Asymptotic Computation of Invariant Manifolds of large Finite Element structures with Geometric Nonlinearities

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Summary. In this contribution we present a method to directly compute asymptotic expansion of invariant manifolds of large finite element models from physical coordinates and their reduced order dynamics on the manifold. We show the accuracy of the reduction method on selected models, exhibiting large rotations and internal resonances. The results obtained with the reduction compared to full-order harmonic balance simulations show that the proposed methodology can reproduce extremely accurately the dynamics of the original systems with a very low computational cost.

Introduction

Slender structures in large amplitude vibration exhibit geometric nonlinear effects that modify their dynamics. To accurately model structures with complex geometries, large finite element models with a high number of degrees of freedom are often required. Numerically, the geometric nonlinear forces affect all the elements in the structure thus making the simulation of such problems very demanding in terms of computational cost. Reduced order models of geometrically nonlinear structures are then an attractive solution to drastically reduce the size of the problem whilst maintaining the accuracy.
In damped systems, slow invariant manifolds are low-dimensional attractors for the dynamics of high-dimensional nonlinear systems, and therefore represents the ideal candidate for model order reduction. However, their computation was until very recently limited to small system written in modal coordinates [1]. Recent developments by the authors [2-4] have proposed the Direct Parametrisation of Invariant Manifolds method (DPIM) to compute invariant manifolds directly in physical coordinates, thus extending the applicability of said model order reduction strategy to large FE structures. A similar work is done in [5].

Direct parametrization of Invariant Manifolds (DPIM)

Mechanical structures in large deformations display geometric nonlinearities in the form of quadratic and cubic restoring forces. When discretised with the finite elements method, the dynamics of such structures can be written as a system of N ordinary differential equations in time, with N the number of degrees of freedom:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{G}(\mathbf{U},\mathbf{U}) + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = \mathbf{0} \tag{1}$$

Invariant manifolds tangent to a given linear *master* subspace of the system are computed as an arbitrary order (o) asymptotic expansion of the physical coordinates $\mathbf{U}$ in terms of the so-called *normal* coordinates $\mathbf{z}$, which are tangent to the linear modal coordinates of the selected *master* mode:

$$\mathbf{U} = \sum_{p=1}^{o} \mathbf{\Psi}^{(p)}(\mathbf{z}) \tag{2}$$

The reduced dynamical model, which represents the dynamics on the computed manifold, is also built as a polynomial function of the normal coordinates in first-order form:

$$\dot{\mathbf{z}} = \sum_{p=1}^{o} \mathbf{f}^{(p)}(\mathbf{z}) \tag{3}$$

The coefficients of the change of coordinates $\mathbf{\Psi}$ and reduced dynamics $\mathbf{f}$ are then found iteratively order by order by assembling a so-called homological equation for each monomial in $\mathbf{z}$, which results in the solution of a series of $N \times N$ linear system of equations.

Numerical Results

Since each homological equation is underdetermined, different styles of parametrisation can be chosen by selecting appropriate resonance conditions, namely the Real Normal Form style (RNF), the Complex Normal Form style (CNF), and the Graph style [3]. The main difference between Normal Form styles and the Graph style is that in the former the manifold is parametrised as an embedding, whereas in the latter it is parametrised as a graph. In most cases, the results provided by each style are very similar but in case the computed manifold presents a folding, the graph style would diverge in

the vicinity of the folding point [6]. Physically, the folding of an invariant manifold can be seen as the point where the nonlinear mode becomes less and less represented by its linear counterpart, at a level such that as the nonlinear modal amplitude continues increasing, then the linear one starts to decrease.
The first nonlinear mode of a cantilever beam is studied here, and it is shown in Fig. 1 that the graph style fails at representing the full system dynamics when the amplitude of oscillations has order of magnitude comparable to the beam length. This failure can be observed both in the non-physical softening effect in the backbone curve and in the divergence of the invariant manifold. On the other hand, both normal form styles can perfectly reproduce the folding in the manifold and the backbone curves obtained with these styles are perfectly overlapped with that obtained from continuation of the full order model, up to very large amplitude oscillations. This exemplifies how powerful the method is in perfectly reproducing the dynamics of a large finite element system with a single oscillator reduced order model.

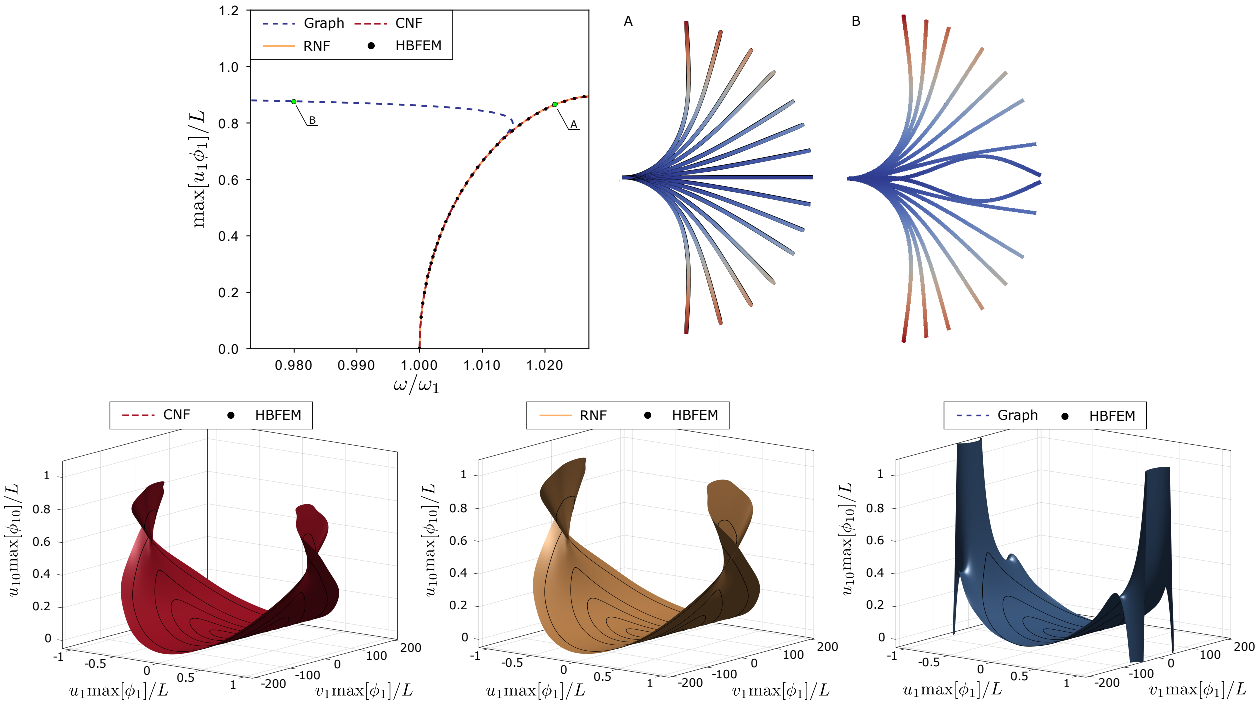

Figure 1: Model reduction of the first nonlinear mode of a cantilever beam up to large amplitude vibration. Backbone curves (top left), reconstruction of reduction at point A and B along the backbone curves (top right), and invariant manifolds (bottom). Three different parametrisation styles are used, and the full order model solution is obtained with harmonic balance method (HBFEM)

Conclusions

Nonlinear modes have become an essential tool for engineers in that they proNonlinear modes have become an essential tool for engineers in that they provide crucial insights on the dynamics of structures. The computation of nonlinear modes and their associated invariant manifolds for model order reduction is of utmost importance for modern industry. In this respect, the extension of nonlinear modes calculations to large scale finite element models is a critical issue that will allow the use of this tool in industrial scale problems. The method proposed here is based on a classical asymptotic expansion of invariant manifolds, but its computation is adapted to the case of FE models with a high number of degrees of freedom. The rigorous mathematical foundation of the method coupled with a thoughtful implementation make it extremely accurate and fast, thus making it the ideal candidate for reduced order modelling of geometrically nonlinear structures in an industrial framework.

References

[1] Touzé C., Vizzaccaro A., Thomas O. (2021) Model order reduction methods for geometrically nonlinear structures: a review of nonlinear techniques. *Nonlinear Dynamics*, **105**, 1237-1272.

[2] Vizzaccaro A., Shen Y., Salles L., Blahoš J., Touzé C., (2021) Direct computation of nonlinear mapping via normal form for reduced-order models of finite element nonlinear structures. *Comput Methods Appl Mech Eng*, **284**, 113957.

[3] Vizzaccaro A., Opreni A., Salles L., Frangi A., Touzé C. (2021, under review) High order direct parametrisation of invariant manifolds for model order reduction of finite element structures: application to large amplitude vibrations and uncovering of a folding point. Submitted to *Nonlinear Dynamics*.

[4] Opreni A., Vizzaccaro A., Frangi A., Touzé C., (2021) Model Order Reduction based on Direct Normal Form: Application to Large FE MEMS Structures Featuring Internal Resonance. *Nonlinear Dynamics*, **105**, 1237-1272.

[5] Jain S., Haller G. (2022, to appear) How to Compute Invariant Manifolds and their Reduced Dynamics in High-Dimensional Finite-Element Models? *Nonlinear Dynamics*, in press.

[6] Ponsioen S., Haller G., (2018) Automated computation of autonomous spectral submanifolds for nonlinear modal analysis. *J Sound Vib*, **420**, 269 – 295.