# Dual Effect in Stochastic Optimization

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# Introduction

During the first part of the course, we have studied open-loop stochastic optimization problems, that is, problems in which the decisions correspond to deterministic variables which minimize a cost function defined as an expectation.

$$\min_{u \in \mathbb{U}^{\mathrm{ad}}} \mathbb{E}\big(j(u, \boldsymbol{W})\big) .$$

We now entre the realm of closed-loop stochastic optimization, that is, the case where on-line information is available to the decision maker. The decisions are thus functions of information and correspond to random variables.

$$\min_{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}} \mathbb{E}\big(j(\boldsymbol{U}, \boldsymbol{W})\big) .$$

# Lecture Outline

# Variables and Constraints

The decision variable $\boldsymbol{U}$ is now a random variable and belongs to a functional space $\mathcal{U}$. A canonical example is: $\mathcal{U} = L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{U})$.

The contraints on the r.v. $\boldsymbol{U}$ may be of different nature:

- point-wise constraints dealing with the possible values of $\boldsymbol{U}$:

$$\boldsymbol{U} \in \mathcal{U}^{\mathrm{po}} = \left\{ \boldsymbol{U} \in \mathcal{U},\ \boldsymbol{U}(\omega) \in U^{\mathrm{ad}}\ \mathbb{P}\text{-a.s.} \right\} ,$$

- risk constraints, such as expectation or probability constraints:

$$\boldsymbol{U} \in \mathcal{U}^{\mathrm{ri}} = \left\{ \boldsymbol{U} \in \mathcal{U},\ \mathbb{P}\big(\Theta(\boldsymbol{U}) \leq \theta\big) \geq \pi \right\} ,$$

- measurability constraints which express the fact that a given amount of information $\boldsymbol{Y}$ is available to the decision maker:

$$\boldsymbol{U} \in \mathcal{U}^{\mathrm{me}} = \left\{ \boldsymbol{U} \in \mathcal{U},\ \boldsymbol{U} \text{ is measurable w.r.t. } \boldsymbol{Y} \right\} .$$

**We will mainly concentrate on the measurability constraints.**

# Compact Formulation of a Closed-Loop Problem

Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, the essential ingredients of a stochastic optimization problem are

- noise $\boldsymbol{W}$: r.v. with values in a measurable space $(\mathbb{W}, \mathcal{W})$,
- decision $\boldsymbol{U}$: r.v. with values in a measurable space $(\mathbb{U}, \mathcal{U})$,
- information $\boldsymbol{Y}$: r.v. with values in a measurable space $(\mathbb{Y}, \mathcal{Y})$,
- a cost function: measurable mapping $j : \mathbb{U} \times \mathbb{W} \to \mathbb{R}$.

The $\sigma$-field generated by $\boldsymbol{W}$ (resp. $\boldsymbol{Y}$) is denoted $\mathcal{F}$ (resp. $\mathcal{G}$).

With all these elements at hand, the problem is set as follows:

$$\min_{\boldsymbol{U} \preceq \boldsymbol{Y}} \; \mathbb{E}\big(j(\boldsymbol{U}, \boldsymbol{W})\big) \; .$$

The notation $\boldsymbol{U} \preceq \boldsymbol{Y}$ (or equivalently $\boldsymbol{U} \preceq \mathcal{G}$) is used to express that the r.v. $\boldsymbol{U}$ is measurable w.r.t. to the $\sigma$-field generated by $\boldsymbol{Y}$.

# Representation of Measurability Constraints

Consider the information structure of the stochastic optimization problem in a compact form, that is, the measurability constraints

$$\boldsymbol{U} \preceq \boldsymbol{Y} .$$

This information structure may be interpreted in different ways.

- From the functional point of view, using Doob's Theorem, the decision $\boldsymbol{U}$ is expressed as a measurable function of $\boldsymbol{Y}$:

  $$\boldsymbol{U} = \varphi(\boldsymbol{Y}) .$$

  In this setting, the decision variable becomes the function $\varphi$.
- From the algebraic point of view, the constraints are expressed in terms of $\sigma$-field, that is,

  $$\sigma\big(\boldsymbol{U}\big) \subset \sigma\big(\boldsymbol{Y}\big) .$$

**Question**: how to take all these representations into account?

# Static Information Structure (SIS)

This is the case when $\mathcal{G} = \sigma(\boldsymbol{Y})$ is fixed, defined independently of the decision $\boldsymbol{U}$. Therefore, the terminology "static" expresses that the $\sigma$-field $\mathcal{G}$ constraining the decision cannot be modified by the decision maker. *It does not imply that no dynamics is present in the problem formulation.*[11]

- If the information $\boldsymbol{Y}$ is defined as a function of the noise $\boldsymbol{W}$, that is, $\boldsymbol{Y} = h(\boldsymbol{W})$, it generates a static information structure.
- Note that it may happen that $\boldsymbol{Y}$ does depend on $\boldsymbol{U}$ whereas the $\sigma$-field $\mathcal{G}$ it generates remains fixed.

Remember from now that SIS will be the "easy" case.

[11] If time is involved in the optimization problem, a decision $\boldsymbol{U}_t$ has to be taken at each time $t$, based on an information $\boldsymbol{Y}_t$, so that a measurability constraint $\boldsymbol{U}_t \preceq \boldsymbol{Y}_t$ is written at each time stage $t$.

# Dynamic Information Structure (DIS) (1)

This is the situation when $\mathcal{G} = \sigma(\boldsymbol{Y})$ depends on $\boldsymbol{U}$. For example, in the case where $\boldsymbol{Y} = h(\boldsymbol{U}, \boldsymbol{W})$, the constraint reads

$$\boldsymbol{U} \preceq h(\boldsymbol{U}, \boldsymbol{W}) ,$$

which yields a (seemingly) implicit measurability constraint.

This is a source of huge complexity for stochastic optimization problems, known under the name of the dual effect of control. Indeed, the decision maker has to take care of the following double effect:

- on the one hand, his decision affects the cost $\mathbb{E}\big(j(\boldsymbol{U}, \boldsymbol{W})\big)$,
- on the other hand, she makes the information more or less constrained, that is, a less or more large admissible set.

# Dynamic Information Structure (DIS) (2)

It will be easier to imagine such problems by explicitly introducing several agents which take decisions based on observations which may depend on decisions of other agents. Those agents may be a priori ordered. Then the notion of causality (who is "upstream" and who is "downstream") becomes relevant, and it turns out that two notions are paramount for the level of difficulty of the problem:

1. who influences the available information of whom?
2. who knows more than whom?

We will illustrate these subtle notions and questions in the case of stochastic optimal control, for which an "agent" takes a decision at each time stage $t$ of the time horizon $\{0, \ldots, T-1\}$.

# Stochastic Optimal Control Problem...

An standard form for a stochastic optimization problem involving a dynamic process $\boldsymbol{X}$ over a time horizon $\{0, \dots, T\}$ is:

$$\min_{(\boldsymbol{U}_0,\dots,\boldsymbol{U}_{T-1},\boldsymbol{X}_0,\dots,\boldsymbol{X}_T)} \mathbb{E}\left(\sum_{t=0}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1}) + K(\boldsymbol{X}_T)\right)$$

subject to the dynamic constraints

$$\boldsymbol{X}_0 = f_{-1}(\boldsymbol{W}_0) \ ,$$

$$\boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1}) \ , \quad t = 0, \dots, T-1 \ .$$

We denote by $\mathcal{F}_t$ the $\sigma$-field generated by noises prior time $t$:

$$\mathcal{F}_t = \sigma\big(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t\big) \ , \quad t = 0, \dots, T \ .$$

**Nonanticipativity:** $\mathcal{F}_t$ is the maximal information available at $t$.

# and a Possible Information Structure

## Information Structure

An observation becomes available at time $t$:

$$\boldsymbol{Z}_t = g_t(\boldsymbol{X}_t, \boldsymbol{W}_t) \ , \quad t = 0, \ldots, T-1 \ .$$

- $\boldsymbol{Z}_t = \boldsymbol{W}_t$: observation of the noise,
- $\boldsymbol{Z}_t = \boldsymbol{X}_t$: observation of the state.

The information available at $t$ is a function of past observations:

$$\boldsymbol{Y}_t = C_t\big(\boldsymbol{Z}_0, \ldots, \boldsymbol{Z}_t\big) \ , \quad t = 0, \ldots, T-1 \ .$$

- $\boldsymbol{Y}_t = (\boldsymbol{Z}_0, \ldots, \boldsymbol{Z}_t)$: perfect memory.

## Information Constraints

$$\boldsymbol{U}_t \preceq \boldsymbol{Y}_t \quad \Longleftrightarrow \quad \begin{cases} \boldsymbol{U}_t &= \varphi_t\big(\boldsymbol{Y}_t\big) \\ \sigma\big(\boldsymbol{U}_t\big) &\subset \sigma\big(\boldsymbol{Y}_t\big) \end{cases} \ , \quad t = 0, \ldots, T-1 \ .$$

# Remarks About the Information Structure (1)

The problem is formulated in the Decision-Hazard framework: the decision $\boldsymbol{U}_t$ at time $t$ must be chosen before $\boldsymbol{W}_{t+1}$ occurs.

In this setting, the information $\boldsymbol{Y}_t$ has the following structure:

$$
\begin{aligned}
\boldsymbol{Y}_t &= C_t(\boldsymbol{Z}_0, \ldots, \boldsymbol{Z}_t) \\
&= C_t\big(g_0(\boldsymbol{X}_0, \boldsymbol{W}_0), \ldots, g_t(\boldsymbol{X}_t, \boldsymbol{W}_t)\big) \\
&= C_t\Big(g_0\big(f_{-1}(\boldsymbol{W}_0), \boldsymbol{W}_0\big), \ldots, g_t\big(f_{t-1}(\boldsymbol{X}_{t-1}, \boldsymbol{U}_{t-1}, \boldsymbol{W}_t), \boldsymbol{W}_t\big)\Big) \\
&\;\;\vdots \\
&= h_t(\boldsymbol{U}_0, \ldots, \boldsymbol{U}_{t-1}, \boldsymbol{W}_0, \ldots, \boldsymbol{W}_t) \,.
\end{aligned}
$$

We are in a specific case of Dynamic Information Structure:

- information at time $t$ depends on past noises,
- information at time $t$ depends on (strictly) past controls.

# Remarks About the Information Structure (2)

The information constraints are as follows:

- $\boldsymbol{U}_0 \preceq \sigma\big(h_0(\boldsymbol{W}_0)\big) \subset \sigma\big(\boldsymbol{W}_0\big)$.
- $\boldsymbol{U}_1 \preceq \sigma\big(h_1(\boldsymbol{U}_0, \boldsymbol{W}_0, \boldsymbol{W}_1)\big) \subset \sigma\big(\boldsymbol{W}_0, \boldsymbol{W}_1\big)$.

  $\vdots$

- $\boldsymbol{U}_t \preceq \underbrace{\sigma\big(h_t(\boldsymbol{U}_0, \ldots, \boldsymbol{U}_{t-1}, \boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)\big)}_{\mathcal{G}_t} \subset \underbrace{\sigma\big(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t\big)}_{\mathcal{F}_t}$.

The causality principle is fulfilled (no dependency on the future), but information depends on past controls, so that dual effect is possible and controls $(\boldsymbol{U}_0, \ldots, \boldsymbol{U}_{t-1})$ may be used to make the $\sigma$-field $\mathcal{G}_t$ as large as possible. Otherwise stated,

decisions prior time $t$ allow to transmit information up to time $t$.

# Functional Approach: Dynamic Programming

Assume that the noise process $(\mathbf{W}_0, \dots, \mathbf{W}_T)$ corresponds to a white noise. Then the three following information structures:

1. $\mathbf{Y}_t = (\mathbf{W}_0, \dots, \mathbf{W}_t)$,
2. $\mathbf{Y}_t = (\mathbf{X}_0, \dots, \mathbf{X}_t)$,
3. $\mathbf{Y}_t = \mathbf{X}_t$,

lead to the same optimal solution of the problem. Moreover, this solution can be obtained by solving the Bellman equation:

$$V_T(x) = K(x) \; ,$$

$$V_t(x) = \min_{u \in \mathbb{U}} \mathbb{E}\Big( L_t(x, u, \mathbf{W}_{t+1}) + V_{t+1}\big(f_t(x, u, \mathbf{W}_{t+1})\big)\Big) \; ,$$

$\rightsquigarrow$ $\mathbf{U}_t^\sharp = \varphi_t^\sharp(\mathbf{X}_t)$: functional approach applied to this specific DIS.

However, in the general case, the solution of the stochastic optimal control problem is not known (see Witsenhausen counterexample).

# Algebraic Approach: Stochastic Programming

- There are cases where a stochastic optimal control problem corresponds to a static information structure, for example

$$\boldsymbol{Y}_t = h_t(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t) \ , \quad t = 0, \dots, T-1 \ .$$

- There are also cases where the r.v. $\boldsymbol{Y}_t$ depends on the controls whereas the associated $\sigma$-fields remain fixed.

In all these situations, it is possible to use the algebraic approach and to look for the solution of the problem in terms of random variables satisfying fixed measurability contraints:

$$\sigma\big(\boldsymbol{U}_t\big) \subset \sigma\big(\boldsymbol{Y}_t\big) \ , \quad t = 0, \dots, T-1 \ .$$

- **First issue**: characterize the class of problems that can be solved by this approach (lack of dual effect).
- **Second issue**: obtain a finite approximation of the problem, and more specifically discretize the information constraints.

# A Simple Linear Quadratic Control Problem

Consider $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space, and six real-valued random variables related by the following dynamic equations:

$$
\begin{aligned}
\boldsymbol{X}_0 &= \boldsymbol{W}_0 \ , \\
\boldsymbol{X}_1 &= \boldsymbol{X}_0 + \boldsymbol{U}_0 \ , \\
\boldsymbol{X}_2 &= \boldsymbol{X}_1 - \boldsymbol{U}_1 \ .
\end{aligned}
$$

The optimization problem under consideration is

$$
\min_{\boldsymbol{U}_0 \preceq \boldsymbol{Y}_0 \, , \, \boldsymbol{U}_1 \preceq \boldsymbol{Y}_1} \ \mathbb{E}\left( k^2 \boldsymbol{U}_0^2 + \boldsymbol{X}_2^2 \right) \ ,
$$

$\boldsymbol{Y}_0$ and $\boldsymbol{Y}_1$ being the information available at $t = 0$ and $t = 1$.

We will examine different choices for the information structure.

# Problem Transformation

By an interchange of minimization and (conditional) expectation, the initial problem is equivalent to

$$\min_{\boldsymbol{U}_0 \preceq \boldsymbol{Y}_0} \mathbb{E}\Big(k^2 \boldsymbol{U}_0^2 + \min_{u_1 \in \mathbb{R}} \mathbb{E}\big((\boldsymbol{X}_1 - u_1)^2 \mid \boldsymbol{Y}_1\big)\Big) .$$

The $\arg\min$ of the inner optimization problem is the conditional expectation $\mathbb{E}\big(\boldsymbol{X}_1 \mid \boldsymbol{Y}_1\big)$, and the associated optimal cost is, by definition, the conditional variance:

$$\mathrm{Var}\left(\boldsymbol{X}_1 \mid \boldsymbol{Y}_1\right) = \mathbb{E}\big(\boldsymbol{X}_1^2 \mid \boldsymbol{Y}_1\big) - \Big(\mathbb{E}\big(\boldsymbol{X}_1 \mid \boldsymbol{Y}_1\big)\Big)^2 .$$

The solution of the initial problem is thus

$$\boldsymbol{U}_0^\sharp = \arg\min_{\boldsymbol{U}_0 \preceq \boldsymbol{Y}_0} \mathbb{E}\Big(k^2 \boldsymbol{U}_0^2 + \mathrm{Var}\left(\boldsymbol{X}_0 + \boldsymbol{U}_0 \mid \boldsymbol{Y}_1\right)\Big) ,$$

$$\boldsymbol{U}_1^\sharp = \mathbb{E}\big(\boldsymbol{X}_0 + \boldsymbol{U}_0^\sharp \mid \boldsymbol{Y}_1\big) .$$

# Information Patterns (1)

## Full Noise Observation

$$\boldsymbol{Y}_0 = \boldsymbol{W}_0 \quad , \quad \boldsymbol{Y}_1 = \boldsymbol{W}_0 \ .$$

$\boldsymbol{X}_0 = \boldsymbol{W}_0$ and $\boldsymbol{U}_0 \preceq \boldsymbol{Y}_0$ imply that $\boldsymbol{X}_1 = \boldsymbol{X}_0 + \boldsymbol{U}_0 \preceq \boldsymbol{Y}_1$, so that

$$\mathrm{Var}\left(\boldsymbol{X}_1 \mid \boldsymbol{Y}_1\right) = 0 \ .$$

We thus deduce that $\boldsymbol{U}_0^\sharp = 0$ and $\boldsymbol{U}_1^\sharp = \boldsymbol{X}_0$.

## Full State Observation

$$\boldsymbol{Y}_0 = \boldsymbol{X}_0 \quad , \quad \boldsymbol{Y}_1 = (\boldsymbol{X}_0, \boldsymbol{X}_1) \ .$$

Obviously we have $\boldsymbol{X}_1 \preceq \boldsymbol{Y}_1$, so that $\boldsymbol{U}_0^\sharp = 0$ and $\boldsymbol{U}_1^\sharp = \boldsymbol{X}_0$.

Note that this result remains true in the Markovian case $\boldsymbol{Y}_1 = \boldsymbol{X}_1$.

# Information Patterns (2)

## Classical Information Pattern (Noisy Observation of the State)

$$\boldsymbol{Y}_0 = \boldsymbol{X}_0 \quad , \quad \boldsymbol{Y}_1 = (\boldsymbol{X}_0, \boldsymbol{X}_1 + \boldsymbol{W}_1) \ .$$

We have $\boldsymbol{X}_1 = \boldsymbol{X}_0 + \boldsymbol{U}_0 \preceq \boldsymbol{X}_0 \preceq \boldsymbol{Y}_1$, so that $\boldsymbol{U}_0^\sharp = 0$ and $\boldsymbol{U}_1^\sharp = \boldsymbol{X}_0$.

## State-Control Observation

$$\boldsymbol{Y}_0 = \boldsymbol{X}_0 \quad , \quad \boldsymbol{Y}_1 = (\boldsymbol{U}_0, \boldsymbol{X}_1 + \boldsymbol{W}_1) \ .$$

It is shown that the problem admits only $\varepsilon$-optimal solutions, that is $\boldsymbol{U}_0 = \varepsilon \boldsymbol{Y}_0 = \varepsilon \boldsymbol{X}_0$ and $\boldsymbol{U}_1 = \boldsymbol{U}_0 / \varepsilon = \boldsymbol{X}_0$.

- For $\varepsilon > 0$, $\boldsymbol{U}_0 = \varepsilon \boldsymbol{X}_0 \Rightarrow \sigma(\boldsymbol{U}_0) = \sigma(\boldsymbol{X}_0)$, hence the result.
- For $\varepsilon = 0$, $\boldsymbol{U}_0 \equiv 0$ so that $\sigma(\boldsymbol{Y}_1) = \sigma(\boldsymbol{X}_0 + \boldsymbol{W}_1) \neq \sigma(\boldsymbol{X}_0)$.

# Information Patterns (3)

## Witsenhausen Counterexample

$$\boldsymbol{Y}_0 = \boldsymbol{X}_0 \quad , \quad \boldsymbol{Y}_1 = \boldsymbol{X}_1 + \boldsymbol{W}_1 \ .$$

An optimal solution exists, but its expression is unknown!

**Intuitive point of view**. The information $\boldsymbol{Y}_0$ available at time $t = 0$ is forgotten at time $t = 1$. The decision $\boldsymbol{U}_0$ may try to transmit information at time $t = 1$ (dual effect).
For example, using the feedback law $\boldsymbol{U}_0 = \alpha \boldsymbol{X}_0$, $\alpha \gg 0$, we have

$$\begin{aligned} \boldsymbol{Y}_1 &= \boldsymbol{X}_1 + \boldsymbol{W}_1 \\ &= (1+\alpha)\boldsymbol{X}_0 + \boldsymbol{W}_1 \\ &\approx (1+\alpha)\boldsymbol{X}_0 \ . \end{aligned}$$

But such a feedback is expensive $\rightsquigarrow$ tradeoff information/cost.

1. Closed Loop Stochastic Optimization Problems
   - Stochastic Optimization Formulation
   - Stochastic Optimal Control Problem
   - Witsenhausen's Counterexample

2. Dual Effect in Stochastic Optimization
   - Tools for Information Handling
   - Dual Effect Free Stochastic Optimization
   - Dual Effect for Stochastic Optimal Control Problems

# Classical Measurability Framework

The standard measurability theory makes use of $\sigma$-fields.

**Definition**

A $\sigma$-field on $\Omega$ is a nonempty collection $\mathcal{A}$ of subsets of $\Omega$ which is stable under complementation and countable union (and hence under countable intersection).

The following fundamental result is due to **J. Doob**.

**Theorem**

*Consider two mappings $H_i : (\Omega, \mathcal{A}) \rightarrow (\mathbb{Y}_i, \mathcal{Y}_i)$, $i = 1, 2$. Assume that $\mathbb{Y}_1$ is a separable complete metric space. The mapping $H_1$ is measurable w.r.t. $H_2$ if and only if there exists a measurable mapping $\mathfrak{f} : \mathrm{im} H_2 \rightarrow \mathrm{im} H_1$ such that $H_1 = \mathfrak{f} \circ H_2$.*

# Information and $\sigma$-Fields

As noted by [Dubra & Echenique, 2004], and contrarily to a widely accepted idea, the use of $\sigma$-fields as the informational content of a signal is not without raising serious problems.

**Example**

On $\Omega = [0,1]$, consider the two partitions $\mathcal{C}^\flat = \big\{[0,1/2], ]1/2,1]\big\}$ and $\mathcal{C}^\sharp = \big\{\{\omega\}\big\}_{\omega\in[0,1]}$ (complete partition). Of course, the set $\mathcal{C}^\sharp$ gives more information that $\mathcal{C}^\flat$. Since $\mathcal{C}^\flat$ is a finite partition, the $\sigma$-field it generates is $\sigma(\mathcal{C}^\flat) = \big\{\emptyset, [0,1], [0,1/2], ]1/2,1]\big\}$. It can be seen that the $\sigma$-field generated by $\mathcal{C}^\sharp$ is made of subsets of $\Omega$ which are either countable or whose complement is countable...
To summarize, we have that $\mathcal{C}^\flat \preceq \mathcal{C}^\sharp$, whereas $\sigma(\mathcal{C}^\flat)$ and $\sigma(\mathcal{C}^\sharp)$ are not comparable.

# Partition Fields or $\pi$-Fields

The inclusion order on $\sigma$-fields is thus not compatible with the order on partitions: using $\sigma$-fields in order to express information may be tricky.[12] We now claim that partition fields are adequate to represent information.

**Definition**

A partition field (or $\pi$-field) on $\Omega$ is a nonempty collection $\mathcal{G}$ of subsets of $\Omega$ which is stable under complementation and unlimited union (and hence under unlimited intersection).

A $\pi$-field may be a large collection of subsets: the $\pi$-field generated by all singletons of $\Omega$ is the collection of all subsets of $\Omega$, that is, $2^{\Omega}$. Partition fields are not used in Probability Theory because they are generally too large to support a probability law.

[12] However, the interest of $\sigma$-fields is that they can support a probability law.

# Properties of Partition Fields

**Definition**

Consider a collection $\mathcal{G}$ of subsets of $\Omega$. An atom of $\mathcal{G}$ is a subset $G \in \mathcal{G}$ such that $K \in \mathcal{G}$ and $K \subset G$ imply that $K = \emptyset$ or $K = G$.

**Theorem**

*Consider $\mathcal{G}$ a $\pi$-field of $\Omega$. The atoms of $\mathcal{G}$ form a partition of $\Omega$, denoted by $\mathrm{part}(\mathcal{G})$, which generates $\mathcal{G}$:*

$$\pi\big(\mathrm{part}(\mathcal{G})\big) = \mathcal{G} \ .$$

**Theorem**

*Let $\mathcal{G}$ and $\mathcal{G}'$ be two $\pi$-fields of $\Omega$. The $\pi$-field $\mathcal{G}$ is finer than $\mathcal{G}'$ if and only if every atom of $\mathcal{G}'$ is the union of $\mathcal{G}$-atoms:*

$$\mathcal{G}' \preceq \mathcal{G} \iff \mathrm{part}(\mathcal{G}') \preceq \mathrm{part}(\mathcal{G}) \ .$$

# Measurability w.r.t. $\pi$-Fields (1)

**Definition**

Let $\Omega$ be equipped with a $\pi$-field $\mathcal{G}$, and let $\mathbb{Y}$ be another set equipped with the complete $\pi$-field $\mathcal{Y} = 2^{\mathbb{Y}}$. The mapping $H : \Omega \to \mathbb{Y}$ is said to be measurable w.r.t. $\mathcal{G}$ if the $\pi$-field generated by $H$, that is, $\pi(H) := H^{-1}(\mathcal{Y})$, is such that $\pi(H) \preceq \mathcal{G}$.

**Theorem**

*Consider a mapping $H : \Omega \to \mathbb{Y}$ and a $\pi$-field $\mathcal{G}$ with associated partition $\mathrm{part}(\mathcal{G})$. The two following assertions are equivalent.*

1. *The mapping $H$ is measurable w.r.t. the $\pi$-field $\mathcal{G}$.*
2. *The mapping $H$ is constant over each element of $\mathrm{part}(\mathcal{G})$.*

# Measurability w.r.t. $\pi$-Fields (2)

**Definition**

Consider two mappings $H_i : \Omega \to \mathbb{Y}_i$, $i = 1, 2$. The mapping $H_1$ is said to be measurable w.r.t. the mapping $H_2$ if $\pi(H_1) \preceq \pi(H_2)$.

**Theorem**

*Consider two mappings $H_i : \Omega \to \mathbb{Y}_i$, $i = 1, 2$. The following conditions are equivalent characterizations of the fact that $H_1$ is measurable w.r.t. $H_2$.*

1. $\forall(\omega, \omega') \in \Omega \times \Omega\ , \quad H_2(\omega) = H_2(\omega') \Rightarrow H_1(\omega) = H_1(\omega')$
2. $\exists!\ \mathfrak{f} : \mathrm{im} H_2 \to \mathrm{im} H_1$. *such that* $H_1 = \mathfrak{f} \circ H_2$.

Similar conditions are available in the case where $H_1$ and $H_2$ are equivalent mappings: $H_1 \equiv H_2 \;\Leftrightarrow\; (H_1 \preceq H_2 \text{ and } H_2 \preceq H_1)$.

# Measurability w.r.t. $\pi$-Fields (3)

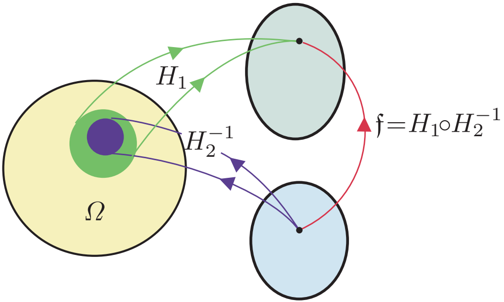


Figure: Measurability relation $H_1 \preceq H_2$

# Problem Statement (1)

From now, the framework is the one of measurability w.r.t. $\pi$-fields and not w.r.t. $\sigma$-fields. Let $\Omega$ be a set equipped with the $\pi$-field $\mathcal{G}$.

The r.v. entering a stochastic optimization problem are

- a noise $\boldsymbol{W} : \Omega \to \mathbb{W}$: mapping with values in a space $\mathbb{W}$,
- a decision $\boldsymbol{U} : \Omega \to \mathbb{U}$: mapping with values in a space $\mathbb{U}$; the set of all possible decisions is denoted by $\mathcal{U}$,
- a information $\boldsymbol{Y} : \Omega \to \mathbb{Y}$: mapping with values in a space $\mathbb{Y}$; the information is given as a function $h : \mathbb{U} \times \mathbb{W} \to \mathbb{Y}$:

$$\boldsymbol{Y} = h\big(\boldsymbol{U}, \boldsymbol{W}\big) ,$$

  so that the information depends on the control (DIS).

The information constraints of the problem write $\boldsymbol{U} \preceq \boldsymbol{Y}$ so that the admissible set upon which the optimization problem relies is

$$\mathcal{U}^{\mathrm{ad}} = \big\{ \boldsymbol{U} \in \mathcal{U} \ , \ \ \boldsymbol{U} \preceq \boldsymbol{Y} \big\} .$$

# Problem Statement (2)

The $\pi$-field generated by $\boldsymbol{Y}$ depends on $\boldsymbol{U}$: dual effect holds true.

We want to characterize the greatest set $\mathcal{U}^{\mathrm{nde}} \subset \mathcal{U}^{\mathrm{ad}}$ such that the information $\pi$-field generated by any $\boldsymbol{U} \in \mathcal{U}^{\mathrm{nde}}$ remains fixed:

$$\pi\big(h(\boldsymbol{U}, \boldsymbol{W})\big) = \pi\big(h(\boldsymbol{U}', \boldsymbol{W})\big) \quad \forall (\boldsymbol{U}, \boldsymbol{U}') \in \mathcal{U}^{\mathrm{nde}} \times \mathcal{U}^{\mathrm{nde}} .$$

This condition is equivalently formulated as[13]

$$h\big(\boldsymbol{U}(\omega), \boldsymbol{W}(\omega)\big) = h\big(\boldsymbol{U}(\omega'), \boldsymbol{W}(\omega')\big) \iff h\big(\boldsymbol{U}'(\omega), \boldsymbol{W}(\omega)\big) = h\big(\boldsymbol{U}'(\omega'), \boldsymbol{W}(\omega')\big) \quad \forall (\omega, \omega') \in \Omega \times \Omega .$$

[13] Recall that, in the $\pi$-field formalism, we have

$$\boldsymbol{U} \preceq \boldsymbol{Y} \iff \Big(\boldsymbol{Y}(\omega) = \boldsymbol{Y}(\omega') \Rightarrow \boldsymbol{U}(\omega) = \boldsymbol{U}(\omega) \ \ \forall (\omega, \omega') \in \Omega \times \Omega\Big) .$$

# No Open-Loop Dual Effect

As a minimal requirement, we assume that the problem is such that all constant decision variables (open-loop controls) lead to the same information structure. We denote this set by

$$\perp_{\mathcal{U}} = \left\{ \boldsymbol{U} \in \mathcal{U} \;,\;\; \boldsymbol{U}(\omega) = \boldsymbol{U}(\omega') \quad \forall (\omega, \omega') \in \Omega \times \Omega \right\} .$$

Of course we have: $\perp_{\mathcal{U}} \subset \mathcal{U}^{\mathrm{ad}}$.

**Definition**

There is No Open-Loop Dual Effect (NOLDE) for the stochastic system with observation function $h : \mathbb{U} \times \mathbb{W} \to \mathbb{Y}$ if we have

$$\pi\big(h(\boldsymbol{U}, \boldsymbol{W})\big) = \pi\big(h(\boldsymbol{U}', \boldsymbol{W})\big) \quad \forall (\boldsymbol{U}, \boldsymbol{U}') \in \perp_{\mathbb{U}} \times \perp_{\mathbb{U}} .$$

Otherwise stated, the NOLDE property means that any mapping in the collection $\left\{ h(u, \boldsymbol{W}) \right\}_{u \in \mathbb{U}}$ generates the same $\pi$-field.

# Characterization of $\mathcal{U}^{\mathrm{nde}}$

Admissible set: $\mathcal{U}^{\mathrm{ad}} = \{\boldsymbol{U} \in \mathcal{U} \,,\ \ \boldsymbol{U} \preceq h(\boldsymbol{U}, \boldsymbol{W})\}$.

We assume the NOLDE property, and we denote by $\zeta : \Omega \to \mathbb{Y}$ the mapping such that $\zeta(\omega) = h(u_0, \boldsymbol{W}(\omega))$ for a given $u_0 \in \mathbb{U}$.

- The no dual effect set $\mathcal{U}^{\mathrm{nde}}$ is given by

$$\mathcal{U}^{\mathrm{nde}} = \{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}} \,,\ \ h(\boldsymbol{U}, \boldsymbol{W}) \equiv \zeta\} \,,$$

- We define the fixed information set $\mathcal{U}^{\zeta}$ by

$$\mathcal{U}^{\zeta} = \{\boldsymbol{U} \in \mathcal{U} \,,\ \ \boldsymbol{U} \preceq \zeta\} \,.$$

**Theorem**

*Under the NOLDE assumption, the set $\mathcal{U}^{\mathrm{nde}}$ is characterized by*

$$\mathcal{U}^{\mathrm{nde}} = \mathcal{U}^{\mathrm{ad}} \cap \mathcal{U}^{\zeta} \,.$$

# Proof

1. $\mathcal{U}^{\mathrm{nde}} \subset \mathcal{U}^{\mathrm{ad}} \cap \mathcal{U}^{\zeta}$.

   Let $\boldsymbol{U} \in \mathcal{U}^{\mathrm{nde}}$.

   - By definition, $\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}$, and $\boldsymbol{U} \in \mathcal{U}^{\zeta}$ since $h(\boldsymbol{U}, \boldsymbol{W}) \equiv \zeta$.

2. $\mathcal{U}^{\mathrm{ad}} \cap \mathcal{U}^{\zeta} \subset \mathcal{U}^{\mathrm{nde}}$.

   Let $\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}} \cap \mathcal{U}^{\zeta}$.

   - Assume that $h(\boldsymbol{U}, \boldsymbol{W})(\omega) = h(\boldsymbol{U}, \boldsymbol{W})(\omega')$. Since $\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}$, we have that $\boldsymbol{U}(\omega) = \boldsymbol{U}(\omega')$. Denoting by $u$ this common value, we have $h(u, \boldsymbol{W}(\omega)) = h(u, \boldsymbol{W}(\omega'))$ and hence $\zeta(\omega) = \zeta(\omega')$. We deduce that $\zeta \preceq h(\boldsymbol{U}, \boldsymbol{W})$.
   - Assume that $\zeta(\omega) = \zeta(\omega')$. Since $\boldsymbol{U} \in \mathcal{U}^{\zeta}$, we have that $\boldsymbol{U}(\omega) = \boldsymbol{U}(\omega')$. Denoting by $u$ this common value, we have $h(u, \boldsymbol{W}(\omega)) = h(u, \boldsymbol{W}(\omega'))$ and $h(\boldsymbol{U}, \boldsymbol{W})(\omega) = h(\boldsymbol{U}, \boldsymbol{W})(\omega')$. We deduce that $h(\boldsymbol{U}, \boldsymbol{W}) \preceq \zeta$.

Ultimately, we obtain $h(\boldsymbol{U}, \boldsymbol{W}) \equiv \zeta$, that is, $\boldsymbol{U} \in \mathcal{U}^{\mathrm{nde}}$. □

# Examples

1. Let $\Omega = \mathbb{W} = \mathbb{U} = \mathbb{R}$, $\boldsymbol{W} = \mathrm{Id}_{\Omega}$ and $h(u, w) = w$.
   No open-loop dual effect holds, $\zeta$ being the identity mapping, and we have
   $$\mathcal{U}^{\zeta} = \mathcal{U} \quad \text{and} \quad \mathcal{U}^{\mathrm{ad}} = \mathcal{U} \ .$$
2. Let $\Omega = \mathbb{W} = \mathbb{U} = \mathbb{R}$, $\boldsymbol{W} = \mathrm{Id}_{\Omega}$ and $h(u, w) = u$.
   No open-loop dual effect holds, $\zeta$ being a constant mapping, and we have
   $$\mathcal{U}^{\zeta} = \perp_{\mathcal{U}} \quad \text{and} \quad \mathcal{U}^{\mathrm{ad}} = \mathcal{U} \ .$$
3. Let $\Omega = \mathbb{W} = \mathbb{U} = \mathbb{R}$, $\boldsymbol{W} = \mathrm{Id}_{\Omega}$ and $h(u, w) = u - w$.
   No open-loop dual effect holds, $\zeta$ being the identity mapping, and we have
   $$\mathcal{U}^{\zeta} = \mathcal{U} \quad \text{but} \quad \mathcal{U}^{\mathrm{ad}} \neq \mathcal{U} \ .$$

Indeed, $\boldsymbol{U}_0 = \boldsymbol{W}$ is such that $h\big(\boldsymbol{U}_0, \boldsymbol{W}\big) = 0$, hence $\boldsymbol{U}_0 \notin \mathcal{U}^{\mathrm{ad}}$.

# Problem Statement (1)

We now consider a stochastic optimal control problem defined on $\{0, \ldots, T\}$. Measurability is defined w.r.t. partition fields, so that the set $\Omega$ is equipped with a $\pi$-field $\mathcal{G}$.

- The noise $\boldsymbol{W} = (\boldsymbol{W}_0, \ldots, \boldsymbol{W}_T) : \Omega \to \mathbb{W}^{T+1}$ is made up of the sequence of noises at each time $t$.
- The decision $\boldsymbol{U} = (\boldsymbol{U}_0, \ldots, \boldsymbol{U}_{T-1}) : \Omega \to \mathbb{U}^T$ is made up of the sequence of decisions at each time $t$, and the set of decisions is denoted by $\mathcal{U}^T$.
- An information $\boldsymbol{Y}_t$ is available at each time $t$, and is defined by a function $h_t : \mathbb{U}^T \times \mathbb{W}^{T+1} \to \mathbb{Y}$:

$$\boldsymbol{Y}_t = h_t\big(\boldsymbol{U}, \boldsymbol{W}\big) .$$

The information constraints are gathered in the admissible set

$$\mathcal{U}^{\mathrm{ad}} = \big\{\boldsymbol{U} \in \mathcal{U}^T , \ \ \boldsymbol{U}_t \preceq \boldsymbol{Y}_t , \ \ t = 0, \ldots, T-1\big\} .$$

# Problem Statement (2)

We denote by $\perp_{\mathcal{U}^T}$ the set of open-loop controls: $\boldsymbol{U} \in \perp_{\mathcal{U}^T}$ if each decision variable $\boldsymbol{U}_t$ is a constant mapping.

Assuming the NOLDE property at each $t = 0 \dots T-1$, we denote by $\zeta_t : \Omega \to \mathbb{Y}$ the mapping such that $\zeta_t(\omega) = h_t(u_0, \boldsymbol{W}(\omega))$ for any given $u_0 \in \mathbb{U}$.

- The no dual effect set $\mathcal{U}^{\mathrm{nde}}$ is defined as

$$\mathcal{U}^{\mathrm{nde}} = \left\{ \boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}} \ , \ \ h_t(\boldsymbol{U}, \boldsymbol{W}) \equiv \zeta_t \ , \ \ t = 0, \dots, T-1 \right\} .$$

- The fixed information set $\mathcal{U}^{\zeta}$ is defined as

$$\mathcal{U}^{\zeta} = \left\{ \boldsymbol{U} \in \mathcal{U}^{T} \ , \ \ \boldsymbol{U}_t \preceq \zeta_t \ , \ \ t = 0, \dots, T-1 \right\} .$$

**Question**: does $\mathcal{U}^{\mathrm{nde}} = \mathcal{U}^{\mathrm{ad}} \cap \mathcal{U}^{\zeta}$ still holds?

# Precedence and Memory Relations

1. Precedence binary relation. We denote by $[[t]]$ the smallest set of time stages $\tau \in \{0, \dots, T-1\}$, such that the mapping $h_t$ functionally depends on $u_\tau$, and we introduce the notation

$$\tau \,\mathfrak{P}\, t \iff \tau \in [[t]] \,.$$

Otherwise stated, $\tau \,\mathfrak{P}\, t$ means that the decision variable at time $\tau$ influences the information variable at time $t$.

2. Memory binary relation. We denote by $\langle\langle t \rangle\rangle$ the greatest set of time stages $\tau \in \{0, \dots, T-1\}$, such that $h_\tau \preceq h_t$ (these mappings are defined on $\mathbb{U}^T \times \mathbb{W}^{T+1}$), and we introduce the notation

$$\tau \,\mathcal{M}\, t \iff \tau \in \langle\langle t \rangle\rangle \,.$$

Otherwise stated, $\tau \,\mathcal{M}\, t$ means that the information available at time $\tau$ is remembered at time $t$.

# A Meaningful Inclusion

**Definition**

We say that the precedence binary relation $\mathfrak{P}$ is included in the memory binary relation $\mathfrak{M}$ if

$$\tau \in [[t]] \implies \tau \in \langle\langle t \rangle\rangle \quad \forall (\tau, t) \in \{0, \dots, T-1\}^2 \ .$$

We denote this property by $\mathfrak{P} \subset \mathfrak{M}$. It is equivalent to:

$$[[t]] \subset \langle\langle t \rangle\rangle \quad \forall t \in \{0, \dots, T-1\} \ .$$

This property means that if a “agent” $\tau$ influences another “agent” $t$, then the information of “agent” $\tau$ is available to “agent” $t$.
From an intuitive point of view, agent $\tau$ has no reason to influence agent $t$ in order to transmit information because agent $t$ already knows the information of agent $\tau$: there is no need of dual effect.

# Main Theorem

**Theorem**

*Let us assume that no open-loop dual effect holds true and that the precedence binary relation $\mathfrak{P}$ is included in the memory binary relation $\mathfrak{M}$. Then*

$$\mathcal{U}^{\mathrm{nde}} = \mathcal{U}^{\mathrm{ad}} \cap \mathcal{U}^{\zeta} \; .$$

**Sketch of proof.**

1. $\mathcal{U}^{\mathrm{nde}} \subset \mathcal{U}^{\mathrm{ad}} \cap \mathcal{U}^{\zeta}$: obvious.
2. $\mathcal{U}^{\mathrm{ad}} \cap \mathcal{U}^{\zeta} \subset \mathcal{U}^{\mathrm{nde}}$: let $\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}} \cap \mathcal{U}^{\zeta}$.
   - Assume $h_t\big(\boldsymbol{U}, \boldsymbol{W}\big)(\omega) = h_t\big(\boldsymbol{U}, \boldsymbol{W}\big)(\omega')$; then it also holds true for any $\tau \in \langle\langle t \rangle\rangle$, so that $\boldsymbol{U}(\omega) = \boldsymbol{U}(\omega')$ for any $\tau \in [[t]]$ and $\zeta_t(\omega) = \zeta_t(\omega')$; hence $\zeta_t \preceq h_t\big(\boldsymbol{U}, \boldsymbol{W}\big)$.
   - Assume $\zeta_t(\omega) = \zeta_t(\omega')$; ......hence $h_t\big(\boldsymbol{U}, \boldsymbol{W}\big) \preceq \zeta_t$. □

# Application to a More Specific Problem (1)

We consider the causal Decision-Hazard information structure:

$$\boldsymbol{Y}_t = h_t(\boldsymbol{U}_0, \ldots, \boldsymbol{U}_{t-1}, \boldsymbol{W}_0, \ldots, \boldsymbol{W}_t) ,$$

and we moreover assume perfect memory:

$$h_\tau \preceq h_t \quad \forall \tau \leq t .$$

Then we have the two following properties:

1. $[[t]] \subset \{0, \ldots, t-1\}$,
2. $\langle\langle t \rangle\rangle \supset \{0, \ldots, t\}$,

so that the precedence relation is included in the memory relation.
Assuming the NOLDE property, the last theorem applies.

**Remark**. The perfect memory property, defined on $\mathbb{U}^T \times \mathbb{W}^{T+1}$, means that

$$h_t(u, w) = h_t(u', w') \quad \Rightarrow \quad h_\tau(u, w) = h_\tau(u', w') \quad \forall \tau \leq t .$$

It is far stronger than some kind of "open-loop perfect memory".

# Application to a More Specific Problem (2)

In this specific case, we obtain the much better following result.

**Theorem**

*Assume that both the NOLDE and the perfect memory properties hold true for the information structure under consideration. Then we have that*

$$\mathcal{U}^{\mathrm{nde}} = \mathcal{U}^{\mathrm{ad}} = \mathcal{U}^{\zeta} \ .$$

**Sketch of proof.** The proof is done by induction on the two sets:

$$\mathcal{U}_t^{\mathrm{ad}} = \left\{ \boldsymbol{U} \in \mathcal{U}^T, \ \boldsymbol{U}_\tau \preceq \boldsymbol{Y}_\tau \ \forall \tau \leq t \right\} , \ \mathcal{U}_t^{\zeta} = \left\{ \boldsymbol{U} \in \mathcal{U}^T, \ \boldsymbol{U}_\tau \preceq \boldsymbol{\zeta}_\tau \ \forall \tau \leq t \right\} .$$

- It is obvious that $\mathcal{U}_0^{\mathrm{ad}} = \mathcal{U}_0^{\zeta}$.
- Assuming that $\mathcal{U}_\tau^{\mathrm{ad}} = \mathcal{U}_\tau^{\zeta} \ \ \forall \tau \leq t$, one can prove that $\mathcal{U}_{t+1}^{\mathrm{ad}} = \mathcal{U}_{t+1}^{\zeta}$.

Ultimately, we obtain $\mathcal{U}^{\mathrm{ad}} = \mathcal{U}_{T-1}^{\mathrm{ad}} = \mathcal{U}_{T-1}^{\zeta} = \mathcal{U}^{\zeta}$. □

# Application to a More Specific Problem (3)

As a conclusion, for a rather popular information structure, and assuming both NOLDE and perfect memory properties, we have:

(a) $\mathcal{U}^{\mathrm{ad}} = \mathcal{U}^{\mathrm{nde}}$. Every admissible decision variable belongs to the no dual effet set, so that there is no optimality loss to restrict the optimization process to $\mathcal{U}^{\mathrm{nde}}$.

(b) $\mathcal{U}^{\zeta} = \mathcal{U}^{\mathrm{nde}}$ every decision variable mesurable w.r.t. $\zeta$ belongs to the no dual effet set, so that the original problem can be solved using a fixed information structure.

**Example**. Consider the following additive information structure:

$$h_t(u_0, \ldots, u_{t-1}, w_0, \ldots, w_t) = h_{1,t}(u_0, \ldots, u_{t-1}) + h_{2,t}(w_0, \ldots, w_t) \, .$$

It always exhibits the NOLDE property. Assuming perfect memory, the previous theorem applies, so that we are in a good position to numerically solve the stochastic optimization problem.