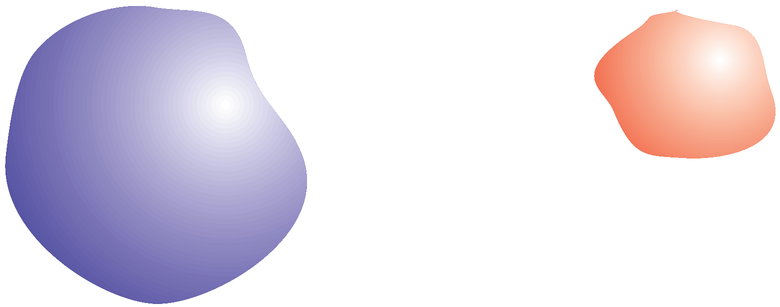
Le problème des 2 corps perturbé

Problème à 2 corps perturbé

Le problème à deux corps est académique ! (Newton ...)

En réalité :



ou plutôt ...

$$\overrightarrow{F} = \overrightarrow{F_{Kep}} + \text{corrections}$$
$$\Downarrow$$
$$\text{Trajectoire} = \text{Conique} + \text{corrections}$$

Trajectoire elliptique : $\{a, e, i, \omega, \Omega, \tau\}$

Perturbation $\Rightarrow$ éléments osculateurs elliptiques :

$$\{a(t), e(t), i(t), \omega(t), \Omega(t), \tau(t)\}$$

Quelles sont les équations (différentielles) dont les éléments osculateurs sont solutions.

Éléments elliptiques

2 corps sphériques, dans R_C galiléen,

$$\ddot{x} := \frac{d^2x}{dt^2} = -\mu\frac{x}{r^3} \qquad avec \qquad r = \sqrt{x^2+y^2+z^2}$$

$$+id\ y, z$$

$\Rightarrow$ 6 EDO non linéaires du premier ordre.

Cauchy : $\{x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0\}$ $\Rightarrow$ Solution unique

$x = x\left(x_o, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o; t\right)$ + idem y, z

plus généralement, on a toujours

$$x = x\left(\{C_i\}\,; t\right) \text{ + idem } y, z \quad \text{avec } 1 \leq i \leq 6 \tag{1}$$

Habituellement, $\{C_i\}$: 6 constantes issues de $\{x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0\}$

Astronomie : $\{C_i\} = \{a, e, i, \omega, \Omega, \tau\}$.

Problème perturbé

Hypothèse de séparation : $\overrightarrow{F_{pert}}$ = $^{t}\left[\mathrm{X}, \mathrm{Y}, \mathrm{Z}\right]$

$$\ddot{x} := \frac{d^2x}{dt^2} = -\mu\frac{x}{r^3} + X\left(x, y, z, \dot{x}, \dot{y}, \dot{z}; t\right) \quad \text{+ idem } \; y, z \tag{2}$$

Equations planétaires de Lagrange : Variation des constantes
On cherche 6 fonctions $C_i(t)$ telles que

$$x = x\left(\left\{C_i\left(t\right)\right\}; t\right) \quad \text{+ idem } \; y, z \quad \text{ avec } 1 \leq i \leq 6$$

soit

$$\dot{x} := \frac{dx}{dt} = \frac{\partial x}{\partial t} + \sum_{i=1}^{6} \frac{\partial x}{\partial C_i}\frac{dC_i}{dt} \quad \text{+ idem } \; y, z$$

Initialement : 3 inconnues $x\left(t\right), y\left(t\right)$ et $z\left(t\right) \Longrightarrow$ 6 nouvelles fonctions: $C_i\left(t\right)$

$\Downarrow$

possibilité de 3 contraintes indépendantes sur les $C_i\left(t\right)$

Jauge de Lagrange

On choisit

$$\sum_{i=1}^{6} \frac{\partial x}{\partial C_i} \frac{dC_i}{dt} = \sum_{i=1}^{6} \frac{\partial y}{\partial C_i} \frac{dC_i}{dt} = \sum_{i=1}^{6} \frac{\partial z}{\partial C_i} \frac{dC_i}{dt} = 0$$

il reste donc

$$\dot{x} = \frac{\partial x}{\partial t} \qquad \dot{y} := \frac{\partial y}{\partial t} \qquad \dot{z} = \frac{\partial z}{\partial t} \tag{3}$$

Equations du mouvement :

$$\ddot{x} = \frac{\partial^2 x}{\partial t^2} + \sum_{i=1}^{6} \dot{C}_i \frac{\partial \dot{x}}{\partial C_i} \quad id\ y, z$$

Hypothèse de séparation

$$\frac{\partial^2 x}{\partial t^2} + \sum_{i=1}^{6} \dot{C}_i \frac{\partial \dot{x}}{\partial C_i} = -\mu \frac{x}{r^3} + X \quad id\ y, z \tag{4}$$

on identifie donc

$$\sum_{i=1}^{6} \dot{C}_i \frac{\partial \dot{x}}{\partial C_i} = X \quad id\ y, z$$

Crochets de Lagrange

Système à résoudre

$$\sum_{i=1}^{6} \dot{C}_i \frac{\partial x}{\partial C_i} = 0 \quad \Big|\Big| \quad \sum_{i=1}^{6} \dot{C}_i \frac{\partial \dot{x}}{\partial C_i} = X \qquad id\ y, z$$

pour tout j fixé ces équations se réecrivent

$$-\sum_{i=1}^{6} \dot{C}_i \frac{\partial x}{\partial C_i} \frac{\partial \dot{x}}{\partial C_j} = 0 \quad \Big|\Big| \quad \sum_{i=1}^{6} \dot{C}_i \frac{\partial \dot{x}}{\partial C_i} \frac{\partial x}{\partial C_j} = X \frac{\partial x}{\partial C_j} \qquad id\ y, z$$

en faisant la somme de toutes ces équations on obtient

$$\sum_{i=1}^{6} \dot{C}_i \, [C_j, C_i] = X \frac{\partial x}{\partial C_j} + Y \frac{\partial y}{\partial C_j} + Z \frac{\partial z}{\partial C_j} \tag{5}$$

$\Rightarrow$ Crochets de Lagrange

$$\begin{aligned}
[C_j, C_i] = \; & \left(\frac{\partial x}{\partial C_j} \frac{\partial \dot{x}}{\partial C_i} - \frac{\partial \dot{x}}{\partial C_j} \frac{\partial x}{\partial C_i} \right) \\
+ \; & \left(\frac{\partial y}{\partial C_j} \frac{\partial \dot{y}}{\partial C_i} - \frac{\partial \dot{y}}{\partial C_j} \frac{\partial y}{\partial C_i} \right) \\
+ \; & \left(\frac{\partial z}{\partial C_j} \frac{\partial \dot{z}}{\partial C_i} - \frac{\partial \dot{z}}{\partial C_j} \frac{\partial z}{\partial C_i} \right)
\end{aligned}$$

Potentiel perturbateur

Hypothèse : la force pertubatrice dérive d'un potentiel

$$\exists R \qquad / \qquad \left\{ X = \frac{\partial R}{\partial x}, Y = \frac{\partial R}{\partial y}, Z = \frac{\partial R}{\partial z} \right\}$$

les équations donnant C_i sont donc

$$1 \leq j \leq 6, \qquad \sum_{i=1}^{6} \dot{C}_i \left[C_j, C_i \right] = \frac{\partial R}{\partial C_i} \tag{6}$$

Théorème de Poisson-Lagrange

$$J_{ji} := \left[C_j, C_i \right] = Cste$$

$\boxed{\text{Dm}}$

$$\begin{aligned} J_{ji} &:= \left(\frac{\partial x}{\partial C_j} \frac{\partial \dot{x}}{\partial C_i} - \frac{\partial \dot{x}}{\partial C_j} \frac{\partial x}{\partial C_i} \right) + id\left(y, \dot{y}\right) + id\left(z, \dot{z}\right) \\ &= \frac{\partial\left(x, \dot{x}\right)}{\partial\left(C_j, C_i\right)} + id\left(y, \dot{y}\right) + id\left(z, \dot{z}\right) \\ &= J_x + J_y + J_z \end{aligned}$$

ainsi

$$\frac{\partial J_{ji}}{\partial t} = \frac{\partial J_x}{\partial t} + \frac{\partial J_y}{\partial t} + \frac{\partial J_z}{\partial t}$$

$$\begin{aligned}\frac{\partial J_x}{\partial t} \quad = & \quad \frac{\partial}{\partial C_j}\left(\frac{\partial x}{\partial t}\right)\frac{\partial \dot{x}}{\partial C_i} - \frac{\partial \dot{x}}{\partial C_j}\frac{\partial}{\partial C_i}\left(\frac{\partial x}{\partial t}\right) \\ + & \quad \frac{\partial}{\partial C_i}\left(\frac{\partial \dot{x}}{\partial t}\right)\frac{\partial x}{\partial C_j} - \frac{\partial}{\partial C_j}\left(\frac{\partial \dot{x}}{\partial t}\right)\frac{\partial x}{\partial C_i}\end{aligned}$$

on a choisi (voir (3))

$$\frac{\partial x}{\partial t} = \frac{dx}{dt} = \dot{x}$$

ainsi

$$\frac{\partial J_x}{\partial t} = \frac{\partial x''}{\partial C_i}\frac{\partial x}{\partial C_j} - \frac{\partial x''}{\partial C_j}\frac{\partial x}{\partial C_i}, \qquad x'' = \frac{\partial^2 x}{\partial t^2}$$

mais (4) nous indique alors que

$$x'' = -\mu\frac{x}{r^3} = \frac{\partial V}{\partial x}, \qquad V = \frac{\mu}{r}$$

finalement

$$\frac{\partial J_x}{\partial t} = \frac{\partial x}{\partial C_j}\frac{\partial V'_{C_i}}{\partial x} - \frac{\partial x}{\partial C_i}\frac{\partial V'_{C_j}}{\partial x}, \qquad V'_{C_{k=1,2,3}} := \frac{\partial V}{\partial C_k}$$

idem J_y et J_z si bien que

$$\frac{\partial J_{ji}}{\partial t} = \frac{\partial V'_{C_i}}{\partial C_j} - \frac{\partial V'_{C_j}}{\partial C_i} = \frac{\partial^2 V}{\partial C_j \partial C_i} - \frac{\partial^2 V}{\partial C_i \partial C_j} = 0 \qquad \blacksquare$$

Équations planétaires

Equations planétaires de Lagrange

Comment calculer les 36 crochets J_{ij} ?

$$i = 1, ..., 6 \qquad \{C_i\} = \{a, e, \tau, \Omega, \omega, i\}$$

$\Rightarrow$ Exemple : Calcul de $[a, e]$

Changement de variable $\xi = r\cos f$ et $\eta = r\sin f$

$$\forall i \neq j = 1, 2, 3 \qquad [C_i, C_j] = \frac{\partial\left(\xi, \dot{\xi}\right)}{\partial\left(C_i, C_j\right)} + \frac{\partial\left(\eta, \dot{\eta}\right)}{\partial\left(C_i, C_j\right)} \tag{7}$$

Pour $t \approx \tau$, $E \ll 1$ l'équation de Kepler donne :

$$\sin E \approx E \approx \frac{n\left(t-\tau\right)}{1-e}, \qquad n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

d'ou l'on tire

$$\cos E \approx 1 - \frac{1}{2}\frac{n^2\left(t-\tau\right)^2}{\left(1-e\right)^2}$$

Calcul de $[a, e]$

$$\xi = r\cos f = a\cos E - ae$$
$$\eta = r\sin f = a\sin E\sqrt{1-e^2}$$

on obtient finalement

$$\xi = a\left[1 - \frac{n^2(t-\tau)^2}{2(1-e)^2}\right] \qquad \dot{\xi} = -\frac{an^2(t-\tau)}{(1-e)^2}$$

$$\Rightarrow$$

$$\eta = na\sqrt{\frac{1+e}{1-e}}(t-\tau) \qquad \dot{\eta} = na\sqrt{\frac{1+e}{1-e}}$$

ainsi au périgée ($\tau = \tau$)

$$\left(\frac{\partial\xi}{\partial a}\right)_\tau = 1-e \qquad \left(\frac{\partial\dot{\xi}}{\partial a}\right)_\tau = 0$$

$$\left(\frac{\partial\xi}{\partial e}\right)_\tau = -a \qquad \left(\frac{\partial\dot{\xi}}{\partial e}\right)_\tau = 0$$

$$\Longrightarrow \left.\frac{\partial\left(\xi,\dot{\xi}\right)}{\partial(a,e)}\right|_\tau = 0$$

de même

$$\left.\frac{\partial(\eta,\dot{\eta})}{\partial(a,e)}\right|_\tau = 0$$

On obtient donc pour $t = \tau$ $\quad [a, e] = 0 \quad t = \tau \ \rightarrow \forall t$: Thm LP

Don des autres ...

$$L_{13} := [a,\tau] = \frac{n^2 a}{2} \qquad L_{24} := [e,\Omega] = \frac{na^2 e\cos i}{\sqrt{1-e^2}}$$
$$L_{14} := [a,\Omega] = -\frac{na\cos i\sqrt{1-e^2}}{2} \qquad L_{25} := [e,\omega] = \frac{na^2 e}{\sqrt{1-e^2}}$$
$$L_{15} := [a,\omega] = -\frac{na\sqrt{1-e^2}}{2} \qquad L_{46} := [\Omega,i] = na^2\sin i\sqrt{1-e^2}$$

Le système donnant les $\{C_i\}$ se réduit alors a

$$\mathbb{L}.\left[\dot{a},\dot{e},\dot{\tau},\dot{\Omega},\dot{\omega},\dot{i}\right]^T = \left[\frac{\partial R}{\partial a},\frac{\partial R}{\partial e},\frac{\partial R}{\partial \tau},\frac{\partial R}{\partial \Omega},\frac{\partial R}{\partial \omega},\frac{\partial R}{\partial \iota}\right]^T$$

avec

$$\mathbb{L} = \operatorname{antisym}\left(L_{ij}\right)_{\substack{1\leq i\leq 6\\ 1\leq j\leq 6}}$$

$L^{-1} \Rightarrow$ équations planétaires de Lagrange

Equations planétaires de Lagrange

$$
\left[\begin{array}{lcl}
\dot{a} & = & -\left(\dfrac{2}{n^2 a}\right)\dfrac{\partial R}{\partial \tau} \\[2ex]
\dot{e} & = & -\left(\dfrac{1-e^2}{n^2 a^2 e}\right)\dfrac{\partial R}{\partial \tau} - \left(\dfrac{\sqrt{1-e^2}}{n^2 a e}\right)\dfrac{\partial R}{\partial \omega} \\[2ex]
\dot{i} & = & \left(\dfrac{\cot i}{n a^2 \sqrt{1-e^2}\sin i}\right)\dfrac{\partial R}{\partial \omega} - \left(\dfrac{1}{n a^2 \sqrt{1-e^2}\sin i}\right)\dfrac{\partial R}{\partial \Omega} \\[2ex]
\dot{\Omega} & = & \left(\dfrac{1}{n a^2 \sqrt{1-e^2}\sin i}\right)\dfrac{\partial R}{\partial i} \\[2ex]
\dot{\omega} & = & \left(\dfrac{\sqrt{1-e^2}}{n a^2 e}\right)\dfrac{\partial R}{\partial e} - \left(\dfrac{\cot i}{n a^2 \sqrt{1-e^2}}\right)\dfrac{\partial R}{\partial i} \\[2ex]
\dot{\tau} & = & \left(\dfrac{2}{n^2 a}\right)\dfrac{\partial R}{\partial a} + \left(\dfrac{1-e^2}{n^2 a^2 e}\right)\dfrac{\partial R}{\partial e}
\end{array}\right.
$$

Cas particuliers

1. R=cste ($\overrightarrow{F_P} = 0$) : $\{a, e, \tau, \Omega, \omega, i\}$=cste
2. $R = R(r)$

$$\frac{\partial R}{\partial C_i} = \left|\overrightarrow{F_P}\right| \frac{\partial r}{\partial C_i}$$

mais $r = a\left(1 - e\cos E\right)$, $\quad E - e\sin E = n\left(t - \tau\right)$ ainsi

$$\frac{\partial r}{\partial \Omega} = \frac{\partial r}{\partial \omega} = \frac{\partial r}{\partial i} = 0$$

Equations planétaires : $\dfrac{di}{dt} = 0$ et $\dfrac{d\Omega}{dt} = 0$

i et Ω ne sont pas affectés et a, e, τ, ω vérifient :

$$\left\{\begin{array}{lcl} \dfrac{da}{dt} & = & -\left(\dfrac{2}{n^2 a}\right) R_\tau \\ \\ \dfrac{de}{dt} & = & -\left(\dfrac{1-e^2}{n^2 a^2 e}\right) R_\tau \end{array}\right. \qquad \left\{\begin{array}{lcl} \dfrac{d\omega}{dt} & = & \left(\dfrac{\sqrt{1-e^2}}{n a^2 e}\right) R_e \\ \\ \dfrac{d\tau}{dt} & = & \left(\dfrac{2}{n^2 a}\right) R_a + \left(\dfrac{1-e^2}{n^2 a^2 e}\right) R_e \end{array}\right.$$

Eléments de Delaunay

Hypothèse: $\{C_i\} = \{C_1, C_2, C_3, C_4, C_5, C_6\} = \{\{\alpha_l\}, \{\beta_l\}\}_{1\leq l\leq 3}$

$\{C_i\}$ est décomposable en deux séries distinctes d'éléments α et β conjugués tels que

$$[\alpha_l, \alpha_m] = [\beta_l, \beta_m] = 0$$

$$[\alpha_l, \beta_m] = \delta_{lm} = \begin{cases} 0 & \text{si } l \neq m \\ 1 & \text{si } l = m \end{cases}$$

Equations planétaires :

$$\begin{cases} \dfrac{d\alpha_l}{dt} = -\dfrac{\partial R}{\partial \beta_l} = [\alpha_l, R] \\ \\ \dfrac{d\beta_l}{dt} = +\dfrac{\partial R}{\partial \alpha_l} = [\beta_l, R] \end{cases} \quad \text{avec} \quad 1 \leq l \leq 3 \qquad (8)$$

Ces éléments existent (Delaunay), par exemple :

$$\begin{aligned} \alpha_1 &= \tfrac{\mu}{2a} = -\xi & \beta_1 &= \tau \\ \alpha_2 &= \sqrt{\mu a\left(1-e^2\right)} = C & \beta_2 &= \omega \\ \alpha_3 &= \sqrt{\mu a\left(1-e^2\right)}\cos i = C\cos i & \beta_3 &= \Omega \end{aligned}$$

Naissance de la mécanique analytique.