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EVALUATION OF 3-D SINGULAR AND NEARLY SINGULAR INTEGRALS IN GALERKIN BEM FOR THIN LAYERS*

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Abstract. An explicit method for the evaluation of singular and near-singular integrals arising in three-dimensional Galerkin BEM is presented. It is based on a recursive reduction of the dimension of the integration domain leading to a linear combination of one-dimensional regular integrals, which can be exactly evaluated. This method has appealing properties in terms of reliability, precision, and flexibility. The results we present here are devoted to the case of thin layers for the Helmholtz equation, a situation where the panels are close and parallel, known to be difficult in terms of accuracy. Nevertheless, the method applies as well to two-dimensional BEM, secant planes, or even volume integral equations. A MATLAB implementation of the formulas presented here is available online.

Key words. Helmholtz equation, Galerkin BEM, integral equations, singular integral, homogeneity

AMS subject classifications. 65R20, 65N38, 32A55

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1. Introduction.

The discretization of scattering problems by variational boundary element methods (BEMs) leads to the evaluation of such elementary integrals (see [6]) as

(1.1)
$$\text{(i)} \int_{S\times T} G\,(x,y)\; v\,(x)\, w\,(y)\; dx\, dy \quad \text{and} \quad \text{(ii)} \int_{S\times T} \frac{\partial}{\partial n_y} G\,(x,y)\; v\,(x)\, w\,(y)\; dx\, dy,$$

where v and w are polynomial basis functions, G is the Green kernel, and S and T are two plane polygons from the discretization of the boundary. Note that (i) is related to the single layer potential and (ii) to the double layer potential. Due to the singularity of the kernel, the numerical evaluation of the elementary integrals may lead to inaccurate results when the panels S and T are close to each other. Two situations are worthy of interest: when S and T are identical, or have a common edge or vertex, and when they are logically remote but geometrically close (see Figure 1.1). This second case is the most difficult because these large off-diagonal terms must be evaluated with a high accuracy, although no real singularity occurs.

In this paper we focus on the case of triangular panels located in the same or in parallel planes and on the singular part of the kernel of the three-dimensional (3-D) Helmholtz equation, actually the kernel of the Laplace equation. We assume the basis functions are constant on each triangle. The extension to other plane polygons is an easy one, and the extension to affine basis functions is in progress. The case of triangles in secant planes has been considered in a previous paper [2], and an extended version will be the subject of a subsequent paper.

More specifically, the Green function of the 3-D Helmholtz equation is written as

$$G(x,y) = -\frac{1}{4\pi}\frac{1}{\|x-y\|} + H\left(\|x-y\|\right),$$

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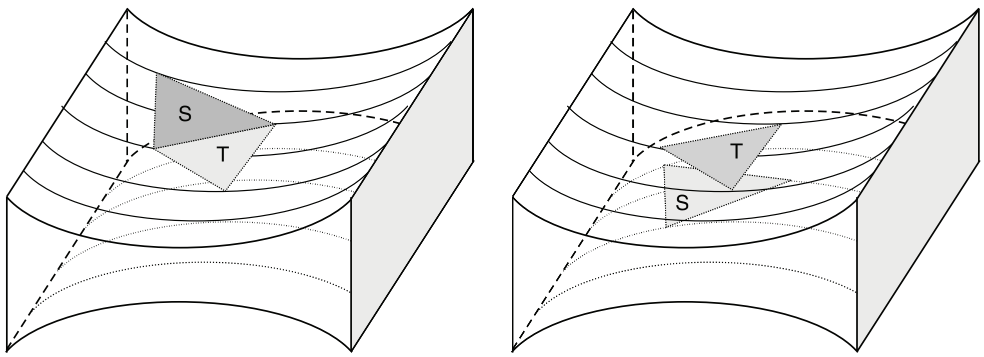

FIG. 1.1. *Close triangles.*

$$\nabla_y G(x,y) = -\frac{1}{4\pi}\frac{x-y}{\|x-y\|^3} - \frac{k^2}{8\pi}\frac{x-y}{\|x-y\|} + (x-y)\,K\left(\|x-y\|\right).$$

where H and K are analytical functions.

In what follows, we give explicit expressions of the singular parts

$$(1.2)\qquad I = \int_{S\times T} \frac{1}{\|x-y\|}\,dx\,dy \quad \text{and} \quad J_\zeta = \int_{S\times T} \frac{x-y}{\|x-y\|^{1+\zeta}}\,dx\,dy, \quad \zeta \in \{0,2\},$$

of the single and double layer potentials, which allow the evaluation of (1.1), the remainder being regular enough to be approximated by classical cubature methods. It is worth noticing that the double layer potential is actually less singular than the integrand of J_2, due to the orthogonality of the normal to the tangent plane.

Our method of calculation relies upon an integration formula for homogeneous functions, which leads us to separate the cases of parallel planes and secant planes. In principle, it could be possible to obtain the first one from the second one, but the limiting process involved leads to an indeterminate form of a very high complexity, which makes it practically intractable. The case of almost parallel planes deserves special care, as it is necessary to choose between two strategies: using the general formula at the risk of losing accuracy due to the indeterminate form, or approximating the planes by parallel ones. An indication of the accuracy of the results can be obtained by comparing these two methods.

The formulas we obtain are rather intricate in the general case. This is not a real drawback, as very few coefficients are large enough to be worthy of such precise calculation. Moreover the formulas simplify drastically when the triangles are identical or share an edge or a vertex. For the sake of compactness, no numerical results are provided, but a MATLAB implementation of the formulas presented is available online (see [5]).

1.1. Others methods. The question of an accurate evaluation of such integrals as I or J_ζ has been previously addressed by a limited number of methods. Most methods rely upon a singular change of variables similar to Duffy's [1], which makes the integrand regular. Various geometric situations have been studied by Sauter [7] (see also Sauter and Schwab [6]), who devises specific changes of variables and then uses a numerical cubature.

The near-singular case has been treated by Nintcheu Fata [3], whose method consists in devising an explicit formula for the inner integral, then performing a

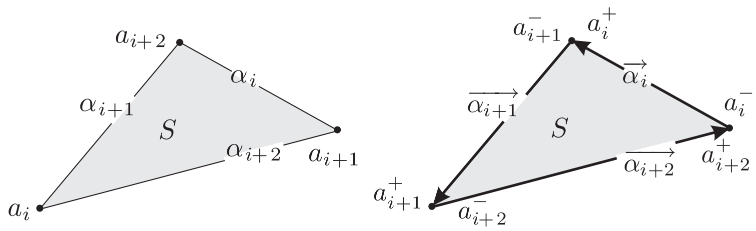

FIG. 1.2. *General notation.*

regularizing change of variables, and finally calculating the outer integral by a cubature formula. A similar scheme has been implemented by Scuderi [8], [9].

The method we present here seems to be the only one avoiding the use of a cubature for the singular part of the Green kernel. The final formulas we obtain may seem rather cumbersome; however, it must be noted that no other method is simple (see [6]).

It is worth pointing out that in the case of self-influence coefficients, when applied to the Green kernel of Laplace equation, the method of Sauter and Schwab (presented in [6]) produces an integral over a hypercube which can be explicitly evaluated and provides a result quite similar to formula (2.11).

1.2. Integration of homogeneous functions. Our calculations rely upon the following formulas (see [4]), the proofs of which are given in Appendix B.

- Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a homogeneous function of degree q. Then if $q + n \neq 0$,

$$(q+n)\int_{\Omega} f(z)\, dz = \int_{\partial\Omega} (z\,|\overrightarrow{\nu}\,)\, f(z)\, \partial z, \tag{1.3}$$

  where $\overrightarrow{\nu}$ is the exterior normal to Ω, ∂z is the surface element on $\partial\Omega$, and $(z|\overrightarrow{\nu})$ is the inner scalar product.
- Let $g : \mathbb{R}^n \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a homogeneous function of degree q, referred to as a homogeneous function with parameter. Then, provided that $\frac{1}{d_0^{q+n}} \int_\Omega g(z, d_0)dz \longrightarrow 0$ when $d_0 \to +\infty$, we have

$$\int_{\Omega} g(z,d)\, dz = d^{q+n} \int_{\partial\Omega} (z\,|\overrightarrow{\nu}\,) \int_d^{+\infty} \frac{g(z,u)}{u^{q+n+1}}\, du\, \partial z. \tag{1.4}$$

1.3. Organization of the paper. A natural increase in the geometric complexity leads us to consider first the case of identical panels, then coplanar triangles, and finally triangles located in parallel planes.

The whole process consists in a recursive reduction of the dimension of the domain of integration by formula (1.3) or (1.4) in such a way that one obtains a linear combination of one-dimensional (1-D) integrals with regular integrands. Finally the choice is open to evaluate these integrals explicitly or numerically.

1.4. Notations. The vertices of S are denoted by $a_i, i = 1, 3$, those of T by $b_j, j = 1, 3$, the opposite side to a_i by α_i and the opposite side to b_j by β_j. We use $|\alpha_i|$ the length of the side $\alpha_i, \overrightarrow{\alpha_i}$ vector $a_{i+2} - a_{i+1} = a_i^+ - a_i^-$ and $\overrightarrow{\alpha_i}' = \overrightarrow{\alpha_i} / |\alpha_i|$, and similarly for T (see Figure 1.2).

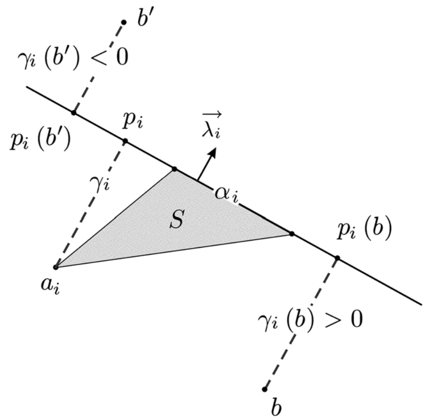

FIG. 1.3. *Distances in S.*

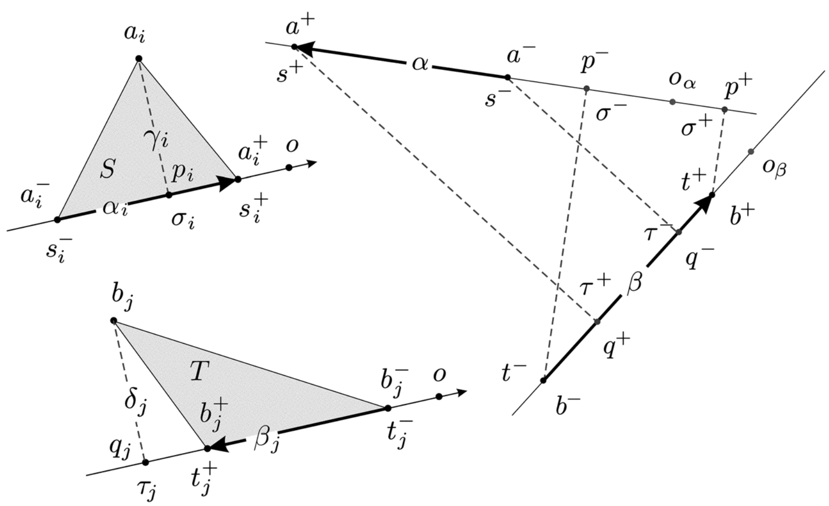

FIG. 1.4. *Abscissas.*

1.4.1. Projections. Various kinds of projections and distances appear in our formulas: the projection of x on α is denoted by $p(x)$, the projection of y on β is denoted by $q(y)$, and one puts $p^\pm = p(b^\pm), q^\pm = q(a^\pm)$. More specifically, p_i is the projection on α_i and q_j the projection on $\beta_j, \overrightarrow{\lambda_i}$ is the exterior normal to S along $\alpha_i, \gamma_i(x) = (p_i - x|\overrightarrow{\lambda_i})$ the signed distance from x to α_i and $\delta_j(y)$ the signed distance from y to β_j. Finally we define $p_i = p_i(a_i)$, $q_j = q_j(b_j), \gamma_i = \gamma_i(a_i) > 0$, and $\delta_j = \delta_j(b_j) > 0$ (see Figure 1.3).

1.4.2. Abscissas. On side α (resp., β) the abscissa s (resp., t) is defined with respect to an origin o_α (resp., o_β) and a unitary direction vector $\overrightarrow{\alpha}' = \overrightarrow{\alpha}/|\alpha|$ (resp., $\overrightarrow{\beta}' = \overrightarrow{\beta}/|\beta|$). The abscissas of the ends $a^\pm$ and $b^\pm$ are, respectively, denoted by $s^\pm$ and $t^\pm$, and the abscissas of $p^\pm$ and $q^\pm$ are denoted by $\sigma^\pm$ and $\tau^\pm$. The same notation with index i or j applies for a side α_i or β_j of S or T (see Figure 1.4).

2. Coplanar triangles. In this section, we consider only the single layer potential, as the double layer potential actually vanishes. We begin with the simple case

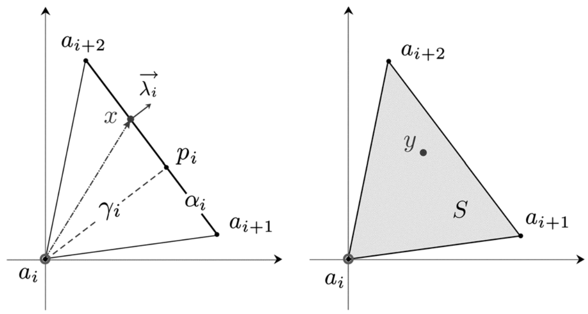

FIG. 2.1. *The boundary of $S \times S$.*

where $S = T$, then consider the case of two coplanar triangles in arbitrary position, and finally the special cases where they share a side or a vertex.

2.1. Self-influence coefficient. Assuming S and T are identical, we reduce integral I to 1-D integrals and then obtain an explicit formula.

2.1.1. Reduction to dimension 3. We apply formula (1.3) to integral I defined by (1.2) with $\Omega = S \times S$ so $n = 4$, and with $z = (x, y)$, $f(z) = \|x - y\|^{-1}$ from which $q = -1$, finally one has

$$I = \frac{1}{3}\int_{\partial(S\times S)} \frac{((x,y)\,|\overrightarrow{\nu}\,)}{\|x-y\|}\,\partial(x,y) = \frac{2}{3}\sum_{i=1,3}\int_{\alpha_i\times S}\frac{\left(x\,\middle|\overrightarrow{\lambda_i}\right)}{\|x-y\|}\,dx\,dy, \tag{2.1}$$

as $\partial(S\times S) = \bigcup_{i=1,3}(\alpha_i\times S)\cup(S\times\alpha_i)$ and $((x,y)\,|\overrightarrow{\nu}\,)_{|\alpha_i\times S} = (x|\overrightarrow{\lambda_i})_{|\alpha_i}$.

With a_i as origin, one has $(x|\overrightarrow{\lambda_{i+1}}) = (x|\overrightarrow{\lambda_{i+2}}) = 0$ and $(x|\overrightarrow{\lambda_i}) = \gamma_i$ (see Figure 2.1). Formula (2.1) reduces then to

$$I = \frac{2}{3}\gamma_i\,U(\alpha_i, S), \quad \text{where } U(\alpha, S) = \int_{\alpha\times S}\frac{1}{\|x-y\|}\,dx\,dy \tag{2.2}$$

is the influence coefficient between the segment α and the triangle S.

2.1.2. Reduction to dimension 2. The process can be repeated, provided that one takes some common point, for example, $a_i^- = a_{i+1}$, to α_i and S as origin, so that function $\|x-y\|^{-1}$ is homogeneous on $\alpha_i \times S$. With $n = 3$, one obtains

$$U(\alpha_i, S) = \frac{1}{2}\int_{\partial(\alpha_i\times S)}((x,y)\,|\overrightarrow{\nu}\,)\,\frac{1}{\|x-y\|}\,\partial(x,y)\,, \tag{2.3}$$

where $\overrightarrow{\nu}$ is now the exterior normal to $\partial(\alpha_i\times S) = \left((a_i^-\cup a_i^+)\times S\right)\cup(\alpha_i\times\partial S)$. From Figure 2.2, it is clear that

$$((x,y)\,|\overrightarrow{\nu}\,)_{|a_i^-\times S} = ((x,y)\,|\overrightarrow{\nu}\,)_{|\alpha_i\times\alpha_i} = ((x,y)\,|\overrightarrow{\nu}\,)_{|\alpha_i\times\alpha_{i+2}} = 0,$$

and as a consequence

$$U(\alpha_i, S) = \frac{|\alpha_i|}{2}P(a_{i+2}, S) + \frac{\gamma_{i+1}}{2}Q(\alpha_i, \alpha_{i+1}), \quad \text{where} \tag{2.4}$$

$$P(a, S) = \int_S \frac{1}{\|a-y\|}\,dy \quad \text{and} \quad Q(\alpha,\beta) = \int_{\alpha\times\beta}\frac{1}{\|x-y\|}\,dx\,dy. \tag{2.5}$$

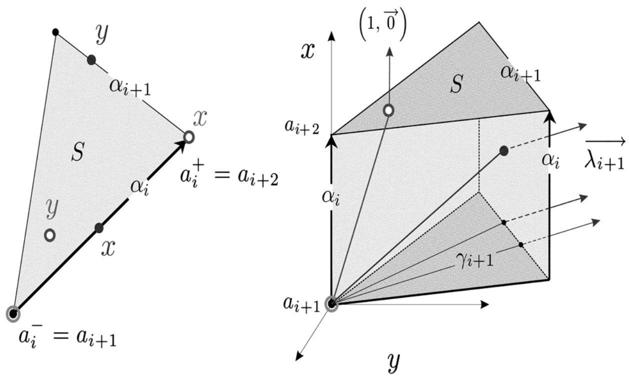

FIG. 2.2. *The boundary of* $\alpha_i \times S$.

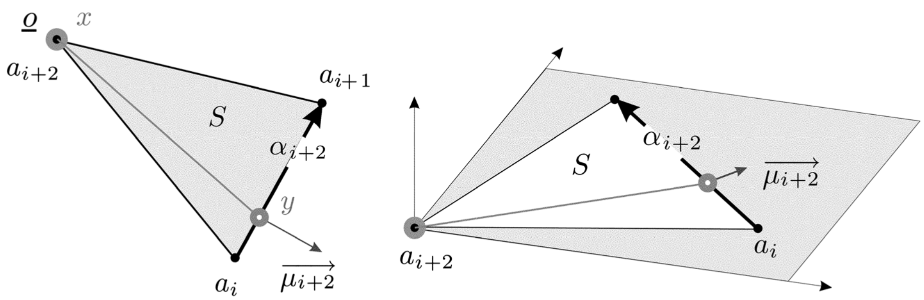

FIG. 2.3. *Calculation of* $P\left(a_{i+2}, S\right)$.

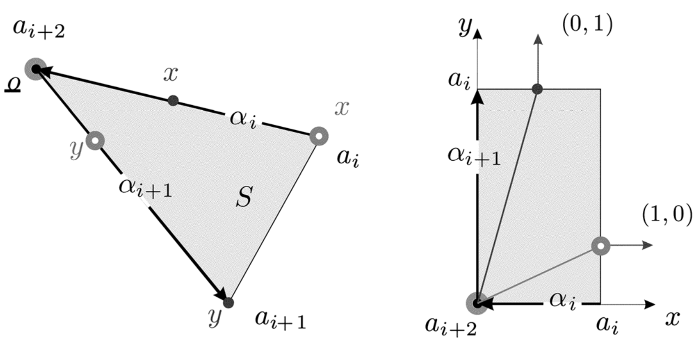

FIG. 2.4. *Calculation of* $Q\left(a_{i+2}, S\right)$.

Actually $P\left(a, S\right)$ is the potential in a from S and $Q\left(\alpha, \beta\right)$ is the influence coefficient between α and β.

2.1.3. Reduction to dimension 1. In the same way, with a_{i+2} as the origin (see Figures 2.3 and 2.4), one obtains

$$P\left(a_{i+2}, S\right) = \gamma_{i+2} R\left(a_{i+2}, \alpha_{i+2}\right), \quad \text{where } R\left(a, \beta\right) = \int_{\beta} \frac{1}{\|a - y\|} dy \tag{2.6}$$

$$\text{and } Q\left(\alpha_i, \alpha_{i+1}\right) = \left|\alpha_i\right| R\left(a_{i+1}, \alpha_{i+1}\right) + \left|\alpha_{i+1}\right| R\left(a_i, \alpha_i\right), \tag{2.7}$$

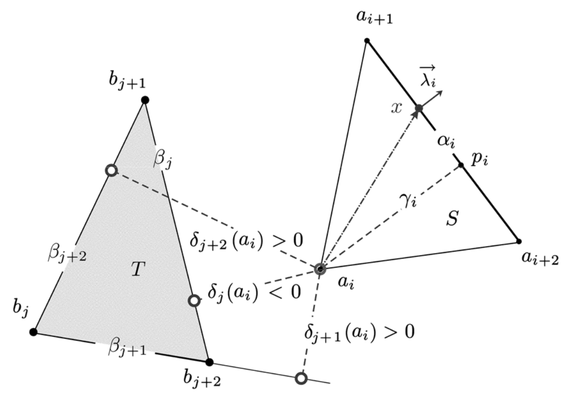

FIG. 2.5. *Coplanar triangles.*

At this stage, we emphasize that the integrand in (2.6) is regular, provided that point a does not belong to segment β, which is the case for $R(a_i, \alpha_i)$. From (2.4), (2.6), and (2.7), with $|S|$ the area of the triangle S,

$$I = \frac{2}{3}|S| \sum_{i=1,3} \gamma_i R(a_i, \alpha_i). \tag{2.8}$$

2.1.4. Explicit formula. Although explicit calculation has not been necessary up to this point, it is required for this last step. With the notation of Figure 1.4 one has

$$R(b,\alpha) = \int_\alpha \frac{1}{\sqrt{\|b-p(b)\|^2 + \|p(b)-x\|^2}} dx = \int_{s^- - \sigma(b)}^{s^+ - \sigma(b)} \frac{1}{\sqrt{\gamma^2(b) + s^2}} ds \tag{2.9}$$

$$= \sum_{k=\pm} k \operatorname{arcsinh}\left(\frac{s^k - \sigma(b)}{\gamma(b)}\right), \tag{2.10}$$

and finally this very simple formula:

$$I = \frac{2}{3}|S| \sum_{i=1,3} \gamma_i \sum_{k=\pm} k \operatorname{arcsinh}\left(\frac{s_i^k - \sigma_i}{\gamma_i}\right). \tag{2.11}$$

2.2. Coplanar arbitrary triangles. We consider now the general case where S differs from T. This section is similar to the previous one; the extra difficulty lies in the fact that when one chooses some vertex of one triangle as origin, it is no longer a vertex of the other (see Figure 2.5).

2.2.1. Reduction to dimension 3. With a_i as origin, formula (2.2) becomes

$$I = \frac{1}{3}\gamma_i U(\alpha_i, T) + \frac{1}{3} \sum_{j=1,3} \delta_j(a_i) U(\beta_j, S). \tag{2.12}$$

2.2.2. Reduction to dimension 2. With $a_{i+1} = a_i^-$ as origin, formula (2.4) takes the following form:

(2.13)

$$U(\alpha_i, T) = \int_{\alpha_i \times T} \frac{1}{\|x-y\|} dx\, dy = \frac{|\alpha_i|}{2} P(a_{i+2}, T) + \frac{1}{2} \sum_{j=1,3} \delta_j(a_{i+1}) Q(\alpha_i, \beta_j);$$

in the same way,

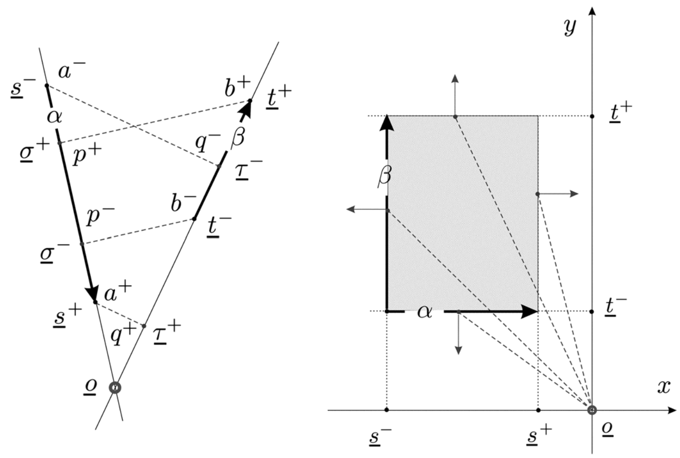

FIG. 2.6. *Calculation of Q: secant supports.*

$$U\left(\beta_j, S\right) = \frac{|\beta_j|}{2} P\left(b_{j+1}, S\right) + \frac{1}{2} \sum_{i=1,3} \gamma_i\left(b_{j+2}\right) Q\left(\beta_j, \alpha_i\right). \tag{2.14}$$

The whole formula reads as

$$\begin{aligned} I = {} & \frac{1}{6} \sum_{i=1,3} \gamma_i\left(o\right) |\alpha_i| \, P\left(a_{i+2}, T\right) + \frac{1}{6} \sum_{j=1,3} \delta_j\left(o\right) |\beta_j| \, P\left(b_{j+1}, S\right) \\ & + \frac{1}{6} \sum_{i=1,3} \sum_{j=1,3} \left(\delta_j\left(o\right) \gamma_i\left(b_{j+2}\right) + \gamma_i\left(o\right) \delta_j\left(a_{i+1}\right)\right) Q\left(\beta_j, \alpha_i\right). \end{aligned} \tag{2.15}$$

2.2.3. Calculation of P. The following formulas are similar to (2.6):

$$P\left(a, T\right) = \sum_{j=1,3} \delta_j\left(a\right) R\left(a, \beta_j\right) \text{ and } P\left(b, S\right) = \sum_{i=1,3} \gamma_i\left(b\right) R\left(b, \alpha_i\right), \tag{2.16}$$

2.2.4. Calculation of Q: Secant supports. Two cases must be considered separately for the evaluation of $Q(\alpha, \beta)$: when the supports of α and β intersect, and when they are parallel. We choose the intersection of the supports of α and β as the new origin $\underline{o}$; the respective abscissas of $a^\pm, b^\pm, p^\pm$, and $q^\pm$ with respect to this origin are denoted by $\underline{s}^\pm, \underline{t}^\pm, \underline{\sigma}^\pm$, and $\underline{\tau}^\pm$ (see Figure 2.6). Similarly to (2.7), one has

$$Q\left(\alpha, \beta\right) = \sum_{k=\pm} k \underline{s}^k R\left(a^k, \beta\right) + \sum_{\ell=\pm} \ell \underline{t}^\ell R\left(b^\ell, \alpha\right). \tag{2.17}$$

If there is a common end to α and β, say $a^k = b^l$, the formula simplifies to

$$Q\left(\alpha, \beta\right) = -k \underline{s}^{-k} R\left(a^{-k}, \beta\right) - \ell \underline{t}^{-\ell} R\left(b^{-\ell}, \alpha\right) = |\alpha| \, R\left(a^{-k}, \beta\right) + |\beta| \, R\left(b^{-\ell}, \alpha\right). \tag{2.18}$$

2.2.5. Calculation of Q: Parallel supports. When α and β are parallel, no origin makes the integrand homogeneous with respect to the integration variable. We are thus led to use formula (1.4) instead of (1.3). Accordingly, we denote by d the distance between α and β, by $\widetilde{\alpha}$ the projection of α on the support of β, and we

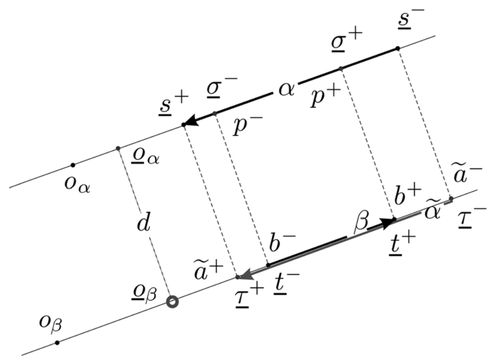

FIG. 2.7. *Calculation of Q: parallel segments.*

choose as new origins on α and β the ends $\underline{o}_\alpha$ and $\underline{o}_\beta$ of an orthogonal segment (see Figure 2.7). With $\overrightarrow{\alpha}' = \mu \overrightarrow{\beta}', \mu = \pm 1$, we have

$$\sigma^\ell - s^k = \underline{\sigma}^\ell - \underline{s}^k = \mu\left(\underline{t}^\ell - \underline{\tau}^k\right) = \mu\left(t^\ell - \tau^k\right), \tag{2.19}$$

$$\underline{t}^\ell - \mu\sigma^\ell = \mu\underline{\sigma}^\ell - \mu\sigma^\ell = \mu\left(\underline{s}^\ell - s^\ell\right) = \mu\left(\underline{s}^k - s^k\right). \tag{2.20}$$

By formula (1.4)

$$\begin{aligned} Q(\alpha,\beta) &= \int_{\widetilde{\alpha},\times\beta} \frac{1}{\sqrt{d^2 + \|\widetilde{x} - y\|^2}} d\widetilde{x}\, dy \\ &= d\int_{\partial(\widetilde{\alpha},\times\beta)} \left((x,y)\,|\,\overrightarrow{\nu}\right) \int_d^{+\infty} \frac{du}{u^2\sqrt{u^2 + \|\widetilde{x} - y\|^2}}\, \partial(\widetilde{x}, y). \end{aligned} \tag{2.21}$$

The corresponding formula to (2.17) is thus

$$Q(\alpha,\beta) = \sum_{k=\pm} k\underline{s}^k \mathsf{R}\left(\widetilde{a}^k, \beta, d\right) + \sum_{\ell=\pm} \ell\underline{t}^\ell \mathsf{R}\left(b^\ell, \widetilde{\alpha}, d\right) \quad \text{with} \tag{2.22}$$

$$\begin{aligned} \mathsf{R}\left(\widetilde{a}^k, \beta, d\right) &= \int_\beta \mathrm{f}_1(d, \left\|\widetilde{a}^k - y\right\|; 0)\, dy, \\ \mathrm{f}_1(d,\eta;0) &= d\int_d^{+\infty} \frac{du}{u^2\left(u^2+\eta^2\right)^{1/2}} = \frac{\sqrt{d^2+\eta^2} - d}{\eta^2}. \end{aligned} \tag{2.23}$$

As a consequence

$$\begin{aligned} \mathsf{R}\left(\widetilde{a}^k, \beta, d\right) &= \mu\sum_{\ell=\pm} \ell \frac{d - \left\|b^\ell - a^k\right\|}{\sigma^\ell - s^k} + \mu\sum_{\ell=\pm} \ell \operatorname{arcsinh}\left(\frac{\sigma^\ell - s^k}{d}\right), \\ \mathsf{R}\left(b^\ell, \widetilde{\alpha}, d\right) &= \sum_{k=\pm} k \frac{d - \left\|b^\ell - a^k\right\|}{s^k - \sigma^\ell} + \sum_{k=\pm} k \operatorname{arcsinh}\left(\frac{s^k - \sigma^\ell}{d}\right). \end{aligned} \tag{2.24}$$

2.2.6. Calculation of Q: Almost parallel supports. When the intersection $\underline{o}$ of the supports of α and β is excessively remote, then $\underline{s}^k$ and $\underline{t}^l$ in formula (2.17) take very large values as compared to $|\alpha|$ and $|\beta|$, resulting in an indeterminate form and a

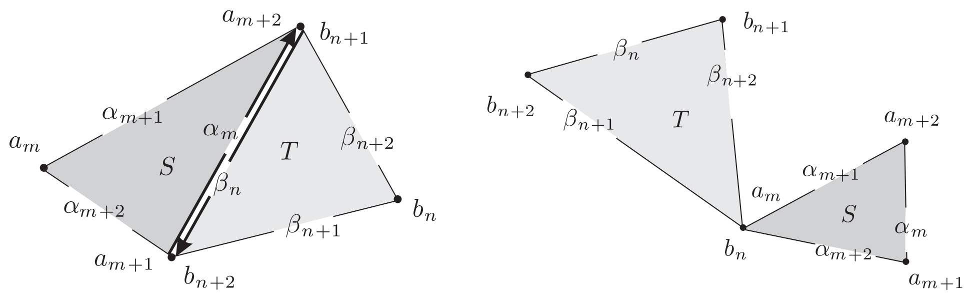

FIG. 2.8. *Triangle with one or two common vertices.*

cancellation of significant digits. The formal expression of (2.17) with respect to the angle between α and β is actually intractable due to the complexity of the interaction between the various terms involved; the strategy we adopt consists in replacing α and β by two close segments which are actually parallel once $|\underline{s}^k|$ and $|\underline{t}^l|$ exceed some prescribed value.

2.3. Coplanar triangles; cases with simplifications. When S differs from T, a common situation leading to significant simplifications is that of one common vertex or side (see Figure 2.8).

2.3.1. Common side. When $\alpha_m = \beta_n$ and the numbering of the vertices satisfies $a_{m+1} = b_{n+2}$ and $a_{m+2} = b_{n+1}$, then (2.15) simplifies to

$$I = \frac{1}{3}\left(|S|\, P\left(a_m, T\right) + |T|\, P\left(b_n, S\right)\right) + \frac{1}{6}\left(\gamma_{m+1}\delta_{n+1} Q\left(\alpha_{m+1}, \beta_{n+1}\right) + \delta_{n+2}\gamma_{m+2} Q\left(\beta_{n+2}, \alpha_{m+2}\right)\right). \tag{2.25}$$

2.3.2. Common vertex. Assume that $a_m = b_n$; then (2.15) is replaced by

$$I = \frac{1}{3}\left(|S|\, P\left(a_{m+2}, T\right) + |T|\, P\left(b_{n+2}, S\right)\right) + \frac{1}{6}\gamma_m \sum_{j\neq n} \delta_j\left(a_{m+1}\right) Q\left(\alpha_m, \beta_j\right) + \frac{1}{6}\delta_n \sum_{i\neq m} \gamma_i\left(b_{n+1}\right) Q\left(\beta_n, \alpha_i\right) + \frac{1}{6}\left(\gamma_m \delta_n\left(a_{m+1}\right) + \delta_n \gamma_m\left(b_{n+1}\right)\right) Q\left(\alpha_m, \beta_n\right). \tag{2.26}$$

3. Triangles in parallel planes. This section is devoted to the case of triangles belonging to different parallel planes. The main difference with the coplanar case lies in the fact that we must use formula (1.4) from the beginning of the reduction process, leading to more intricate calculations. The reduction will be made in an abstract way, leading to the introduction of such intermediate functions as f_a, F_a, or Φ_a^b. The explicit expressions of both the 1-D integrands and the final formulas will be postponed to the end of the reduction process (see Appendix A), allowing a greater flexibility, in this case an inversion of the order of integration. By $\widehat{T}$ we denote the orthogonal projection of T on the plane of S, and by $\overrightarrow{v}$ the vector of the projection (see Figure 3.1). The decomposition $x - y = (x - \widehat{y}) + (\widehat{y} - y) = (x - \widehat{y}) + \overrightarrow{v}$ leads to the splitting of integral J_ζ into two integrals:

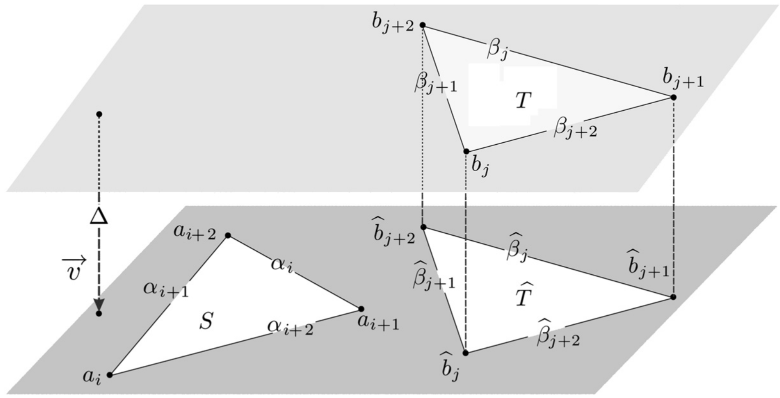

FIG. 3.1. *Triangles in parallel planes.*

$$J_\zeta = \int_{S\times\widehat{T}} \frac{x-\widehat{y}}{\left(\Delta^2+\|x-\widehat{y}\|^2\right)^{(1+\zeta)/2}}\,dx\,d\widehat{y} + \overrightarrow{v}\int_{S\times\widehat{T}} \frac{1}{\left(\Delta^2+\|x-\widehat{y}\|^2\right)^{(1+\zeta)/2}}\,dx\,d\widehat{y},$$

say $J_\zeta = J^a_\zeta + \overrightarrow{v}\,J^s_\zeta$, where J^a_ζ is antisymmetric according to the exchange of S and $\widehat{T}$, and J^s_ζ is symmetrical; one notes that $J^s_0 = I$.

This section consists of two parts: the simplest case of superposed triangles will be discussed first, then the case of triangles in general position.

3.1. Superposed triangles. In this case $\widehat{T} = S$ and we notice that $J^a_\zeta = 0$. The reduction process is very similar to that for the self-influence coefficient; the notation has been chosen to highlight this analogy.

3.1.1. Reduction to dimension 3. With a_i as origin, $n = 4, z = (x, y)$, and $f(z,d) = \left(||x-y||^2 + d^2\right)^{-(1+\zeta)/2}$, from which $q = -1-\zeta$, formula (1.4) gives

$$J^s_\zeta = 2\gamma_i \int_{\alpha_i\times S} \Delta^{3-\zeta} \int_\Delta^{+\infty} \frac{du}{u^{4-\zeta}\left(u^2+\|x-\widehat{y}\|^2\right)^{(1+\zeta)/2}} dx d\widehat{y}. \tag{3.1}$$

We introduce the abstract function f_a to transfer calculation in Appendix A; so with

$$\mathrm{f}_a(\Delta,\eta;\zeta) = \Delta^{a-\zeta}\int_\Delta^{+\infty} \frac{du}{u^{a+1-\zeta}\left(u^2+\eta^2\right)^{(1+\zeta)/2}}, \tag{3.2}$$

as in (2.2), one obtains

$$J^s_\zeta = 2\gamma_i \int_{\alpha_i\times S} \mathrm{f}_3(\Delta, \|x-\widehat{y}\|\,;\zeta) dx d\widehat{y} = 2\gamma_i \mathrm{U}_3(\alpha_i, S, \Delta\,;\,\zeta), \tag{3.3}$$

$$\text{where } \mathrm{U}_3(\alpha, S, \Delta\,;\,\zeta) = \int_{\alpha\times S} \mathrm{f}_3(\Delta, \|x-\widehat{y}\|\,;\zeta) dx d\widehat{y}. \tag{3.4}$$

3.1.2. Reduction to dimension 2. With a_{i+1} as origin (see Figure 2.2) and according to formula (1.4), one obtains

$$\mathrm{U}_3(\alpha_i, S, \Delta\,;\,\zeta) = |\alpha_i| \int_S \Delta^{2-\zeta} \int_\Delta^{+\infty} \frac{\mathrm{f}_3(v, \|a_{i+2} - \widehat{y}\|\,;\zeta)}{v^{3-\zeta}} dv d\widehat{y}$$
$$+ \gamma_{i+1} \int_{\alpha_i \times \alpha_{i+1}} \Delta^{2-\zeta} \int_\Delta^{+\infty} \frac{\mathrm{f}_3(v, \|a_{i+2} - \widehat{y}\|\,;\zeta)}{v^{3-\zeta}} dv dx d\widehat{y}.$$

As in (2.4), one obtains

$$\mathrm{U}_3(\alpha_i, S, \Delta\,;\,\zeta) = |\alpha_i|\, \mathrm{P}_3(a_{i+2}, S, \Delta\,;\,\zeta) + \gamma_{i+1}\, \mathrm{Q}_3(\alpha_i, \alpha_{i+1}, \Delta\,;\,\zeta), \text{ where} \tag{3.5}$$

$$\mathrm{F}_a(\Delta, \eta; \zeta) = \Delta^{a-1-\zeta} \int_\Delta^{+\infty} \frac{\mathrm{f}_a(v, \eta; \zeta)}{v^{a-\zeta}} dv, \tag{3.6}$$

$$\mathrm{P}_3(a, S, \Delta\,;\,\zeta) = \int_S \mathrm{F}_3(\Delta, \|a - \widehat{y}\|\,;\zeta) d\widehat{y},$$

$$\mathrm{Q}_3(\alpha_i, \alpha_{i+1}, \Delta\,;\,\zeta) = \int_{\alpha_i \times \alpha_{i+1}} \mathrm{F}_3(\Delta, \|x - \widehat{y}\|\,;\zeta) dx d\widehat{y}. \tag{3.7}$$

3.1.3. Reduction of $\mathbf{P_3(a, S, \Delta\,;\,\zeta)}$. We use formula (1.4) with a_{i+2} as origin:

$$\mathrm{P}_3(a_{i+2}, S, \Delta\,;\,\zeta) = \gamma_{i+2} \int_{\alpha_{i+2}} \Delta^{1-\zeta} \int_\Delta^{+\infty} \frac{\mathrm{F}_3(w, \|a_{i+2} - \widehat{y}\|\,;\zeta)}{w^{2-\zeta}} dw d\widehat{y}.$$

Then, as in (2.6),

$$\mathrm{P}_3(a_{i+2}, S, \Delta\,;\,\zeta) = \frac{\gamma_{i+2}}{6} \int_{\alpha_{i+2}} \Phi_3^1 (\Delta, \|a_{i+2} - \widehat{y}\|\,;\zeta)\, d\widehat{y} = \frac{\gamma_{i+2}}{6} \mathrm{R}_3^1 (a_{i+2}, \alpha_{i+2}, \Delta; \zeta)\,, \tag{3.8}$$

with

$$\Phi_a^c (\Delta, \eta; \zeta) = 6\Delta^{c-\zeta} \int_\Delta^{+\infty} \frac{\mathrm{F}_a(w, \eta; \zeta)}{w^{c+1-\zeta}} dw, \tag{3.9}$$

$$\mathrm{R}_3^1 (a, \alpha, \Delta; \zeta) = \int_\alpha \Phi_3^1 (\Delta, \|a - x\|\,; \zeta)\, dx \tag{3.10}$$

(see formulas (A.2) and (A.7)).

3.1.4. Reduction of $\mathbf{Q_3(\alpha_i, \alpha_{i+1}, \Delta\,;\,\zeta)}$. We choose a_{i+2} as origin: the intersection of α_i and α_{i+1}. As in (2.7)

$$\mathrm{Q}_3(\alpha_i, \alpha_{i+1}, \Delta\,;\,\zeta) = \frac{|\alpha_i|}{6} \mathrm{R}_3^1 (a_{i+1}, \alpha_{i+1}, \Delta; \zeta) + \frac{|\alpha_{i+1}|}{6} \mathrm{R}_3^1 (a_i, \alpha_i, \Delta; \zeta)\,. \tag{3.11}$$

3.1.5. Final result for superposed triangles. One obtains

$$J_\zeta = \frac{2}{3} |S|\, \overrightarrow{v} \sum_{i=1,3} \gamma_i \mathrm{R}_3^1 (a_i, \alpha_i, \Delta; \zeta)\,, \tag{3.12}$$

where $|S|$ is the area of S. We note the similarity between (3.12) and (2.8), this last expression being the limit of (3.12) when $\Delta \to 0$ and $\zeta = 0$.

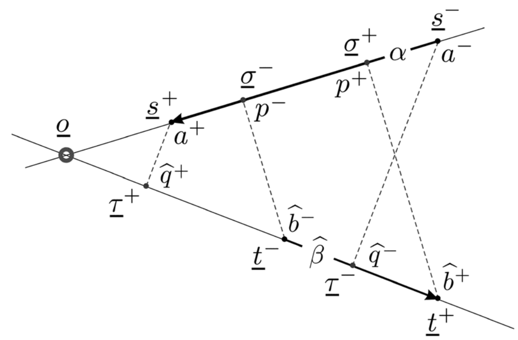

FIG. 3.2. *Segments with secant projections.*

3.2. Shifted triangles. We have treated the case of superposed triangles; as for coplanar triangles, we treat the other configurations for triangles in parallel planes ($S \neq \widehat{T}$). The general case (see Figure 3.1) follows both from sections 2.2 and 3.1, the main difference being that the antisymmetric part J^a_ζ no longer vanishes.

3.2.1. Symmetric part. Let's recall its expression:

$$J^s_\zeta = \int_{S\times\widehat{T}} \frac{1}{\left(\Delta^2 + \|x-\widehat{y}\|^2\right)^{(1+\zeta)/2}}\, dx\, d\widehat{y}.$$

Reduction to dimension 3. As in (2.12) and (3.3), we choose a_i as origin and note that $q+n = 3-\zeta$. Then using formula (1.4), one obtains

$$J^s_\zeta = \gamma_i \mathrm{U}_3(\alpha_i, \widehat{T}, \Delta\,;\,\zeta) + \sum_{j=1}^{3} \widehat{\delta}_j(a_i)\, \mathrm{U}_3(\widehat{\beta}_j, S, \Delta\,;\,\zeta). \tag{3.13}$$

Reduction to dimension 2. As in (3.5), the reduction of U_3 involves the functions P_3 and Q_3. Choosing a_{i+1} as origin, as in (2.14) we obtain

$$\mathrm{U}_3(\alpha_i, \widehat{T}, \Delta\,;\,\zeta) = |\alpha_i|\, \mathrm{P}_3(a_{i+2}, \widehat{T}, \Delta\,;\,\zeta) + \sum_{j=1}^{3} \widehat{\delta}_j(a_{i+1})\, \mathrm{Q}_3(\alpha_i, \widehat{\beta}_j, \Delta\,;\,\zeta). \tag{3.14}$$

Reduction of $\mathrm{P}_3(a_{i+2}, \widehat{T}, \Delta\,;\,\zeta)$. As in (2.16) and (3.8), we have

$$\mathrm{P}_3(a_{i+2}, \widehat{T}, \Delta\,;\,\zeta) = \sum_{j=1}^{3} \widehat{\delta}_j(a_{i+2}) \mathrm{R}^1_3(a_{i+2}, \widehat{\beta}_j, \Delta\,;\,\zeta). \tag{3.15}$$

Reduction of $\mathrm{Q}_3(\alpha_i, \widehat{\beta}_j, \Delta\,;\,\zeta)$*: Secant supports.* When the supports of α_i and $\widehat{\beta}_j$ are secant (see Figure 3.2), we choose the intersection as origin, and, as previously,

$$\mathrm{Q}_3(\alpha_i, \widehat{\beta}_j, \Delta\,;\,\zeta) = \sum_{k=\pm} k\, \underline{s}^k\, \mathrm{R}^1_3(a^k_i, \widehat{\beta}_j, \Delta\,;\,\zeta) + \sum_{l=\pm} l\, \underline{t}^l\, \mathrm{R}^1_3(\widehat{b}^l_j, \alpha_i, \Delta\,;\,\zeta). \tag{3.16}$$

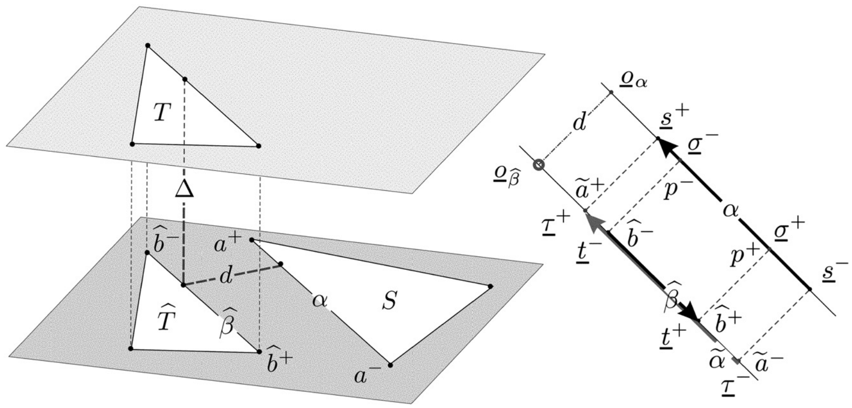

FIG. 3.3. *Segments with parallel projections.*

Reduction of $\mathrm{Q}_3(\alpha_i, \widehat{\beta}_j, \Delta\,;\zeta)$*: parallel supports.* When the supports of α_i and β_j are parallel (see Figure 3.3), as in section 2.2.5, we define $\widetilde{\alpha}_i$ and d, respectively, as the projection of α_i on the support of $\widehat{\beta}_j$ and the distance between α_i and $\widehat{\beta}_j$. Two parameters (Δ and d) occur now, a situation which is not addressed by formula (1.4). The trick consists in defining $\lambda = d/\Delta$, from which it follows that $\|x-y\|^2 = \|\widetilde{x}-\widehat{y}\|^2 + (1+\lambda^2)\Delta^2, x \in \alpha_i, y \in \beta_j$. With $\widehat{b}_j^-$ as origin, by (1.4) one obtains

$$\begin{aligned}
\mathrm{Q}_3(\alpha_i, \widehat{\beta}_j, \Delta\,;\zeta) &= \int_{\widetilde{\alpha}_i \times \widehat{\beta}_j} \mathrm{F}_3(\Delta, \sqrt{\lambda^2\Delta^2 + \|\widetilde{x}-\widehat{y}\|^2}\,;\zeta)\, d\widetilde{x}\, d\widehat{y} \\
&= \sum_{k=\pm} k\, \underline{s}^k \int_{\widehat{\beta}_j} \Delta^{1-\zeta} \int_{\Delta}^{+\infty} \frac{\mathrm{F}_3(u, \sqrt{\lambda^2 u^2 + \|\widetilde{a}_i^k - \widehat{y}\|^2}\,;\zeta)}{u^{2-\zeta}}\, du\, d\widehat{y} \\
&\quad + \sum_{l=\pm} l\, \underline{t}^l \int_{\widetilde{\alpha}_i} \Delta^{1-\zeta} \int_{\Delta}^{+\infty} \frac{\mathrm{F}_3(u, \sqrt{\lambda^2 u^2 + \|\widetilde{x} - \widehat{b}_j^l\|^2}\,;\zeta)}{u^{2-\zeta}}\, du\, d\widetilde{x}.
\end{aligned}$$

We define $\widetilde{\Phi}_a^c$ similarly to Φ_a^c (see (3.9)):

$$\widetilde{\Phi}_a^c(\Delta, d, \eta\,;\zeta) = \Delta^{c-\zeta} \int_{\Delta}^{+\infty} \frac{\mathrm{F}_a(u, \sqrt{\left(\frac{d}{\Delta}\right)^2 u^2 + \eta^2}\,;\zeta)}{u^{c+1-\zeta}}\, du, \quad \text{so} \tag{3.17}$$

$$\begin{aligned}
\mathrm{Q}_3(\alpha_i, \widehat{\beta}_j, \Delta\,;\zeta) = &\sum_{k=\pm} k\, \underline{s}^k \int_{\widehat{\beta}_j} \widetilde{\Phi}_3^1(\Delta, d, \|\widetilde{a}_i^k - \widehat{y}\|\,;\zeta)\, d\widehat{y} \\
&+ \sum_{l=\pm} l\, \underline{t}^l \int_{\widetilde{\alpha}_i} \widetilde{\Phi}_3^1(\Delta, d, \|\widetilde{x} - \widehat{b}_j^l\|\,;\zeta)\, d\widetilde{x}.
\end{aligned} \tag{3.18}$$

Finally we obtain a reduced formula similar to (2.22) with

$$\widetilde{\mathrm{R}}_3^1(\widehat{b}, \widetilde{\alpha}, \Delta, d\,;\zeta) = \int_{\widetilde{\alpha}} \widetilde{\Phi}_3^1(\Delta, d, \|\widetilde{x} - \widehat{b}\|\,;\zeta)\, d\widetilde{x},$$

$$\mathrm{Q}_3(\alpha_i, \widehat{\beta}_j, \Delta\,;\zeta) = \sum_{k=\pm} k\, \widetilde{s}^k\, \tilde{\mathrm{R}}_3^1(\widetilde{a}_i^k, \widehat{\beta}_j, \Delta, d\,;\zeta) + \sum_{l=\pm} l\, \widehat{t}^l\, \tilde{\mathrm{R}}_3^1(\widehat{b}_j^l, \widetilde{\alpha}_i, \Delta, d\,;\zeta) \tag{3.19}$$

(see formulas (A.4) and (A.10)).

3.2.2. Antisymmetric part. Let's recall that

$$J_\zeta^a = \int_{S\times\widehat{T}} \frac{x-\widehat{y}}{\left(\Delta^2+\|x-\widehat{y}\|^2\right)^{(1+\zeta)/2}}\,dx\,d\widehat{y}.$$

Reduction to dimension 3. By (1.4) and with a_i as origin (see Figure 3.1)

$$J_\zeta^a = \gamma_i \int_{\alpha_i\times\widehat{T}} (x-\widehat{y})\Delta^{4-\zeta} \int_\Delta^{+\infty} \frac{du}{u^{5-\zeta}\,(u^2+\|x-\widehat{y}\|^2)^{(1+\zeta)/2}}\,dx\,d\widehat{y}$$
$$+\sum_{j=1}^{3}\widehat{\delta}_j(a_i) \int_{S\times\widehat{\beta}_j} (x-\widehat{y})\Delta^{4-\zeta} \int_\Delta^{\infty} \frac{du}{u^{5-\zeta}\,(u^2+\|x-\widehat{y}\|^2)^{(1+\zeta)/2}}\,dx\,d\widehat{y},$$

which can be written by (3.2) as

$$J_\zeta^a = \gamma_i\,\mathcal{U}_4(\alpha_i,\widehat{T},\Delta\,;\,\zeta) - \sum_{j=1}^{3}\widehat{\delta}_j(a_i)\,\mathcal{U}_4(\widehat{\beta}_j,S,\Delta\,;\,\zeta), \text{ where} \tag{3.20}$$

$$\mathcal{U}_4(\alpha,\widehat{T},\Delta\,;\,\zeta) = \int_{\alpha\times\widehat{T}} (x-\widehat{y})\mathrm{f}_4(\Delta,\|x-\widehat{y}\|\,;\,\zeta)\,dx\,d\widehat{y}.$$

It is worthwhile to remark the presence of the sign $-$ in (3.20), a consequence of the antisymmetry.

Reduction to dimension 2. We continue the reduction process with a_{i+1} as origin and $q+n=3-\zeta$. By (3.6) and as in (3.14)

$$\begin{aligned}\mathcal{U}_4(\alpha_i,\widehat{T},\Delta\,;\,\zeta) &= |\alpha_i| \int_{\widehat{T}} (a_{i+2}-\widehat{y})\mathrm{F}_4(\Delta,\|a_{i+2}-\widehat{y}\|\,;\,\zeta)\,d\widehat{y} \\ &\quad + \sum_{j=1}^{3}\widehat{\delta}_j(a_{i+1}) \int_{\alpha_i\times\widehat{\beta}_j} (x-\widehat{y})\mathrm{F}_4(\Delta,\|x-\widehat{y}\|\,;\,\zeta)\,dx\,d\widehat{y}, \\ &= |\alpha_i|\,\mathcal{P}_4(a_{i+2},\widehat{T},\Delta\,;\,\zeta) + \sum_{j=1}^{3}\widehat{\delta}_j(a_{i+1})\;\mathcal{Q}_4(\alpha_i,\widehat{\beta}_j,\Delta\,;\,\zeta),\end{aligned} \tag{3.21}$$

where

$$\mathcal{P}_4(a_{i+2},\widehat{T},\Delta\,;\,\zeta) = \int_{\widehat{T}} (a_{i+2}-\widehat{y})\mathrm{F}_4(\Delta,\|a_{i+2}-\widehat{y}\|\,;\,\zeta)\,d\widehat{y},$$

$$\mathcal{Q}_4(\alpha,\widehat{\beta}_j,\Delta\,;\,\zeta) = \int_{\alpha\times\widehat{\beta}_j} (x-\widehat{y})\mathrm{F}_4(\Delta,\|x-\widehat{y}\|\,;\,\zeta)\,dx\,d\widehat{y}. \tag{3.22}$$

Reduction of $\mathcal{P}_4(a_{i+2},\widehat{T},\Delta\,;\,\zeta)$. As in (3.15), applying (1.4) with a_{i+2} as origin and $q+n=2-\zeta$, we obtain

$$\mathcal{P}_4(a_{i+2},\widehat{T},\Delta\,;\,\zeta) = \sum_{j=1}^{3}\widehat{\delta}_j(a_{i+2})\mathfrak{R}_4^2(a_{i+2},\widehat{\beta}_j,\Delta\,;\,\zeta), \text{ where} \tag{3.23}$$

$$\begin{aligned}\mathfrak{R}_4^2(a_{i+2},\widehat{\beta},\Delta\,;\,\zeta) &= \int_{\widehat{\beta}} (a_{i+2}-\widehat{y})\Phi_4^2(\Delta,\|a_{i+2}-\widehat{y}\|\,;\,\zeta)\,d\widehat{y} \\ &= \overrightarrow{w}\,\mathrm{R}_4^2(\widetilde{a},\widehat{\beta},\Delta\,;\,\zeta) + \mathfrak{R}_4^2(\widetilde{a},\widehat{\beta},\Delta\,;\,\zeta),\end{aligned} \tag{3.24}$$

where $\widetilde{a}$ is the projection of a on the support of β and $\overrightarrow{w} = a - \widetilde{a}$ (see Figure 3.3). Explicit expressions of R_4^2 and $\mathfrak{R}_4^2$ are provided by (A.8) and (A.9).

Reduction of $\mathcal{Q}_4$: Secant supports. The procedure is the same as for Q_3:

$$\mathcal{Q}_4(\alpha_i, \widehat{\beta}_j, \Delta\,;\zeta) = \sum_{k=\pm} k\,\underline{s}^k\,\mathfrak{R}_4^2(a_i^k, \widehat{\beta}_j, \Delta\,;\zeta) - \sum_{l=\pm} l\,\underline{t}^l\,\mathfrak{R}_4^2(\widehat{b}_j^l, \alpha_i, \Delta\,;\zeta). \tag{3.25}$$

Reduction of $\mathcal{Q}_4$: Parallel supports. We proceed as in (3.19) with $x - \widehat{y} = \overrightarrow{w} + (\widetilde{x} - \widehat{y})$, which leads to

$$\begin{aligned}\mathcal{Q}_4(\alpha_i, \widehat{\beta}_j, \Delta\,;\zeta) &= \overrightarrow{w} \int_{\widetilde{\alpha_i}\times\widehat{\beta}_j} \mathrm{F}_4(\Delta, \sqrt{d^2 + \|\widetilde{x} - \widehat{y}\|^2}\,;\zeta)\,d\widetilde{x}\,d\widehat{y} \\ &\quad + \int_{\widetilde{\alpha_i}\times\widehat{\beta}_j} (\widetilde{x} - \widehat{y})\mathrm{F}_4(\Delta, \sqrt{d^2 + \|\widetilde{x} - \widehat{y}\|^2}\,;\zeta)\,d\widetilde{x}\,d\widehat{y} \\ &= \overrightarrow{w}\,\widetilde{\mathrm{Q}}_4(\widetilde{\alpha}_i, \widehat{\beta}_j, \Delta, d\,;\zeta) + \widetilde{\mathfrak{Q}}_4(\widetilde{\alpha}_i, \widehat{\beta}_j, \Delta, d\,;\zeta)\end{aligned} \tag{3.26}$$

where

$$\widetilde{\mathrm{Q}}_4(\widetilde{\alpha}_i, \widehat{\beta}_j, \Delta, d\,;\zeta) = \int_{\widetilde{\alpha_i}\times\widehat{\beta}_j} \mathrm{F}_4^3(\Delta, \sqrt{d^2 + \|\widetilde{x} - \widehat{y}\|^2}\,;\zeta)\,d\widetilde{x}\,d\widehat{y},$$

$$\widetilde{\mathfrak{Q}}_4(\widetilde{\alpha}_i, \widehat{\beta}_j, \Delta, d\,;\zeta) = \int_{\widetilde{\alpha_i}\times\widehat{\beta}_j} (\widetilde{x} - \widehat{y})\mathrm{F}_4^3(\Delta, \sqrt{d^2 + \|\widetilde{x} - \widehat{y}\|^2}\,;\zeta)\,d\widetilde{x}\,d\widehat{y}.$$

The reduction of $\widetilde{\mathrm{Q}}_4$ and $\widetilde{\mathfrak{Q}}_4$ will be carried out with $\widehat{b}_j^-$ as origin and $d = \lambda\Delta$.

- The antisymmetric term $\widetilde{\mathfrak{Q}}_4$: by (1.4) with $q = -\zeta$, similarly to (3.18), we have

$$\begin{aligned}\widetilde{\mathfrak{Q}}_4(\widetilde{\alpha}_i, \widehat{\beta}_j, \Delta, d\,;\zeta) &= \sum_{k=\pm} k\,\underline{s}^k \int_{\widehat{\beta}_j} (\widetilde{a}_i^k - \widehat{y})\,\widetilde{\Phi}_4^2(\Delta, d, \|\widetilde{a}_i^k - \widehat{y}\|\,;\zeta)\,d\widehat{y} \\ &\quad - \sum_{l=\pm} l\,\underline{t}^l \int_{\widetilde{\alpha}_i} (\widehat{b}_j^l - \widetilde{x})\,\widetilde{\Phi}_4^2(\Delta, d, \|\widetilde{x} - \widehat{b}_j^l\|\,;\zeta)\,d\widetilde{x},\end{aligned}$$

  where the function $\widetilde{\Phi}_a^c$ has been defined in (3.17). So, finally

$$\widetilde{\mathfrak{Q}}_4(\widetilde{\alpha}_i, \widehat{\beta}_j, \Delta, d\,;\zeta) = \sum_{k=\pm} k\,\underline{s}^k\,\tilde{\mathfrak{R}}_4^2(\widetilde{a}_i^k, \widehat{\beta}_j, \Delta, d\,;\zeta) - \sum_{l=\pm} l\,\underline{t}^l\,\tilde{\mathfrak{R}}_4^2(\widehat{b}_j^l, \widetilde{\alpha}_i, \Delta, d\,;\zeta), \tag{3.27}$$

  where

$$\tilde{\mathfrak{R}}_4^2(\widetilde{a}_i^k, \widehat{\beta}_j, \Delta, d\,;\zeta) = \int_{\widehat{\beta}_j} (\widetilde{a}_i^k - \widehat{y})\,\widetilde{\Phi}_4^2(\Delta, d, \|\widetilde{a}_i^k - \widehat{y}\|\,;\zeta)\,d\widehat{y} \tag{3.28}$$

  (see (A.6) and (A.12)).
- The symmetric term $\widetilde{\mathrm{Q}}_4$ is handled in the same way:

$$\widetilde{\mathrm{Q}}_4(\widetilde{\alpha}_i, \widehat{\beta}_j, \Delta, d\,;\zeta) = \sum_{k=\pm} k\,\underline{s}^k\,\widetilde{\mathrm{R}}_4^1(\widetilde{a}_i^k, \widehat{\beta}_j, \Delta, d\,;\zeta) + \sum_{l=\pm} l\,\underline{t}^l\,\widetilde{\mathrm{R}}_4^1(\widehat{b}_j^l, \widetilde{\alpha}_i, \Delta, d\,;\zeta), \tag{3.29}$$

where

$$\widetilde{\mathrm{R}}_4^1(\widetilde{a}_i^k, \widehat{\beta}_j, \Delta, d\,;\,\zeta) = \int_{\widehat{\beta}_j} \widetilde{\Phi}_4^1(\Delta, d, \|\widetilde{a}_i^k - \widehat{y}\|\,;\,\zeta)\, d\widehat{y}$$

(see (A.6) and (A.11). One finally obtains

(3.30)

$$\mathcal{Q}_4(\alpha_i, \widehat{\beta}_j, \Delta\,;\,\zeta) = \sum_{k=\pm} k\, \underline{s}^k \left[\overrightarrow{w}\; \widetilde{\mathrm{R}}_4^1(\widetilde{a}_i^k, \widehat{\beta}_j, \Delta, d\,;\,\zeta) + \tilde{\mathfrak{R}}_4^2(\widetilde{a}_i^k, \widehat{\beta}_j, \Delta, d\,;\,\zeta)\right] + \sum_{l=\pm} l\, \underline{t}^l \left[\overrightarrow{w}\; \widetilde{\mathrm{R}}_4^1(\widehat{b}_j^l, \widetilde{\alpha}_i, \Delta, d\,;\,\zeta) - \tilde{\mathfrak{R}}_4^2(\widehat{b}_j^l, \widetilde{\alpha}_i, \Delta, d\,;\,\zeta)\right].$$

3.2.3. Calculation of *I*. As $J_0^s = I$, with a_i as origin we obtain

$$I = \gamma_i \mathrm{U}_3(\alpha_i, \widehat{T}, \Delta\,;\,0) + \sum_{j=1}^{3} \widehat{\delta}_j(a_i)\mathrm{U}_3(\widehat{\beta}_j, S, \Delta\,;\,0) = J_0^s, \tag{3.31}$$

where U_3 has been defined in (3.4).

4. Conclusion. We provided formulas to evaluate three singular and near-singular integrals arising in 3-D Galerkin BEM when triangles are coplanar or in parallel planes. The final formulas are easy to use and to implement in a code, despite this; a MATLAB implementation is available online so that the reader may test and compare these formulas. We have only shown here a small part of the available formulas. As a matter of fact, formulas for triangles in secant planes and for edge elements in the framework of Maxwell equations have been successfully implemented in an integral equations code, resulting in better accuracy and reliability. The current developments relate to piecewise linear potentials, generalization to volumic potential, and cubature formulas for the regular part of the Green kernel.

Appendix A. Final explicit expressions.

A.1. Calculation of $\boldsymbol{\Phi_a^c}$. From formulas (3.2), (3.6), and (3.9) one has

$$\Phi_a^c(\Delta, \eta\,;\,\zeta) = \int_{w=\Delta}^{+\infty} \int_{v=w}^{+\infty} \int_{u=v}^{+\infty} g(u, v, w, \Delta, \eta) du dv dw,$$

where $g(u, v, w, \Delta, \eta) = 6\Delta^{c-\zeta} \frac{w^{a-c-2}}{u^{a+1-\zeta}(u^2+\eta^2)^{(1+\zeta)/2}}$. A change of of the integration variables (see Figure A.1) leads to

$$\Phi_a^c(\Delta, \eta\,;\,\zeta) = \int_{u=\Delta}^{+\infty} \frac{6\Delta^{c-\zeta}}{u^{a+1-\zeta}(u^2+\eta^2)^{(1+\zeta)/2}} \int_{v=\Delta}^{u} \int_{w=\Delta}^{v} w^{a-c-2} dw dv du,$$

and more specifically

$$\Phi_a^{a-2}(\Delta, \eta\,;\,\zeta) = 3\Delta^{a-2-\zeta} \int_{\Delta}^{+\infty} \frac{(u-\Delta)^2}{u^{a+1-\zeta}(u^2+\eta^2)^{(1+\zeta)/2}} du. \tag{A.1}$$

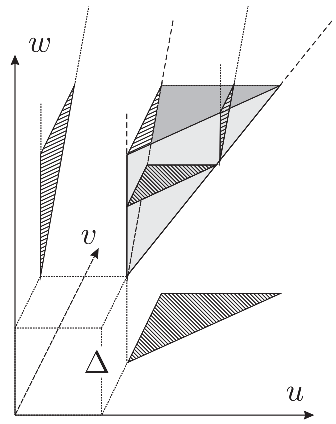

Fig. A.1. *Domains of integration.*

Explicit results.

(A.2)

$$\Phi_3^1(\Delta,\eta\,;0)=\frac{3\Delta^2}{\eta^3}\operatorname{arcsinh}\left(\frac{\eta}{\Delta}\right)+\frac{\sqrt{\Delta^2+\eta^2}-3\Delta}{\eta^2}-\frac{2\Delta^2\left(\sqrt{\Delta^2+\eta^2}-\Delta\right)}{\eta^4},$$

$$\Phi_3^1(\Delta,\eta\,;2)=\frac{6(\sqrt{\Delta^2+\eta^2}-\Delta)}{\eta^4}+\frac{3}{\Delta\eta^2}-\frac{6}{\eta^3}\operatorname{arcsinh}\left(\frac{\eta}{\Delta}\right),$$

(A.3)

$$\Phi_4^2(\Delta,\eta\,;0)=\frac{(2\eta^2+23\Delta^2)\sqrt{\Delta^2+\eta^2}}{8\eta^4}-\frac{4\Delta^3}{\eta^4}+\frac{3\Delta^2(3\Delta^2-4\eta^2)}{8\eta^5}\operatorname{arcsinh}\left(\frac{\eta}{\Delta}\right),$$

$$\Phi_4^2(\Delta,\eta\,;2)=\frac{3(8\Delta-5\sqrt{\Delta^2+\eta^2})}{2\eta^4}-\frac{3(3\Delta^2-2\eta^2)}{2\eta^5}\operatorname{arcsinh}\left(\frac{\eta}{\Delta}\right).$$

A.2. Calculation of $\widetilde{\boldsymbol{\Phi}}_{\boldsymbol{a}}^{\boldsymbol{c}}$. The evaluation of $\widetilde{\Phi}_a^c$ defined by (3.17) involves F_a:

$$\mathrm{F}_3(\Delta,\eta\,;0)=\frac{\sqrt{\Delta^2+\eta^2}}{6\eta^2}-\frac{\Delta^2}{2\eta^3}\operatorname{arcsinh}\left(\frac{\eta}{\Delta}\right)+\frac{2\Delta^2(\sqrt{\Delta^2+\eta^2}-\Delta)}{3\eta^4},$$

$$\mathrm{F}_3(\Delta,\eta\,;2)=\frac{1}{\eta^3}\operatorname{arcsinh}\left(\frac{\eta}{\Delta}\right)-\frac{2(\sqrt{\Delta^2+\eta^2}-\Delta)}{\eta^4},$$

$$\mathrm{F}_4(\Delta,\eta\,;0)=\frac{\sqrt{\Delta^2+\eta^2}}{12\eta^2}-\frac{3\Delta^4}{8\eta^5}\operatorname{arcsinh}\left(\frac{\eta}{\Delta}\right),-\frac{\Delta^2(7\sqrt{\Delta^2+\eta^2}-16\Delta)}{24\eta^4},$$

$$\mathrm{F}_4(\Delta,\eta\,;2)=\frac{\sqrt{\Delta^2+\eta^2}-4\Delta}{2\eta^4}+\frac{3\Delta^2}{2\eta^5}\operatorname{arcsinh}\left(\frac{\eta}{\Delta}\right).$$

For the sake of simplicity, we introduce the following notation:

$$T_1=\operatorname{arcsinh}\left(\frac{\sqrt{d^2+s^2}}{\Delta}\right),\ T_2=\operatorname{arcsinh}\left(\frac{s}{\sqrt{\Delta^2+d^2}}\right),\ T_3=\arctan\left(\frac{s}{d}\right),$$

$$T_4 = \frac{\sqrt{\Delta^2+d^2+s^2}-\Delta}{d^2+s^2},\ T_5 = \frac{\sqrt{\Delta^2+d^2+s^2}-\sqrt{\Delta^2+d^2}}{d^2},$$
$$T_6 = \operatorname{arctanh}\left(\frac{\Delta^2+d^2+isd}{\Delta\sqrt{\Delta^2+d^2+s^2}}\right),\ T_7 = \operatorname{arcsinh}\left(\frac{d}{\Delta}\right),$$
$$T_8 = T_3 + \mathcal{I}\mathrm{m}(T_6) - \frac{\pi}{2}\operatorname{sgn}(s),\ T_9 = \frac{\sqrt{\Delta^2+d^2}-\Delta}{d^2}$$

Explicit results.

(A.4)
$$\widetilde{\Phi}_3^1(\Delta,d,s\,;\,0) = \frac{3\Delta^2}{s^2\sqrt{d^2+s^2}}T_1 - \frac{3\Delta^2}{ds^2}T_7 + \frac{d^2}{s^2}T_5 - \frac{2\Delta^2}{s^2}T_4 + \frac{2\Delta^2}{s^2}T_9,$$
$$\widetilde{\Phi}_3^1(\Delta,d,s\,;\,2) = \frac{6}{ds^2}T_7 - \frac{6}{s^2\sqrt{d^2+s^2}}T_1 + \frac{6d^2}{s^2(d^2+s^2)}T_5 - \frac{6}{d^2+s^2}T_9,$$

(A.5)
$$\widetilde{\Phi}_4^2(\Delta,d,s\,;\,0) = -\frac{3\Delta^4T_1}{4d^2(d^2+s^2)^{3/2}} - \frac{d^4+6d^2\Delta^2-3\Delta^4}{4s^3d^2}T_2 + \frac{\Delta^2}{s^2}T_4 + \frac{2\Delta^3T_8}{ds^3} + \frac{(\Delta^2+d^2+s^2)^{3/2}-4\Delta^3}{4s^2(d^2+s^2)},$$
$$\widetilde{\Phi}_4^2(\Delta,d,s\,;\,2) = \frac{3\Delta^2T_1}{d^2(d^2+s^2)^{3/2}} + \frac{3(d^2-\Delta^2)}{d^2s^3}T_2 - \frac{3T_4}{s^2} + \frac{3\Delta}{s^2(d^2+s^2)} - \frac{6\Delta T_8}{ds^3},$$

(A.6)
$$\widetilde{\Phi}_4^1(\Delta,d,s\,;\,0) = -\frac{3\Delta^4}{4d^3s^2}T_7 + \frac{3\Delta^4}{s^2(d^2+s^2)^{3/2}}T_1 + \frac{d^2}{2s^2}T_5 + \frac{5\Delta^2}{4s^2}T_4 - \frac{\Delta^2(5d^2+8s^2)}{4s^2(d^2+s^2)}T_9 + \frac{3\Delta^2\sqrt{\Delta^2+d^2}}{d^2(d^2+s^2)},$$
$$\widetilde{\Phi}_4^1(\Delta,d,s\,;\,2) = \frac{\Delta^2}{2d^3s^2}T_7 - \frac{\Delta^2}{2s^2(d^2+s^2)^{3/2}}T_1 - \frac{d^2}{2s^2(d^2+s^2)}T_5 + \frac{1}{2(d^2+s^2)}T_9 - \frac{\Delta}{2d^2(d^2+s^2)}.$$

A.3. One-dimensional integrals.

A.3.1. One parameter. We denote by $\mathcal{K}$ and $\mathfrak{K}$ the respective primitives of R_a^c and $\mathfrak{R}_a^c$; then with $\chi^{\pm} = s^{\pm} - \sigma$, one obtains

$$\mathrm{R}_a^c(b,\alpha,\Delta\,;\,\zeta) = \Big[\mathcal{K}(s,\Delta,d\,;\,\zeta)\Big]_{s=\chi^+}^{\chi^-},\ \mathfrak{R}_a^c(b,\alpha,\Delta\,;\,\zeta) = -\frac{\vec{\alpha}}{\|\vec{\alpha}\|}\Big[\mathfrak{K}(s,\Delta,d\,;\,\zeta)\Big]_{s=\chi^+}^{\chi^-}.$$

Explicit results.

(A.7)
$$\mathrm{R}_3^1(a,\beta,\Delta\,;\,0) = \left[\frac{3s\Delta^2T_1}{\sqrt{d^2+s^2}d^2} + \frac{(d^2-3\Delta^2)T_2}{d^2} - \frac{s\Delta^2T_4}{d^2} - \frac{\Delta(3d^2-\Delta^2)T_8}{d^3}\right]_{s=\chi^+}^{\chi^-},$$
$$\mathrm{R}_3^1(a,\beta,\Delta\,;\,2) = \left[-\frac{6sT_1}{d^2\sqrt{d^2+s^2}} + \frac{3sT_4}{d^2} + \frac{6T_2}{d^2} + \frac{3(d^2-\Delta^2)T_8}{\Delta d^3}\right]_{s=\chi^+}^{\chi^-},$$

(A.8)

$$\mathrm{R}_4^2(a,\beta,\Delta\,;\,0)=\Bigg[-\frac{3s\Delta^2(4d^4+4d^2s^2-2s^2\Delta^2-3d^2\Delta^2)T_1}{8d^4(d^2+s^2)^{3/2}}-\frac{2\Delta^3}{d^3}T_8+\frac{(d^4+6d^2\Delta^2-3\Delta^4)T_2}{4d^4}+\frac{h^2s}{8d^2}\left(13T_4-\frac{3\Delta}{d^2+s^2}\right)\Bigg]_{s=\chi^+}^{\chi^-},$$

$$\mathrm{R}_4^2(a,\beta,\Delta\,;\,2)=\Bigg[\frac{3s\,T_1}{d^2\sqrt{d^2+s^2}}\left(1-\frac{h^2}{d^2}-\frac{h^2}{2(d^2+s^2)}\right)+\frac{3s}{2d^2}\left(\frac{\Delta}{d^2+s^2}-3T_4\right)-\frac{3(d^2-\Delta^2)}{d^4}T_2+\frac{6\Delta}{d^3}T_8\Bigg]_{s=\chi^+}^{\chi^-},$$

(A.9)

$$\mathfrak{R}_4^2(a,\beta,\Delta\,;\,0)=\Bigg[\frac{3\Delta^2(4(d^2+s^2)-\Delta^2)T_1}{8(d^2+s^2)^{3/2}}+\frac{2\sqrt{\Delta^2+d^2+s^2}-13\Delta^2T_4}{8}+\frac{3\Delta^3}{8(d^2+s^2)}\Bigg]_{s=\chi^-}^{\chi^+},$$

$$\mathfrak{R}_4^2(a,\beta,\Delta\,;\,2)=\Bigg[\frac{3}{2}\left(3T_4-\frac{\Delta}{d^2+s^2}\right)-\frac{3(2d^2+2s^2-\Delta^2)}{2(d^2+s^2)^{3/2}}T_1\Bigg]_{s=\chi^-}^{\chi^+s}.$$

A.3.2. Two parameters. One recalls

$$\widetilde{\mathrm{R}}_a^c(b,\alpha,\Delta,d\,;\,\zeta)=\int_\alpha\widetilde{\Phi}_a^c(\Delta,d,\|b-x\|\,;\,\zeta)dx,$$
$$\widetilde{\mathfrak{R}}_a^c(b,\alpha,\Delta,d\,;\,\zeta)=\int_\alpha(x-b)\widetilde{\Phi}_a^c(\Delta,d,\|b-x\|\,;\,\zeta)dx.$$

Explicit results.

(A.10)

$$\widetilde{\mathrm{R}}_3^1(a,\beta,\Delta,d\,;\,0)=\Bigg[\frac{3\Delta^2T_7}{ds}+\frac{(d^2+3\Delta^2)T_2}{d^2}-\frac{(d^2-2\Delta^2)}{s}T_5-\frac{2\Delta^3T_8}{d^3}-\frac{3\Delta^2\sqrt{d^2+s^2}}{sd^2}T_1\Bigg]_{s=\chi^-}^{\chi^+},$$

$$\widetilde{\mathrm{R}}_3^1(a,\beta,\Delta\,;\,2)=\Bigg[\frac{6\sqrt{d^2+s^2}T_1}{sd^2}-\frac{6T_7}{sd}-\frac{6T_2}{d^2}-\frac{6T_5}{s}+\frac{6\Delta T_8}{d^3}\Bigg]_{s=\chi^-}^{\chi^+},$$

(A.11)

$$\widetilde{\mathrm{R}}_4^1(a,\beta,\Delta,d\,;\,0)=\Bigg[\frac{3\Delta^4T_7}{4d^3s}-\frac{3\Delta^4(2s^2+d^2)}{4d^4s\sqrt{d^2+s^2}}T_1+\frac{d^4+3\Delta^4}{2d^4}T_2-\frac{2d^2+5\Delta^2}{4s}T_5+\frac{2\Delta^3T_8}{d^3}\Bigg]_{s=\chi^-}^{\chi^+},$$

$$\widetilde{\mathrm{R}}_4^1(a,\beta,\Delta\,;\,2)=\Bigg[-\frac{3\Delta^2T_7}{d^3s}+\frac{3\Delta^2(2s^2+d^2)}{d^4s\sqrt{d^2+s^2}}T_1-\frac{6\Delta^2T_2}{d^4}+\frac{3T_5}{s}-\frac{6\Delta T_8}{d^3}\Bigg]_{s=\chi^-}^{\chi^+},$$

(A.12)

$$\widetilde{\mathfrak{R}}_4^2(a,\beta,\Delta,d\,;\,0) = \left[\frac{d^4+6\Delta^2 d^2-3\Delta^4}{4sd^2}T_2 + \frac{3\Delta^4 T_1}{4d^2\sqrt{d^2+s^2}} - \frac{2\Delta^3 T_8}{ds} + \frac{\sqrt{\Delta^2+d^2+s^2}}{4} + \frac{3\Delta^3}{4d^2}\operatorname{arcsinh}\left(\frac{\Delta}{\sqrt{d^2+s^2}}\right) - \frac{3\Delta^3}{4d^2}\mathrm{Re}\,(T_6)\right]_{s=\chi^-}^{\chi^+},$$

$$\widetilde{\mathfrak{R}}_4^2(a,\beta,\Delta\,;\,2) = \left[-\frac{3\Delta^2 T_1}{d^2\sqrt{d^2+s^2}} - \frac{3(d^2-\Delta^2)}{sd^2}T_2 + \frac{6\Delta T_8}{ds}\right]_{s=\chi^-}^{\chi^+}.$$

Appendix B. Integration of homogeneous functions. Our method for handling the singular part of the Green function relies on an integration formula for homogeneous functions. It has the advantage of reducing an integral over an n-D domain into an integral over its boundary. Although it is a classical result, for the sake of definiteness, we present a detailed proof. We denote by Ω this bounded domain in $\mathbb{R}^n$; $y=(z,x)$ is an element of $\Omega\times\mathbb{R}$ and g is a positively homogeneous function of degree q:

$$g(\lambda y) = \lambda^q g(y) \quad \forall\lambda>0 \tag{B.1}$$

B.1. Euler's formula. We differentiate the previous formula with respect to λ and put $\lambda=1$; then

$$(\nabla_z\, g\,(z,x)\,|z\,) + x\frac{\partial g}{\partial x}(z,x) = qg(z,x). \tag{B.2}$$

Now with $I(x)=\int_\Omega g(z,x)\,dz$, by formula (B.2),

$$qI(x) = xI'(x) + \int_\Omega (\nabla_z\, g\,(z,x)\,|z\,)\;dz; \tag{B.3}$$

a Green's formula leads to the fundamental result:

$$(q+n)I(x) = \int_{\partial\Omega}(\overrightarrow{z}\,|\overrightarrow{\nu}\,)\,g(z,x)\,\partial z + xI'(x), \tag{B.4}$$

where ∂z denotes the surface measure on $\partial\Omega$.

B.2. The case without parameter. Assume I is continuous at 0 and $xI'(x)\to 0$ when $x\to 0$; then if $q+n\neq 0$,

$$(q+n)I(0) = \int_{\partial\Omega}(\overrightarrow{z}\,|\overrightarrow{\nu}\,)\,g(z,0)\,\partial z; \tag{B.5}$$

in the same way, if g does not depend on x, with $f\,(z)=g(z,0)$ according to (B.4), we obtain

$$(q+n)I = \int_{\partial\Omega}(\overrightarrow{z}\,|\overrightarrow{\nu}\,)\,f(z)\,\partial z. \tag{B.6}$$

An additional trick is necessary to handle the case where $q+n=0$.

B.3. The case with a parameter. The solution of the differential equation (B.4) is given by

$$K(x) = K(x_0) + \int_x^{x_0} \frac{du}{u^{q+n+1}} \int_{\partial\Omega} (\overrightarrow{z} \,|\, \overrightarrow{\nu}\,)\, g(z,u)\, \partial z,$$

where $\operatorname{sgn}(x_0) = \operatorname{sgn}(x)$ in order to avoid the singularity of u^{-q-n-1} at the origin. By inversion of the order of integration

$$K(x) = K(x_0) + \int_{\partial\Omega} (\overrightarrow{z} \,|\, \overrightarrow{\nu}\,) \int_x^{x_0} \frac{g(z,u)}{u^{q+n+1}} du\, \partial z;$$

then

$$I(x) = \left(\frac{x}{x_0}\right)^{q+n} I(x_0) + x^{q+n} \int_{\partial\Omega} (\overrightarrow{z} \,|\, \overrightarrow{\nu}\,) \int_x^{x_0} \frac{g(z,u)}{u^{q+n+1}} du\, \partial z. \tag{B.7}$$

Finally,

$$I(x) = x^{q+n} \int_{\partial\Omega} (\overrightarrow{z} \,|\, \overrightarrow{\nu}\,) \int_x^{\infty \operatorname{sgn} x} \frac{g(z,u)}{u^{q+n+1}} du\, \partial z, \tag{B.8}$$

provided that

$$\frac{1}{x_0^{q+n}} I(x_0) \longrightarrow 0 \quad \text{when } x_0 \to \infty \operatorname{sgn}(x).$$

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