Exact and converging bounds for SDDP

V. Leclère, P. Carpentier, J-P. Chancelier, A. Lenoir, F. Pacaud

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We are interested in multistage stochastic optimization problems of the form

$$\min_{\pi} \mathbb{E} \left( \sum_{t=0}^{T-1} L_t(X_t, U_t, \xi_t) + K(X_T) \right)$$

s.t. \( X_{t+1} = f_t(X_t, U_t, \xi_t) \)

\( U_t = \pi_t(X_t, \xi_t) \)

where

- \( x_t \) is the state of the system,
- \( u_t \) is the control applied at time \( t \),
- \( \xi_t \) is the noise happening between time \( t \) and \( t + 1 \), assumed to be time-independent,
- \( \pi \) is the policy.
By the white noise assumption, this problem can be solved by **Dynamic Programming**, where the Bellman functions satisfy

\[
\begin{align*}
V_T(x) &= K(x) \\
\hat{V}_t(x, \xi) &= \min_{u_t \in \mathcal{U}} L_t(x, u_t, \xi) + V_{t+1} \circ f_t(x, u_t, \xi) \\
V_t(x) &= \mathbb{E}\left(\hat{V}_t(x, \xi_t)\right)
\end{align*}
\]

Indeed, \( \pi \) is an optimal policy if

\[
\pi_t(x, \xi) \in \arg\min_{u_t \in \mathcal{U}} \left\{ L_t(x, u_t, \xi) + V_{t+1} \circ f_t(x, u_t, \xi) \right\}
\]
Bellman operator

For any time $t$, and any function $R : \mathbb{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ we define

$$\hat{T}_t(R)(x, \xi) := \min_{u_t \in U} L_t(x, u_t, \xi) + R \circ f_t(x, u_t, \xi)$$

and

$$\mathcal{T}_t(R)(x) := \mathbb{E}\left[\hat{T}_t(R)(x, \xi)\right].$$

Thus the Bellman equation simply reads

$$\begin{cases}
V_T &= K \\
V_t &= \mathcal{T}_t(V_{t+1})
\end{cases}$$

Incidentally, $R$ induce a policy $\pi^R_t(x, \xi)$
Bellman operator

For any time $t$, and any function $R : X \rightarrow \mathbb{R} \cup \{+\infty\}$ we define

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For any time $t$, and any function $R : \mathbb{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ we define

$$\hat{T}_t(R)(x, \xi) := \min_{u_t \in U} L_t(x, u_t, \xi) + R \circ f_t(x, u_t, \xi)$$

and

$$T_t(R)(x) := \mathbb{E}\left[\hat{T}_t(R)(x, \xi)\right].$$

Thus the Bellman equation simply reads

$$\begin{cases} V_T = K \\ V_t = T_t(V_{t+1}) \end{cases}$$

Incidentally, $R$ induce a policy $\pi^R_t(x, \xi)$
Under linear dynamics, and convex costs, the SDDP algorithm iteratively constructs polyhedral outer approximations of $V_t$.

More precisely, at iteration $k$

- We have polyhedral functions $V^k_t(\cdot) = \max_{\kappa \leq k} \langle \lambda^\kappa_t, \cdot \rangle + \beta^\kappa_t$, such that $V^k_t \leq V_t$.

- **Forward pass**: We simulate the dynamical system, along one scenario, according to policy $\pi V^k$, yielding a trajectory $\{x^k_t\}_{t \in [0,T]}$.

- **Backward pass**: We compute cuts $x \mapsto \langle \lambda^{k+1}_t, \cdot \rangle + \beta^{k+1}_t \leq V_t$ along this trajectory, and update our outer approximations.
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4. Numerical results
SDDP strengths

- SDDP is a widely used algorithm in the energy community, with multiple applications in:
  - mid and long term water storage management problem,
  - long-term investment problems,
  - ...

- Recent works have presented extensions of the algorithm to:
  - deal with some non-convexity,
  - treat risk-averse or distributionally robust problems,
  - incorporate integer variables.

- Multiple numerical improvements have been proposed:
  - cut selection
  - regularization
  - multi-cut or $\varepsilon$-resolution
SDDP weaknesses

There are still some gaps in our knowledge of this approach:

- there is no convergence speed guaranteed,
- regularization methods are not mature yet,
- there is no good stopping test.
SDDP Stopping test

- Exact lower bound of the problem: $V_0^k(x_0)$.
- Upper-bound estimated by Monte-Carlo simulation yielding costly statistical stopping tests (Pereira Pinto (1991) or Shapiro (2011)).
- Alternative statistical tests have been proposed (see Homem de Mello et al (2011)).
- Exact upper-bound computation has been proposed by Philpott et al (2013) but without any proof of convergence, leading to possibly not converging stopping tests.
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4 Numerical results
An operator $\mathcal{B} : F(\mathbb{R}^{nx}) \rightarrow F(\mathbb{R}^{nx})$ is said to be a linear Bellman operator (LBO) if it is defined as follows:

$$
\mathcal{B}(R) : x \mapsto \inf_{(u,y)} \mathbb{E} \left[ c^\top u + R(y) \right]
\text{subject to }Tx + \mathcal{W}_u(u) + \mathcal{W}_y(y) \leq h
$$

where $\mathcal{W}_u : \mathcal{L}^0(\mathbb{R}^{nu}) \rightarrow \mathcal{L}^0(\mathbb{R}^{nc})$ and $\mathcal{W}_y : \mathcal{L}^0(\mathbb{R}^{nx}) \rightarrow \mathcal{L}^0(\mathbb{R}^{nc})$ are two linear operators. We denote $S(R)(x)$ the set of $y$ that are part of optimal solutions to the above problem. We also define $\mathcal{G}(x)$

$$
\mathcal{G}(x) := \{(u,y) \mid Tx + \mathcal{W}_u(u) + \mathcal{W}_y(y) \leq h\}.
$$
Examples

- **Linear point-wise operator:**

\[
W : \mathcal{L}^0(\mathbb{R}^{nx}) \rightarrow \mathcal{L}^0(\mathbb{R}^{nc}) \\
(\omega \mapsto y(\omega)) \mapsto (\omega \mapsto Ay(\omega))
\]

Such an operator allows to encode **almost sure constraints**.

- **Linear expected operator:**

\[
W : \mathcal{L}^0(\mathbb{R}^{nx}) \rightarrow \mathcal{L}^0(\mathbb{R}^{nc}) \\
(\omega \mapsto y(\omega)) \mapsto (\omega \mapsto A\mathbb{E}(y))
\]

Such an operator allows to encode **constraints in expectation**.
Relatively Complete Recourse and cuts

**Definition (Relatively Complete Recourse)**

We say that the pair \((B, R)\) satisfy a *relatively complete recourse* (RCR) assumption if for all \(x \in \text{dom}(G)\) there exists admissible controls \((u, y) \in G(x)\) such that \(y \in \text{dom}(R)\).

**Cut**

If \(R\) is proper and polyhedral, with RCR assumption, then \(B(R)\) is a proper polyhedral function.

Furthermore, computing \(B(R)(x)\) consists of solving a linear problem which also generates a supporting hyperplane of \(B(R)\), that is, a pair \((\lambda, \beta) \in \mathbb{R}^{nx} \times \mathbb{R}\) such that

\[
\begin{cases}
\langle \lambda, \cdot \rangle + \beta \leq B(R)(\cdot) \\
\langle \lambda, x \rangle + \beta = B(R)(x).
\end{cases}
\]
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4 Numerical results
Consider a *compatible* sequence of LBO \( \{B_t\}_{t \in [0, T-1]} \), that is, such that all admissible controls of \( B_t \) lead to admissible states of \( B_{t+1} \).

Consider a sequence of functions such that

\[
\begin{align*}
R_T &= K \\
R_t &= B_t(R_{t+1}) \\
&\forall t \in [0, T-1]
\end{align*}
\]

Then, the abstract SDDP algorithm generates a sequence of lower polyhedral approximations of \( R_t \). In a forward pass it simulates a trajectory of states, along which the approximation is refined in the backward pass.
Consider a compatible sequence of LBO $\{B_t\}_{t \in [0, T-1]}$, that is, such that all admissible controls of $B_t$ lead to admissible states of $B_{t+1}$.

Consider a sequence of functions such that

$$\begin{align*}
R_T &= K \\
R_t &= B_t(R_{t+1}) \quad \forall t \in [0, T - 1]
\end{align*}$$

Then, the abstract SDDP algorithm generates a sequence of lower polyhedral approximations of $R_t$. In a forward pass it simulates a trajectory of states, along which the approximation is refined in the backward pass.
Abstract SDDP

$t=0 \rightarrow x \rightarrow t=1 \rightarrow x \rightarrow t=2 \rightarrow K$

Final Cost $R_2 = K$
Real Bellman function $R_1 = T_1(R_2)$
Abstract SDDP

Real Bellman function $R_0 = T_0(R_1)$
Abstract SDDP

Lower polyhedral approximation \( K \) of \( K \)
Abstract SDDP

Lower polyhedral approximation $R_1 = T_t(K)$ of $R_1$
Abstract SDDP

Lower polyhedral approximation $R_0 = T_t(R_1)$ of $R_0$
Assume that we have lower polyhedral approximations of $R_t$. 

$t=0$  
$t=1$  
$t=2$  

$x$  

$x$  

$x$  

$R_0$  

$R_1$  

$K$
Thus we have a lower bound on the value of our problem.
We apply $\pi^R_{0}^{(2)}$ to $x_0$ and obtain $X^{(2)}_1$.
Abstract SDDP

We apply $\pi_0 R_1^{(2)}$ to $x_0$ and obtain $X_1^{(2)}$
Abstract SDDP

Draw a random realisation $x_{1}^{(2)}$ of $X_{1}^{(2)}$
We apply $\pi_1^{(2)}$ to $\mathbf{x}_1^{(2)}$ and obtain $\mathbf{X}_2^{(2)}$
We apply $\pi_1^{(2)}$ to $x_1^{(2)}$ and obtain $X_2^{(2)}$
Abstract SDDP

Draw a random realisation $x_2^{(2)}$ of $X_2^{(2)}$
Abstract SDDP

Compute a cut for $K$ at $x^{(2)}_2$
Add the cut to $R_2^{(2)}$ which gives $R_2^{(3)}$
A new lower approximation of $R_1$ is $T_1(R_2^{(3)})$
Abstract SDDP

We only compute the face active at $x_1^{(2)}$
Add the cut to $R_1^{(2)}$ which gives $R_1^{(3)}$
Abstract SDDP

A new lower approximation of $R_0$ is $T_0(R_1^{(3)})$. 
Abstract SDDP

We only compute the face active at $x_0$. 

\[ R_0(x_0) \]
\[ R_1 \]
\[ K \]
We only compute the face active at $x_0$
Abstract SDDP

We obtain a new lower bound
Abstract SDDP

We obtain a new lower bound
Data: Initial point $x_0$
Set $R_t^{(0)} = -\infty$
for $k \in \mathbb{N}$ do
  // Forward Pass: compute a set of trial points $\{x_t^k\}_{t \in [0,T]}$
  Set $x_0^k = x_0$;
  for $t : 0 \rightarrow T$ do
    select $x_{t+1}^k \in S_t(R_{t+1}^k)(x_t^k)$;
    draw a realisation $x_{t+1}^k$ of $x_{t+1}^k(\omega^k)$;
  end
  // Backward Pass: refine the lower-approx at trial points
  Set $R_{T}^{k+1} = K$;
  for $t : T - 1 \rightarrow 0$ do
    $\beta_{t}^{k+1} = B_t(R_{t+1}^{k+1})(x_t^k)$;  // computing cut coefficients
    $\lambda_{t}^{k+1} \in \partial B_t(R_{t+1}^{k+1})(x_t^k)$;
    $\beta_{t}^{k+1} := \theta_{t}^{k+1} - \langle \lambda_{t}^{k+1}, x_t^k \rangle$;
    set $C_{t}^{k+1} : x \mapsto \langle \lambda_{t}^{k+1}, x \rangle + \beta_{t}^{k+1}$;  // new cut
    $R_{t}^{k+1} := \max \{ R_t^k, C_t^{k+1} \}$;  // update lower approximation
  end
Abstract SDDP convergence

Theorem

Assume that $\Omega$ is finite, $R(x_0)$ is finite, and $\{\mathcal{B}_t\}_t$ is compatible. Further assume that, for all $t \in [0, T]$ there exists compact sets $X_t$ such that, for all $k$, $x_t^k \in X_t$ (e.g. $\mathcal{B}_t$ have compact domain).

Then, $(\underline{R}_t^k)_{k \in \mathbb{N}}$ is a non-decreasing sequence of lower approximations of $R_t$, and $\lim_k \underline{R}_0^k(x_0) = R_0(x_0)$, for $t \in [0, T - 1]$.

Further, the cuts coefficients generated remain in a compact set.
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4 Numerical results
Fenchel transform of LBO

**Theorem**

Assume that the pair \((\mathcal{B}, R)\) satisfy the RCR assumption, \(R\) being proper polyhedral, and \(\mathcal{B}\) compact (i.e. \(\mathcal{G}\) is compact valued with compact domain).

Then \(\mathcal{B}(R)\) is a proper function and we have that

\[
[\mathcal{B}(R)]^* = \mathcal{B}^{\dagger}(R^*)
\]

where \(\mathcal{B}^{\dagger}\) is an explicitly given LBO.
More precisely we have

\[
B^\dagger(Q) : \lambda \mapsto \inf_{\mu \in \mathcal{L}^0(\mathbb{R}^n), \nu \in \mathcal{L}^0(\mathbb{R}^n)} \mathbb{E} \left[ -\mu^\top h + Q(\nu) \right]
\]

\[
s.t. \quad T^\top \mathbb{E}[^\mu] + \lambda = 0
\]

\[
\mathcal{W}_u^\dagger(\mu) = \mathcal{C}
\]

\[
\mathcal{W}_y^\dagger(\mu) = \nu
\]

\[
\mu \leq 0,
\]
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4 Numerical results
Recursion over dual value function

Denote $D_t := V_t^*$.

Theorem

Then

$$
\begin{cases}
D_T = K^*, \\
D_t = B_t^+ (D_{t+1}) & \forall t \in [0, T - 1]
\end{cases}
$$

This is a Bellman recursion on $D_t$ instead of $V_t$. 
Recursion over dual value function

Denote $\mathcal{D}_t := V_t^*$. 

**Theorem**

Then

$$
\begin{align*}
\mathcal{D}_T &= K^* , \\
\mathcal{D}_t &= B_{t,L_{t+1}}^\dagger (\mathcal{D}_{t+1}) & \forall t \in [0, T - 1] \\
\end{align*}
$$

where $B_{t,L_{t+1}}^\dagger := B_t^\dagger + \mathbb{I}\|\lambda_{t+1}\|_\infty \leq L_{t+1}$.

This is a **Bellman recursion** on $\mathcal{D}_t$ instead of $V_t$. 
Recursion over dual value function

Denote $D_t := V^*_t$.

**Theorem**

Then

$$
\begin{align*}
D_T &= K^*, \\
D_t &= B^\dagger_{t,L_{t+1}}(D_{t+1}) \quad \forall t \in [0, T - 1]
\end{align*}
$$

where $B^\dagger_{t,L_{t+1}} := B^\dagger_t + \Pi \|\lambda_{t+1}\|_\infty \leq L_{t+1}$.

This is a Bellman recursion on $D_t$ instead of $V_t$.

Further, under easy technical assumptions, $\{B^\dagger_{t,L_{t+1}} \mid t \in [0, T]\}$ is a compatible sequence of LBOs, where $V_t$ is $L_t$-Lipschitz.
Data: Initial primal point $x_0$, Lipschitz bounds $\{L_t\}_{t \in [0,T]}$

for $k \in \mathbb{N}$ do

  // Forward Pass : compute a set of trial points $\{\lambda_t^{(k)}\}_{t \in [0,T]}$
  Compute $\lambda_0^{(k)} \in \arg \max_{\|\lambda_0\|_\infty \leq L_0} \left\{ x_0^\top \lambda_0 - D_0^k(\lambda_0) \right\}$;

  for $t : 0 \to T$ do
    select $\lambda_{t+1}^{(k)} \in \arg \min B_t^{\perp}(D_{t+1}^k)(\lambda_t^{(k)})$;
    and draw a realization $\lambda_{t+1}^{(k)}$ of $\lambda_{t+1}^{(k)}$;
  end

  // Backward Pass : refine the lower-approx at trial points
  Set $D_T^k = K^*$.

  for $t : T - 1 \to 0$ do
    $\theta_{t+1}^{(k)} := B_{t,L_{t+1}}^{\perp}(D_{t+1}^k)(\lambda_t^{(k)})$;  // computing cut coefficients
    $\bar{x}_{t+1}^{(k)} \in \partial B_{t,L_{t+1}}^{\perp}(D_{t+1}^k)(\lambda_t^{(k)})$;
    $\beta_t^{(k+1)} := \theta_t^{(k+1)} - \langle \lambda_t^{(k)}, \bar{x}_{t+1}^{(k)} \rangle$;
    $C_t^{k+1} : \lambda \mapsto \langle \bar{x}_{t+1}^{(k)}, \lambda \rangle + \beta_t^{(k+1)}$;
    $D_t^{k+1} = \max(D_t^k, C_t^{k+1})$;  // update lower approximation
  end

  If some stopping test is satisfied STOP;

end
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4 Numerical results
Converging upper bound and stopping test

We have

\[ V^k_t \leq V_t \]

and

\[ D^k_t \leq D_t \implies (D^k_t)^* \geq (D_t^*) = V_t^{**} = V_t \]

Finally, we obtain

\[ V_0(x_0) \leq V_0(x_0) \leq \overline{V}_0(x_0). \]

Using the convergence of the abstract SDDP algorithm we show that this bounds are converging, yielding converging deterministic stopping tests.
Converging upper bound and stopping test

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$$V^k_t \leq V_t$$

and

$$D^k_t \leq D_t \implies (D^k_t)^* \geq (D_t^*) = V^{**}_t = V_t$$

Finally, we obtain

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Using the convergence of the abstract SDDP algorithm we show that this bounds are converging, yielding converging deterministic stopping tests.
Link between primal and dual approximations
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4. **Numerical results**
A converging strategy - with guaranteed payoff

Theorem

Let \( C_{t}^{IA,k}(x) \) be the expected cost of the strategy \( \pi \overline{V}_{t}^{k} \) when starting from state \( x \) at time \( t \).

We have,

\[
C_{t}^{IA,k}(x) \leq \overline{V}_{t}^{k}(x), \quad \lim_{k} C_{t}^{IA,k}(x) = V_{t}(x)
\]

Thus, the inner-approximation yields a new converging strategy, and we have an upper-bound on the (expected) value of this strategy.
Inner Approximation

- $\bar{V}_t^k := [D_t^k]^*$ which is lower than $V_t$ on $X_t$
- Or

$$\bar{V}_t^k(x) = \min_{\sigma \in \Delta} \left\{ - \sum_{\kappa=1}^{k} \sigma_\kappa \beta_{t}^\kappa \bigg| \sum_{\kappa=1}^{k} \sigma_\kappa \bar{X}_t^\kappa = x \right\}$$

- The inner approximation can be computed by solving

$$\bar{V}_t^{k+1}(x) = \sup_{\lambda, \theta} x^\top \lambda - \theta$$

$$s.t. \quad \theta \geq \langle x_t^i, \lambda \rangle + \beta_t^\kappa \quad \forall \kappa \in [1, k].$$
Inner Approximation - regularized

\[ \overline{V}^k_t := \left[ D^k_t \right]^* \mathbb{D}(L_t \| \cdot \|_1) \] which is lower than \( V_t \) on \( X_t \)

Or

\[ \overline{V}^k_t(x) = \min_{y \in \mathbb{R}^{nx}, \sigma \in \Delta} \left\{ L_t \| x - y \|_1 - \sum_{\kappa=1}^{k} \sigma_\kappa \beta_\kappa^t \bigg| \sum_{\kappa=1}^{k} \sigma_\kappa x^\kappa_t = y \right\} \]

The inner approximation can be computed by solving

\[ \overline{V}^{k+1}_t(x) = \sup_{\lambda, \theta} x^\top \lambda - \theta \]

s.t. \( \theta \geq \langle x^i_t, \lambda \rangle + \beta_\kappa^t \) \quad \forall \kappa \in [1, k] .

\[ \| \lambda \|_\infty \leq L_t \]
Numerical results

The diagram illustrates the convergence of different optimization methods over iterations. The x-axis represents the number of iterations, ranging from 0 to 1000, and the y-axis shows the values of the objective function, which range from 2980000 to 3100000.

The following lines are plotted:
- **Blue line**: Dual Upper Bound (Dual UB)
- **Red line**: Primal Lower Bound (Primal LB)
- **Black line**: Monte Carlo Optimization Algorithm (MC OA)
- **Yellow line**: Monte Carlo Iterative Algorithm (MC IA)
- **Gray line**: Confidence interval at 97.5% confidence level
## Stopping test

<table>
<thead>
<tr>
<th>$\varepsilon$ (%)</th>
<th>Dual stopping test</th>
<th>Statistical stopping test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ it.</td>
<td>CPU time</td>
<td>$n$ it.</td>
</tr>
<tr>
<td>2.0</td>
<td>156</td>
<td>183s</td>
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<tr>
<td>1.0</td>
<td>236</td>
<td>400s</td>
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<tr>
<td>0.5</td>
<td>388</td>
<td>1116s</td>
</tr>
<tr>
<td>0.1</td>
<td>$&gt;1000$</td>
<td>.</td>
</tr>
</tbody>
</table>

**Table**: Comparing dual and statistical stopping criteria for different accuracy levels $\varepsilon$. 
Conclusion

- We extend the SDDP algorithm to an abstract framework.
- Leveraging Fenchel conjugate we are able to show a dynamic recursion between dual Bellman value functions.
- We can apply SDDP to this dual recursion.
- This yields a converging exact upper bound on the value of the original problem, hence giving exact and converging stopping tests.
- This also yields a converging strategy with guaranteed payoff.

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Mario VF Pereira and Leontina MVG Pinto. 
Multi-stage stochastic optimization applied to energy planning. 

Tito Homem-de Mello, Vitor L De Matos, and Erlon C Finardi. 
Sampling strategies and stopping criteria for stochastic dual dynamic programming: a case study in long-term hydrothermal scheduling. 

Andrew Philpott, Vitor de Matos, and Erlon Finardi. 
On solving multistage stochastic programs with coherent risk measures. 

Alexander Shapiro. 
Analysis of stochastic dual dynamic programming method. 

Pierre Girardeau, Vincent Leclere, and Andrew B Philpott. 
On the convergence of decomposition methods for multistage stochastic convex programs. 
Bibliography

