Model order reduction methods based on normal form for geometrically nonlinear structures: a direct approach

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Summary. Normal form approach applied to nonlinear vibratory systems defines a nonlinear mapping from modal coordinates to normal coordinates that are linked to the invariant manifolds of the system, thus allowing one to derive efficient reduced-order models. In the context of finite element discretisation, the drawback of this approach is the need to compute first the complete linear basis together with the modal nonlinear coupling coefficients, that can be obtained thanks to a non-intrusive method (*e.g.* StEP method). We propose here a direct method to compute the coefficients of the normal form and the reduced-order models without the need of prior computation of the modal basis. The method gives the nonlinear modes of a structure with an exact formulation up to the third order directly from the system in physical coordinates. This feature enables to circumvent the requirement of having the system in modal basis, thus rendering the method applicable to any discretised structure. Moreover, its non-intrusive implementation is possible in any finite element software.

Introduction

Normal form method has been recognized as a way to evaluate the nonlinear normal modes (NNMs) of a geometrically nonlinear system [1, 2]. More specifically, it defines a near-identity transformation in order to cancel as much as possible nonlinear terms and express the dynamics in an invariant-based span of the phase space. The usual starting point is thus an expression of the dynamics of the system in its modal basis, which is needed in order to apply general formulas as those given *e.g.* in [2, 3], that also contains the quadratic and cubic nonlinear coupling coefficients expressed in the modal basis. Provided that *all* the modes of the system have been computed together with the nonlinear coupling coefficients, then normal form allows one to obtain efficient reduced order model which are exact up to the order of the normal form, being computed theoretically from asymptotic expansions.
For small systems, the transformation from physical to modal coordinates is a straightforward operation, but for a generic finite element (FE) structure, the nonlinear coefficients are not explicitly available and they might be extracted from a FE software in a non-intrusive manner. The non-intrusive evaluation of these coefficients can be typically attained with methods such Stiffness Evaluation Procedure (StEP) [4], that allows to transform the geometrically nonlinear forces from physical basis to modal basis. More involved direct methods can also be used but with the need of getting inside the FE code, and compute at the element level the nonlienar stiffness [5].
However, for systems with many degrees of freedom, a full transformation would be computationally infeasible. In practical application, the modal basis is normally truncated up to a number of modes much smaller than the number of degrees of freedom (dofs). Due to this truncation, the exactness of the reduced-order model (ROM) is then subjected to a convergence study that can be cumbersome since numerous couplings with high-frequency modes exist in geometrically nonlinear structures. These inherent difficulties create severe limitations for a direct application of the normal form method to geometrically nonlinear structures discretized with finite elements.
The aim of this work is to remove this limitation by proposing a non-modal normal form approach, so that the general formula can be applied directly to the system expressed in physical coordinates. A third-order manifold in displacement and velocity coordinates is built with general expressions given directly from the FE dofs. All the proposed calculations can be realised in a non-intrusive manner. In this contribution, we first restrict ourselves to the motions on a single NNM, but the formula can easily be extended in order to build ROMs with an arbitrary number of modes.

Classical Normal form approach

The classical normal form approach is based on a near-identity transformation from modal to normal coordinates. Given a second-order dynamical system with quadratic and cubic nonlinearities, the equation of motion in modal basis reads:

$$\ddot{\mathbf{x}} + \mathbf{\Omega}^2\mathbf{x} + \mathbf{g}(\mathbf{x},\mathbf{x}) + \mathbf{h}(\mathbf{x},\mathbf{x},\mathbf{x}) = \mathbf{0}, \tag{1}$$

with $\mathbf{\Omega}^2$ the diagonal matrix of squared radian eigenfrequencies. The system can be expressed in first-order formulation by defining $\mathbf{y} = \dot{\mathbf{x}}$. The reduction of the dynamics up to third order to one single mode, say the one with label p, is done through a nonlinear mapping from the modal coordinates $\mathbf{x}$ and $\mathbf{y}$ to the normal coordinates of the nonlinear mode R_p and S_p. This nonlinear mapping is defined by:

$$\mathbf{x}(t) = \mathbf{e}_p R_p(t) + \mathbf{a}_{pp} R_p^2(t) + \mathbf{b}_{pp} S_p^2(t) + \mathbf{r}_{ppp} R_p^3(t) + \mathbf{u}_{ppp} R_p(t) S_p^2(t), \tag{2}$$

$$\mathbf{y}(t) = \mathbf{e}_p S_p(t) + \mathbf{c}_{pp} R_p(t) S_p(t) + \mathbf{m}_{ppp} S_p^3(t) + \mathbf{n}_{ppp} S_p(t) R_p^2(t), \tag{3}$$

with $\mathbf{e}_p$ the vector with entries $e_p^r = \delta_{rp}$, and δ_{rp} the usual Kronecker delta. The coefficients from the nonlinear change of coordinates are given in [2] for the undamped case, they read, for the first two ones:

$$\mathbf{a}_{pp} = -(2\omega_p^2\mathbf{I} - \mathbf{\Omega}^2)\,(4\omega_p^2\mathbf{I} - \mathbf{\Omega}^2)^{-1}\,(\mathbf{\Omega}^2)^{-1}\,\mathbf{g}(\mathbf{e}_p, \mathbf{e}_p), \tag{4}$$

$$\mathbf{b}_{pp} = -(2\mathbf{I})\,(4\omega_p^2\mathbf{I} - \mathbf{\Omega}^2)^{-1}\,(\mathbf{\Omega}^2)^{-1}\,\mathbf{g}(\mathbf{e}_p, \mathbf{e}_p). \tag{5}$$

All the other coefficients $\mathbf{c}_{pp}$, $\mathbf{r}_{ppp}$, $\mathbf{u}_{ppp}$, $\mathbf{m}_{ppp}$, $\mathbf{n}_{ppp}$ can be deduced in a vectorial form from the term-by-term entries given in [2]. Upon such a nonlinear mapping, the system of equation (1) will simply reduce to:

$$\ddot{R}_p + \omega_p^2 R_p + \left(h_{ppp}^p + A_{ppp}^p\right) R_p^3 + B_{ppp}^p R_p S_p^2 \;=\; 0\,, \tag{6}$$

with h_{ppp}^p, A_{ppp}^p, and B_{ppp}^p defined as:

$$h_{ppp}^p = \mathbf{e}_p \cdot \mathbf{h}(\mathbf{e}_p, \mathbf{e}_p, \mathbf{e}_p)\;, \qquad A_{ppp}^p = \mathbf{e}_p \cdot \mathbf{g}(\mathbf{e}_p, \mathbf{a}_{pp})\;, \qquad B_{ppp}^p = \mathbf{e}_p \cdot \mathbf{g}(\mathbf{e}_p, \mathbf{a}_{pp})\;.$$

This equation defines, up to third order, the dynamics on the p^{th} invariant manifold of the phase space and creates an efficient reduced-order model that predicts the correct type of nonlinearity.

Direct Normal form approach

The method developed in this article is an extension of the normal form approach suitable for systems for which the modal basis is not available. Given a second-order system in physical coordinates:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} + \mathbf{G}(\mathbf{X}, \mathbf{X}) + \mathbf{H}(\mathbf{X}, \mathbf{X}, \mathbf{X}) = \mathbf{0}, \tag{7}$$

it is possible to demonstrate that by operating the nonlinear mapping defined by:

$$\mathbf{X}(t) = \boldsymbol{\phi}_p R_p(t) + \boldsymbol{\alpha}_{pp} R_p^2(t) + \boldsymbol{\beta}_{pp} S_p^2(t) + \boldsymbol{\rho}_{ppp} R_p^3(t) + \boldsymbol{\upsilon}_{ppp} R_p(t) S_p^2(t), \tag{8}$$

$$\mathbf{Y}(t) = \boldsymbol{\phi}_p S_p(t) + \boldsymbol{\gamma}_{pp} R_p(t) S_p(t) + \boldsymbol{\mu}_{ppp} S_i p^3(t) + \boldsymbol{\nu}_{ppp} S_p(t) R_p^2(t), \tag{9}$$

with $\boldsymbol{\phi}_p$ the p-th mode, and with the coefficients reading as:

$$\boldsymbol{\alpha}_{pp} = -(2\omega_p^2\mathbf{I} - \mathbf{O}^2)\,(4\omega_p^2\mathbf{I} - \mathbf{O}^2)^{-1}\,(\mathbf{O}^2)^{-1}\,\mathbf{M}^{-1}\mathbf{G}(\boldsymbol{\phi}_p, \boldsymbol{\phi}_p), \tag{10}$$

$$\boldsymbol{\beta}_{pp} = -(2\mathbf{I})\,(4\omega_p^2\mathbf{I} - \mathbf{O}^2)^{-1}\,(\mathbf{O}^2)^{-1}\,\mathbf{M}^{-1}\mathbf{G}(\boldsymbol{\phi}_p, \boldsymbol{\phi}_p), \tag{11}$$

where:

$$\mathbf{O}^2 = \mathbf{M}^{-1}\mathbf{K}, \tag{12}$$

and with the vectors $\boldsymbol{\gamma}_{pp}$, $\boldsymbol{\rho}_{pp}$, $\boldsymbol{\upsilon}_{pp}$, $\boldsymbol{\mu}_{pp}$, $\boldsymbol{\nu}_{pp}$ defined similarly, then the system (7) will also reduce to Eq. (6). Moreover, H_{ppp}^p, A_{ppp}^p, and B_{ppp}^p can also be directly computed in physical coordinates from:

$$H_{ppp}^p = \boldsymbol{\phi}_p \cdot \mathbf{H}(\boldsymbol{\phi}_p, \boldsymbol{\phi}_p, \boldsymbol{\phi}_p)\;, \qquad A_{ppp}^p = \boldsymbol{\phi}_p \cdot \mathbf{G}(\boldsymbol{\phi}_p, \boldsymbol{\alpha}_{pp})\;, \qquad B_{ppp}^p = \boldsymbol{\phi}_p \cdot \mathbf{G}(\boldsymbol{\phi}_p, \boldsymbol{\beta}_{pp})\;.$$

It can be noticed that the structure of the equations of the direct approach reflects that of the classical formulation but here only mode p is computed. In fact, by comparing Eqs. (10)-(11) and (4)-(5) it is possible to observe that the full eigenvalues matrix $\mathbf{\Omega}^2$ is substituted by the matrix $\mathbf{O}^2$ that only requires the inversion of the mass matrix. Moreover, the quadratic and cubic nonlinear forces in modal coordinates are no longer required as the method uses their expression in physical coordinates. All these formulas can be computed in a non-intrusive manner using a standard FE code so that efficient ROM can be computed following these general formulas, inheriting from all the good properties of NNMs as invariant manifolds.

Conclusions

A direct normal form approach is presented in this study. The method can be extended to multiple nonlinear modes thus being suitable for the case of internal resonances, and giving an efficient procedure to build directly efficient ROM in a third-order approximations of invariant manifolds. The reduced order model provided is exact up to third order because all the modes are automatically included in the basis although their computation is not explicit. Numerical example on FE structures will be presented at the conference.

References

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