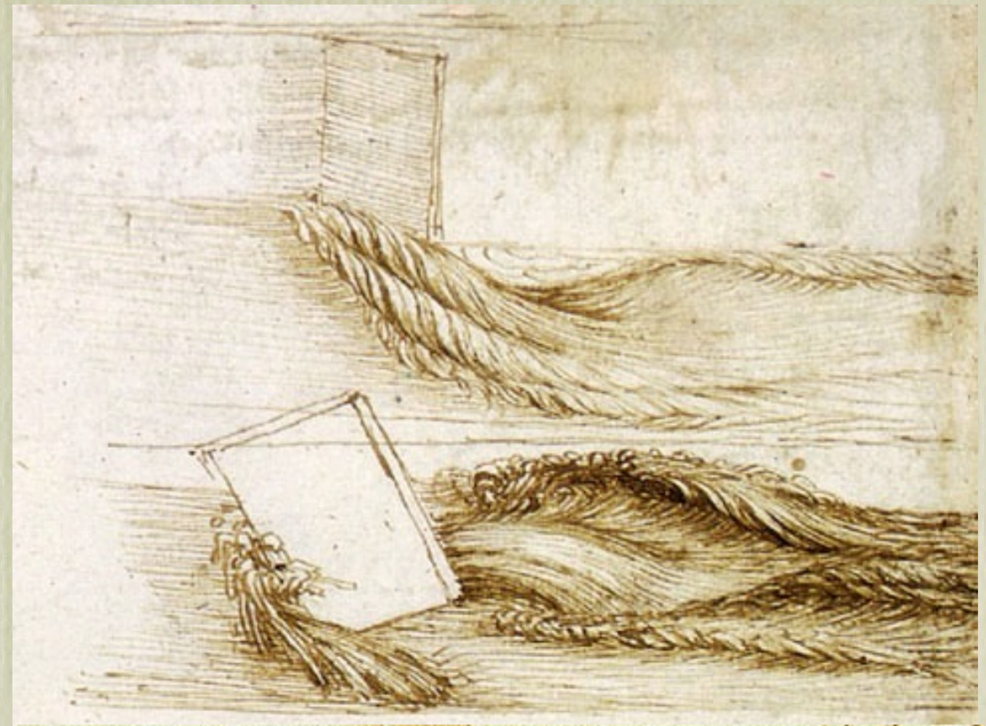
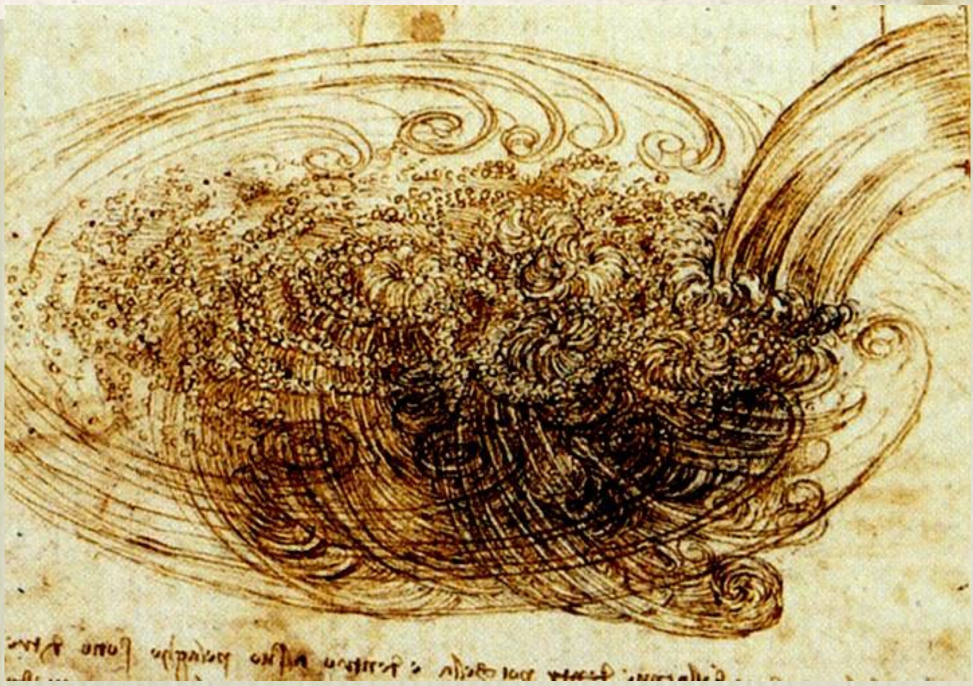
Turbulence: old non linear problem!

- turbulence is present in many situations (fluid dynamics, plasmas)
- «chaotic» motion with high fluctuations when the linear (laminar) flow becomes unstable
- in general, fully nonlinear system (no small parameters).



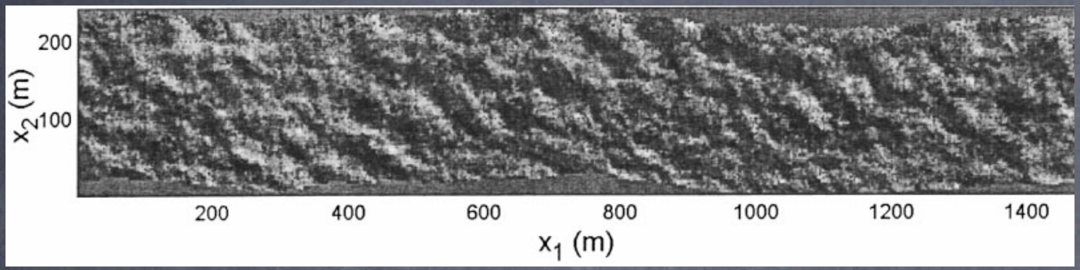


Wave or and Weak turbulence?

- developed originally for water waves (ocean waves, Hasselmann equations)
- stationary observed states: turbulence and statistical description of wave systems?
- for waves, the linear order is crucial since one can make an expansion analysis with a small parameter (wave amplitude)
- assuming weak nonlinearities, a kinetic equation for the wave amplitudes can be obtained using asymptotic closures
- although experimental evidences, many questions remain (ocean waves, range of validity, mathematical proof ...)

Experimental evidence of Weak Turbulence in gravity (ocean) waves

Y. Toba (1973); Hwang et al. (2000).



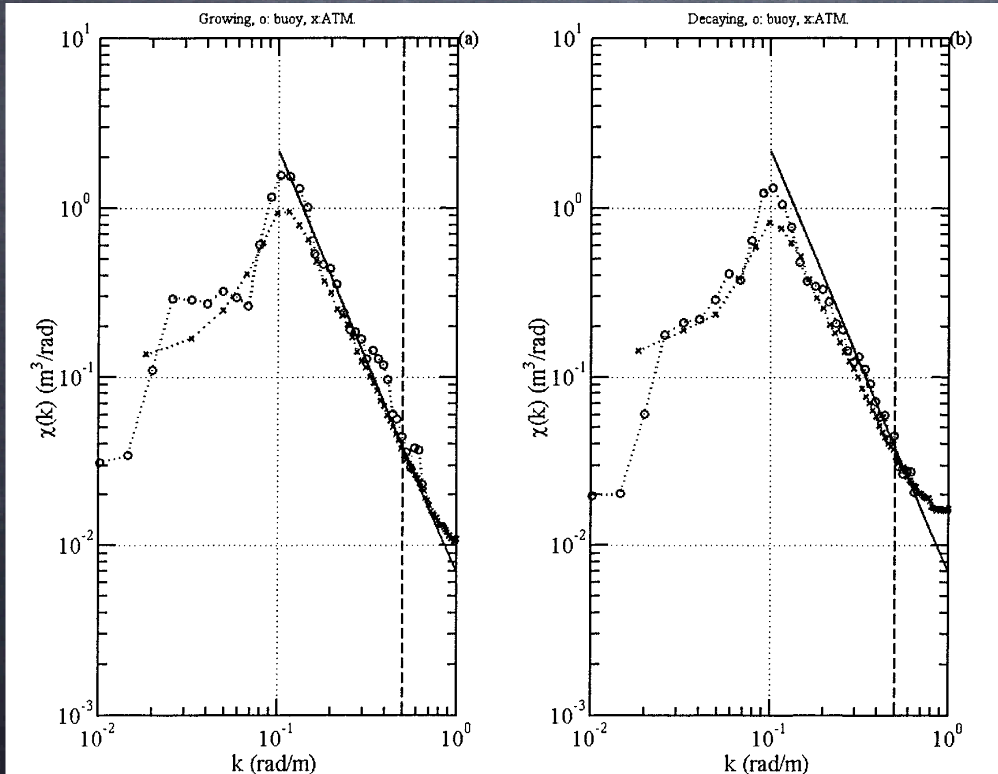


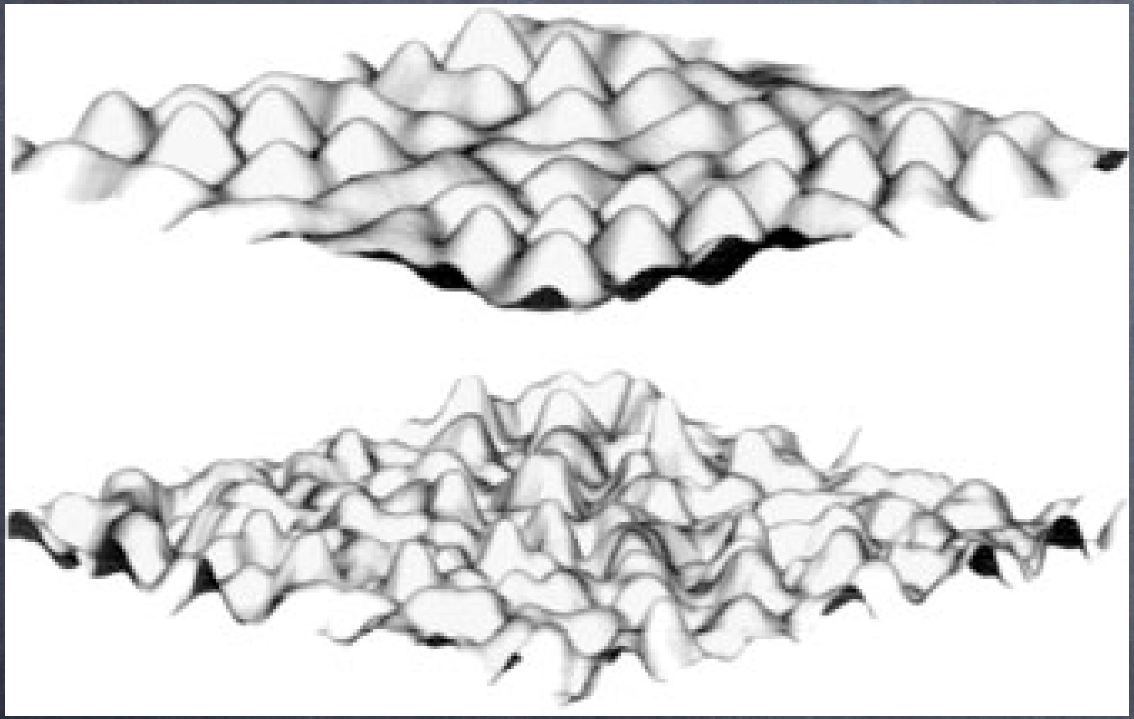
FIG. 8. A comparison of the omnidirectional spectra measured by ATM (crosses) and offshore buoy (ID 44014) (circles). (a) Average of the first 2 hours of data—quasi-steady condition, and (b) average of the last 2 hours of data—decaying wave field. Solid curves: $\chi(k) = 0.06u_* g^{-0.5} k^{-2.5}$ (Phillips 1985).

$$\langle |\zeta_{\boldsymbol{k}}|^2 \rangle = \frac{P^{1/3} g^{1/2}}{k^{5/2}}$$

slope: -5/2

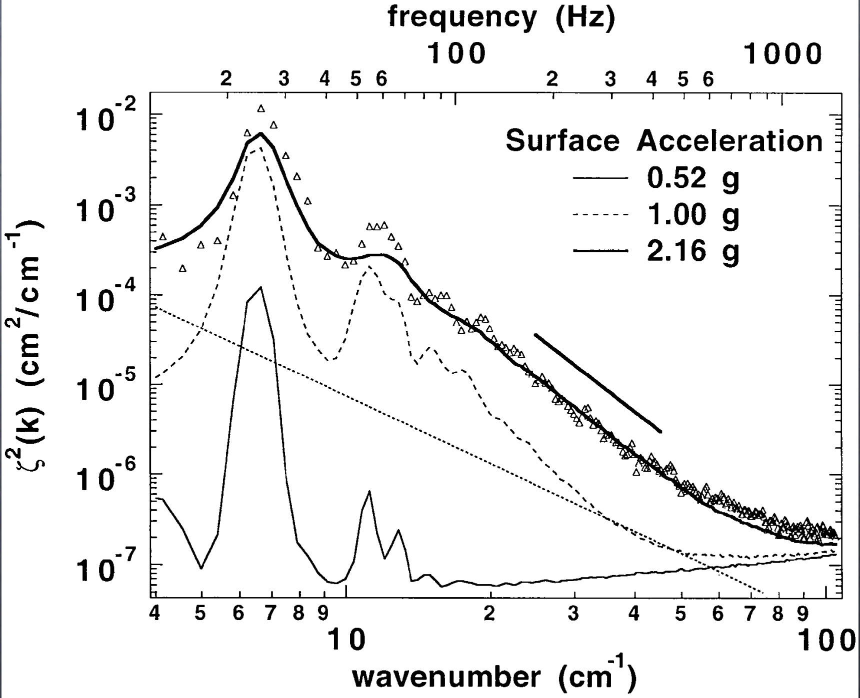
Experimental evidence of Weak Turbulence in capillary waves

Wright, Budakian & Putterman (1996); Henry, Alstrøm & Levinsen (2000); Brazhnikov, Kolmakov, Levchenko & Mezhov-Deglin (2002).

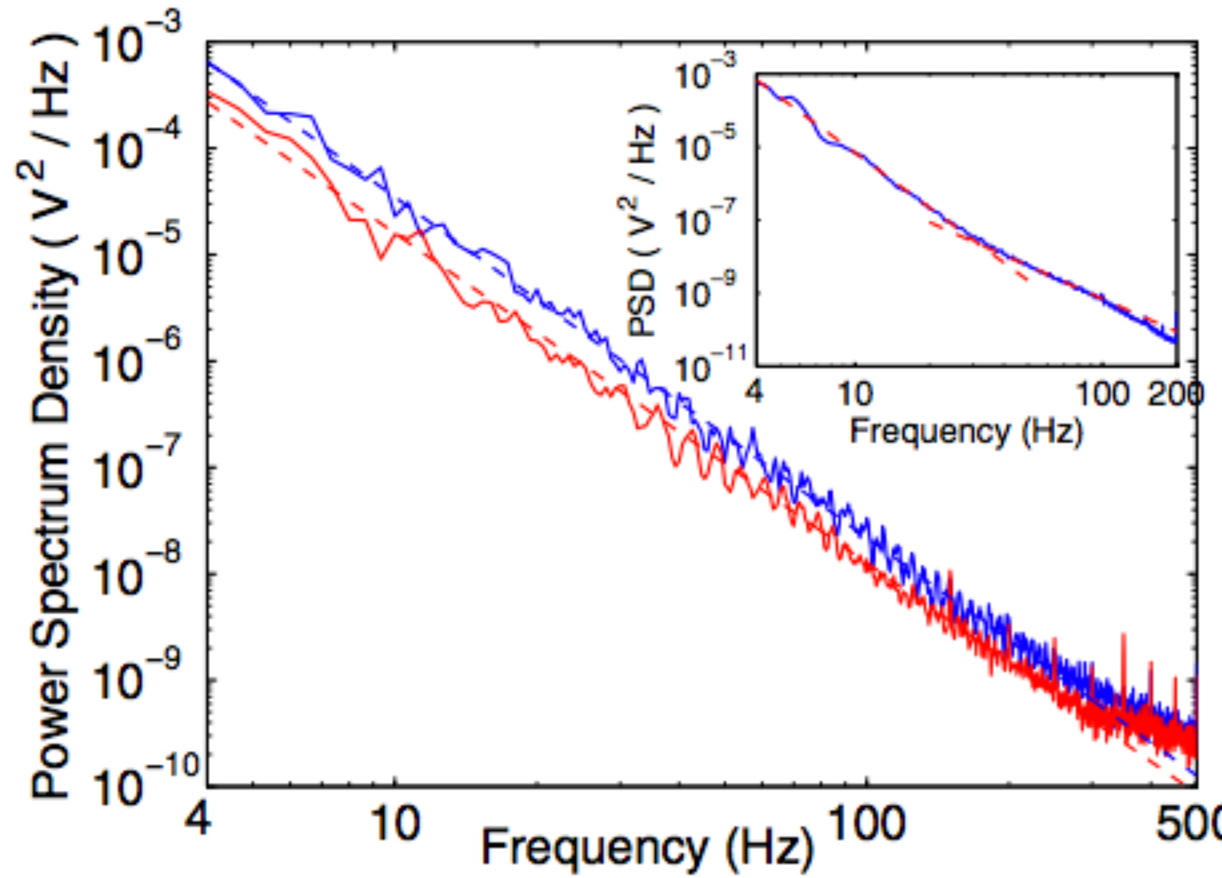


from Wright et al.

$$\langle |\zeta_{\boldsymbol{k}}|^2 \rangle = C \frac{P^{1/2}\rho^{1/4}}{\sigma^{3/4}} \frac{1}{k^{17/4}}$$



slope: -4.2



C. Falcon, E. Falcon, U. Bortolozzo and S. Fauve,
"Capillary wave turbulence on a spherical fluid surface
in low gravity", Europhysics Letters 86, 14002 (2009).

Wave turbulence in elastic plate

- elastic plates have dispersive waves and geometrical nonlinearities which suggest that wave turbulence can apply

- predicted theoretically in 2006 using classical wave turbulence arguments and shown in numerical simulations

- spectra of direct cascade of energy experimentally obtained with important differences with the theory.

- let know hear a Kolmogorov spectrum

Experiment movie by
N. Mordant (LPS-ENS)

Elastic plates

$$\rho \frac{\partial^2 \zeta}{\partial t^2} = -\frac{Eh^2}{12(1-\sigma^2)}\Delta^2 \zeta + \{\zeta, \chi\}$$

$$\frac{1}{E}\Delta^2 \chi = -\frac{1}{2}\{\zeta, \zeta\}$$

$$\omega_k = \sqrt{\frac{Eh^2}{12(1-\sigma^2)\rho}}\,|k|^2.$$

where E is the Young Modulus, h the plate thickness and:

$$\{f, g\} = f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy}$$

and $\frac{1}{2}\{\zeta, \zeta\} = \zeta_{yy}\zeta_{xx} - \zeta_{xy}^2$ is the Gaussian curvature

Dimensional analysis (can) give information!

- for usual turbulence: dimensional analysis predicts the spectra!

$$\mathcal{E} = \int E_k dk \qquad \frac{\partial}{\partial t} E_k = -\frac{\partial}{\partial k} P$$

$$E_k = C P^{2/3} k^{-5/3}$$

- for wave turbulence the dispersion relation makes a new number enter!

- nonlinear wave interaction helps (often) to solve the problem.

$$E_k = P^{1/3} \sqrt{\frac{E}{\rho}} k^{-4/3} \Phi_1(k\ell),$$

Wave turbulence framework

- classical wave/weak turbulence machinery applies (DJR 2006)
- the dissipationless equation has a Hamiltonian structure

$$H = \int \left[\frac{h^2 E}{24(1-\sigma^2)} (\Delta\zeta)^2 - \frac{1}{2E} (\Delta\chi)^2 - \frac{1}{2} \chi\{\zeta, \zeta\} \right]$$

$$H[\zeta, \chi] = h \int \left(\frac{\rho}{2} \dot{\zeta}^2 + \frac{E h^2}{24(1-\sigma^2)} (\Delta\zeta)^2 + \frac{E}{8} \left[\Delta^{-1}\zeta, \zeta \right]^2 \right) dr$$

Using Fourier transform

$$H = h\int\left[\frac{1}{2\rho h^2}p_{k_1}p_{-k_1} + \frac{h^2E}{24(1-\sigma^2)}{k_1}^4\zeta_{k_1}\zeta_{-k_1}\right]d^2k_1 + \frac{h}{(2\pi)^2}\int T_{k_1k_2;k_3k_4}\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\zeta_{k_4}\delta(k_1+k_2+k_3+k_4)d^2k_{123}$$

With

$$p_k = \rho h\dot{\zeta}_k$$

$$T_{k_1k_2;k_3k_4} = \frac{E}{8}\left(\frac{1}{2|k_1+k_2|^4} + \frac{1}{2|k_3+k_4|^4}\right)(k_1\times k_2)^2(k_3\times k_4)^2$$

Canonical variables: $\zeta_k = \frac{X_k}{\sqrt{2}}(A_k + A^*_{-k})$

$$X_k = \frac{1}{\sqrt{\omega_k \rho h}}.$$

$$p_k = -i\frac{X_k^{-1}}{\sqrt{2}}(A_k - A^*_{-k})$$

$$H = \int \omega_k A_k A_k^* dk + \frac{1}{4(2\pi)^2}\int X_{k_1}X_{k_2}X_{k_3}X_{k_4}T_{k_1k_2;k_3k_4}\sum_{s_1s_2s_3s_4} A_{k_1}^{s_1}A_{k_2}^{s_2}A_{k_3}^{s_3}A_{k_4}^{s_4}\delta(k_1+k_2+k_3+k_4)dk_{1234}$$

Equation for the cumulant: hierarchy. Need a closure (asymptotic arguments/random phase approximations).

$$\frac{dA_k^s}{dt} + is\omega_{sk}A_k^s = \varepsilon^2 \sum_{s_1 s_2 s_3} \int L_{kk_1k_2k_3}^{ss_1s_2s_3} A_{k_1}^{s_1} A_{k_2}^{s_2} A_{k_3}^{s_3} \delta(k_1 + k_2 + k_3 - k)\, dk_{123}$$

Cumulant hierarchy equations+wave frequency resonance as asymptotics in time

$$A_k^s = a_k^s e^{is\omega_k t}$$

Multiscale analysis, leading to dirac function in frequency in the cumulant hierarchy equation for the « wave spectrum »

$$n_k = < a_k a_k^* >$$

$$\frac{d}{dt}n(l_2\boldsymbol{p}_2) = \epsilon^4 12\pi l_2 \sum_{s_1 s_2 s_3} \int \left| J^{-l_2 s_1 s_2 s_3}_{-\boldsymbol{p}_2 \boldsymbol{k}_1 \boldsymbol{k}_2 \boldsymbol{k}_3} \right|^2 n(s_1\boldsymbol{k}_1)n(s_2\boldsymbol{k}_2)n(s_3\boldsymbol{k}_3)n(l_2\boldsymbol{p}_2)\left(\frac{l_2}{n(l_2\boldsymbol{p}_2)} - \frac{s_1}{n(s_1\boldsymbol{k}_1)} - \frac{s_2}{n(s_2\boldsymbol{k}_2)} - \frac{s_3}{n(s_3\boldsymbol{k}_3)}\right) \times$$
$$\times \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 - \boldsymbol{p}_2)\delta(l_2\omega(\boldsymbol{p}_2) - s_1\omega(\boldsymbol{k}_1) - s_2\omega(\boldsymbol{k}_2) - s_3\omega(\boldsymbol{k}_3))d\boldsymbol{k}_{123}$$

Energy conservation

$$\mathcal{E} = \sum_{l_1} \int \omega(l_1\boldsymbol{p}_1)n(l_1\boldsymbol{p}_1, t)d\boldsymbol{p}_1$$

But no wave-action conservation!

H-Theorem

$$\mathcal{S}(t) = \sum_{l_1} \int \log[n(l_1\boldsymbol{p}_1, t)]d\boldsymbol{p}_1$$

$$\frac{d\mathcal{S}}{dt} \geq 0$$

2 types of stationnary solutions

- Rayleigh-Jeans equilibrium

$$n_k^{eq} = \frac{T}{\omega_k} = \frac{T}{hck^2}$$

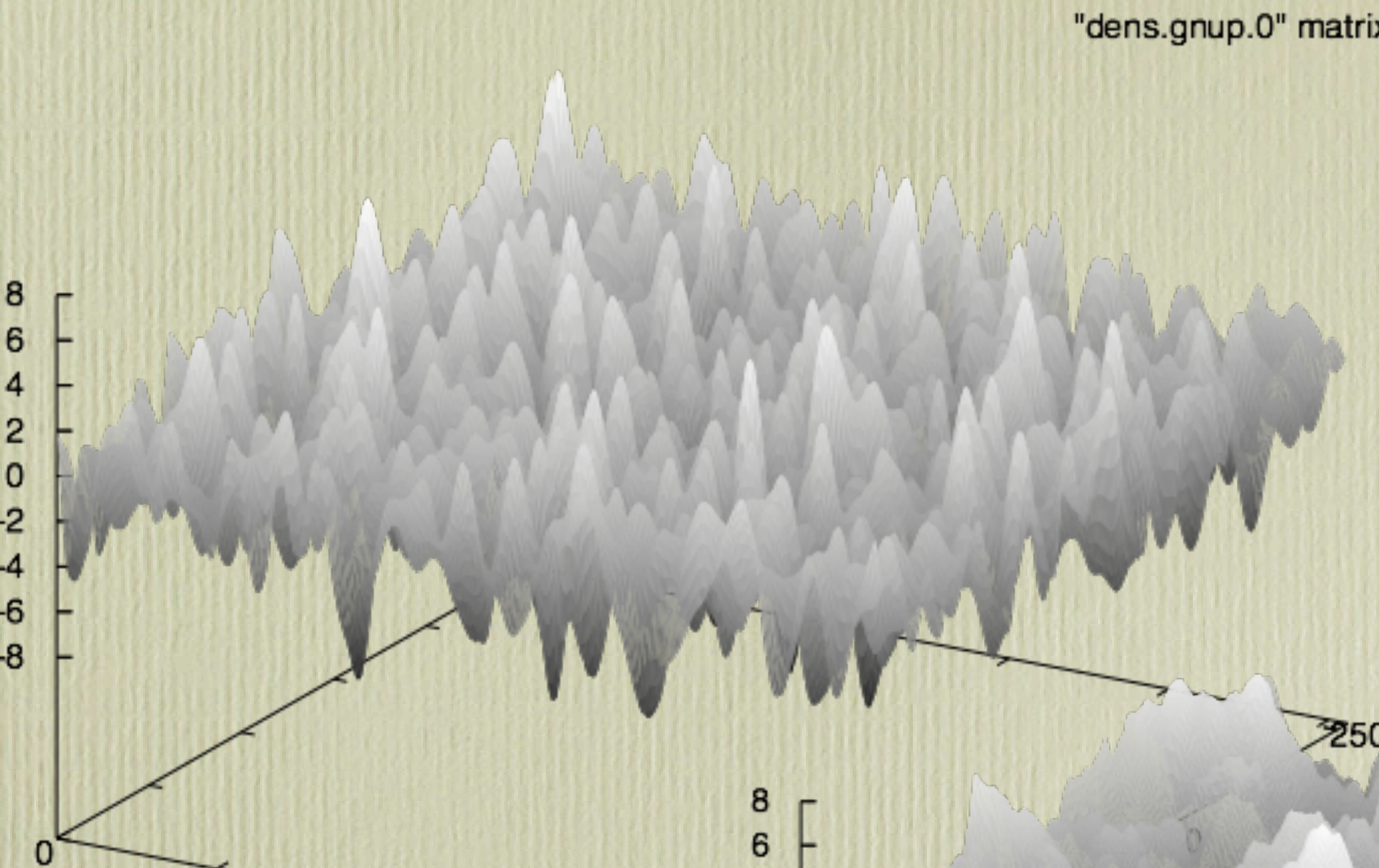
- K-Z cascade of energy

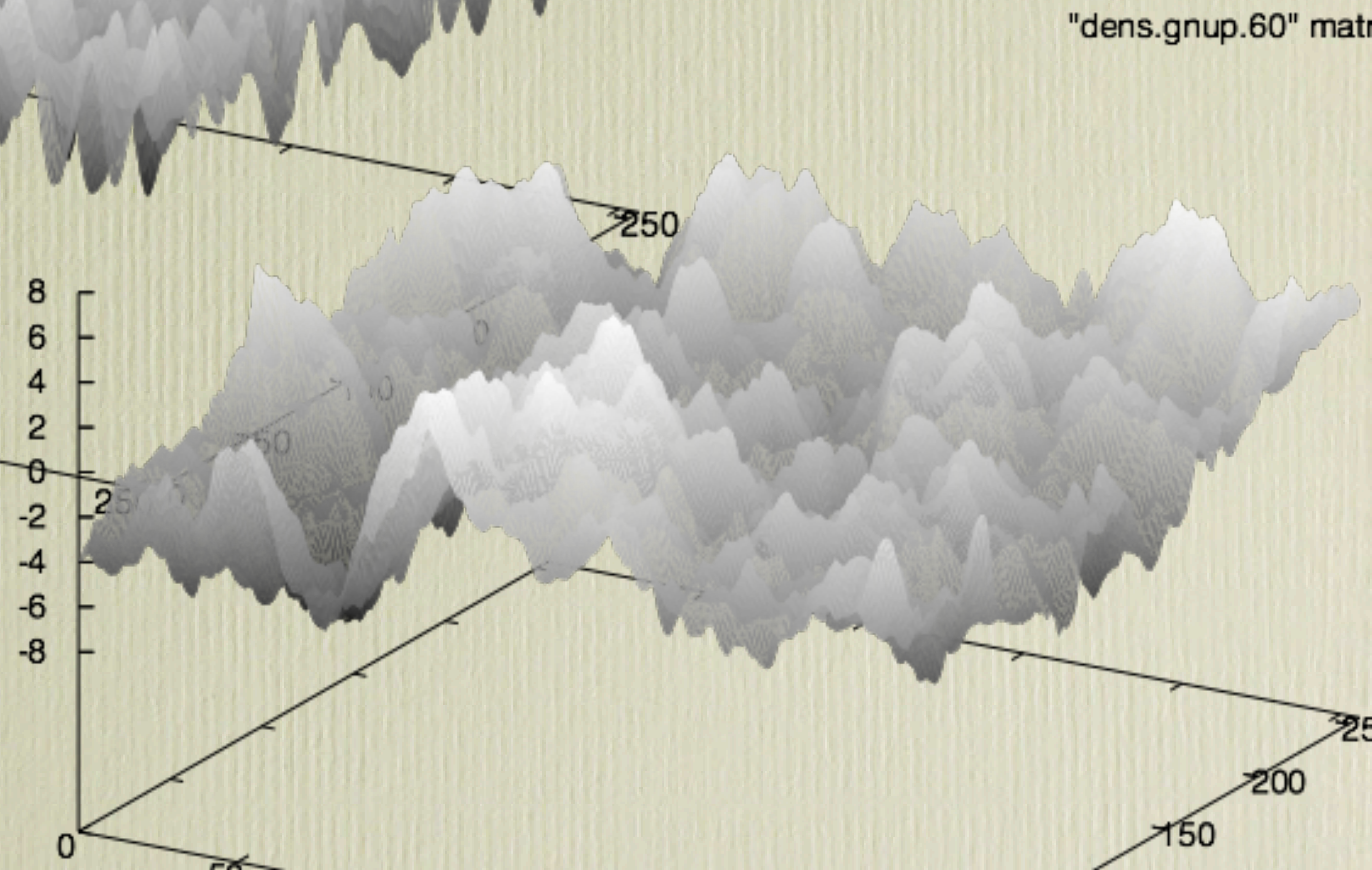
$$n_k^{eq} = C\frac{P^{1/3}}{k^2} log^{1/3}\left(\frac{k}{k_c}\right)$$

Numerical simulations

$$\rho \frac{\partial^2 \zeta}{\partial t^2} = -\frac{Eh^2}{12(1-\sigma^2)} \Delta^2 \zeta + \{\zeta, \chi\} + f_{in} + f_{diss}$$

- pseudo-spectral method
- periodic boundary conditions
- Adams-Bashford scheme

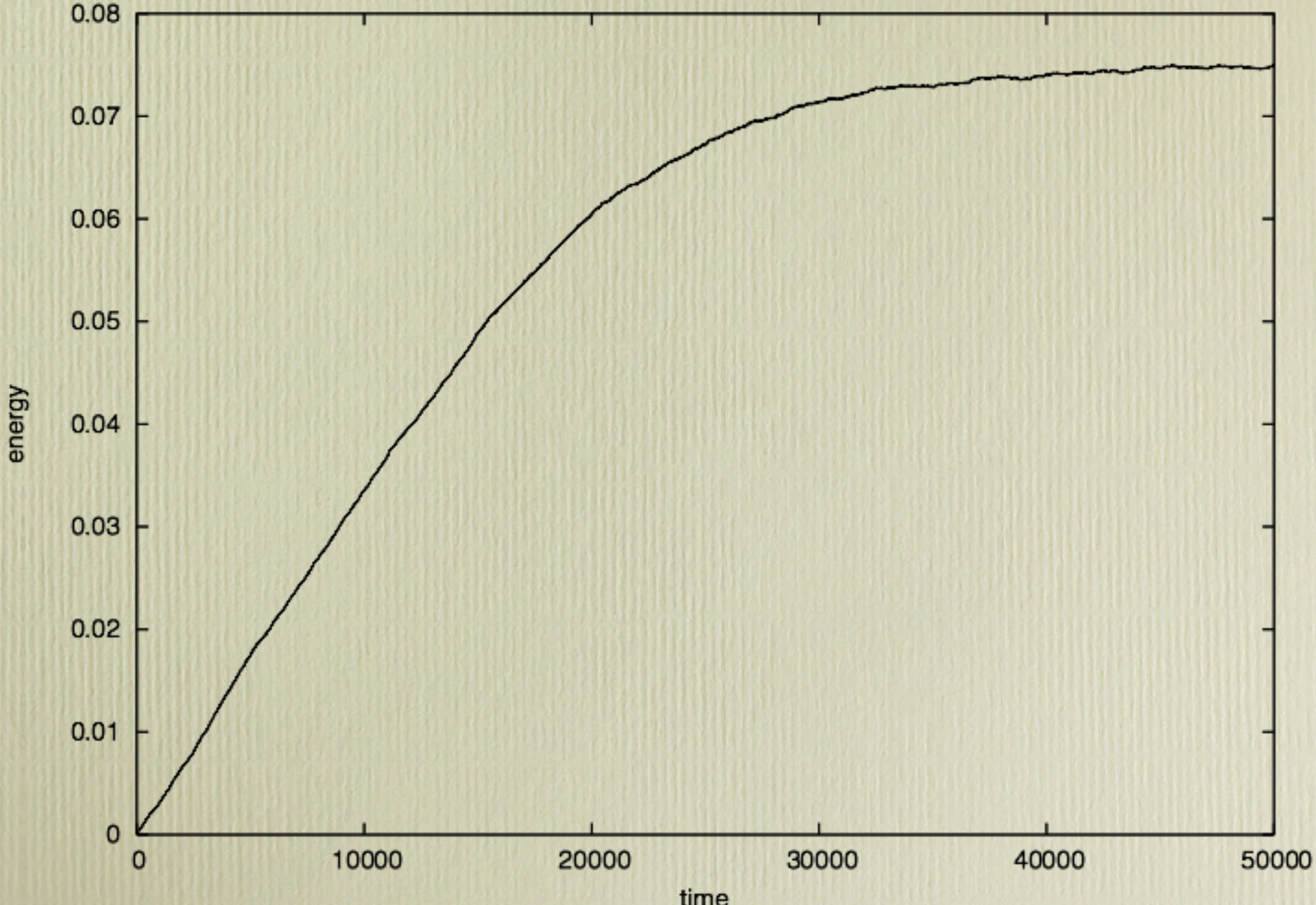

"dens.gnup.0" matrix
8
6
4
2
0
-2
-4
-6
-8
0
50
100
150
200
250

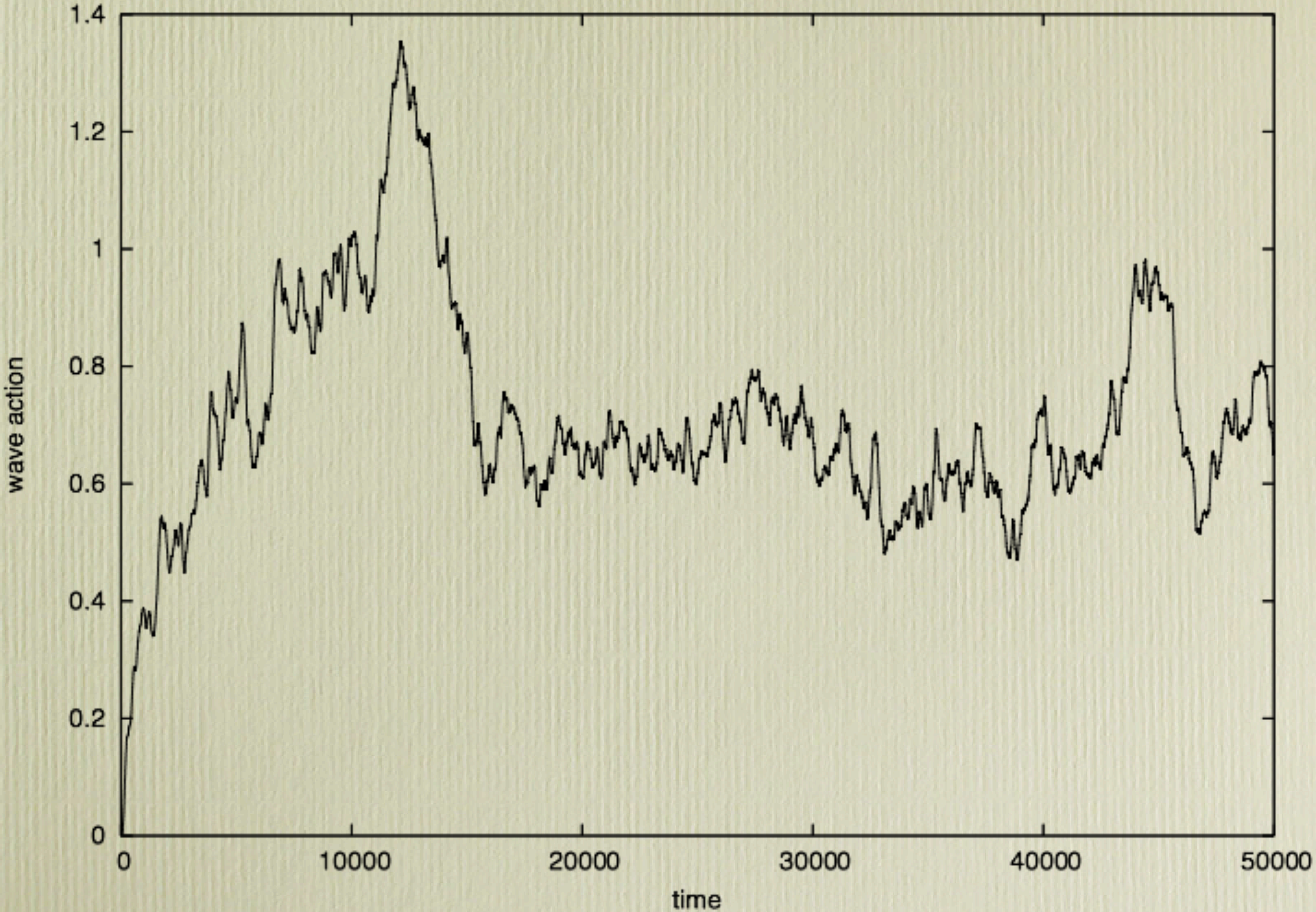

"dens.gnup.60" matr
8
6
4
2
0
-2
-4
-6
-8
0
150
200

Forced turbulence

- injection at large scale (white noise in a narrow wavenumber window)
- dissipation above a critical wavenumber kc
- wave action pumping (or not) at large scale to avoid numerical instability

Stationnary state?

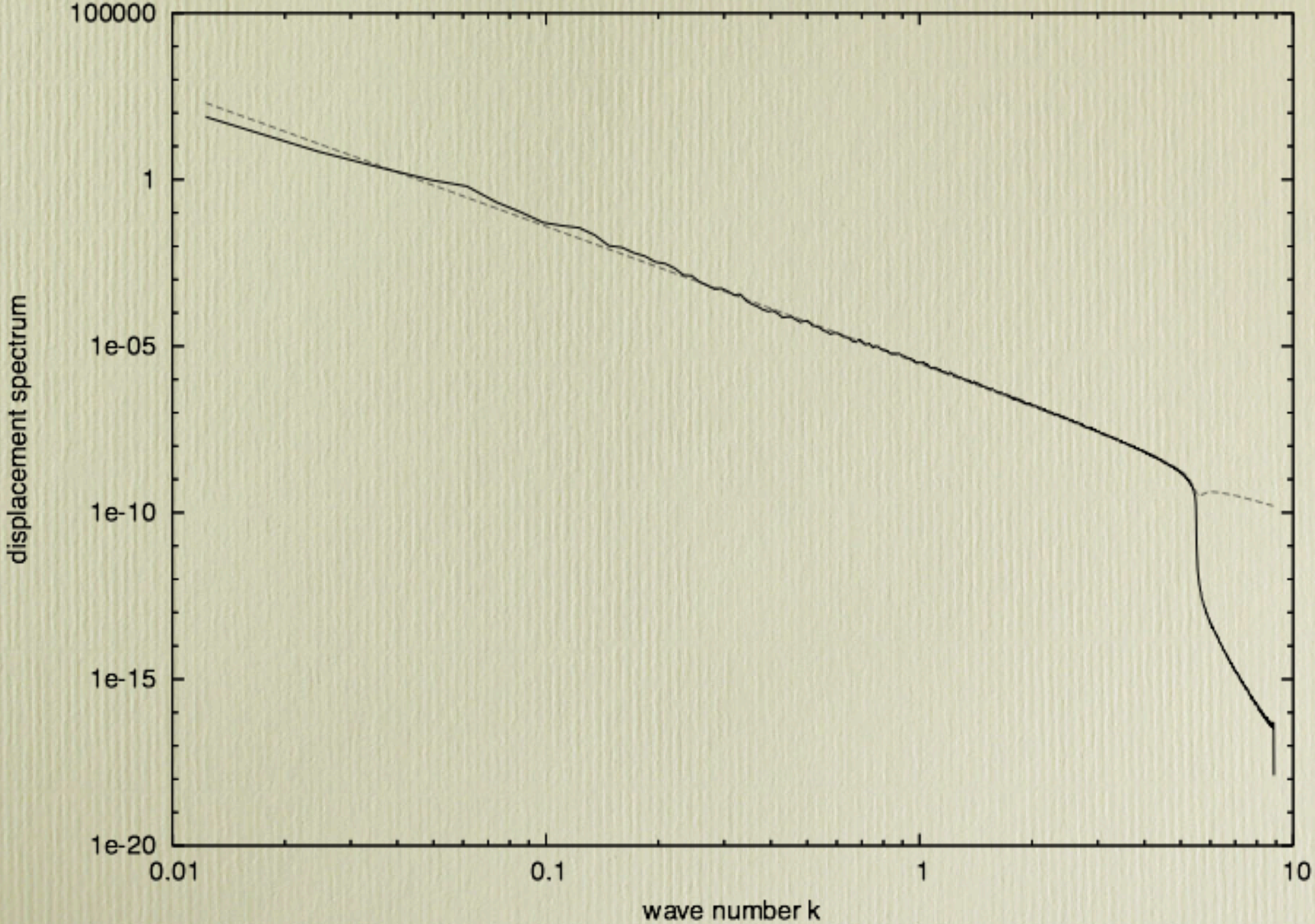


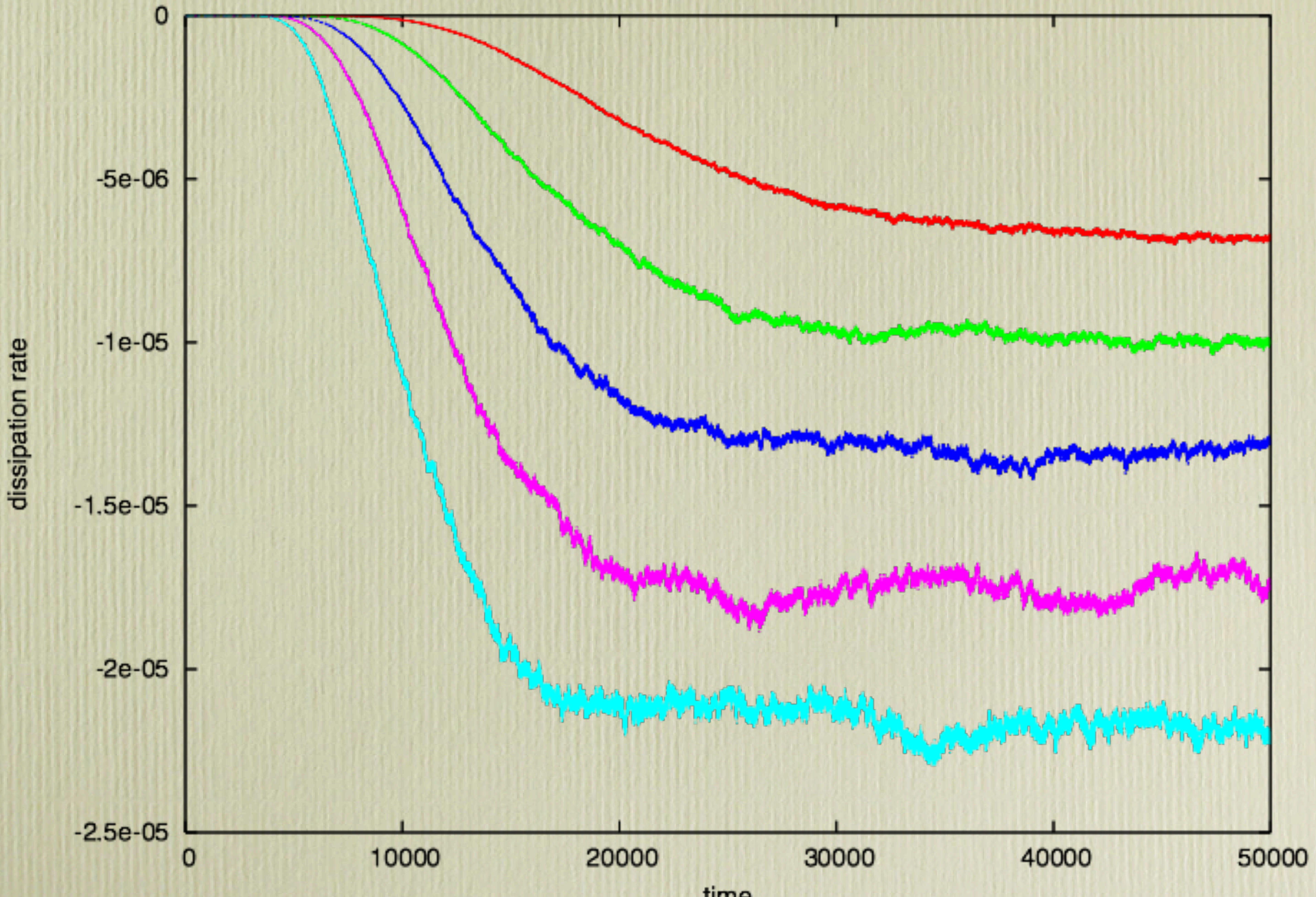


wave action
time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
10000
20000
30000
40000
50000

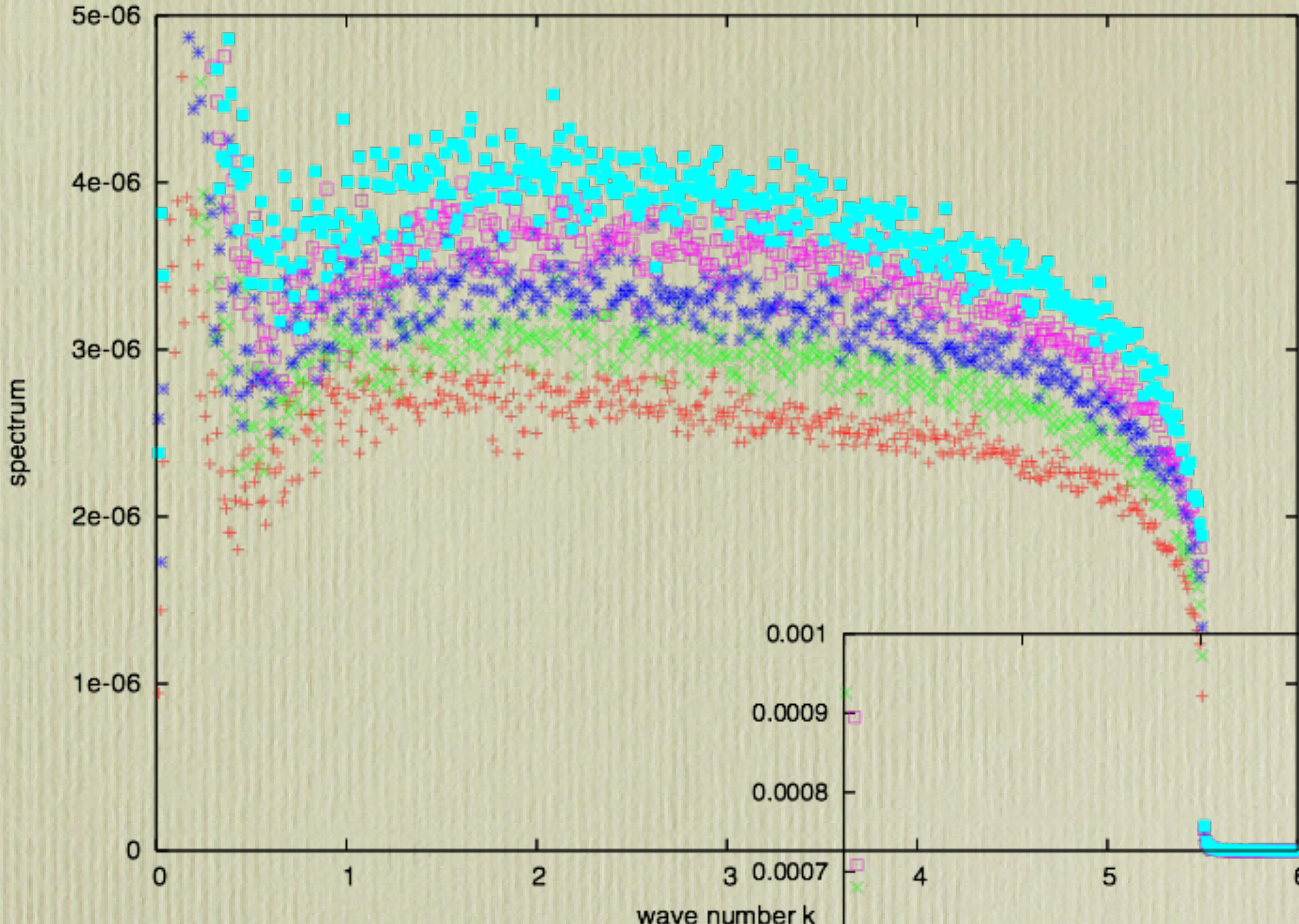
Spectrum

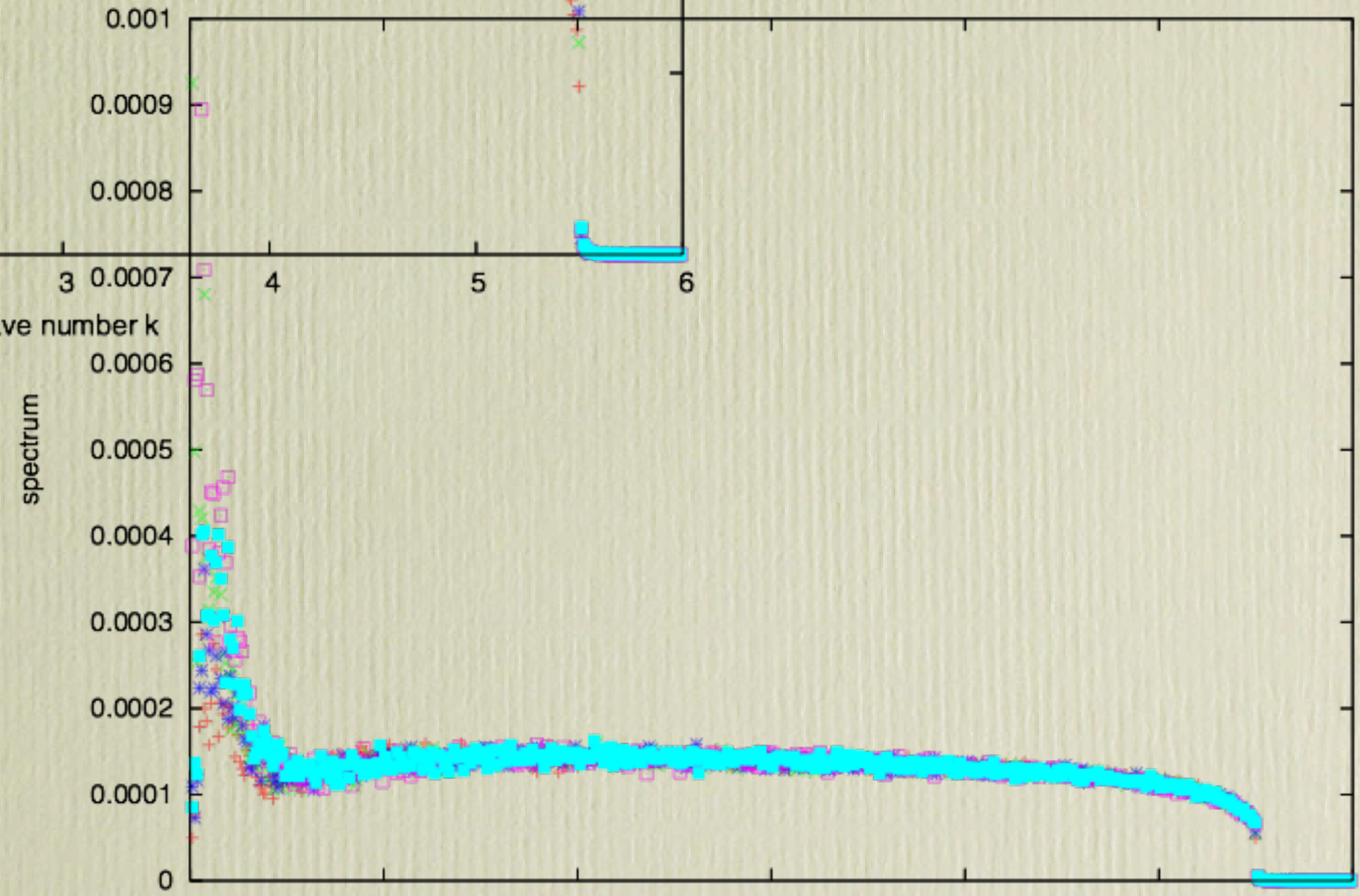
$$<|\zeta_k|^2>=\frac{n_k^{KZ}}{\rho\omega_k}\propto\frac{P^{1/3}log(kd/k)^{1/3}}{k^4}$$





0
-5e-06
-1e-05
-1.5e-05
-2e-05
-2.5e-05
dissipation rate
0
10000
20000
30000
40000
50000
time

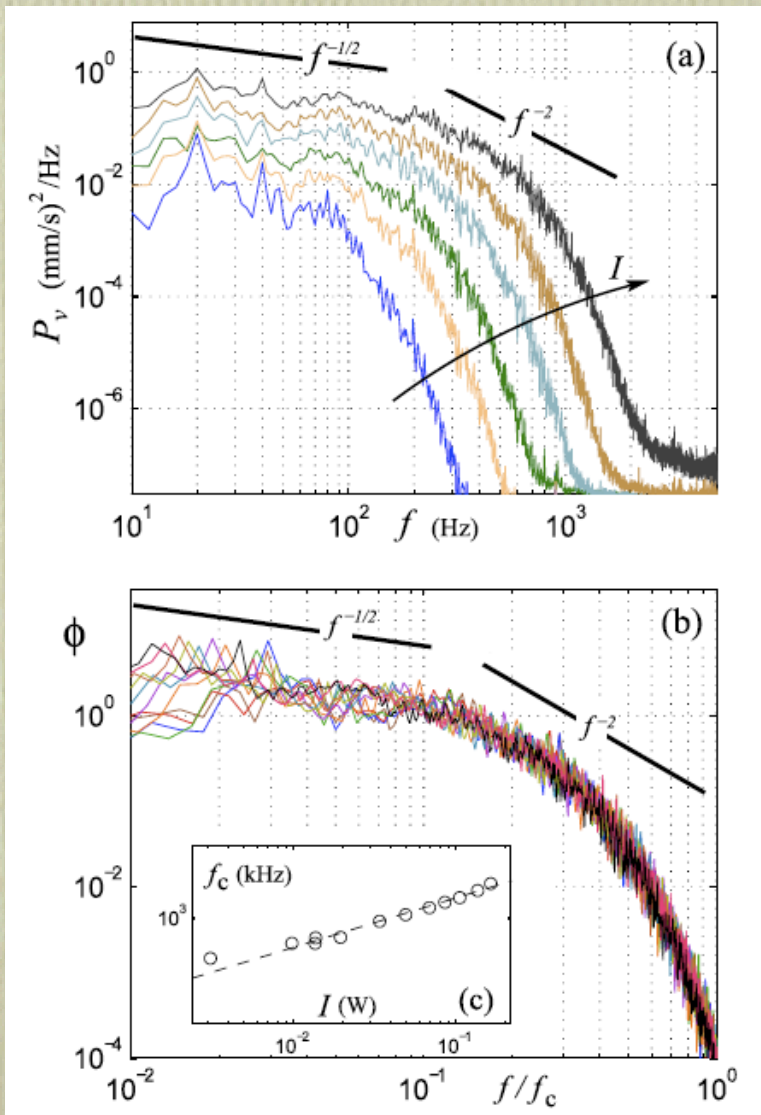

5e-06
4e-06
3e-06
2e-06
1e-06
0
spectrum
0
1
2
3
4
5
6
wave number k


0.001
0.0009
0.0008
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0
spectrum

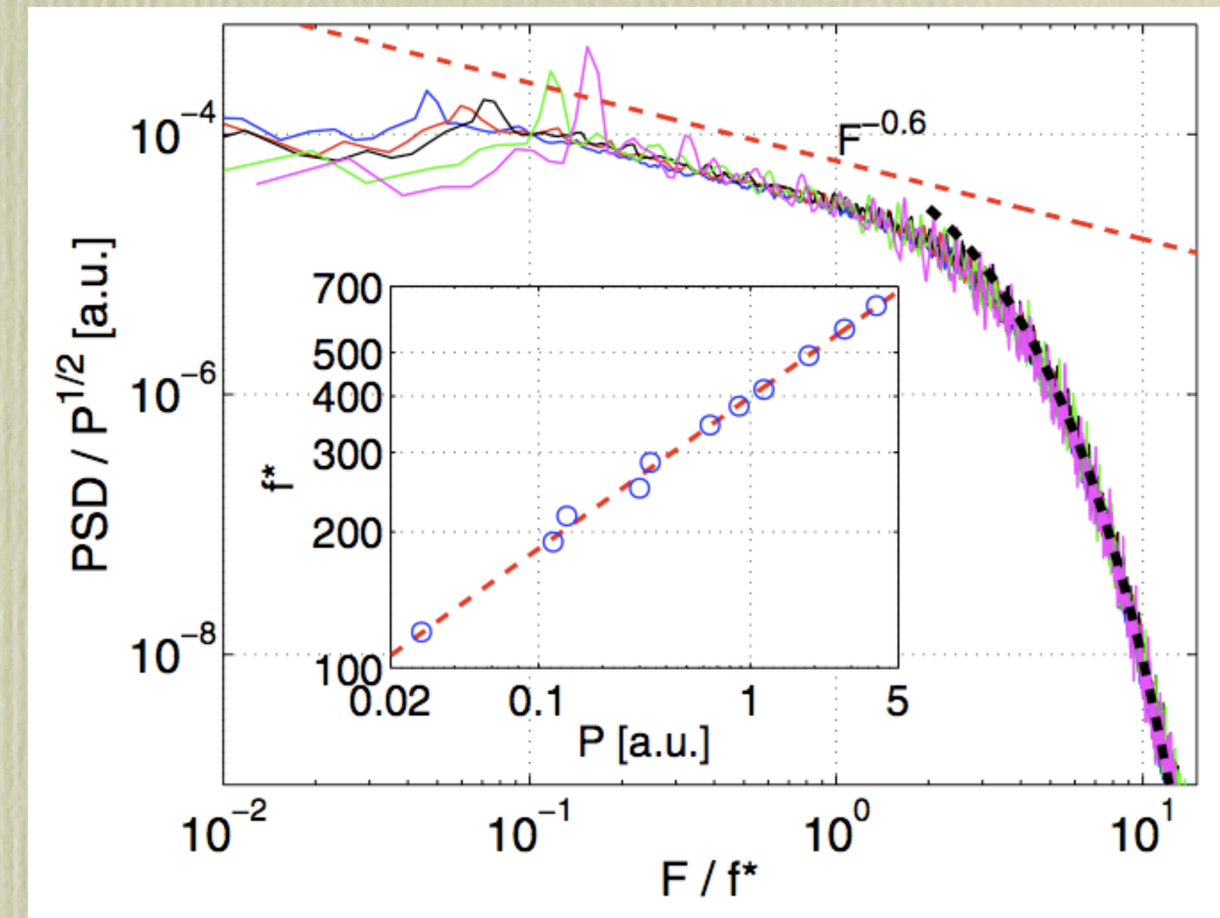
Experiment movie by
N. Mordant (LPS-ENS)

Experiments

$$<|\zeta_\omega|^2> \propto E_0$$

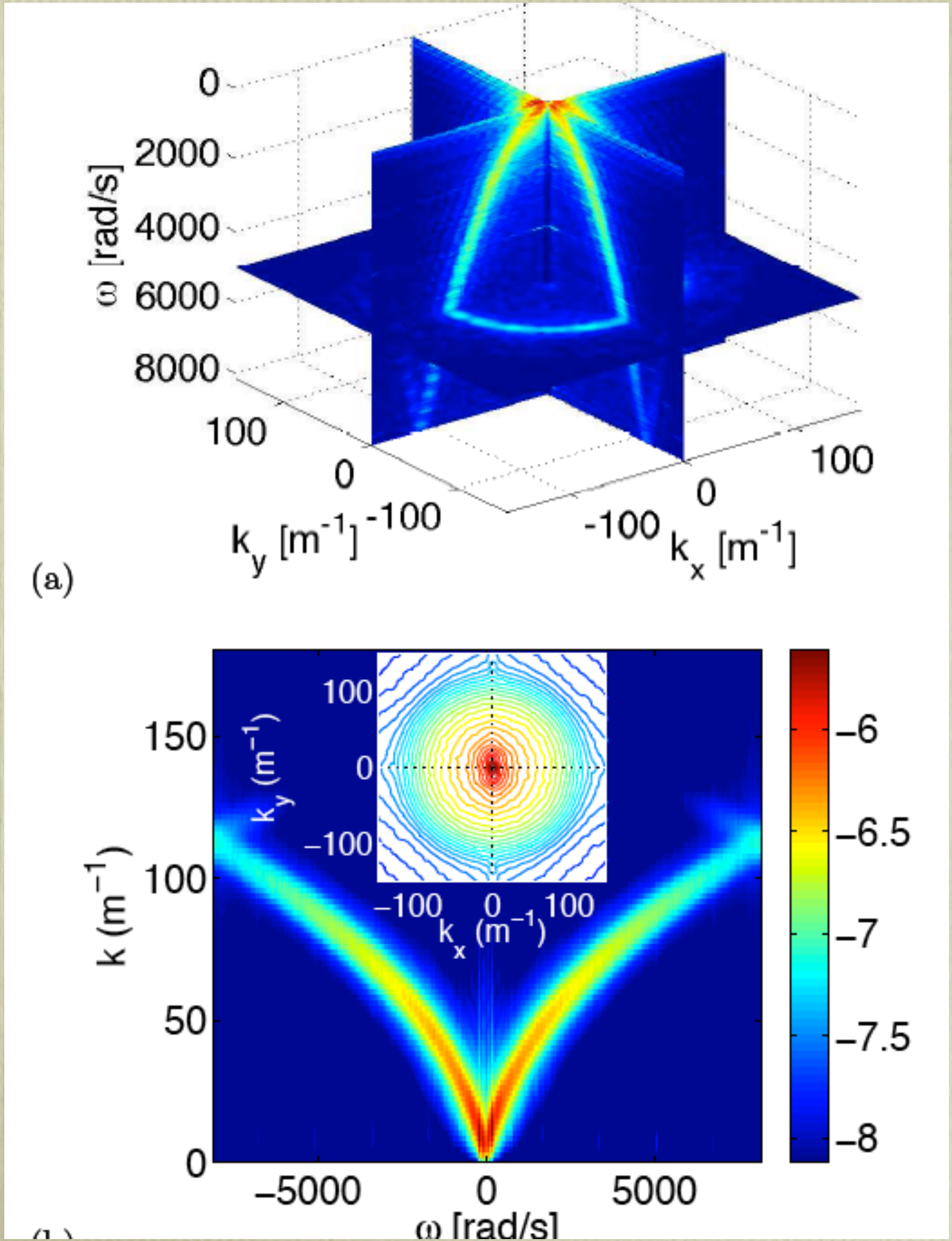


A. Boudaoud, O. Cadot, B. Odille & C. Touzé, PRL 234504 (2008).



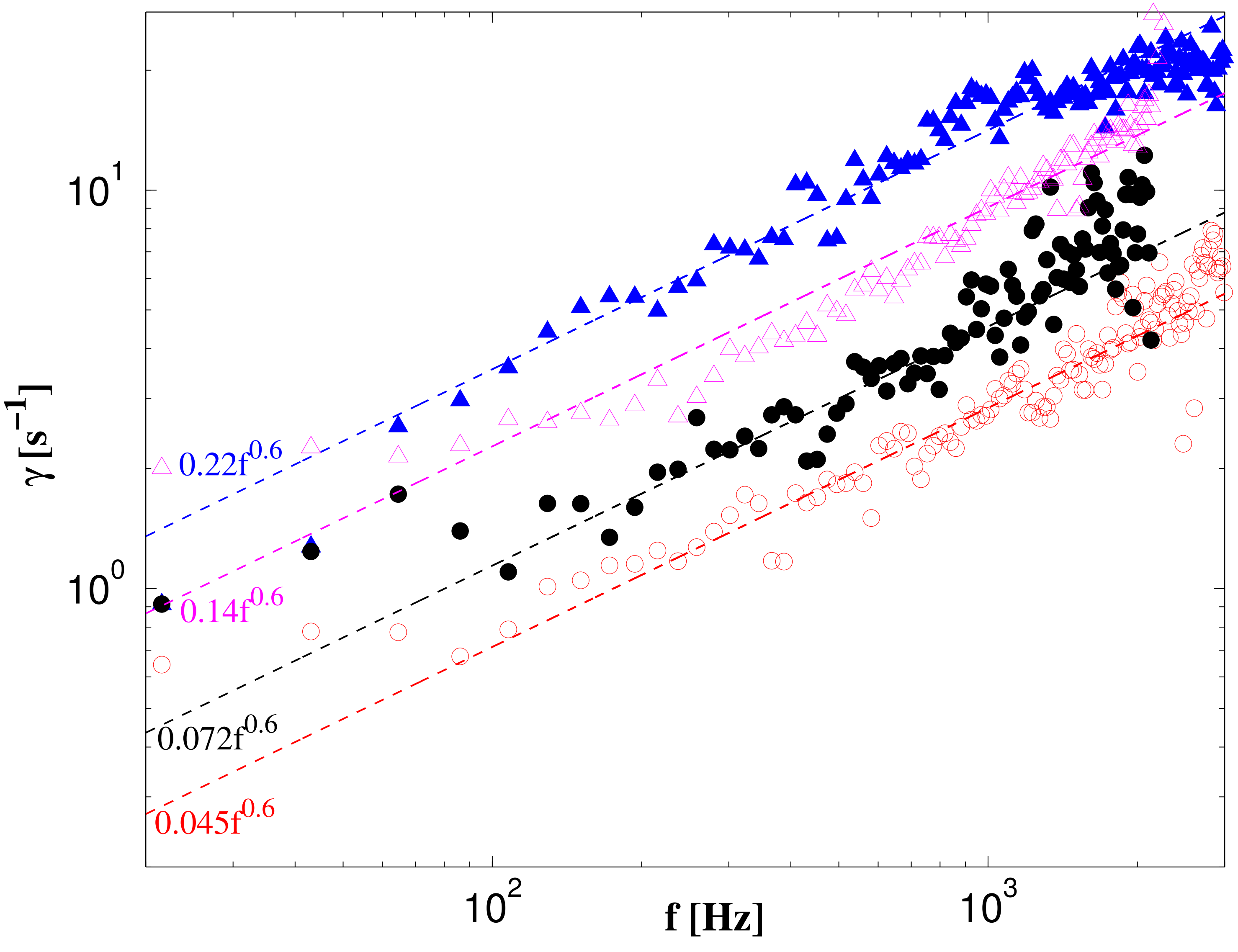
N. Mordant, PRL 234505 (2008).

- experiments show important differences with the theoretical predictions
- dispersion relation (mean curvature, three wave interactions, finite amplitude effects)
- boundary conditions
- an-isotropy-homogenous assumption
- dissipation
- consensus emerges

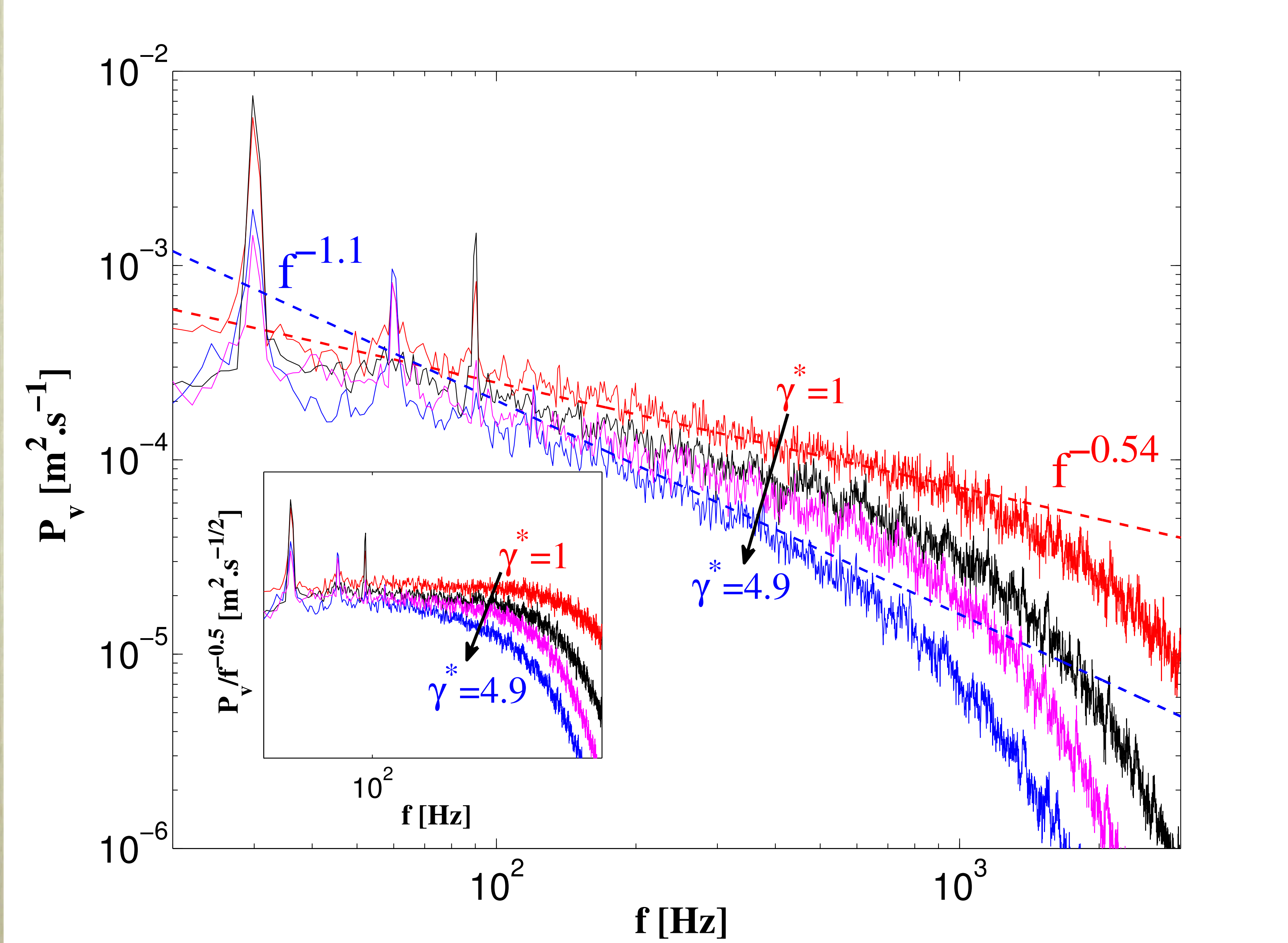


P. Cobelli, P. Petitjeans, A. Maurel, V. Pagneux & N. Mordant, PRL 103, 204301 (2009).

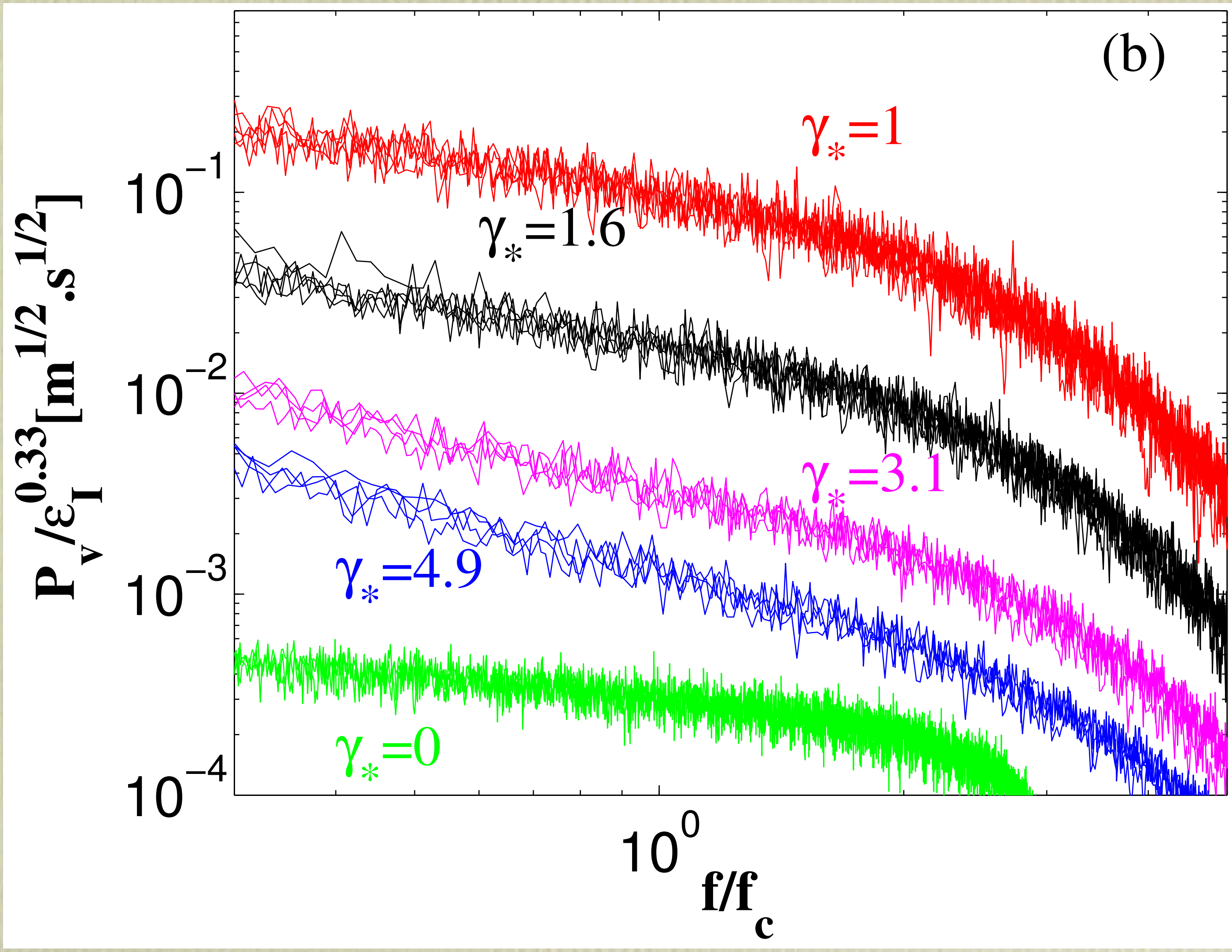
Changing dissipation



Changes slope!



Numerically also!



Phenomenological model

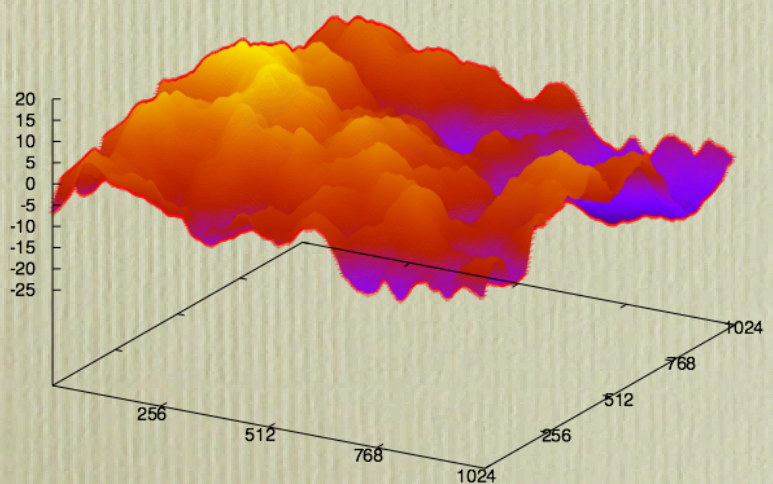
- difficult to handle dissipation in the asymptotic theory (« Hamiltonian » framework)
- phenomenological model (These T. Humbert)
- deduce by seeking local equation with RJ and KZ solutions.

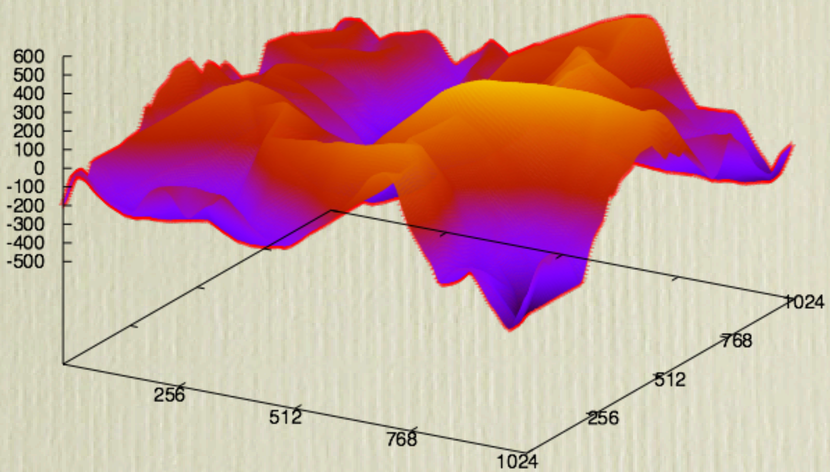
$$\partial_t E_\omega = \partial_\omega(\omega E_\omega^2 \partial_\omega E_\omega)$$

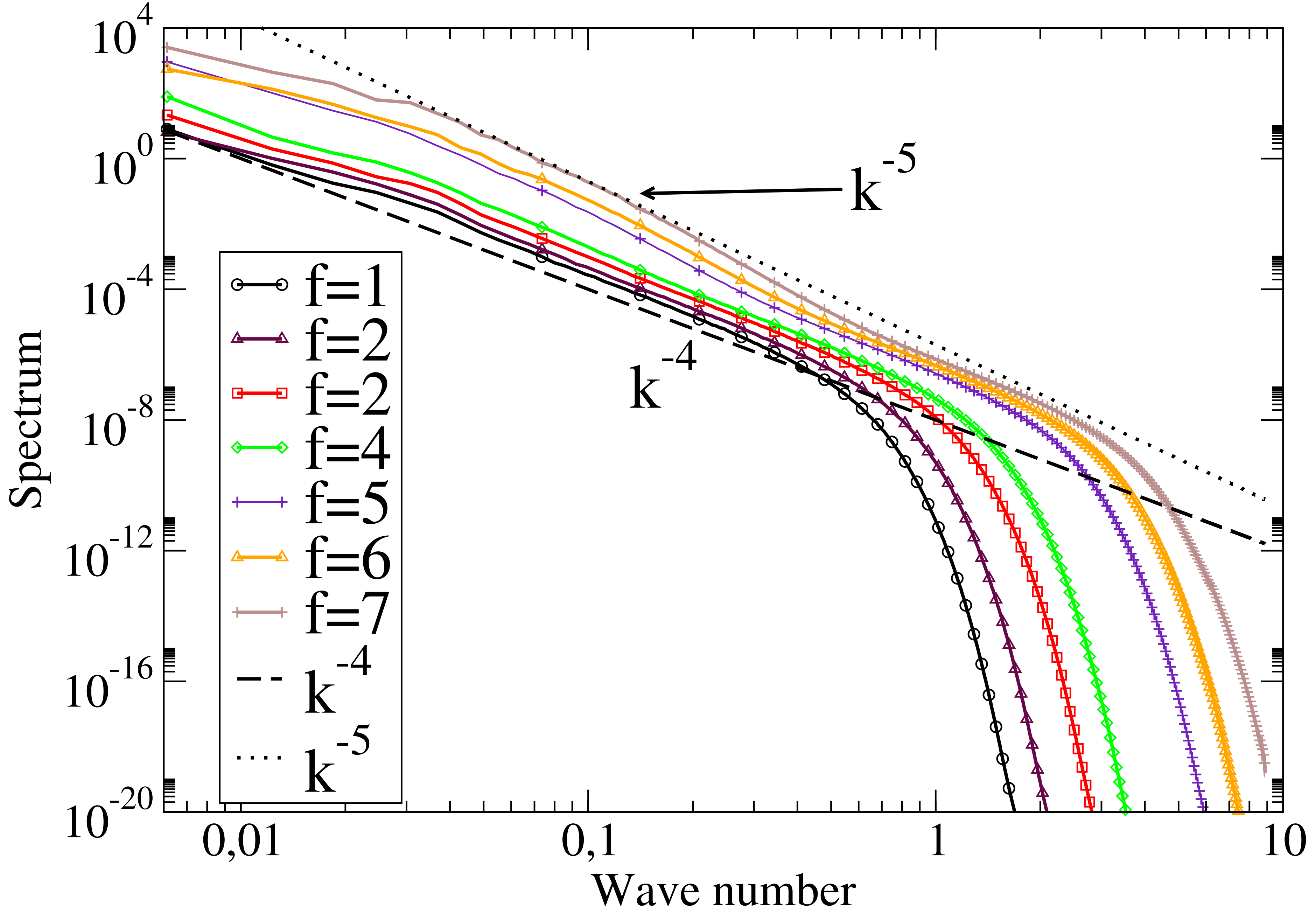
$$\partial_t E_\omega = \partial_\omega(\omega E_\omega^2 \partial_\omega E_\omega) - \hat{\gamma} E_\omega,$$

High forcing

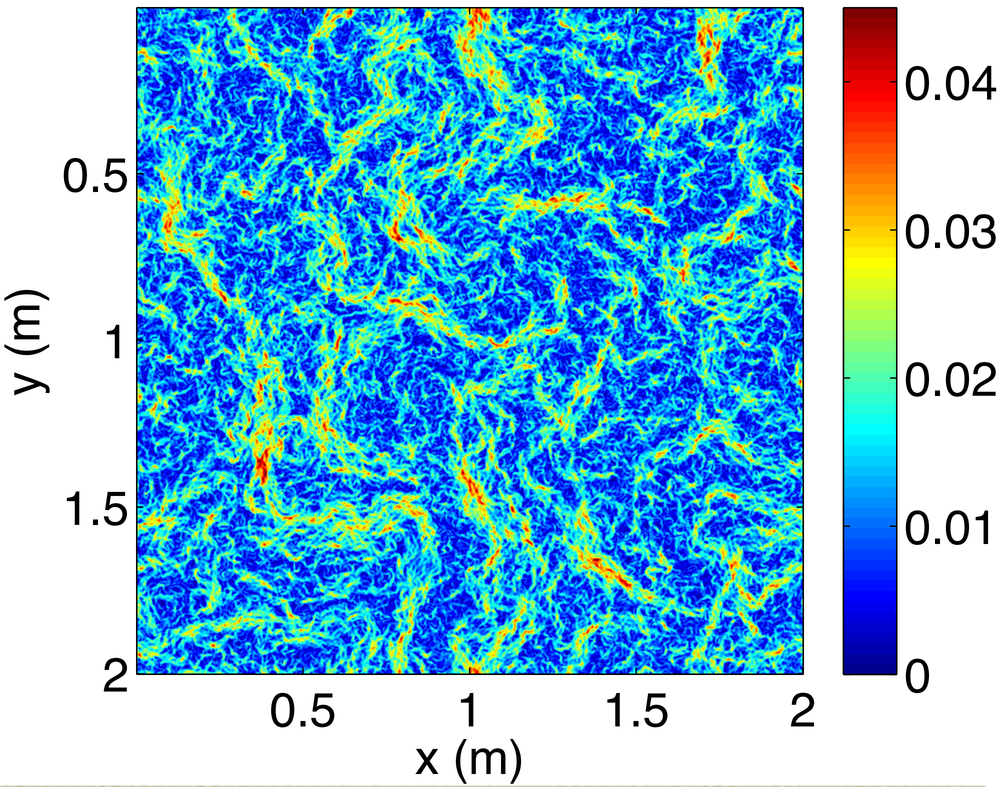
- At high forcing, the spectra are changing and large scale structures appear
- Intermittency?

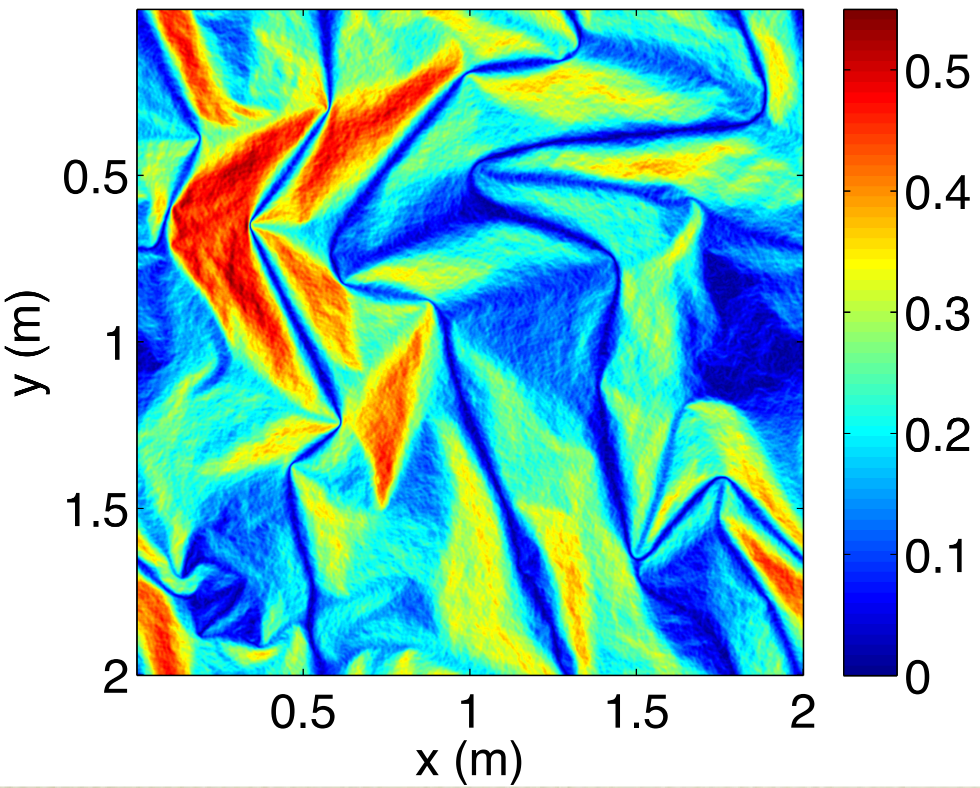






10^4
10^0
10^{-4}
10^{-8}
10^{-12}
10^{-16}
10^{-20}
Spectrum
0,01
0,1
1
10
Wave number
k^{-5}
k^{-4}
f=1
f=2
f=2
f=4
f=5
f=6
f=7
k^{-4}
k^{-5}


0.5
1
1.5
2
y (m)
0.5
1
1.5
2
x (m)
0.04
0.03
0.02
0.01
0

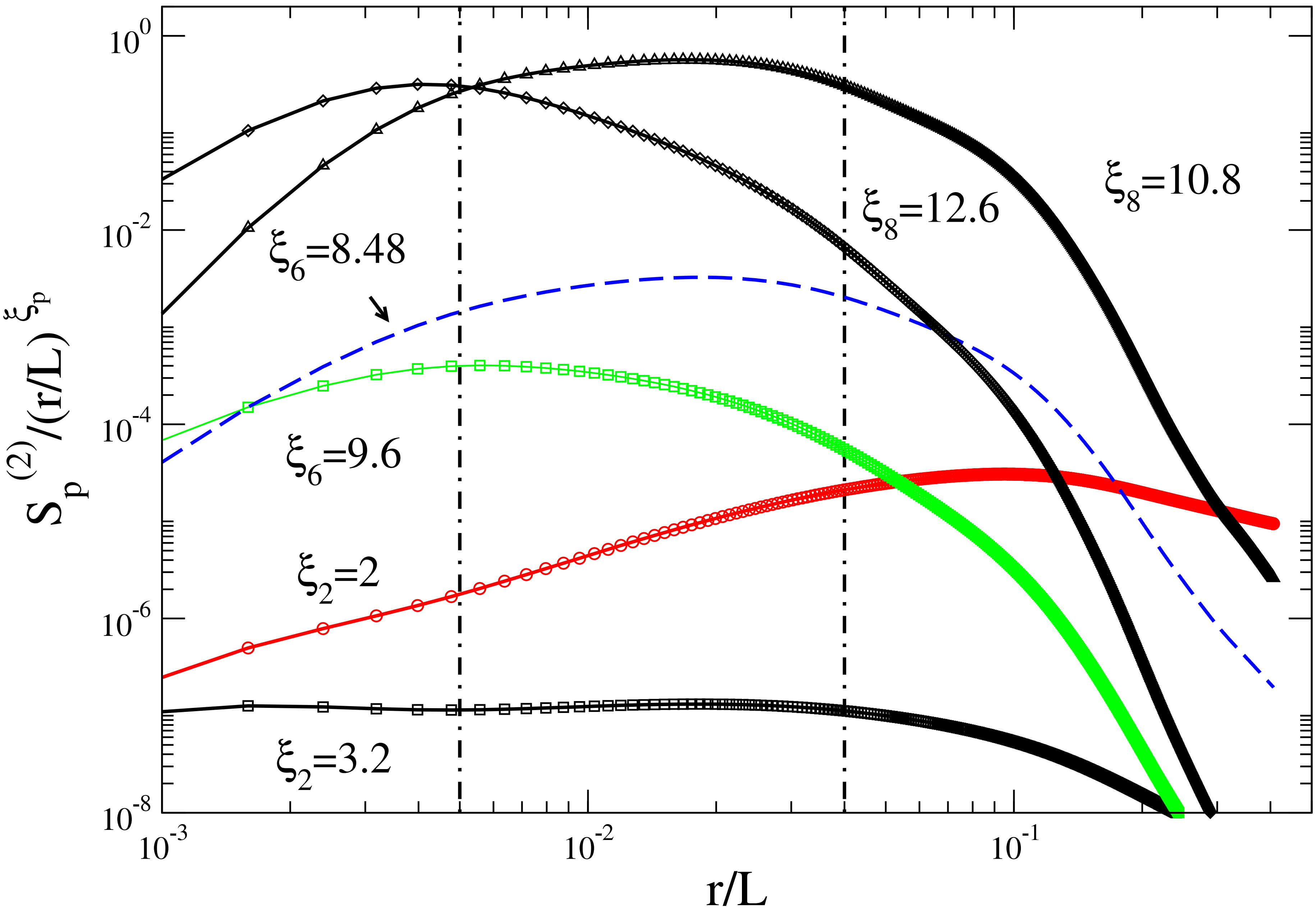

0.5
1
1.5
2
y (m)
0.5
1
1.5
2
x (m)
0.5
0.4
0.3
0.2
0.1
0

Intermittency analysis

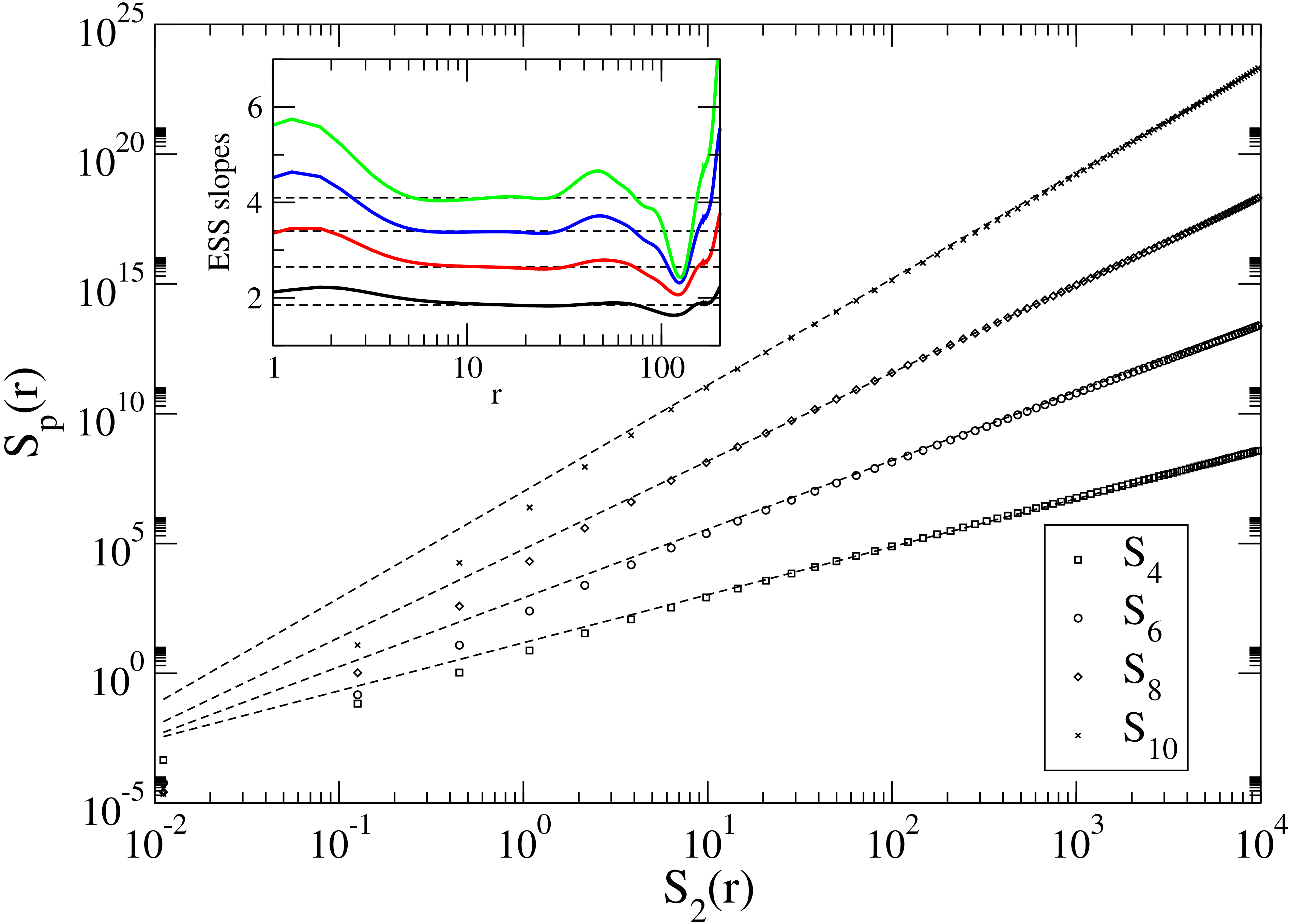
- Structure function on the increments of the displacements
- Intermittency analysed as the difference between results obtained assuming gaussian distribution of fluctuations

$$S_p^2(r) =< |\delta\zeta^2(\mathbf{x}, \mathbf{r})|^p >$$

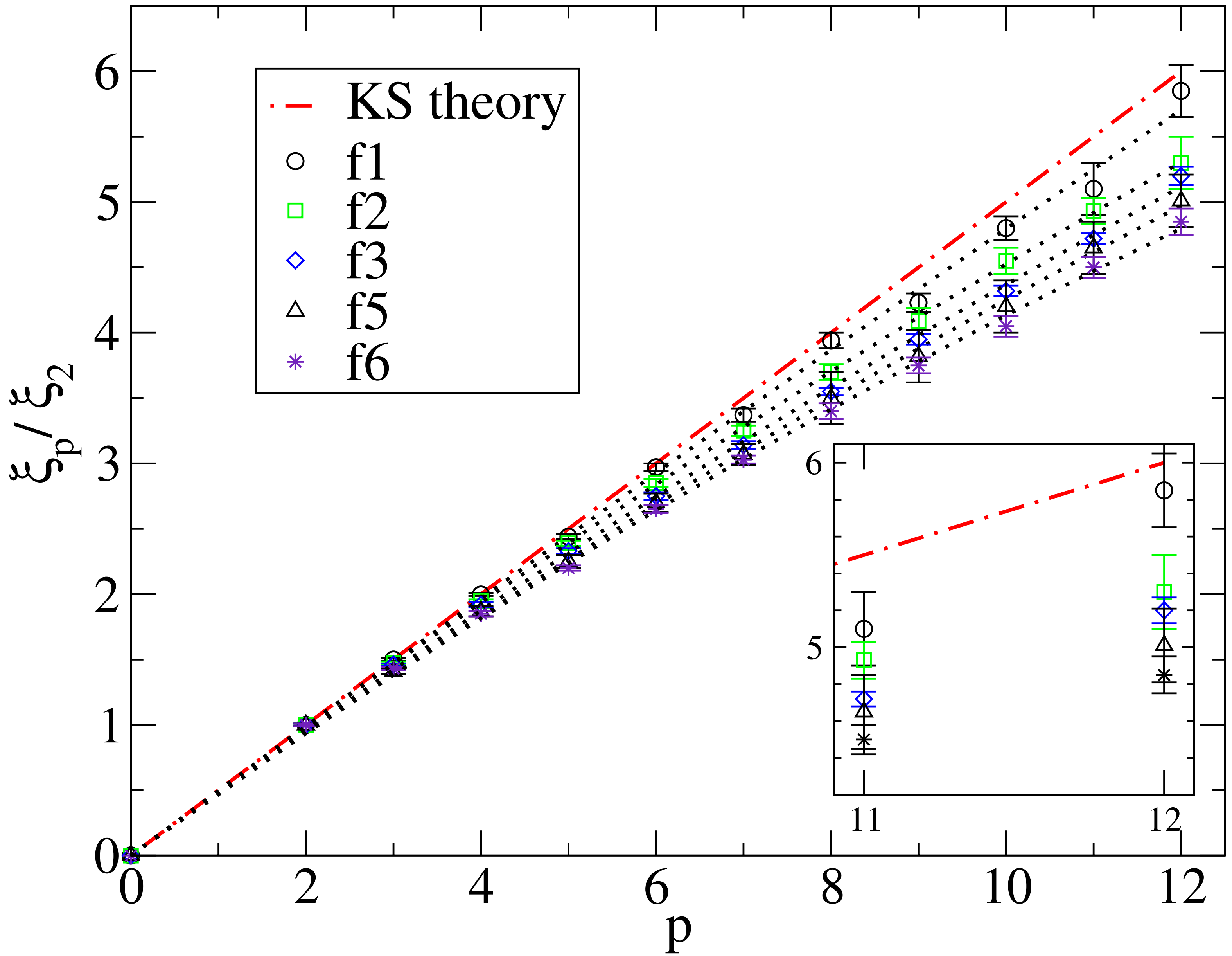
$$\delta\zeta^2(\mathbf{x}, \mathbf{r}) = \zeta(\mathbf{x} + \mathbf{r}) - \mathbf{2}\zeta(\mathbf{x}) + \zeta(\mathbf{x} - \mathbf{r})$$



ξ_8=10.8
ξ_8=12.6
ξ_6=8.48
ξ_6=9.6
ξ_2=2
ξ_2=3.2
$S_p^{(2)}/(r/L)^{\xi_p}$
r/L
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-3}
10^{-2}
10^{-1}



ESS slopes
6
4
2
1
10
100
r
$S_p(r)$
10^{25}
10^{20}
10^{15}
10^{10}
10^{5}
10^{0}
10^{-5}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
$S_2(r)$
S_4
S_6
S_8
S_{10}



KS theory
f1
f2
f3
f5
f6
ξ_p/ξ_2
p
0
1
2
3
4
5
6
0
2
4
6
8
10
12
5
6
11
12

Conclusion

- vibrating plates is a great tool for investigating WT concepts
- WT applies and KZ spectrum is predicted
- dissipation (at all scales) is a good candidate to explain the difference with experiments: need a modified WT theory to account for dissipation at all scales
- inverse cascade-transfer observed
- Intermittency and breakdown of WT