# Non-stationary wave turbulence in elastic plates: a numerical investigation

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*Summary*. Nonlinear (large amplitude) vibrations of thin elastic plates can exhibit strongly nonlinear regimes characterized by a broadband Fourier spectrum and a cascade of energy from the large to the small wavelengths. This particular regime can be properly described within the framework of wave turbulence theory. The dynamics of the local kinetic energy spectrum is here investigated numerically with a finite difference, energy-conserving scheme, for a simply-supported rectangular plate excited pointwise and harmonically. Damping is not considered so that energy is left free to cascade until the highest simulated frequency is reached. The framework of non-stationary wave turbulence is thus appropriate to study quantitatively the numerical results. In particular, numerical simulations show the presence of a front propagating to high frequencies, leaving a steady spectrum in its wake, which has the property of being self-similar. When a finite amount of energy is given at initial state to the plate which is then left free to vibrate, the spectra are found to be in perfect accordance with the log-correction theoretically predicted. When forced vibrations are considered so that energy is continuously fed into the plate, a slightly steeper slope is observed in the low-frequency range of the spectrum. It is concluded that the pointwise forcing introduces an anisotropy that have an influence on the slope of the power spectrum, hence explaining one of the discrepancies reported in experimental studies.

## Introduction

When subjected to large-amplitude external loads, thin plates and shells can exhibit a complex vibratory regime that can be interpreted as turbulence in a solid medium [1, 2, 3]. Wave turbulence (WT) theory then allows the derivation of analytical statistical properties of this regime, as for instance the power spectrum of the vibration characterizing the energy repartition through scales in an idealized case where damping is not considered, and a window of transparency is assumed between the injection and the dissipative scales [1]. Comparison to experiments have however shown discrepancies [2, 3], which have been later attributed mainly to damping [4]. The aim of the present paper is to study numerically the WT regime in an elastic plate with a finite difference, strictly energy-conserving (or, in the lossy case, strictly dissipative) scheme, in a framework that is close to experiments by using realistic boundary conditions as well as a pointwise external forcing. In particular, the effect of forcing is investigated, in order to corroborate experimental observations reported in [5].

## Theoretical considerations

Wave turbulence in thin vibrating plates has been mostly studied within the framework of the von Kármán equations that describe the dynamics of the transverse displacement $w$. The analytical Kolmogorov-Zakharov (KZ) spectrum is found to be [1]:

$$P_{\dot{w}}(f) = \frac{Ch}{(1-\nu^2)^{2/3}}\varepsilon_c^{1/3}\log^{1/3}\left(\frac{f_c^\star}{f}\right); \tag{1}$$

where $\varepsilon_c$ is the constant flux of energy transferred through scales, $P_{\dot{w}}$ refers to the power spectrum of the transverse velocity $\dot{w}$, $h$ is the thickness of the plate, $\nu$ Poisson's ratio of the material, and $C$ a constant. Because the theory is fully inertial, $f_c^\star$ is the frequency at which energy is removed from the system.

In this study, a simply-supported, perfect rectangular plate is considered, with a pointwise harmonic forcing located at an arbitrary point of the plate. A second-order, finite difference method with an energy-conserving time integration scheme is used to solve the dynamical equations. The scheme is fully described in [7] and has already been used in [8] to study the transition to turbulence.

Damping is not considered so that the framework of non-stationary turbulence is used [9]. The simulations show a cascade front developing and leaving in its wake a stationary spectrum, having the property of being self-similar and which can thus be compared to the theoretical results given in Eq. (1).

## Simulation results

A number of simulations have been performed for different geometric configurations of the plate and constant material properties selected as $E$= 200 GPa, $\nu$=0.3 and $\rho$=7860 kg.m$^{-3}$. We first investigate the case of a perfect plate which is continuously forced harmonically, the excitation frequency being selected in the vicinity of the fourth eigenmode. Choice of the excitation frequency is only constrained by the fact that it should be small to activate properly the cascade, but another eigenfrequency could have been selected without changing the results. The displacement of an arbitrary point of the plate is selected for subsequent analysis, and we have checked that homogeneity is verified. Fig. 1(a) shows the spectrogram of a simulation where one can clearly observes the characteristic frequency that increases linearly with time. The spectrum left in the wake of the cascade front is self-similar, and can be rescaled as shown in Fig. 1(b) using the characteristic frequency defined as : $f_c = \frac{\int \langle P_v(f) \rangle f \, df}{\int \langle P_v(f) \rangle \, df}$. As compared to the theoretical prediction given in Eq. (1), one can observe that the spectrum is slightly steeper that the theoretical log-correction.

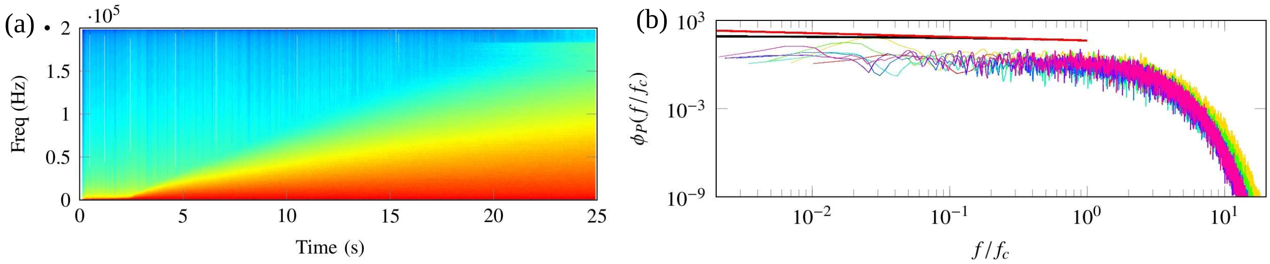


Figure 1: (a) Spectrogram of the velocity for a plate of lateral dimensions $L_x$=0.4m, $L_y$=0.6m, thickness $h$=1mm, harmonically forced at 75 Hz with a constant amplitude of 10 N, sampling rate 400 kHz. (b) Normalized spectra with the characteristic frequency $f_c$ and amplitude $P_v(f_c)$. Continuous black line shows the log correction $\log^{1/3}(\frac{f_c}{f})$ of the KZ spectrum, Eq. (1), dashed red line shows a power law $f^{-\frac{1}{4}}$.

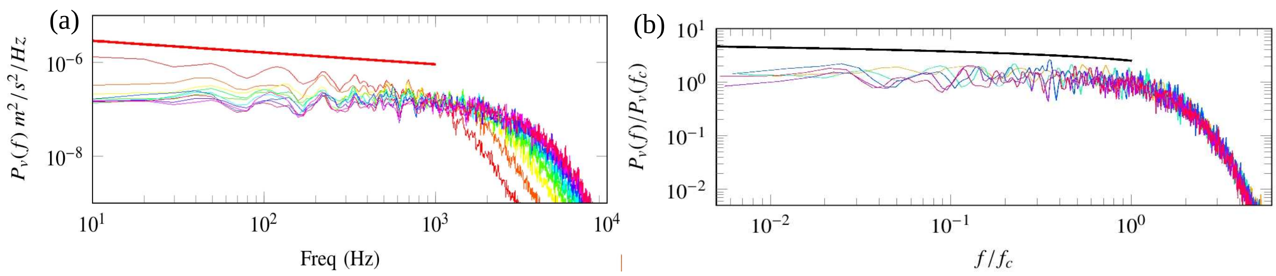


Figure 2: Perfect, undamped plate, $L_x$=0.4m, $L_y$=0.6m, thickness $h$=0.1mm, sampling rate 40 kHz, for which the forcing is stopped after 0.1s. (a) Velocity power spectra averaged over 10s, displayed for time intervals of 30s. The first (red) is computed from 0.1s (i.e. the end of the forcing) to 10.1s. The straight red line corresponds to the power law $f^{-1/4}$. (b) Normalized spectra, black line: log-correction of the KZ spectrum.

The effect of the forcing is further investigated with a simulation where the forcing is stopped after 0.1s, hence giving to the system a finite amount of energy and allowing free vibration. In this case the characteristic frequency evolves as $t^{1/3}$. The flattening of the spectrum is clearly shown in Fig.2(a), and the rescaled spectrum, Fig.2(b), displays now a perfect agreement with the log-correction of the KZ spectrum. This numerical result is in line with the experiments described in [5], and should be interpreted likewise, as a consequence of the anisotropy brought about by the presence of the pointwise forcing at large wavelengths. Further results will be shown at the conference, and in particular other effects such as (static) imperfections of the plate will be addressed.

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