Worst case bounds on the point-wise discretization error in boundary element method for the elasticity problem

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ABSTRACT

In this work, point-wise discretization error is bounded via interval approach for the elasticity problem using interval boundary element formulation. The formulation allows for computation of the worst case bounds on the boundary values for the elasticity problem. From these bounds the worst case bounds on the true solution at any point in the domain of the system can be computed. Examples are presented to demonstrate the effectiveness of the treatment of local discretization error in elasticity problem via interval methods.

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1. Introduction

Most of the problems in engineering mechanics are governed by partial differential equations, to which exact solutions, in general, cannot be obtained due to complexities in the geometry of the system for which the applied boundary conditions must be satisfied. Therefore, numerical methods have been developed to approximate the true solution by a polynomial interpolation between discrete values. The foremost method that emerged is the finite element method (FEM), in which the domain of the system is discretized into elements. These elements consist of polynomial interpolation functions between a finite set of points at which a solution is either known or is to be computed. Another numerical method used to approximate the solutions to partial differential equations is the boundary element method (BEM). In boundary element analysis (BEA), the domain variables are transformed to the boundary variables, thus decreasing the dimension of the problem by one. This allows, in general, decreasing the time necessary for mesh generation or mesh refinement. The domain transformation is performed by the use of fundamental solutions, or Green's functions, to the linear partial differential equations, thus restricting classical BEM to problems for which the fundamental solution is known. The boundary integral equations, resulting from weighted residual formulation, are solved using point collocation methods, in which the residual is set to zero in the domain and exists only on the boundary of the system. To achieve such residual, the weighted

residual function in a weak formulation of the partial differential equation takes the form of the fundamental solution. The transformed boundary integral equations are then solved by approximating the true solution over discrete boundaries, thus introducing the discretization error. Although discretization error estimates have been made for BEM [1,2] the worst case bounds on the local discretization error have been computed only for the Laplace problem [3].

In this work, point-wise discretization error is studied for Navier's equation which describes the elasticity problem. The boundary integral equations, resulting from BEM formulation, are enclosed by interval boundary integral equations, which eventually result in interval linear system of equations. A parametric solver is reviewed that enables the computation of non-naive bounds on the boundary values. Example problems are presented to illustrate the behavior of the discretization error bounds on the solution.

2. Boundary element formulation for navier's equation

The boundary element formulation for the behavior of an isotropic and homogeneous body is discussed in the literature [4–6]. The following section reviews the two dimensional boundary element formulation of Navier's equation. Navier's equation describes the behavior of an elastic solid body, undergoing small deformation, which is subjected to either displacements, forces, or both. The most common situation in practice is for the solid to be subjected to both displacements and forces and it is of interest to the engineers to obtain the solution for the displacement field in

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the entire body. From the obtained displacements other quantities of interest, such as stress, can be derived. Small deformation stress tensor can be defined as:

$$\sigma_{ij} = \frac{\partial F_i}{\partial A_j}, \tag{1}$$

where F is a force in direction i and A is the area in direction j, both in the undeformed configuration. The Navier's equation is:

$$\left.\begin{array}{l} \sigma_{ij,j} + b_i = 0 \quad \text{in } \Omega, \\ u_i = \hat{u}_i \quad \text{on } \Gamma_1, t_i = \hat{t}_i \quad \text{on } \Gamma_2, \\ \bigcup_{i=1}^{2} \Gamma_i = \Gamma \quad \text{and } \bigcap_{i=1}^{2} \Gamma_i = 0, \end{array}\right\} \tag{2}$$

where Ω is the domain of the system, Γ is the boundary of the system, σ_{ij} is the stress tensor, b_i is the vector of body force, u_i is the displacement vector with a forced boundary condition $\hat{u}_i$ on Γ_1, and t_i is the traction vector with a natural boundary condition $\hat{t}_i$ on Γ_2. The traction vector is related to the stress tensor as:

$$\sigma_{ij} n_j = t_i, \tag{3}$$

where n_j is an outward normal vector to the boundary of the system. For every part of the boundary, either a displacement field or a traction field must be known. Although, for most practical cases, the quantities of interest in solving the Navier's equation are the stress tensor and the displacement field in the domain of the system, for BEA, traction and displacement fields are obtained on the boundary of the system. In the following, Betti's reciprocal theorem is used as a starting point of the boundary element formulation for the elasticity problem. The details on obtaining Betti's reciprocal theorem can be found in [6] and are not of essence to the presented work. Betti's reciprocal theorem is stated as:

$$\int_{\Gamma} t_i u_i^* d\Gamma + \int_{\Gamma} b_i u_i^* d\Gamma = \int_{\Gamma} t_i^* u_i d\Gamma + \int_{\Gamma} b_i^* u_i d\Gamma, \tag{4}$$

where the superscript $*$ denotes a weighted residual function either for the displacement, traction, or body force. Equilibrium condition $\sigma_{ij,j}^* = -b_i^*$ is substituted into Eq. (4) resulting in:

$$-\int_{\Gamma} \sigma_{ij,j}^* u_i d\Gamma + \int_{\Gamma} t_i^* u_i d\Gamma = \int_{\Gamma} u_i^* b_i d\Gamma + \int_{\Gamma} u_i^* t_i d\Gamma, \tag{5}$$

In order to decrease the dimension of the integral equation, Eq. (5), such that all terms exist on the boundary of the system, the weighted residual function is set to be the Green's function, which is obtained by applying a point force in direction a_i. This is analogous to applying a concentrated charge or a heat source. The above consideration can be expressed as:

$$\sigma_{ij,j}^* = -\delta(x - \xi) a_i, \tag{6}$$

where ξ is a source point at which a concentrated force is applied, x is a field point at which a response to the concentrated force is observed, and $\delta(x - \xi)$ is the Dirac delta function. Applying the concentrated force results in the fundamental solution:

$$u_i^* = u_{ji}^* a_j, \tag{7}$$
$$t_i^* = t_{ji}^* a_j, \tag{8}$$

where u_{ji}^* and t_{ji}^* are i components of the displacements and tractions, respectively, due to a concentrated force in the j direction, and a_j is a unit vector in the direction of the applied concentrated force and it is the same direction as in Eq. (6). The kernel functions u_{ji}^* and t_{ji}^* are given as:

$$u_{ij}^* = \frac{1}{8\pi(1-v)G} \left[(4v - 3)\ln(r)\delta_{ij} + \frac{(\vec{x} - \vec{\xi}) \cdot \vec{i}}{r} \cdot \frac{(\vec{x} - \vec{\xi}) \cdot \vec{j}}{r} \right], \tag{9}$$

$$q_{ij}^* = \frac{-1}{4\pi(1-v)r} \left\{ \left[(1 - 2v)\delta_{ij} + 2\frac{(\vec{x} - \vec{\xi}) \cdot \vec{i}}{r} \cdot \frac{(\vec{x} - \vec{\xi}) \cdot \vec{j}}{r} \right] \cdot \frac{(\vec{x} - \vec{\xi}) \cdot \vec{n}}{r} - (1 - 2v) \left[\frac{(\vec{x} - \vec{\xi}) \cdot \vec{i}}{r} n_y - \frac{(\vec{x} - \vec{\xi}) \cdot \vec{j}}{r} n_x \right] \right\}, \tag{10}$$

where the operator $\cdot$ denotes a dot product and vectors i and j are unit vectors in directions x and y, respectively. Substituting Eqs. (6)–(8) into Eq. (5) yields:

$$u_i(\xi) a_i + \int_{\Gamma} t_{ji}^* a_j u_i d\Gamma = \int_{\Gamma} u_{ji}^* a_j b_i d\Gamma + \int_{\Gamma} u_{ji}^* a_j t_i d\Gamma, \quad \xi \in \Omega. \tag{11}$$

The indices are exchanged in all the integral terms in Eq. (11) as:

$$u_i(\xi) a_i + \int_{\Gamma} t_{ij}^* a_i u_j d\Gamma = \int_{\Gamma} u_{ij}^* a_i b_j d\Gamma + \int_{\Gamma} u_{ij}^* a_i t_j d\Gamma, \quad \xi \in \Omega. \tag{12}$$

The a_i coefficients are constant and can be canceled out from Eq. (12):

$$u_i(\xi) + \int_{\Gamma} t_{ij}^* u_j d\Gamma = \int_{\Gamma} u_{ij}^* b_j d\Gamma + \int_{\Gamma} u_{ij}^* t_j d\Gamma, \quad \xi \in \Omega. \tag{13}$$

For convenience the body force is assumed to be zero simplifying Eq. (13) to:

$$u_i(\xi) + \int_{\Gamma} t_{ij}^* u_j d\Gamma = \int_{\Gamma} u_{ij}^* t_j d\Gamma, \quad \xi \in \Omega. \tag{14}$$

Since the only entity in Eq. (14) that appears in the domain of the system is a source point ξ, Eq. (14) is integrated such that ξ is enclosed by the circular boundary of radius ε, as $\varepsilon \to 0$. This results in the right side integral vanishing. For constant elements, the left side integral of Eq. (14) results in $-1/2 u_i(\xi)$. Thus on the boundary of the system and for constant elements, Eq. (14) can be rewritten as:

$$\frac{1}{2} u_i(\xi) + \int_{\Gamma} t_{ij}^*(x, \xi) u_j(x) dx = \int_{\Gamma} u_{ij}^*(x, \xi) t_j(x) dx, \quad \xi \in \Gamma. \tag{15}$$

In most cases, the exact solution to Eq. (15) cannot be found. Therefore, Eq. (15) can be approximately solved using numerical methods such as BEM.

In general, boundary integral equations, such as Eq. (15), cannot be solved analytically. To obtain approximate solutions, the boundary integral equation is discretized into boundary elements for which the true solution is approximated by a polynomial interpolation between known values of either u or t. In this work, only boundary elements with constant shape functions are used to generate significant discretization errors. Higher order polynomial approximation is assumed to approximate the true solutions better thus decreasing the discretization error. Constant elements contain one node per element, leading to the following discretization:

$$u(x) = \Phi u_i, \tag{16}$$
$$t(x) = \Phi t_i, \tag{17}$$

where u_i and t_i are the vectors of nodal values of u or t, respectively, at node i, and Φ is the vector of constant shape functions. The discretized Eq. (15) can be written as:

$$\frac{1}{2} u_i + \sum_{\text{Elements}} \int_{\Gamma} t_{ij}^*(x, \xi) \Phi dx u_j = \sum_{\text{Elements}} \int_{\Gamma} u_{ij}^*(x, \xi) \Phi dx t_j. \tag{18}$$

Eq. (18) can be written in a matrix form:

$$Hu = Gt, \tag{19}$$

where matrix H is singular and therefore satisfies the rigid body motion. To obtain a unique solution to Eq. (19) at least one boundary condition in each direction of the problem must be specified for

the displacement. Eq. (19) is then rearranged according to the appropriate boundary conditions and solved as a linear algebra problem:

$$Ax = f. \tag{20}$$

The terms of H and G matrices can either be determined explicitly or are computed numerically using numerical integration schemes. The effects of the integration error and truncation error have been studied [7] and can be implemented to enclose the true solution of Eq. (20). In this work the impact of the discretization error on the solution to Eq. (20) is studied, following the boundary element formulation, using interval methods.

3. Interval analysis

In this paper the discretization error is treated using an interval approach. The following section reviews some of the interval operations. An interval number $\mathbf{x} = [a, b]$ [8,9] is a set of real numbers such that:

$$[a, b] = \{x | a \leqslant x \leqslant b\}, \tag{21}$$

where $(a, b) \in \Re$. Interval variables $\mathbf{x} = [a, b]$ and $\mathbf{y} = [c, d]$ behave according to the following operations:

Addition:

$$\mathbf{x} + \mathbf{y} = [a + c, b + d]. \tag{22}$$

Subtraction:

$$\mathbf{x} - \mathbf{y} = [a - d, b - c]. \tag{23}$$

Multiplication:

$$\mathbf{x} \cdot \mathbf{y} = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}] \tag{24}$$

Division:

$$\frac{\mathbf{x}}{\mathbf{y}} = [a, b] \cdot \left[\frac{1}{d}, \frac{1}{c}\right], \quad 0 \notin \mathbf{y}. \tag{25}$$

Integration of interval-valued function $f(x, \xi)$, which is a class of all possible functions bounded by a given interval is performed as:

$$\int_{\Gamma} f(x, \xi) dx = \left[\int_{\Gamma} \underline{f}(x, \xi) dx, \int_{\Gamma} \bar{f}(x, \xi) dx\right], \quad \xi \in [\underline{\xi}, \bar{\xi}]. \tag{26}$$

Subdistributive property:

$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) \subseteq \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}. \tag{27}$$

One of the major sources of overestimation or underestimation in interval solutions is the subdistributive property of interval numbers. Great emphasis should be made to the correct order of operations in interval analysis. If the correct representation is given by the left term in Eq. (27), expressing the operation by the right term may cause overestimation. If the correct representation is expressed as the right term in Eq. (27), expressing it as the left term may result in inner bounds and the enclosure of the solution may not be guaranteed. This issue will be farther referred to in considering interval kernel functions.

Another source of overestimation occurs due to the dependency of interval numbers, either linear or nonlinear. Linear dependency of interval numbers for $\mathbf{x} = [-1, 1]$ and $\mathbf{y} = [-1, 1]$ can be illustrated as:

$$\mathbf{x} \cdot \mathbf{y} = [-1, 1], \tag{28}$$
$$\mathbf{x} \cdot \mathbf{x} = [0, 1]. \tag{29}$$

Eq. (28) considers all combinations of the product of $\mathbf{x}$ and $\mathbf{y}$ while Eq. (29) considers that every number within the set $\mathbf{x}$ is multiplied

by itself. For engineering problems, this dependency occurs mostly due to the physics of the problem and needs to be considered for sharp solutions. Naive interval application may results in wide and unrealistic bounds. For example if:

$$\mathbf{y} = 6 \cdot \mathbf{x} \cdot \mathbf{x} + 3 \cdot \mathbf{x}, \mathbf{x} = [-1, 1]$$

then the naive bounds for the solution are $\mathbf{y} = [-9, 9]$. However, considering nonlinear interval dependency, the bounds on the solution result in exact bounds $\mathbf{y} = [-0.375, 9]$. Another source of overestimation is the order of operations in interval linear algebra [10–12]. To obtain sharp results, interval operations should be performed last to reduce the overestimation due to the dependency of interval matrix coefficients. The following example demonstrates this consideration.

$$\mathbf{y}_1 = A \cdot (B \cdot \mathbf{x}), \mathbf{y}_2 = (A \cdot B) \cdot \mathbf{x},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}.$$

Performing matrix multiplications results in:

$$\mathbf{y}_1 = \begin{bmatrix} a_{11}(b_{11}\mathbf{x}_1 + b_{12}\mathbf{x}_2) + a_{12}(b_{21}\mathbf{x}_1 + b_{22}\mathbf{x}_2) \\ a_{21}(b_{11}\mathbf{x}_1 + b_{12}\mathbf{x}_2) + a_{22}(b_{21}\mathbf{x}_1 + b_{22}\mathbf{x}_2) \end{bmatrix},$$

$$\mathbf{y}_2 = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21})\mathbf{x}_1 + (a_{11}b_{12} + a_{12}b_{22})\mathbf{x}_2 \\ (a_{21}b_{11} + a_{22}b_{21})\mathbf{x}_1 + (a_{21}b_{12} + a_{22}b_{22})\mathbf{x}_2 \end{bmatrix}.$$

It can be clearly seen that $\mathbf{y}_2$ is sharper then $\mathbf{y}_1$ due to the considered dependency of $\mathbf{x}_1$ and $\mathbf{x}_2$ throughout the rows of $\mathbf{y}_2$. Therefore special care should be given to the order of interval operations to obtain sharp bounds on the solution.

4. Iterative scheme for interval linear system of equations

The following section reviews an iterative method which is used to solve the interval linear system of equations:

$$\mathbf{Ax} = \mathbf{b}. \tag{30}$$

The interval linear system of equations Eq. (30) is solved using Krawczyk iteration [13] based on Brouwer's fixed point theorem [10–12]. One approach of self-validating (SV) methods to find the zero of the function $f(x) = 0, \Re^n \rightarrow \Re^n$ is to consider a fixed point function $g(x) = x$. The transformation between $f(x)$ and $g(x)$ for a nonsingular preconditioning matrix C is:

$$f(x) = 0 \Longleftrightarrow g(x) = x, \tag{31}$$
$$g(x) = x - C \cdot f(x), \tag{32}$$

where the function $g(x)$ is considered as a Newton operator. From Brouwer's fixed point theorem and from:

$$g(\mathbf{x}) \subseteq \mathbf{x} \text{ for some } \mathbf{x} \in \Re^n \tag{33}$$

the following is true:

$$\exists x \in \mathbf{x} : f(x) = 0. \tag{34}$$

This method is used to solve linear system of equations, Eq. (30). The preconditioning matrix C is chosen as $C = A^{-1}$. From Eq. (32) and Eq. (33) it follows that:

$$Cb + (I - CA)\mathbf{x} \subseteq \mathbf{x}. \tag{35}$$

The left hand side of Eq. (35) is the Krawczyk operator. For the iteration to provide finite solution, the preconditioning matrix needs to be proven regular [9,14]. The following proves this condition.
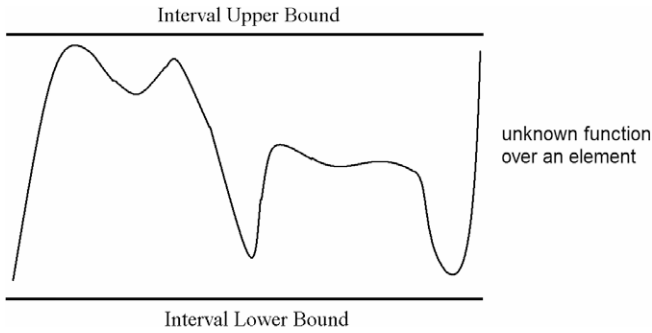
Fig. 1. Constant interval bounds on a function.

Theorem 1 [14]. *Let $A, R \in \Re^{n \times n}, b \in \Re^n,$ and $\mathbf{x} \in \Re^n$ be given. If*

$$Cb + (I - CA)\mathbf{x} \subseteq int(\mathbf{x}) \tag{36}$$

then C and A are regular and the unique solution of $Ax = b$ satisfies $A^{-1}b \in \mathbf{x}$. $int(\mathbf{x})$ refers to the interior of $\mathbf{x}$. However, all terms in Eq. (30) are interval terms, thus the following is a proof for the guarantee of the solution for this equation.

Theorem 2 [14]. *Let $\mathbf{A} \in \Re^{n \times n}, R \in \Re^{n \times n}, \mathbf{b} \in \Re^n,$ and $\mathbf{x} \in \Re^n$ be given. If*

$$C\mathbf{b} + (I - C\mathbf{A})\mathbf{x} \subseteq int(\mathbf{x}) \tag{37}$$

then C and every matrix $A \in \mathbf{A}$ is regular and

$$\sum(\mathbf{A}, \mathbf{b}) = \{x \in \Re^n | \exists A \in \mathbf{A} \exists b \in \mathbf{b} : Ax = b\} \subseteq \mathbf{x}. \tag{38}$$

Eq. (38) guarantees the solution to the interval linear system of equations, Eq. (30). The residual form of Eq. (38) is [9]:

$$C\mathbf{b} - C\mathbf{A}x_0 + (I - C\mathbf{A})\delta \subseteq int(\delta), \tag{39}$$

where $\mathbf{x} = x_0 + \delta$. A good initial guess is $x_0 = C mid(\mathbf{b})$, where $C = mid(\mathbf{A})^{-1}$.

5. Interval boundary element method

Interval Boundary Element Method (IBEM) has been developed to address the impact of the uncertainty in boundary conditions, integration error, and truncation error on the solutions [7]. The IBEM formulation results in the interval linear system of equations:

$$\mathbf{Hu} = \mathbf{Gq}, \tag{40}$$

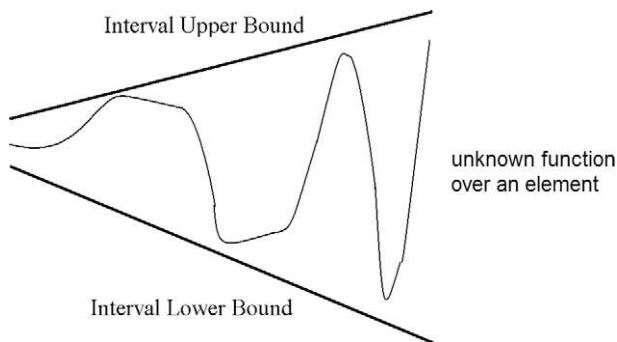which is rearranged according to the boundary conditions yielding:

$$\mathbf{Ax} = \mathbf{b}. \tag{41}$$

Considering a local discretization error in IBEM for Laplace equation [3] results in the interval linear system of equations:

$$\mathbf{H}_1\mathbf{u} + \mathbf{H}_2\mathbf{u} = \mathbf{G}_1\mathbf{q} + \mathbf{G}_2\mathbf{q}, \tag{42}$$

which is rearranged according to the boundary conditions yielding:

$$\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{x} = \mathbf{b}. \tag{43}$$

The treatment of the discetization error in IBEM for elasticity problem is described in the following sections.

6. Point-wise discretization error bounds for boundary element method

The discretization error in the solutions to integral equations results from considering a finite number of collocation points, source points, for which these solutions are computed. In general, the true solutions to integral equations are functions, not discrete values, and therefore the space of the approximate solutions does not cover the space of the true solutions. Boundary integral equations can be obtained by the use of collocation methods resulting in equation of the form of Eq. (15). The boundary integral equations are satisfied exactly only if all the locations of the source point ξ on the boundary are considered. However, to obtain a linear system of equations, a finite number of source points is considered. Moreover, the location of each source points is unique and the solution is considered as a polynomial interpolation between discrete values whose location corresponds to the location of the source point. This allows for the solution of the linear system of equations to be unique. It should be noted that if all non countable source points are considered, the boundary values at all points can be computed, resulting in the true solution. The boundary integral equation can also be evaluated over n sub-domains as expressed by Eq. (18). The unique location of the source point and its correspondence to the point at which the approximate solution is computed must be satisfied for all sub-domains. Eq. (18) is satisfied exactly only if all the locations of the source point are considered. Thus, the discretization error in Eq. (18) is introduced in the same manner as in Eq. (15). In this work the locations of the source point are treated via interval approach. Considering interval bounds ξ on all the possible locations of the source points ξ, allows obtaining an interval solution which encloses the true solution. From the interval bounds on the boundary values for displacements and tractions, the bounds on the true solution for displacements and stresses for any point in the domain can be computed. Eq. (15) is enclosed by an interval boundary integral equation in which the terms $u_{ij}^*(x, \xi)$ and $t_{ij}^*(x, \xi)$ are enclosed by known interval-valued
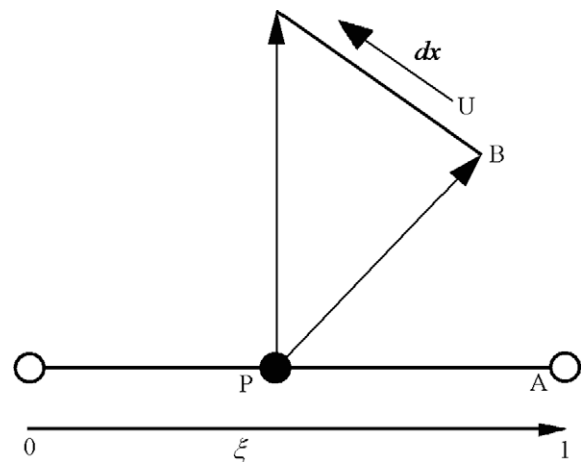

Fig. 2. Linear interval bounds on a function.

Fig. 3. Integration over element B from point P on element A.
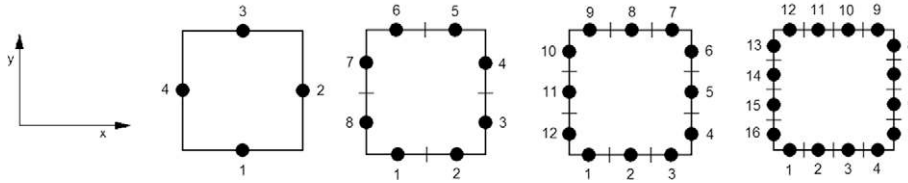
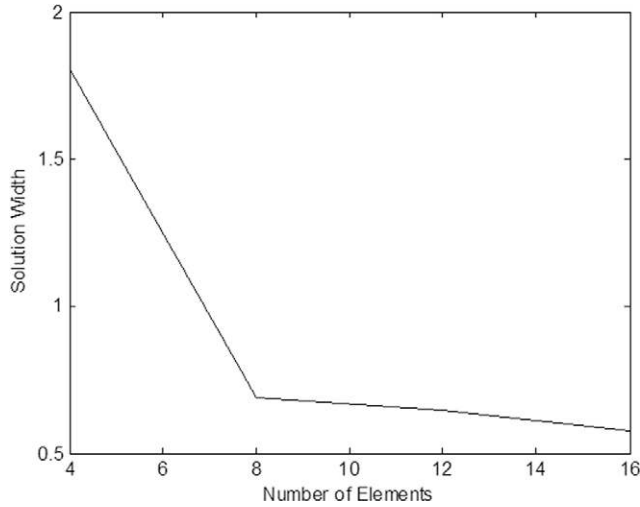**Fig. 4.** Uniform boundary discretization using constant boundary elements.

Fig. 5. Behavior of the width of the interval solution with problem size.
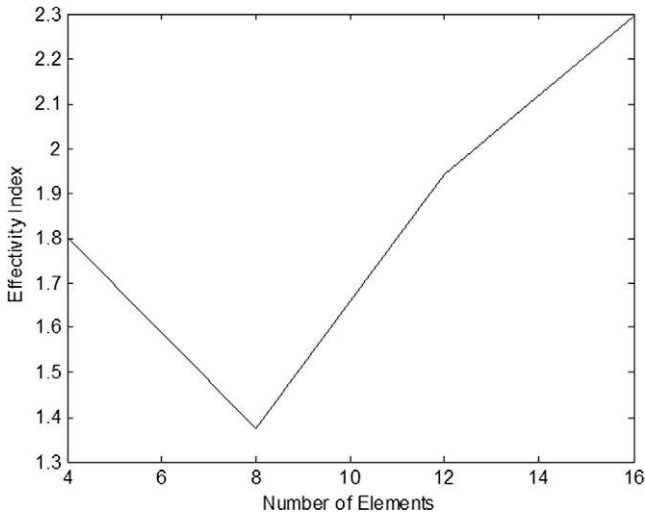

Fig. 6. Behavior of the effectivity index with problem size.

functions. The unknown functions $u_j(x)$ and $t_j(x)$ in Eq. (15) are then bounded by interval values enclosing the true solution.

The integral over the domain can be expressed as the sum of the integrals over the elements, sub-domains, and thus the boundary integral equation must be enclosed on each element for all the locations of the source points. Hence, first boundary Γ is subdivided into n boundary elements and then for each element k the interval values **u** and **t**, that bound the functions $u(x)$ and $t(x)$, are found (Fig. 1). For higher order elements the interval-valued function, of the order of the polynomial approximation, encloses the true solution. The bounding of the function using linear elements is shown (Fig. 2). It is assumed that on all other elements,

except for the single element in consideration, the bounds on all boundary values are known. Also either the bounds on the forced or the natural boundary condition bounds are known for the element in consideration. Then, the remaining boundary value for the single element in consideration is enclosed. The process is repeated for the second element with the assumed bounds for all the other elements, a computed bound for the previously considered element, and either the forced or the natural boundary condition bounds for the second element in consideration. This procedure, known as the interval Gauss–Seidel iteration [9], is performed for all elements until the true solution is enclosed. Mathematically the above statement can be expressed as:

$$\forall k \in \{1, 2, \ldots, n\} \; Assume \; \underline{u_m} \leqslant u_m \leqslant \overline{u_m}, \; \underline{t_m} \leqslant t_m \leqslant \overline{t_m} \; is \; known \; \forall m \neq k.$$
$$Also \; known \; \underline{t_k} \leqslant t_k \leqslant \overline{t_k}. \; Find \; \underline{u_k} \leqslant u_k \leqslant \overline{u_k}$$
$$\forall \xi_k \left| \begin{array}{l} \frac{1}{2} u_{ik}(\xi_k) + \int_{\Gamma_k} t_{ij}^*(x, \xi_k) u_{jk}(x) dx = \\ \sum_{m=1}^{n} \int_{\Gamma_m} u_{ij}^*(x, \xi_k) t_{jm}(x) dx + \int_{\Gamma_k} u_{ij}^*(x, \xi_k) t_{jk}(x) dx - \sum_{m=1}^{n} \int_{\Gamma_m} t_{ij}^*(x, \xi_k) u_{jm}(x) dx \end{array} \right.$$
$$or$$
$$\forall k \in \{1, 2, \ldots, n\} \; Assume \; \underline{u_m} \leqslant u_m \leqslant \overline{u_m}, \underline{t_m} \leqslant t_m \leqslant \overline{t_m} \; is \; known \quad \forall m \neq k.$$
$$Also \; known \; \underline{u_k} \leqslant u_k \leqslant \overline{u_k}. \; Find \; \underline{t_k} \leqslant t_k \leqslant \overline{t_k}$$
$$\forall \xi_k \left| \begin{array}{l} \int_{\Gamma_k} u_{ij}^*(x, \xi_k) t_{jk}(x) dx = \frac{1}{2} u_{ik}(\xi_k) + \int_{\Gamma_k} t_{ij}^*(x, \xi_k) u_{jk}(x) dx + \\ \sum_{m=1}^{n} \int_{\Gamma_m} t_{ij}^*(x, \xi_k) u_{jm}(x) dx - \sum_{m=1}^{n} \int_{\Gamma_m} u_{ij}^*(x, \xi_k) t_{jm}(x) dx. \end{array} \right.$$

$$(44)$$

Each term of the summation in Eq. (44) is represented graphically (Fig. 3). If u or q are specified boundary conditions, the interval integration can be performed explicitly as described in Section 3, Eq. (26). In this work, for computational efficiency purposes, the underlying system of interval equations is solved using Krawczyk iteration, rather than using the interval Gauss–Seidel iteration. This substitution of the method for bounding the unknown boundary
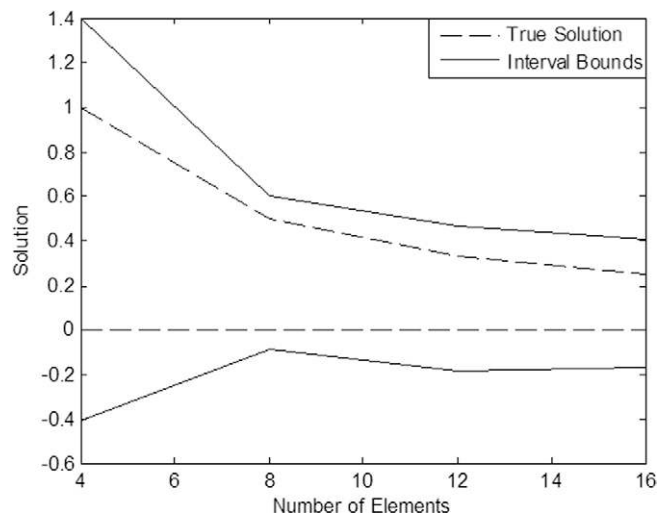

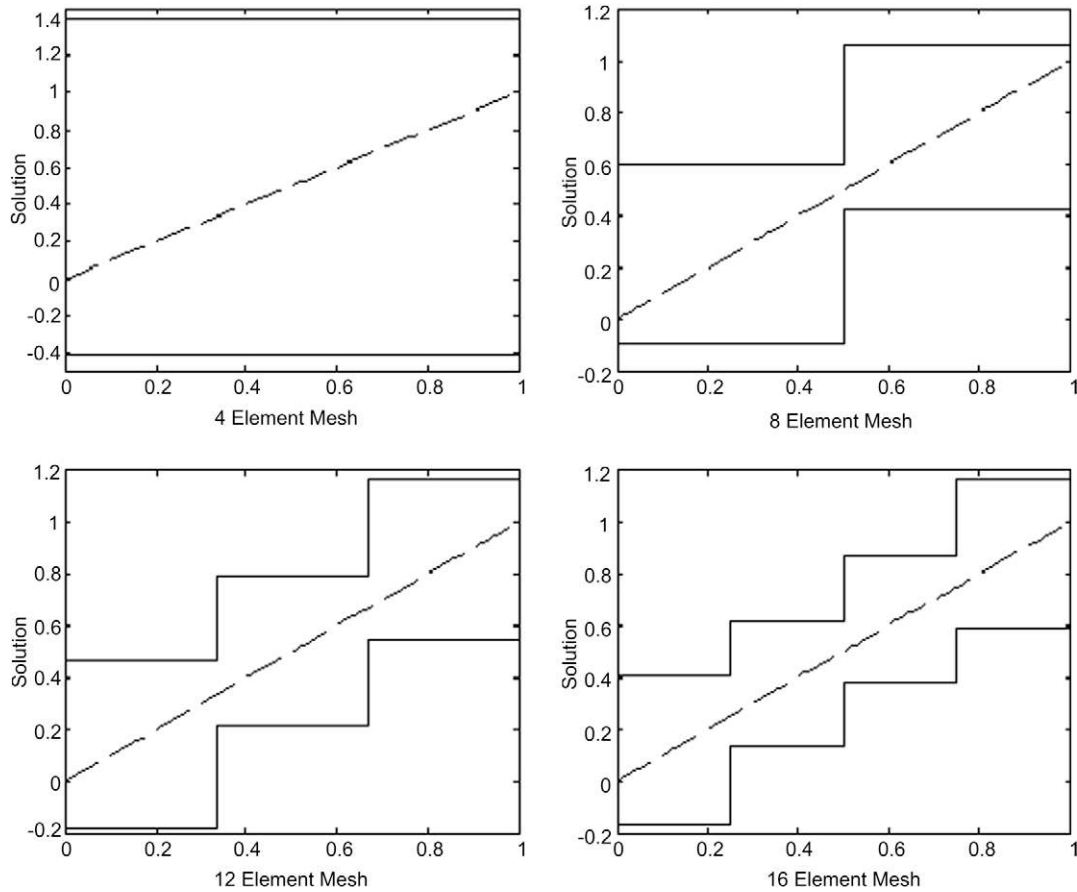**Fig. 7.** Behavior of the interval bounds with problem size.

Fig. 8. Behavior of the interval bounds for the different meshes.

values is justifiable since both of these methods are iterative methods for solving interval linear systems of equations and both obtain guaranteed bounds for the solution. Hence, in the following sections the interval boundary element method (IBEM) formulation is performed such that the resulting interval linear system of equations is of the form of Eq. (30).

Fig. 9. Behavior of the width of the interval solution with parameterization for a 4 element mesh.

7. Interval kernel splitting technique

The analysis of the discretization error requires that the boundary integral equations for each element be bounded for all the locations of the source point ξ. The integral equation in the boundary element formulation has the form of the Fredholm equation of the first kind. Kernel splitting techniques have been used to bound the interval Fredholm equation of the first kind in which the right side is deterministic [15] as:

$$\int_\Gamma \mathbf{a}(x, \xi)u(x)dx = b(\xi). \tag{45}$$

In Eq. (45) $a(x, \xi)$ is the kernel function, $u(x)$ is the unknown variable, which is analogous to displacement and traction vectors, and $b(\xi)$ is the right hand side. However, the interval boundary integral equations considered herein have an interval right hand side, due to the interval valued location of the source point ξ, therefore an interval kernel splitting technique (IKST) is developed [16]. The integral of the product of two functions is bounded considering interval bounds on the unknown value as:

$$\int_\Gamma \mathbf{a}(x, \xi)\mathbf{u}dx \supseteq \int_\Gamma \mathbf{a}(x, \xi)u(x)dx = \mathbf{b}(\xi). \tag{46}$$

To separate the kernels such that the unknown $\mathbf{u}$ can be taken out of the integral on Γ, the left side integral from Eq. (46) is expressed as a sum of the integrals:

$$\int_\Gamma \mathbf{a}(x, \xi)\mathbf{u}dx = \int_{\Gamma_1} \mathbf{a}(x, \xi)\mathbf{u}dx + \int_{\Gamma_2} \mathbf{a}(x, \xi)\mathbf{u}dx \tag{47}$$
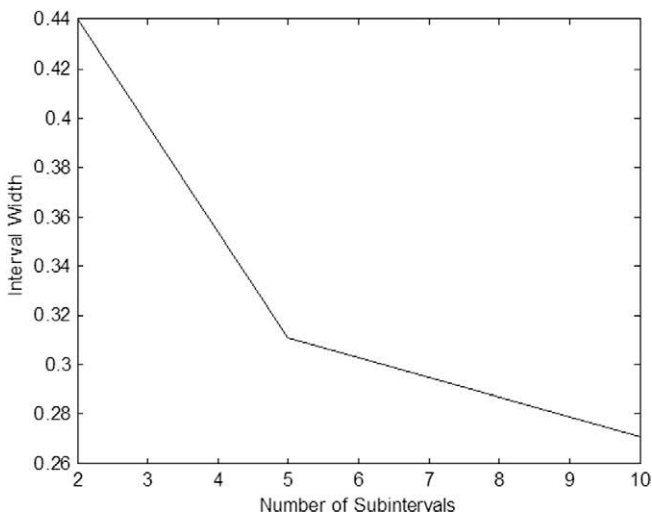
where $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = 0$ and:

$$a(x,\xi) > 0 \text{ or } a(x,\xi) < 0 \quad \text{on } \Gamma_1, \qquad (48)$$

$$a(x,\xi) \in 0 \quad \text{on } \Gamma_2. \qquad (49)$$

The interval kernel is of the same sign on Γ_1, thus **u** can be directly taken out of the integral on Γ_1 as:

$$\int_{\Gamma_1} \mathbf{a}(x,\xi)\mathbf{u}dx = \int_{\Gamma_1} \mathbf{a}(x,\xi)dx\mathbf{u}. \qquad (50)$$

Due to the subdistributive property of interval numbers, Eq. (27), **u** cannot be taken out of the integral on Γ_2. The direct application of the subdistributive property may result in inner bounds on the interval integral as:

$$\int_{\Gamma_2} \mathbf{a}(x,\xi)dx\mathbf{u} \subseteq \int_{\Gamma_2} \mathbf{a}(x,\xi)\mathbf{u}dx. \qquad (51)$$

Hence the interval kernel is bounded by its limits on Γ_2:

$$\int_{\Gamma_2} \mathbf{a}\mathbf{u}dx \supseteq \int_{\Gamma_2} \mathbf{a}(x,\xi)\mathbf{u}dx, \qquad (52)$$

where **a** is defined as:

$$\mathbf{a} = [\min\{\mathbf{a}(x+\varepsilon,\xi)\}, \ \max\{\mathbf{a}(x+\varepsilon,\xi)\}], \qquad (53)$$

$$\boldsymbol{\varepsilon} = [-\varepsilon,\varepsilon], \qquad (54)$$

ε is the tolerance level of the nonlinear solver used to find the zero location of $\mathbf{a}(x,\xi)$. To show that by bounding the kernel on Γ_2 allows **u** to be taken out from the integral on Γ_2, the integral on Γ_2 is expressed as an infinite sum:

$$\int_{\Gamma_2} \mathbf{a}\mathbf{u}dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} (\Delta x \mathbf{a}\mathbf{u}) \Big|_{\Gamma_2} = \lim_{\Delta x \to 0} (n\Delta x \mathbf{a}\mathbf{u}) \Big|_{\Gamma_2}$$

$$= \lim_{\Delta x \to 0} (n\Delta x \mathbf{a})\mathbf{u} \Big|_{\Gamma_2} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} (\Delta x \mathbf{a}) \Big|_{\Gamma_2} \mathbf{u} = \int_{\Gamma_2} \mathbf{a}dx\mathbf{u}. \qquad (55)$$

Thus **u** can be taken out of both integrals on Γ_1 and on Γ_2 and the split interval boundary integral equation becomes:

$$\int_{\Gamma_1} \mathbf{a}(x,\xi)dx\mathbf{u} + \int_{\Gamma_2} \mathbf{a}dx\mathbf{u} \supseteq \int_{\Gamma} \mathbf{a}(x,\xi)\mathbf{u}dx \supseteq \int_{\Gamma} \mathbf{a}(x,\xi)u(x)dx = \mathbf{b}(\xi). \qquad (56)$$

The kernels are bounded for all the elements resulting in interval linear system of equations:

$$\mathbf{A}_1\mathbf{u} + \mathbf{A}_2\mathbf{u} \supseteq \mathbf{b}. \qquad (57)$$

Therefore, the IKST bounds the continuous boundary integral equation for all the locations of the source point ξ and Eq. (57) is guaranteed to be satisfied for all the weighted residual functions.

8. Iterative solver for the interval linear system of equations

The bounding of the original boundary integral equation using IKST results in the interval linear system of equations different from that of Eq. (30). Hence, the algorithm to solve the interval linear system of equations, Eq. (57), must be developed. This section describes the transformation of Eq. (57) to obtain it in the form of Eq. (30). Then, Krawczyk iteration is performed to obtain the guaranteed bounds on the solution. Considering the linear system of equations:

$$\mathbf{A}_{1e}\mathbf{x}_e + \mathbf{A}_{2e}\mathbf{x}_e = \mathbf{b}, \qquad (58)$$

where $\mathbf{A}_{1e} \in \mathbf{A}_1, \mathbf{A}_{2e} \in \mathbf{A}_2, \mathbf{b}_e \in \mathbf{b}, \mathbf{x}_e \in \mathbf{x}$ and A_{1e} is regular $\forall A_{1e}|A_{1e} \in \mathbf{A}_{1e}$. Eq. (58) is pre-multiplied by $\mathbf{A}_{1e}^{-1}$ as:

$$\mathbf{A}_{1e}^{-1}\mathbf{A}_{1e}\mathbf{x}_e + \mathbf{A}_{1e}^{-1}\mathbf{A}_{2e}\mathbf{x}_e = \mathbf{A}_{1e}^{-1}\mathbf{b}_e. \qquad (59)$$

By substituting $\mathbf{A}_{1e}^{-1}\mathbf{A}_{1e} = I, \mathbf{A}_{1e}^{-1}\mathbf{A}_{2e} = \mathbf{A}_{3e}$ and $\mathbf{A}_{1e}^{-1}\mathbf{b}_e = \mathbf{b}_{1e}$, Eq. (59) can be rewritten as:

$$\mathbf{x}_e + \mathbf{A}_{3e}\mathbf{x}_e = \mathbf{b}_{1e}. \qquad (60)$$

Since the first term in Eq. (60) is a deterministic identity matrix premultiplying $\mathbf{x}_e$, the following substitution can be made directly. Letting $I + \mathbf{A}_{3e} = \mathbf{A}_e$ results in:

$$\mathbf{A}_e\mathbf{x}_e = \mathbf{b}_{1e}. \qquad (61)$$

The transformed system of equations is subjected to Krawczyk iteration as described in the previous section.

9. Interval boundary element method considering discretization error

In the preceding formulation, the bounds on the unknown boundary values are found using iterative techniques. The kernel functions, Eqs. (9,10) are enclosed using IKST resulting in the interval system of equations, Eq. (61), which is solved using Krawczyk iteration. The obtained bounds, however, are greatly overestimated since the dependency of interval values was not considered. One reason for this overestimation is that the interval kernels are bounded such that the source point ξ is allowed to vary along the entire element. Thus, for two adjacent elements, two source points are allowed to be at the connecting point between the elements and have the same location, resulting in the reduction of the rank of the system of equations. The unique location of a single source point is also not considered throughout the rows of H and G matrices, which are in $R^{n \times n}$, since each term in both H and G matrices is computed independently due to the complexity of the kernels. Thus, the parameterization of the interval location of the source point, ξ, in the **H** and **G** matrices must be considered in the solver to obtain n independent interval equations and to reduce the overestimation which results from a non-unique location of the source point on any individual element. For convenience, the system is parameterized such that $\xi = \frac{1}{a}[0,1] + b$, where a is the length of the element and b is the starting coordinate of that element. In performing interval matrix products, the value of ξ is decomposed into sub-intervals such that:

$$\bigcup_{i=1}^{n} \xi_i = \xi \quad \text{and} \quad \bigcap_{i=1}^{n} \xi_i = 0. \qquad (62)$$

The parameterized boundary integral equation is bounded by IKST for each subinterval ξ_i, resulting in the linear system of equations:

$$\mathbf{H}_1(\xi_i)\mathbf{u} + \mathbf{H}_2(\xi_i)\mathbf{u} = \mathbf{G}_1(\xi_i)\mathbf{t} + \mathbf{G}_2(\xi_i)\mathbf{t}, \qquad (63)$$

where the kernel is of the same sign for $\mathbf{H}_1(\xi_i)$ and $\mathbf{G}_1(\xi_i)$ and includes zero for $\mathbf{H}_2(\xi_i)$ and $\mathbf{G}_2(\xi_i)$. The system of equations is rearranged according to the boundary conditions as:

$$\mathbf{A}_1(\xi_i)\mathbf{x} + \mathbf{A}_2(\xi_i)\mathbf{x} = \mathbf{b}(\xi_i). \qquad (64)$$

Steps described in the previous section lead to the equation of the form:

$$\mathbf{A}(\xi_i)\mathbf{x} = \mathbf{b}_1(\xi_i). \qquad (65)$$

The initial interval guess is then considered as:

$$\mathbf{x}_0 = A^{-1} \cup_{i=1}^{n} \mathbf{b}_1(\xi_i), \qquad (66)$$

where A is computed for $\xi = 1/2$. The difference between I and the preconditioning matrix A^{-1} post-multiplied by the interval matrix $\mathbf{A}(\xi_i)$ is computed as:

$$\mathbf{I}_d = I - \cup_{i=1}^{n} A^{-1}\mathbf{A}(\xi_i). \qquad (67)$$
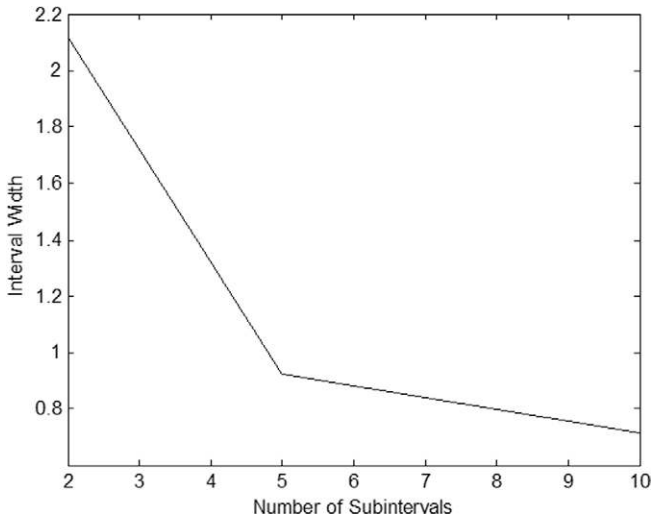
Fig. 10. Behavior of the width of the interval solution with parameterization for an 8 element mesh.
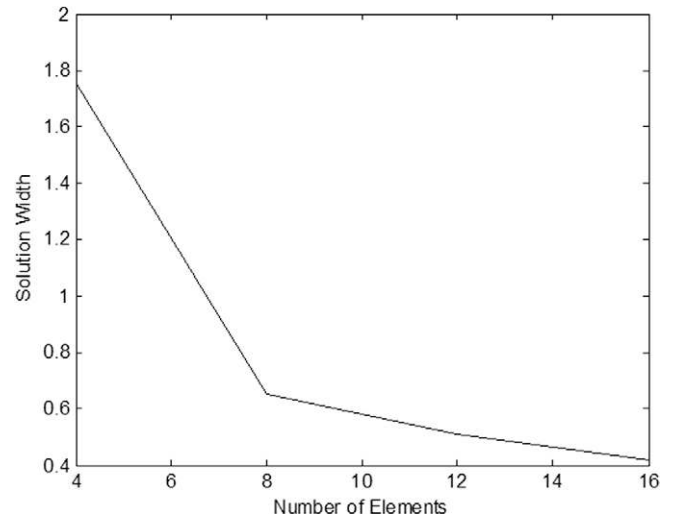

Fig. 14. Behavior of the width of the interval solution with problem size.
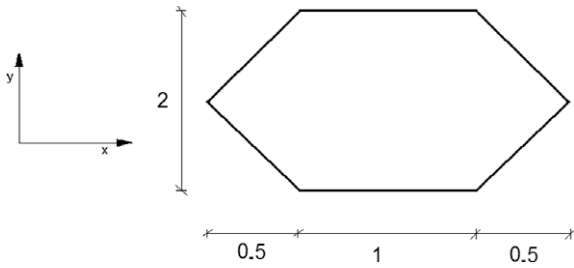

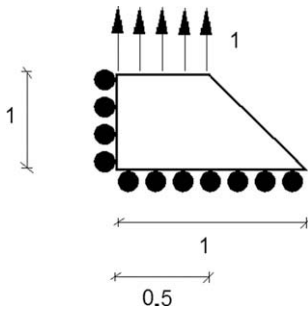**Fig. 11.** Hexagonal plate subjected to a unit displacement.
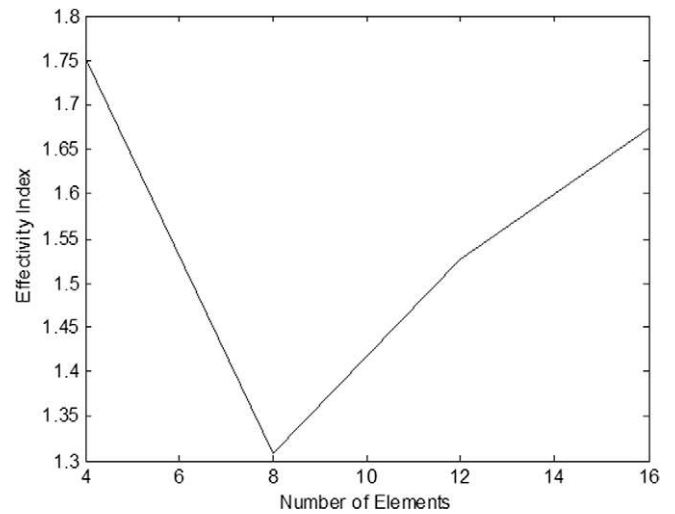

Fig. 15. Behavior of the effectivity index with problem size.

Fig. 12. Symmetry model of the hexagonal plate.

The iteration is performed as:

$$\mathbf{del} = \boldsymbol{\delta}_1, \tag{70}$$

$$\boldsymbol{\delta}_1 = \boldsymbol{\delta} + \mathbf{I}_d\mathbf{del}, \tag{71}$$

$$if\ \mathbf{del} \supset \boldsymbol{\delta}_1, \tag{72}$$

$$\mathbf{X} = \mathbf{X}_0 + \boldsymbol{\delta}_1. \tag{73}$$

For any point n on element k the bounds on the discretization error are found as:

$$\mathbf{E}_{nk}^{\text{Discretization}} = \mathbf{X}_k - x_n \tag{74}$$

where $\mathbf{x}_k$ are the solution bounds over an element k and x_n is the solution from a conventional boundary element analysis for point n.
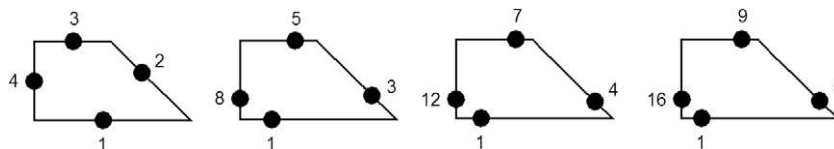
The difference between the solution and the initial guess is computed for each ξ_i pre-multiplied by the preconditioning matrix I, which numerically gave the sharpest results:

$$\boldsymbol{\delta} = \cup_{i=1}^n (\mathbf{b}(\xi_i) - \mathbf{A}_1(\xi_i)\mathbf{x}_0 - \mathbf{A}_2(\xi_i)\mathbf{x}_0). \tag{68}$$

Also:

$$\boldsymbol{\delta}_1 = \boldsymbol{\delta}. \tag{69}$$

Fig. 13. Uniform boundary discretization using constant boundary elements.
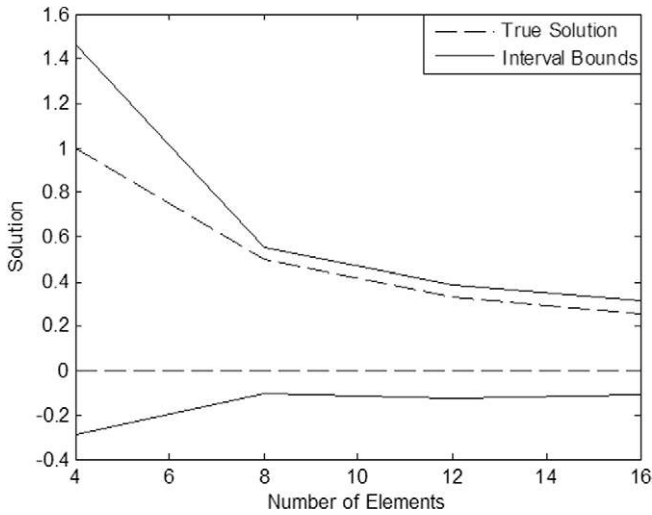
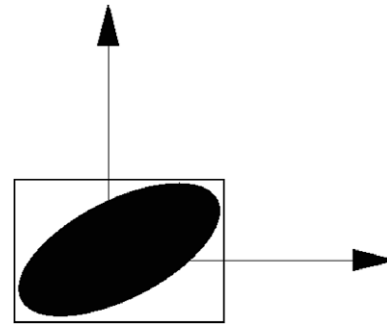Fig. 16. Behavior of the interval bounds with problem size.

Fig. 18. Interval bounds on the solution.

10. Examples

The first example demonstrates the IBEM considering discretization error for the elasticity problem. A unit square domain of the problem as well as the boundary element mesh is shown (Fig. 4). The body has a unit elastic modulus and a zero Poisson ratio. The left and right sides have a zero traction boundary condition; the bottom boundary has a zero displacement boundary condition, while the top boundary has a zero traction condition in the x direction and a unit displacement in the y direction. The behavior of the y direction displacement bounds such as solution width, effectivity index, and solution bounds is depicted (Figs. 5–7) for nodes 2, 3, 4, and 5 on the four respective meshes. The interval bounds, depicted by a solid line enclosing the dashed true solution, for the right edge displacement in the y direction are shown (Fig. 8). The effect of the parameterization for the traction in the x direction on element 1 for the 4 and 8 element meshes is also shown (Figs. 9 and 10).

The second example obtains bounds on the solution, considering the discretization error, of a hexagonal plate subjected to a unit displacement in the y direction at the top and a unit displacement in the −y direction on the bottom (Fig. 11). The body has a unit
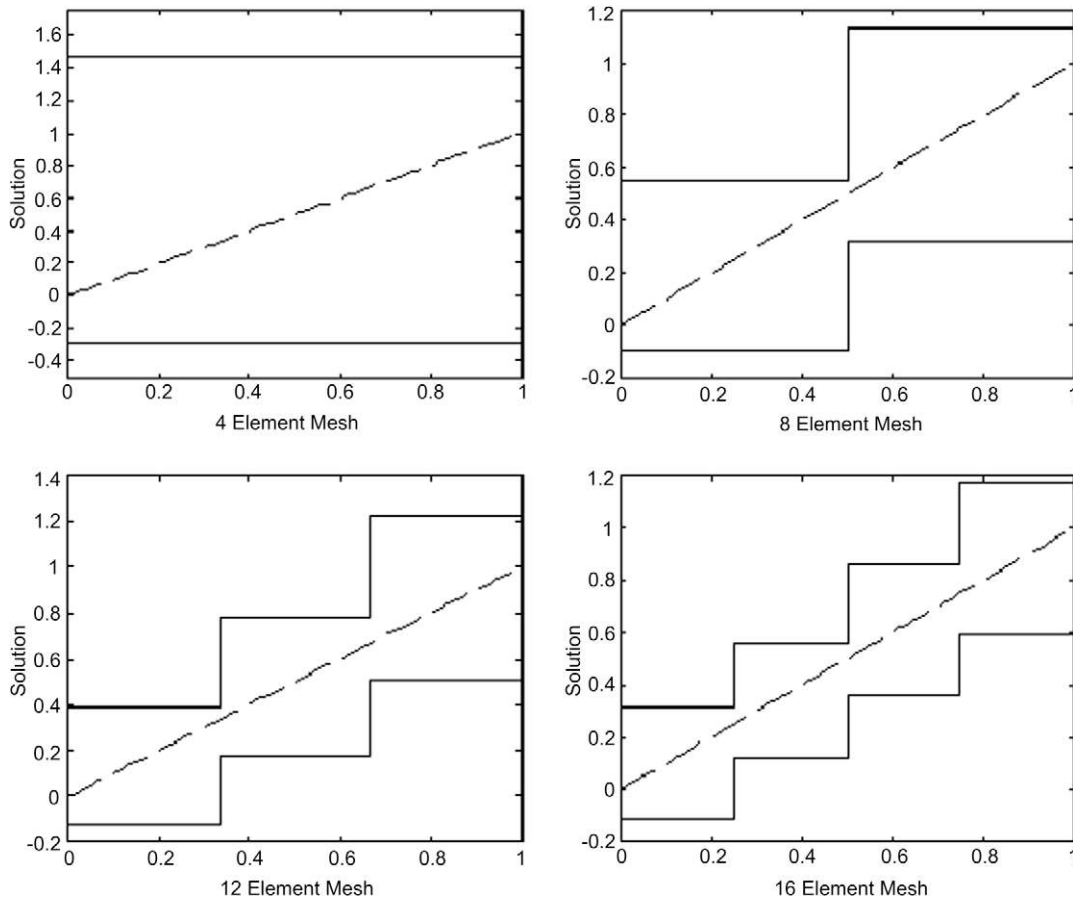

Fig. 17. Behavior of the interval bounds for the different meshes.

elastic modulus and a zero Poisson ratio. A symmetry model is considered, to decrease the computational time, with a unit displacement at the top and is uniformly discretized using constant boundary elements (Figs. 12 and 13). The behavior of the solution width, effectivity index, and solution bounds is depicted (Figs. 14–16) for the displacement in the *y* direction for nodes 4, 8, 12, and 16 on the four respective meshes. The interval bounds, depicted by a solid line enclosing the dashed true solution, for the left edge displacement in the *y* direction are shown (Fig. 17).

11. Conclusion

In this work the discretization error for the elasticity problem is enclosed using interval boundary element method. The interval bounds on the true solution are shown to converge for the meshes considered despite the increase in the effectivity index. The increase in the effectivity index is attributed to the slower convergence of the interval bounds than the true solution. The overestimation in the interval bounds is due to the overestimation of the terms in the interval boundary integral equation using IKST, imperfect parameterization of the location of the source point throughout the rows of the matrices *H* and *G*, and the overestimation in the iterative interval solver. There are two sources of overestimation in the iterative scheme solving the interval system of linear equations. The first one is due to the inherent overestimation when Krawczyk iteration is used to solve interval linear system of equations. This source of overestimation occurs due to the orthogonal multidimensional interval bounds enclosing a true solution which may not be, and in most cases is not, orthogonal and/or oriented in the same direction as the interval bounds (Fig. 18). The second source of overestimation of the interval solver comes from incomplete consideration of the interval parameterization in Eq. (71). Each term in Eq. (71) is parameterized; however, each of these terms must be dealt with in its entirety when operated with. The solution of the linear system of equations must be satisfied for the entire system and thus the interval residual has to be calculated for the entire interval width, not for the length of the subinterval. If the interval residual is computed for the portion of the interval, for instance an interval width corresponding to a subinterval such that a complete interval parameterization can be utilized in Eq. (71), the enclosure in no longer guaranteed.

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