# Finite elements based reduced order models for geometrically nonlinear and piezoelectric thin structures: validation and three-dimensional effects

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*Summary*. This paper presents a general methodology to obtain a reduced order model (ROM) of geometrically nonlinear electromechanical structures with piezoelectric transducers. A standard modal reduction is used and the ROM is built using a finite-elements software thanks to a non-intrusive strategy. In this context, this article focus first on the validation of the proposed reduced order modelling strategy, especially for the piezoelectric part of the ROM, and second to the use of three-dimensional finite elements and associated convergence issues.

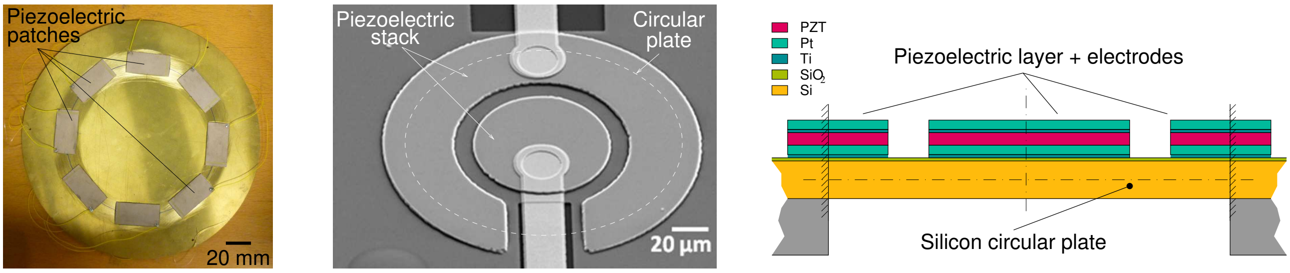


Figure 1: (left) Photograph of two examples of piezoelectric thin structures: a circular plate with eight piezoelectric patches and a micro-plate with two annular piezoelectric patches. (right) cutaway view of the microplate, showing its laminated structure

Geometrical nonlinearities, due to large transverse displacements of thin structures, are encountered in a large range of applications, as long as the thickness of the structure is small corresponding to the other dimensions. On the other hand, the use of piezoelectric materials to actuate or/and sense the vibrations is also widely spread. Applications, among others, range from vibration control [3, 11] to Micro/Nano-Electro Mechanical Systems (M/NEMS) developments, whose purpose can be to master and use the geometrically nonlinear behaviour [12, 14, 10]. This paper focus on the numerical computation of the frequency response, in the frequency domain, of structures with geometrical nonlinearities and equipped with piezoelectric patches. Recent advances in non-intrusive reduced-order finite element modeling of nonlinear geometric structures offer new perspectives for massive nonlinear prediction in structural computation [7]. An application on piezoelectric nanobridges of such a method has been proposed in [6, 15]. Apart those two references, to the knowledge of the authors, the case of both geometrical nonlinearities and piezoelectric electromechanical coupling has been scarcely considered in the past literature. The purpose of this paper is to fill this gap and to propose a non intrusive method able to efficiently compute the coefficients of a modal reduced order model, taking into account both the geometrical nonlinearities and the direct and converse piezoelectric couplings.

Common architectures of thin structures with piezoelectric layers (see examples on Fig. 1) are often laminated, such as beams with complex cross section [13] and laminated beams/plate structures [2, 4, 1]. When modelling such a structure in a finite-element context, one can consider equivalent single layer theories [9, 6] or simply use three-dimensional finite elements (3DFE). Because in our case some layers of the structure are piezoelectric, in most of the cases, only 3DFE are available in commercial codes. In this article, we will consider both approaches.

We consider an elastic structures with $P$ piezoelectric patches discretized by the finite-elements method. The vector containing the mechanical degrees of freedom is denoted by $\boldsymbol{U}(t)$ and $V^{(p)}$ denotes the voltage across the terminals of the $p$-th. piezoelectric patch. As shown in [6], considering a modal expansion of $\boldsymbol{U}(t)$ on $K$ modes $(\boldsymbol{\Phi}_k, \omega_k)$ of the structure with all piezoelectric patches short circuited, $\boldsymbol{U}(t) = \sum_{k=1}^{K} \boldsymbol{\Phi}_k q_k(t)$, one obtains the following system of equations for the modal coordinates $q_k(t)$ and the voltages $V^{(p)}$, $\forall\, k = 1, \dots K, p = 1, \dots P$:

$$
\left\{
\begin{aligned}
&\ddot{q}_k + 2\xi_k\omega_k\dot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{K} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{K} \gamma_{ijl}^k q_i q_j q_l + \sum_{p=1}^{P} \chi_k^{(p)} V^{(p)} + \sum_{p=1}^{P}\sum_{i=1}^{N} \Theta_{ik}^{(p)} q_i V^{(p)} = F_k, && \text{(1a)}\\
&C^{(p)} V^{(p)} - \sum_{k=1}^{K} \chi_k^{(p)} q_k - \sum_{i,j=1}^{K} \frac{1}{2}\Theta_{ij}^{(p)} q_i q_j = Q^{(p)}. && \text{(1b)}
\end{aligned}
\right.
$$

In the above equations, three independant groups of term can be identified. First, the geometrical nonlinearities create quadratic and cubic terms in $q_k$ with coefficients $\beta^k_{ij}$ and $\gamma^k_{ijlk}$. Secondly, the piezoelectric linear coupling creates two terms of coefficients $\chi^{(p)}_k$, which correspond to the exchange of energy between the $k$-th. vibration mode and the $p$-th. piezoelectric patch. Then, both geometrical nonlinearities and piezoelectric coupling are responsible of additional coupling terms, of coefficient $\Theta^{(p)}_{ij}$, not symmetric because of the $1/2$ factor in the second equation. In addition, $C^{(p)}$ is the electric capacitance of the $p$-th. patch, $F_k$ is the mechanical forcing and $Q^{(p)}$ is the electric charge contained in one of the electrodes of the $p$-th. patch.

In this context, this article has several purposes. First, it will be shown that it is possible to extend a particular non intrusive method, the so-called stiffness evaluation procedure (STEP), relying on the static application of prescribed displacements and introduced in [8], to compute the piezoelectric coefficients $\chi^{(p)}_k$ and $\Theta^{(p)}_{ij}$ using a commercial code in a non intrusive way. Then, even if the STEP has been validated in many articles in the past (see [5] and reference therein) to compute the geometrically nonlinear part of the ROM (the quadratic and cubic terms with coefficients $\beta^k_{ij}$ and $\gamma^k_{ijlk}$), it will be shown that using 3DEF is not as straightforward as it could appear at first and that it conducts to strange behaviours of slow convergence as a function of the number $K$ of modes retained in the expansion basis, much slower that in the case of the use of plate or shell two-dimensional finite-elements. Finally, the computation of the coupling coefficients $\chi^{(p)}_k$ and $\Theta^{(p)}_{ij}$ will be validated with comparison to an analytical reference model of a three layers hinged hinged beam with two colloacated piezoelectric patches.

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