Low dissipation entropy fix for positivity preserving Roe's scheme

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Abstract

We present a general formulation for the entropy fix which encompasses both Harten and Hyman entropy fixes and HLLE/M schemes. Such a formulation is found to be a simple tool for benefitting from the properties of different methods so as to obtain a positivity preserving version of entropy fix with low dissipation.

1 Introduction

Roe's linearization [7] represents a widespread method for solving Euler equations. However, as well known, this method may suffer from possible entropy violations and the prediction of unphysical intermediate states in presence of strong rarefactions.

The generation of nonentropic solutions can be prevented by means of suitable entropy fixes. This can be accomplished by writing the upwind scheme so as to put into evidence its numerical dissipation matrix and by operating on this matrix to assure a non zero "viscous" contribution to the numerical flux, as proposed for instance by Harten [4]. It results a very simple and computationally convenient correction to the original Roe's scheme.

To avoid the violation of positivity of density or internal energy is a much more difficult task. The classical Roe's linearization in Jacobian form has not enough degrees of freedom to impose positivity together with consistency with the conservation laws, as demonstrated in [3], at least for a certain class of symmetric Riemann problems. The difficulty is avoided by resorting to a positivity preserving approximation, as the HLLE scheme [2, 3]. Unfortunately the HLLE method is characterized by a numerical dissipation larger than in Roe's scheme, particularly near contact discontinuities. To overcome this drawback in recent years some modifications of the original HLLE scheme have been proposed [3, 8]. An alternative approach is suggested by Dubroca [1], that allows for the required degrees of freedom by departing from the classical Jacobian-based Roe's linearization.

The present work follows a different line. The starting point is the observation that the lack of positivity of Roe's scheme has been explained [3] as the consequence of an underestimation by Roe's approximate solver of the absolute value of the minimum and maximum physical signal velocities. By means of the approach we propose here, the entropy fix can be regarded as a tool to properly correct the numerical signal velocities computed by Roe's method in order to preserve the positivity of the solution. Moreover, since the entropy fix allows introducing additional parameters in the numerical scheme, it guarantees a sufficient number of degrees of freedom to impose the positivity condition, while still retaining the classical Jacobian form in the Roe's linearization.

The outline of this paper is as follows. In section 2 a general formulation of the entropy fix is presented, which encompasses both some of the existing entropy fixes [5, 4, 6] and the HLLE/M schemes. In section 3 we use the established general framework to propose a positivity preserving version of entropy fix with low dissipation, by exploiting the properties of different methods. Some numerical examples are shown in section 4.

2 Unitary framework

2.1 A general expression for the entropy fix

In the Introduction, the approach to the entropy fix based on the notion of a "viscosity" matrix has been recalled. A rather different viewpoint is that adopted by LeVeque in [6], where the entropy fix is presented as a remedy for the difficulties encountered by Roe's approximate solver in case of transonic rarefactions.

LeVeque's approach provides us the guidelines for obtaining a general formulation of the entropy fix in terms of propagating velocities. According to the the usual notation, $\widehat{a}_k = \widehat{a}_k(\boldsymbol{u}_\ell, \boldsymbol{u}_r)$ and $\widehat{\boldsymbol{r}}_k = \widehat{\boldsymbol{r}}_k(\boldsymbol{u}_\ell, \boldsymbol{u}_r)$ will denote here respectively the eigenvalues and the eigenvectors of the Roe's matrix $\widehat{\boldsymbol{A}} = \widehat{\boldsymbol{A}}(\boldsymbol{u}_\ell, \boldsymbol{u}_r)$. The proposed general formulation is expressed by the numerical flux:

$$\boldsymbol{F}(\boldsymbol{u}_\ell, \boldsymbol{u}_r) = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{u}_\ell) + \boldsymbol{f}(\boldsymbol{u}_r)] - \frac{1}{2}\sum_{k=1}^{3} \alpha_k \, q(\widehat{a}_k) \, \widehat{\boldsymbol{r}}_k \,, \tag{1a}$$

where

$$q(\widehat{a}_k) = \begin{cases} \dfrac{[\mathcal{P}(a_{kr}) + \mathcal{N}(a_{k\ell})]\widehat{a}_k - 2\mathcal{P}(a_{kr})\mathcal{N}(a_{k\ell})}{\mathcal{P}(a_{kr}) - \mathcal{N}(a_{k\ell})} + \dfrac{\mathcal{P}(a_{kr})\,\mathcal{N}(a_{k\ell})}{\mathcal{P}(a_{kr}) - \mathcal{N}(a_{k\ell})}\,\sigma_k & \\ & \text{if } \mathcal{P}(a_{kr}) \neq \mathcal{N}(a_{k\ell}) \\ |\widehat{a}_k| & \text{if } \mathcal{P}(a_{kr}) = \mathcal{N}(a_{k\ell}) \,, \end{cases} \tag{1b}$$

with the propagation velocities $a_{k\ell}$ and a_{kr} and the parameter σ_k to be suitably defined. We use $\mathcal{P}(\cdot)$ and $\mathcal{N}(\cdot)$ to denote the positive and negative parts of their respective argument.

Following Harten and Hyman [5], it is possible to demonstrate that in the scalar case, and for $\sigma = 0$, formulation (1) guarantees consistency with the entropy condition if the propagation velocities a_ℓ and a_r satisfy the inequalities:

$$a_\ell \leq a(u_\ell) \qquad \text{and} \qquad a_r \geq a(u_r)\,. \tag{2}$$

The problem of imposing more general conditions on the parameters $a_{k\ell}$, a_{kr} and σ_k to have entropic solutions is still to be investigated.

2.2 Recovering Harten and Hyman entropy fixes

Formulation (1) allows recovering Harten and Hyman entropy fixes and the version of entropy fix proposed by LeVeque by defining properly the velocities $a_{k\ell}$ and a_{kr} and the parameter σ_k. If we choose

$$a_{k\ell} = \widehat{a}_k - \delta_k \qquad \text{and} \qquad a_{kr} = \widehat{a}_k + \delta_k\,, \tag{3}$$

with $\delta_k = \max\{0, \widehat{a}_k - a_k(\boldsymbol{u}_\ell),\, a_k(\boldsymbol{u}_r) - \widehat{a}_k\}$, and we set $\sigma_k = 0$, $\forall k$, we obtain the method derived by Harten and Hyman in [5, p. 243]. With the same definition for $a_{k\ell}$ and a_{kr}, but now setting $\sigma_k = 1$, we find the alternative method also proposed by Harten and Hyman in [5, Remark p. 266]. LeVeque's version of entropy fix [6, p. 151] is obtained by restricting correction (1b) of Roe's scheme to the sonic eigenvalues $k = 1$ and $k = 3$, setting $\sigma_k = 0$, and by defining

$$a_{k\ell} = a_k(\widehat{\boldsymbol{u}}_{k,\ell}) \qquad \text{and} \qquad a_{kr} = a_k(\widehat{\boldsymbol{u}}_{k,r})\,, \tag{4}$$

where, as usual, $\widehat{\boldsymbol{u}}_{1,\ell} = \boldsymbol{u}_\ell$, $\widehat{\boldsymbol{u}}_{1,r} = \widehat{\boldsymbol{u}}_1 = \boldsymbol{u}_\ell + \alpha_1\,\widehat{\boldsymbol{r}}_1 = \widehat{\boldsymbol{u}}_{2,\ell}$, $\widehat{\boldsymbol{u}}_{2,r} = \widehat{\boldsymbol{u}}_2 = \boldsymbol{u}_\ell + \alpha_1\,\widehat{\boldsymbol{r}}_1 + \alpha_2\,\widehat{\boldsymbol{r}}_2 = \widehat{\boldsymbol{u}}_{3,\ell}$, and $\widehat{\boldsymbol{u}}_{3,r} = \boldsymbol{u}_r$.

2.3 Recovering HLLE and HLLEM schemes

We find that HLLE and HLLEM schemes fall in the general formulation (1) of entropy fix just presented. We will use in the following the quantities b_ℓ and b_r introduced by Einfeldt et al. in [3]:

$$b_\ell = \min\{\widehat{a}_1,\, v_\ell - c_\ell\} \qquad \text{and} \qquad b_r = \max\{\widehat{a}_3,\, v_r + c_r\}\,, \tag{5}$$

where v and c denotes respectively the fluid velocity and the sound speed. For both the HLLE and HLLEM methods we have

$$a_{k\ell} = b_\ell \qquad \text{and} \qquad a_{kr} = b_r\,, \quad \forall k\,, \tag{6}$$

and $\sigma_1 = \sigma_3 = 0$. We obtain the HLLE scheme [3, p. 281] imposing $\sigma_2 = 0$, while we recover the HLLEM scheme [3, p. 284] if we set $\sigma_2 = 2\widehat{\delta}$, with $\widehat{\delta} = \frac{\widehat{c}}{\widehat{c}+|\bar{v}|}$, being $\widehat{c} = c(\widehat{h}^{\mathrm{t}}, \widehat{v})$ and $\bar{v} = \frac{b_\ell + b_r}{2}$. Here $\widehat{h}^{\mathrm{t}}$ and $\widehat{v}$ are the well known Roe-average of the total enthalpy per unit mass and of velocity.

Placing the HLLE scheme in the same setting of the classical entropy fix formulations supports the introduction of the idea of *positivity preserving* entropy fix and also suggests how to correct Roe's scheme to impose the positivity.

3 The proposed method

The general formulation (1) is found to be a useful and simple tool to benefit from the properties of the different methods considered here in order to guarantee:

i) consistency with the entropy condition,

ii) positivity,

iii) low numerical dissipation.

Indeed, we can suitably define the quantities $a_{k\ell}$, a_{kr} and σ_k depending on the local solution to assure the aforementioned properties.

For fixed values of $a_{k\ell}$ and a_{kr}, an increase in the slope σ_k implies a lower numerical dissipation. Nevertheless, this cannot be the only criterion to adopt to set properly the value of σ_k, and in particular we still need to investigate the relation of this parameter with the entropy condition, as anticipated in section 2. Therefore, in the following we restrict our analysis to the case $\sigma_k = 0$, $\forall k$, for semplicity.

We start distinguishing the case in which Roe's intermediate states $\widehat{\boldsymbol{u}}_1 = \boldsymbol{u}_\ell + \alpha_1 \widehat{\boldsymbol{r}}_1$ and $\widehat{\boldsymbol{u}}_2 = \boldsymbol{u}_\ell + \alpha_1 \widehat{\boldsymbol{r}}_1 + \alpha_2 \widehat{\boldsymbol{r}}_2$ are physically admissible from the case in which one of them or both are not. The condition discriminating the two cases consists in checking the positivity of the density and internal energy of states $\widehat{\boldsymbol{u}}_1$ and $\widehat{\boldsymbol{u}}_2$.

If the two computed intermediate states are physical, the positivity of the solution is naturally preserved, and we correct Roe's numerical flux by means of (1) only to avoid entropy violations, as usual. In such a case, for $a_{k\ell}$ and a_{kr} we use the definitions (4), as in LeVeque's method. According to the physical interpretation of the entropy fix suggested by LeVeque, this choice of the propagating velocities allows a better approximation of the exact solution of the Riemann problem and is found to introduce the lowest level of numerical viscosity, with respect to the other possible definitions of $a_{k\ell}$ and a_{kr}, for $\sigma_k = 0$.

If on the contrary negative values of density or internal energy or both are detected, definition (4) for the propagation velocities cannot be used, since they depend at least on one not physically admissible state. Moreover, in this case we need to define $a_{k\ell}$ and a_{kr} so as to force a suitable enlargement of the numerical signal velocities, thus avoiding the underestimation of the limiting physical velocities caused by Roe's approximate solver. Following the HLLE idea, we use definition (6) for the propagation velocities. This choice guarantees both consistency with entropy condition and positivity, as demonstrated in [3]. We remark that, if we use a nonzero value for the parameter σ_2, still having $\sigma_1 = \sigma_3 = 0$ and the same definition (6) of $a_{k\ell}$ and a_{kr}, it is in principle possible to find out sufficient conditions on σ_2 guaranteeing positivity. These conditions are presently under investigation.

The proposed version of entropy fix proves to be a positivity preserving correction of Roe's scheme that allows an easy implementation and requires an additional computation of no relevant cost with respect to Roe's method augmented by LeVeque's entropy fix [6].

4 Numerical results

Figure 1 compares the first order numerical results obtained with the presented method and the HLLE/M methods for a Riemann problem proposed in [3], consisting in two symmetric rarefactions. The initial data are $\rho_\ell = 1$, $v_\ell = -2$, $P_\ell = 0.4$ for the left state, $\rho_r = 1$, $v_r = 2$, $P_r = 0.4$ for the right state. The proposed entropy fix allows resolving without difficulties this strong rarefaction test, which causes the failure of Roe's classical scheme [7], and is found to be slightly less dissipative than the HLLE scheme.

Figure 2 shows the solutions computed using the same methods for a Riemann problem obtained from the former, by replacing the value of the pressure of the left state with $P_\ell = 2$. In such a case the problem is non-symmetric. The present method and the HLLEM method, being less dissipative than the HLLE scheme, feature a small undershoot with respect to the exact solution, which does not prevent, however, to compute a positive solution.

In solving Riemann problems different from those implying low density regions, the presented method preserves all the properties of Roe's scheme.

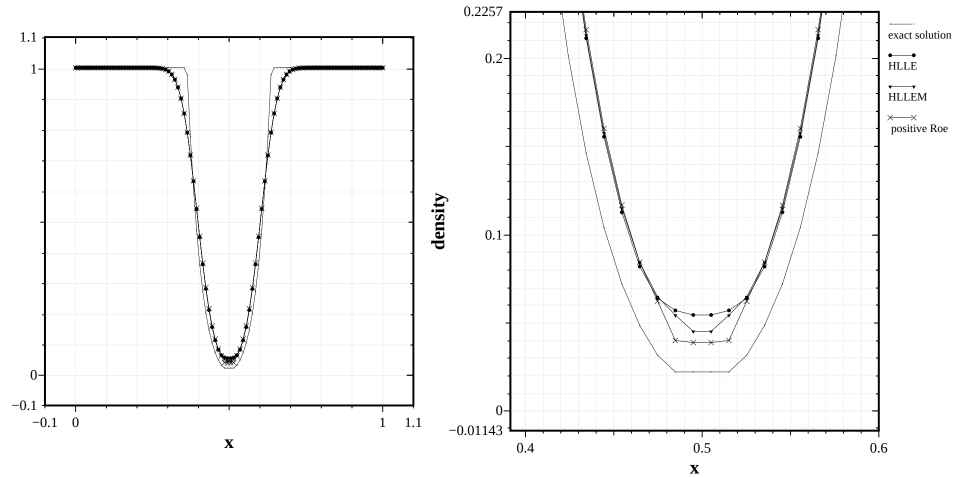

Figure 1: Strong rarefaction test problem [3]. Computed densities for Roe method augmented with the proposed entropy fix in comparison with the exact solution and the solutions by HLLE/M methods.

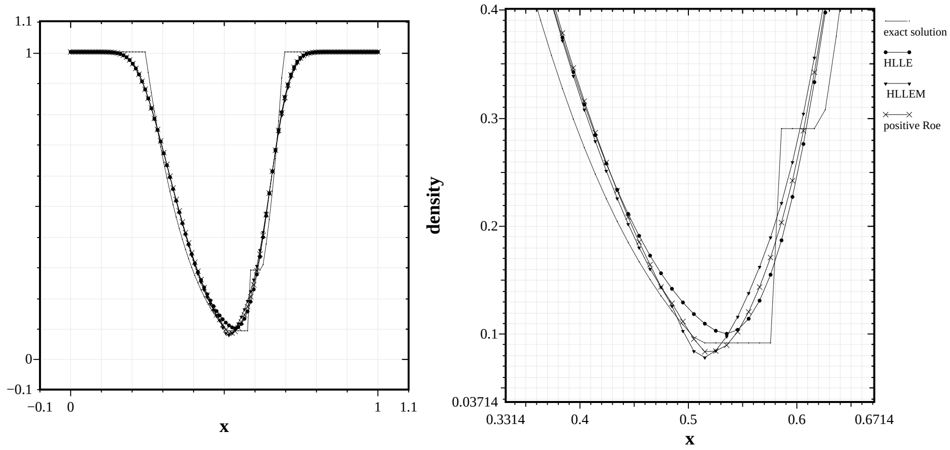

Figure 2: Strong rarefaction problem [3] modified so as to obtain a non-symmetric solution.

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