Introduction to Control Theory

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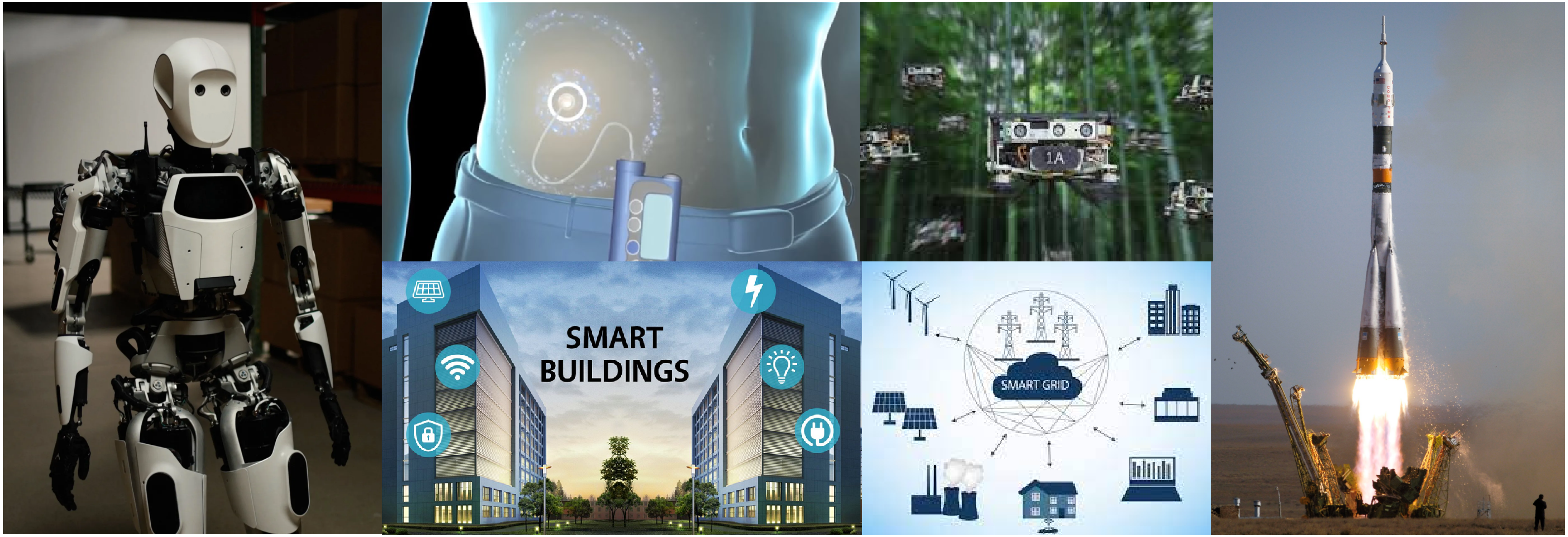
Course grade breakdowns
Labs - 50%
Final project - 50 %

What is a control system?

Link to the video: **Prof. Jeff Hoffman, MIT Open Learning**

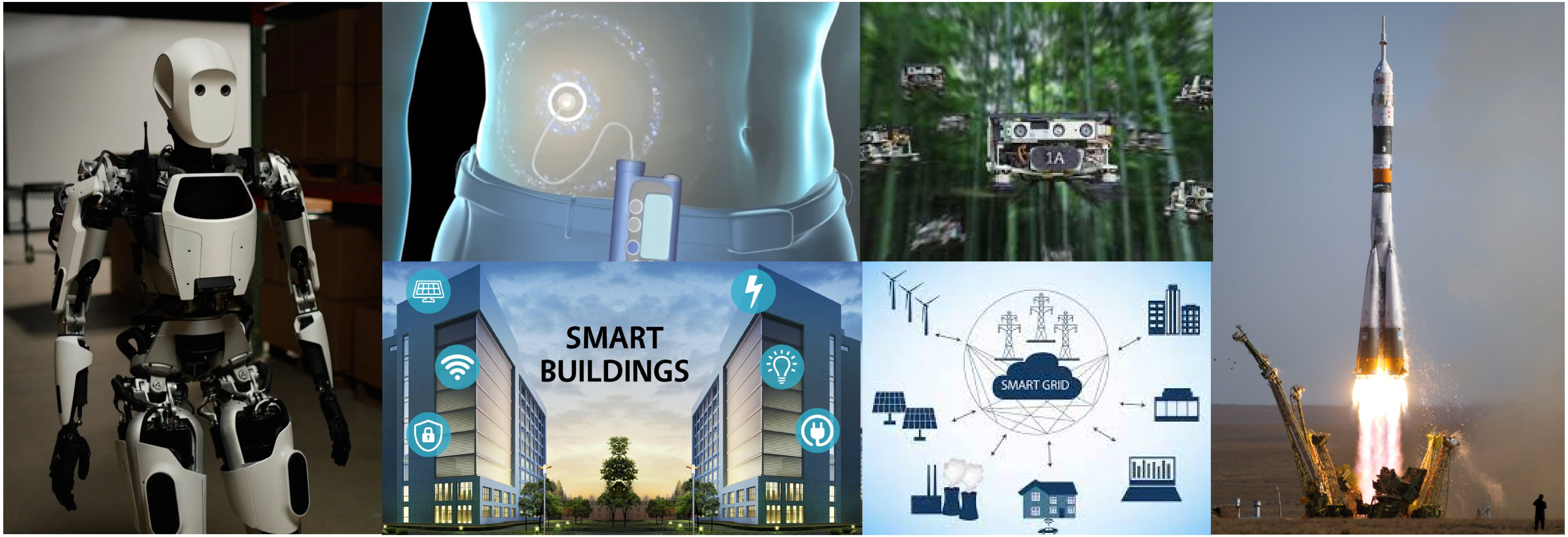
Why automatic control ?



A brief history of control theory…

https://www.youtube.com/watch?v=FD6Fz9cYy5I

“smart” means “automatically controlled”…



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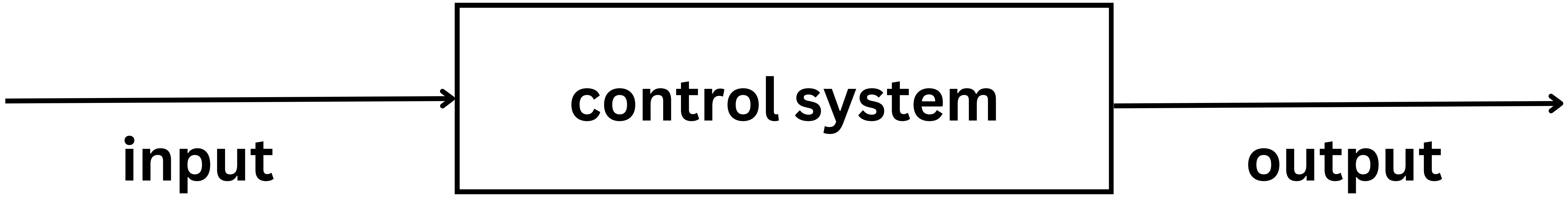
What is a control system?



control system

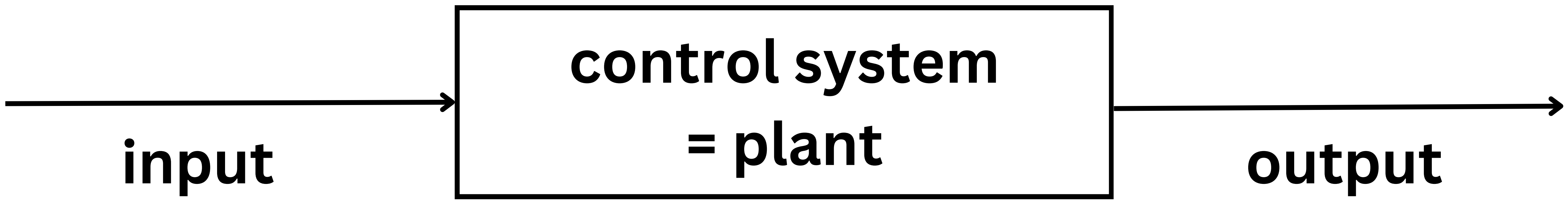
input

output

Control system =

mechanism that alters the future state of the system

What is a control theory?



input

control system
= plant

output

how do I change this ?

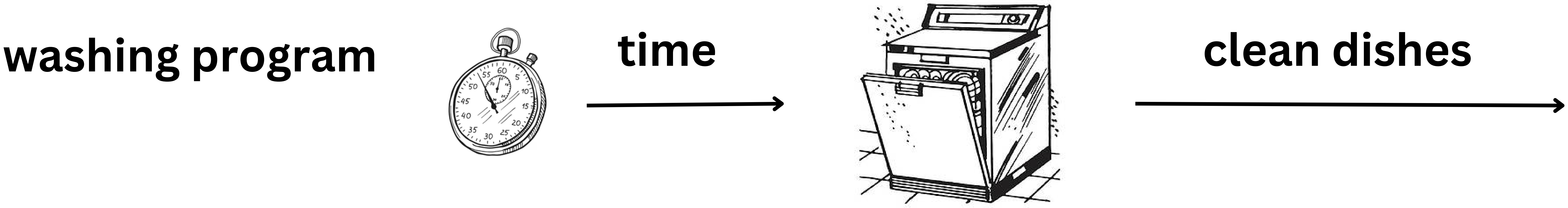
to get what I want?

Control system =
mechanism that alters the future state of the system

Control theory =
a strategy to select appropriate input

Open-loop vs Closed-loop

Open-loop control systems are typically reserved for simple processes that have well-defined input to output behaviors.



Once the user sets the wash timer the dishwasher will run for that set time, regardless of whether the dishes are actually clean or not when it finishes running.

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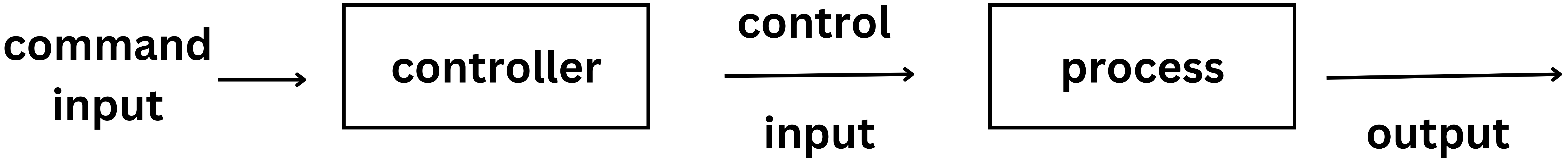
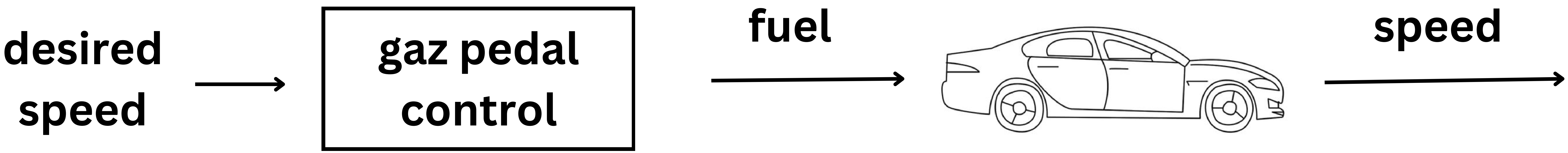


Open-loop vs Closed-loop

For any arbitrary process, though, an open-loop control system is typically not sufficient.



desired speed

gaz pedal control

fuel

speed

Imagine you are trying to move your car with a constant speed

Open-loop vs Closed-loop

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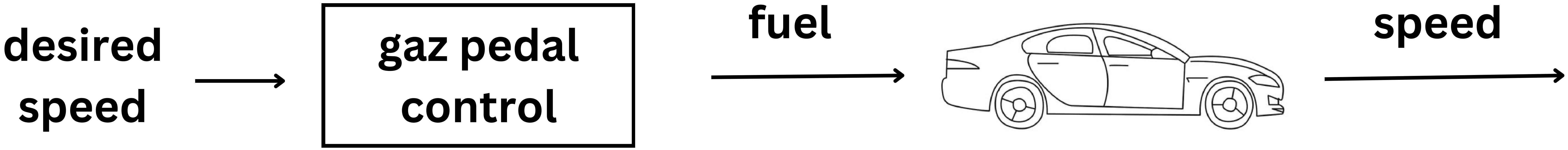
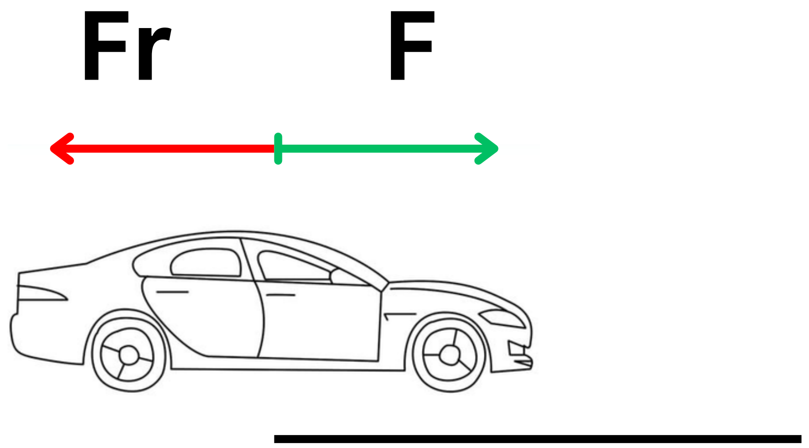


desired speed → gaz pedal control → fuel → speed

Moving flat road you can apply the force F which is balanced by the force of friction Fr at this point



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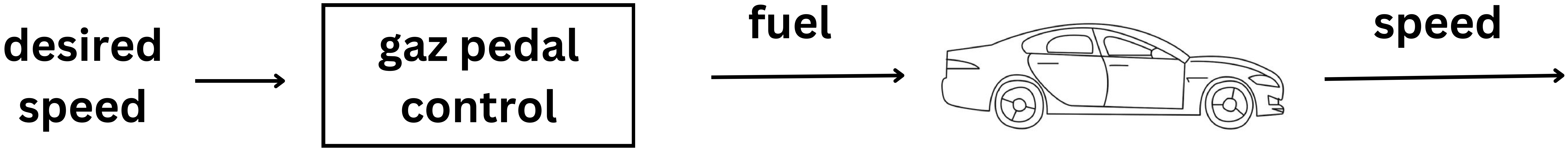

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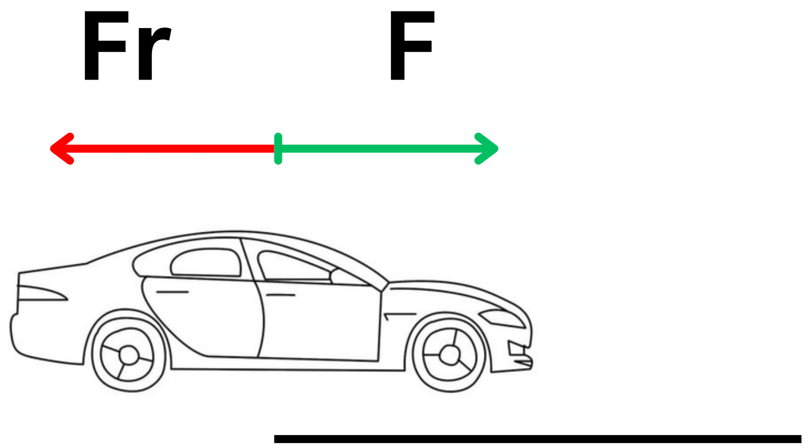

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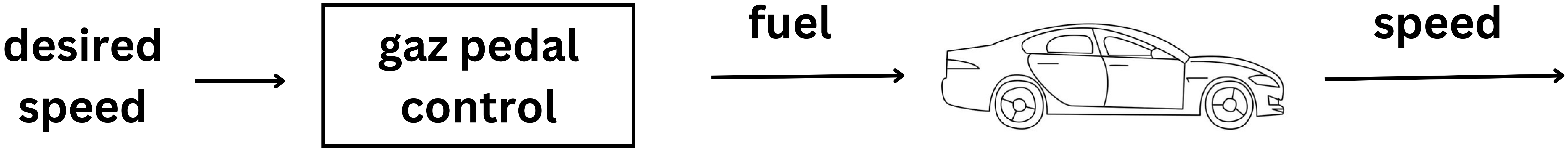


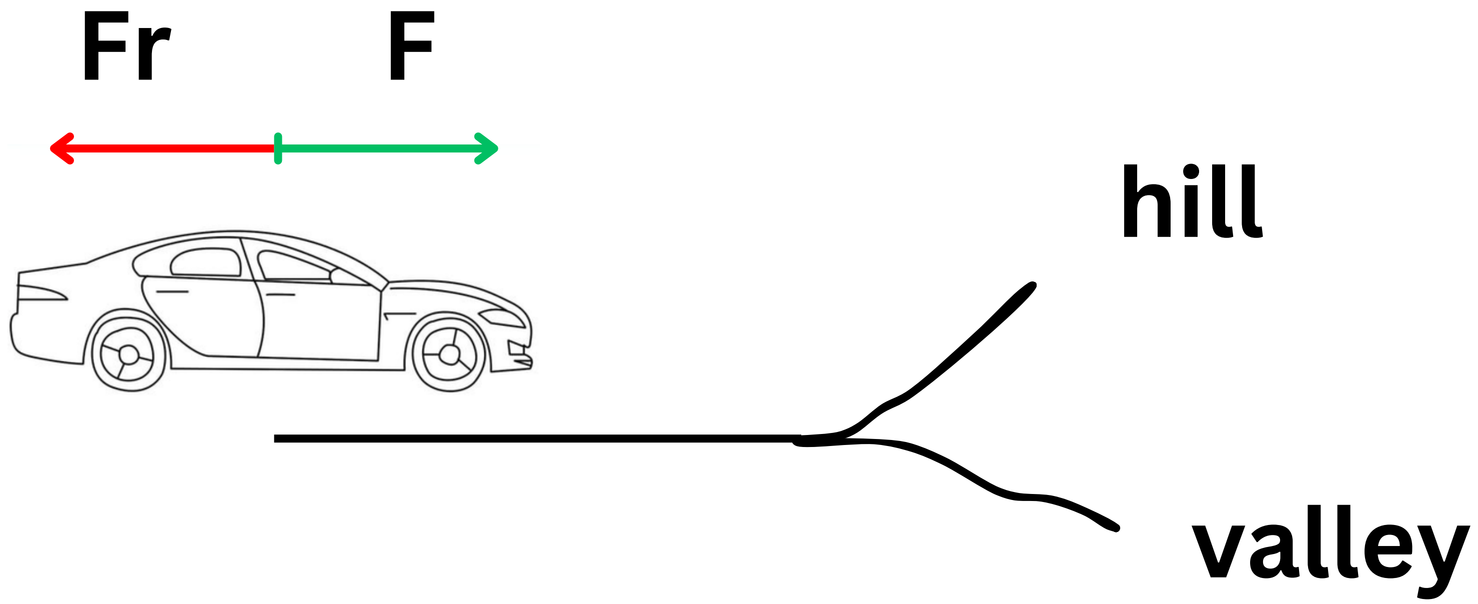
desired speed

gaz pedal control

fuel

speed

but what happens when
the car encounters a hill or a valley?



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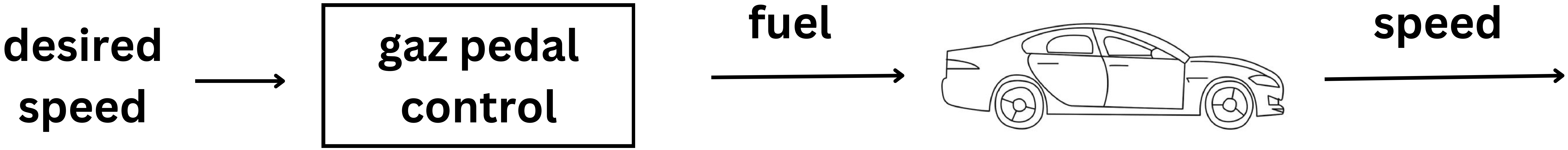

desired speed → gaz pedal control → fuel → speed

to account for road gradient changes you must vary the input to your system with respect to the output

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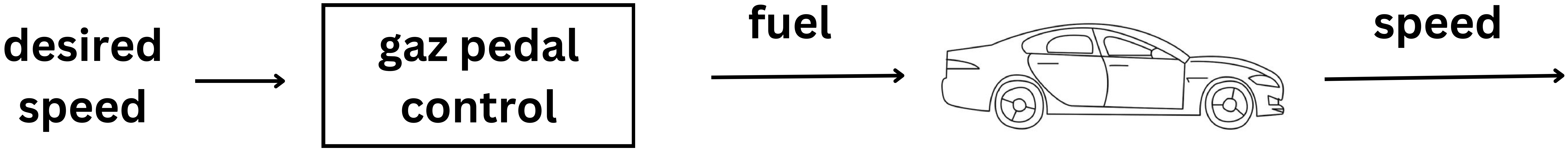

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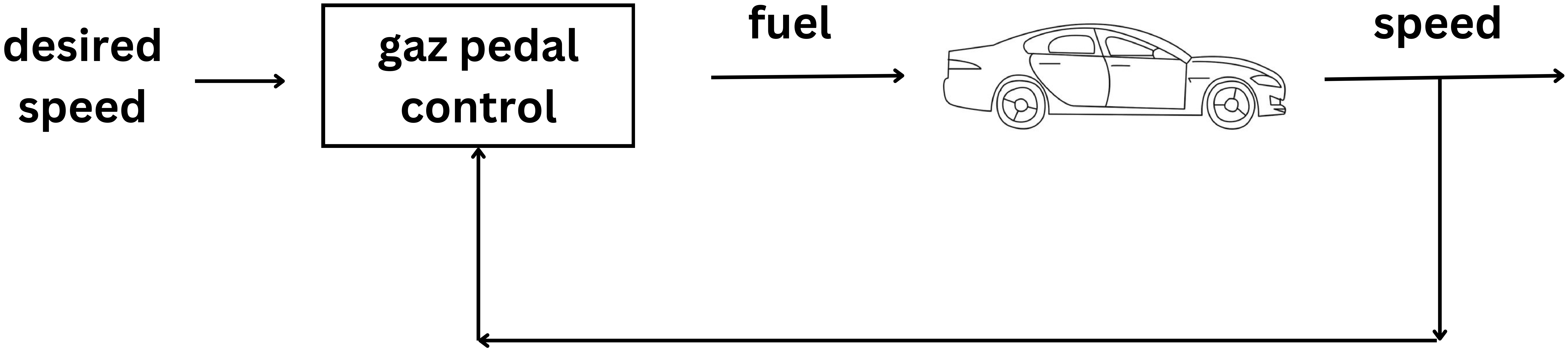

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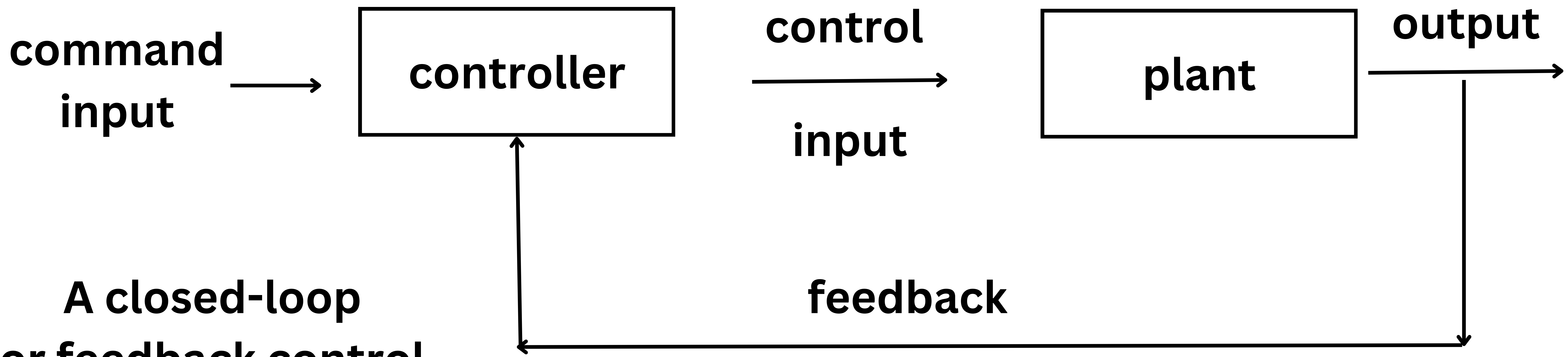


command input

controller

control input

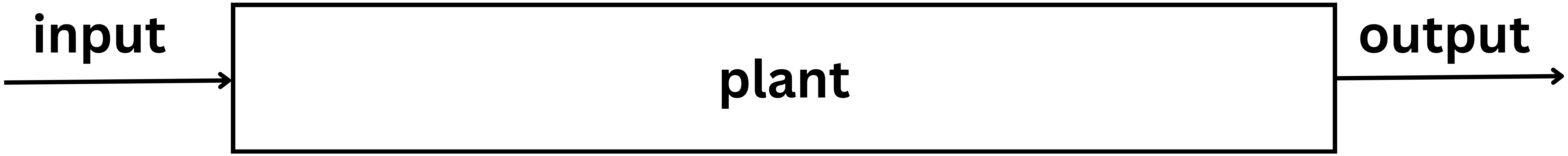
plant

output

feedback

A closed-loop or feedback control system

Modeling

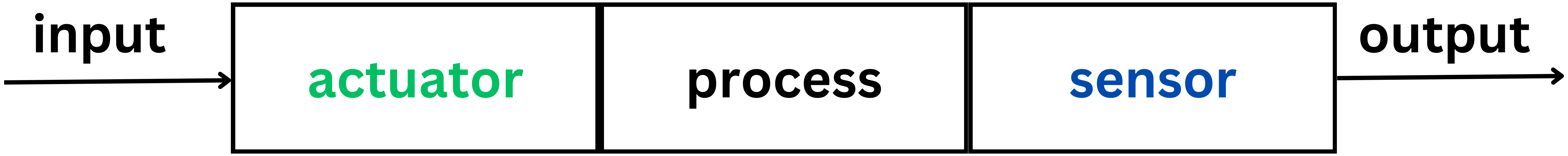


input

plant

output

Modeling



input

actuator

process

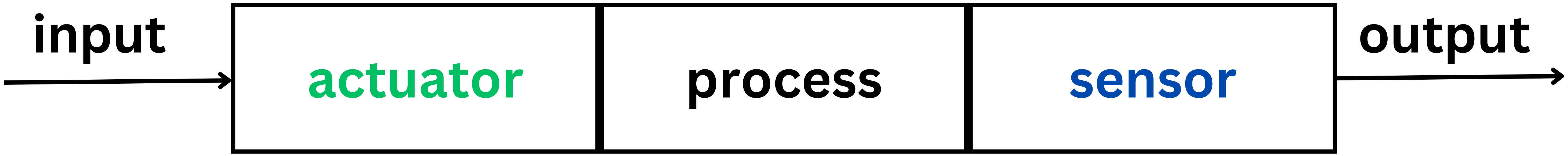
sensor

output

An actuator is a part of a device or machine that convertы energy, often electrical, air, or hydraulic, into mechanical force. It is the component in any machine that enables movement.

A sensor is a device that produces an output signal for the purpose of sensing a physical phenomenon.

Modeling


input

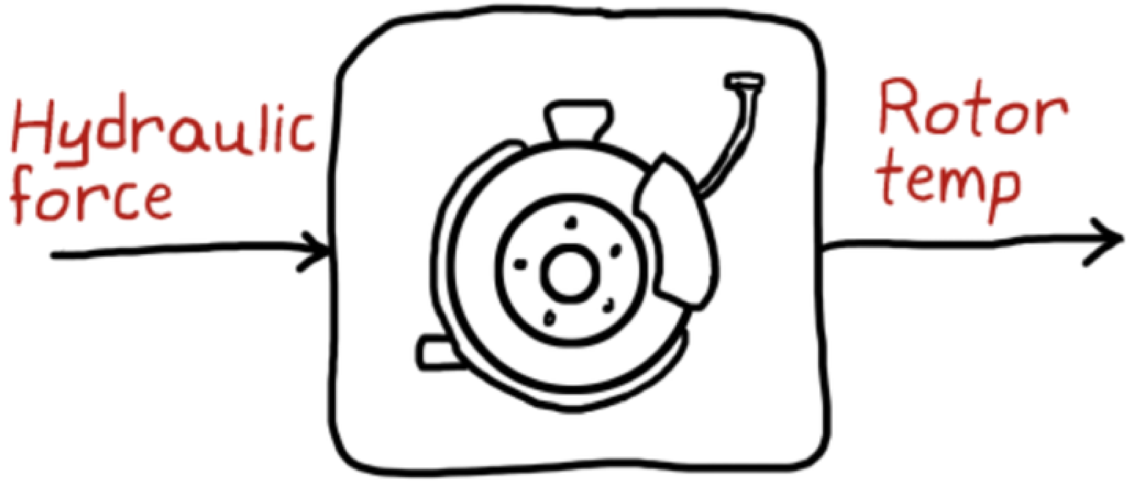
actuator

process

sensor

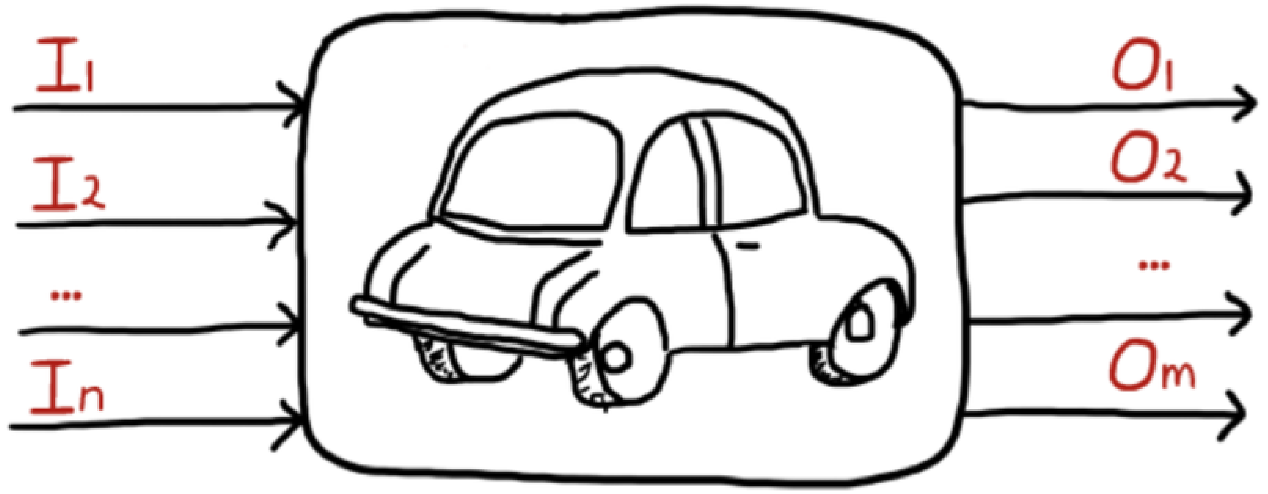
output

SISO



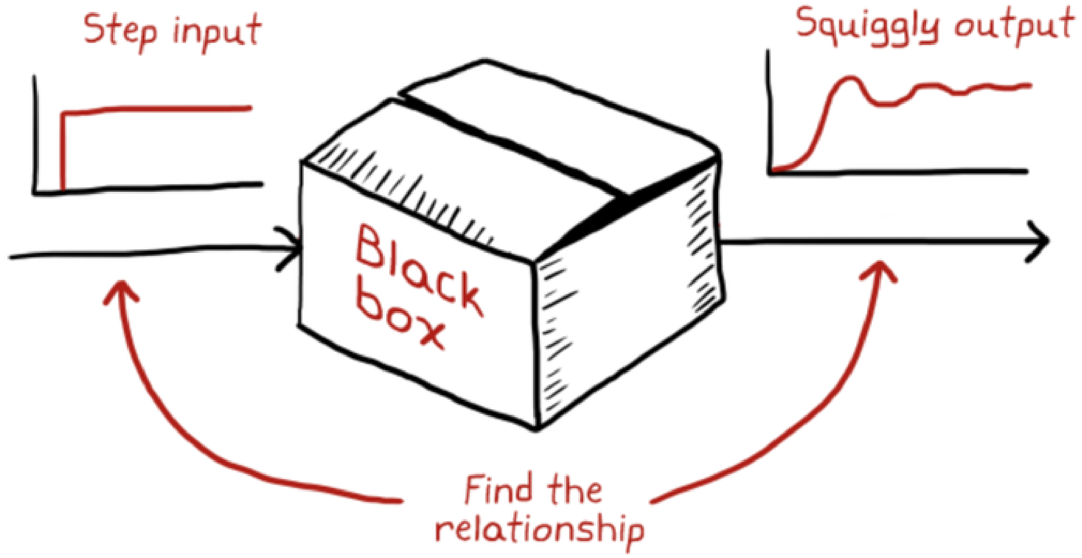
Single Input Single Output

MIMO

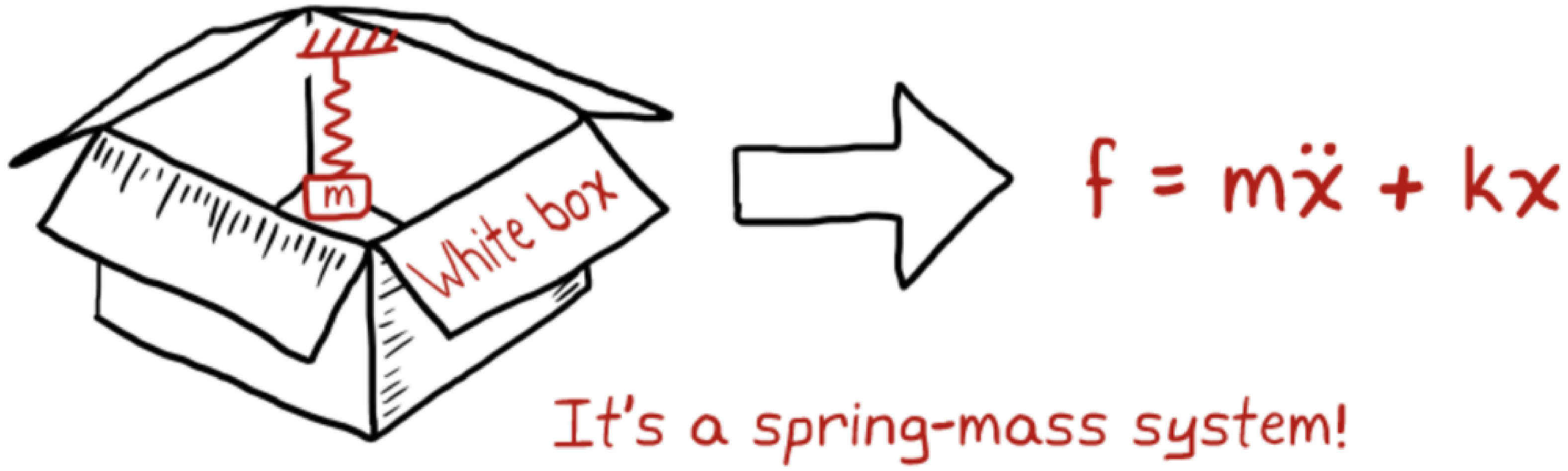


Multiple Inputs Multiple Outputs

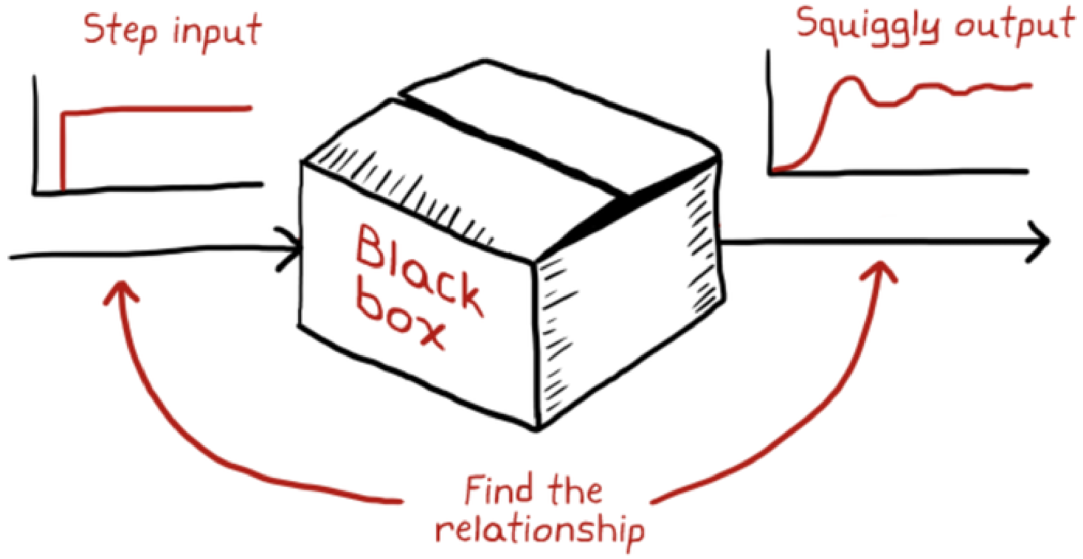
Modeling



A **black box** model receives inputs and produces outputs but its workings are unknowable.
For example: neural networks

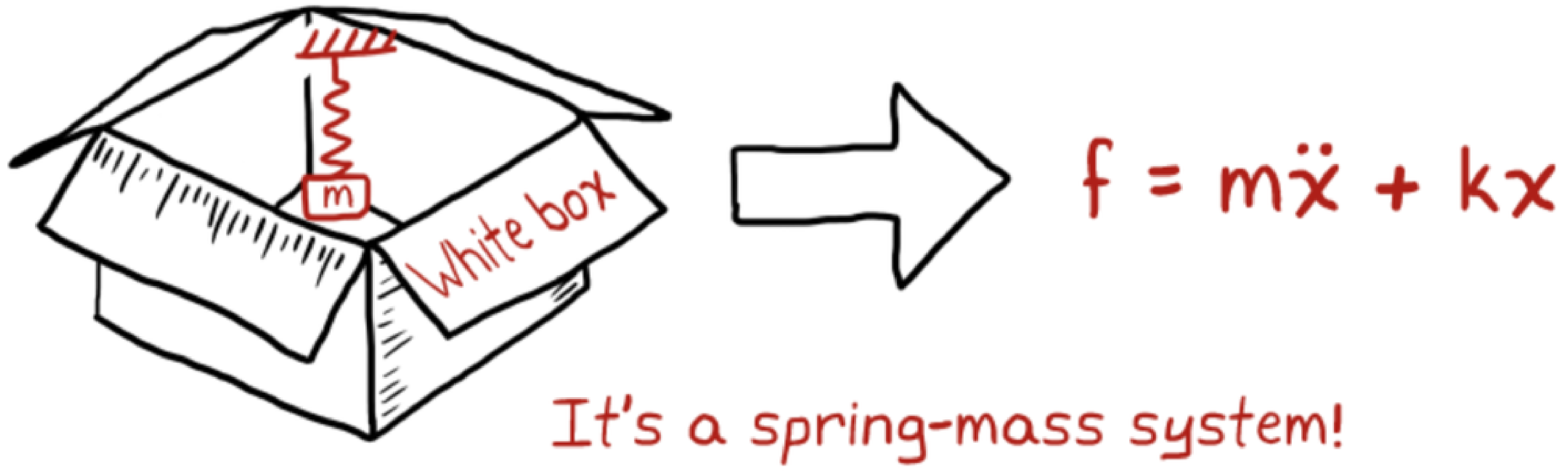


Modeling

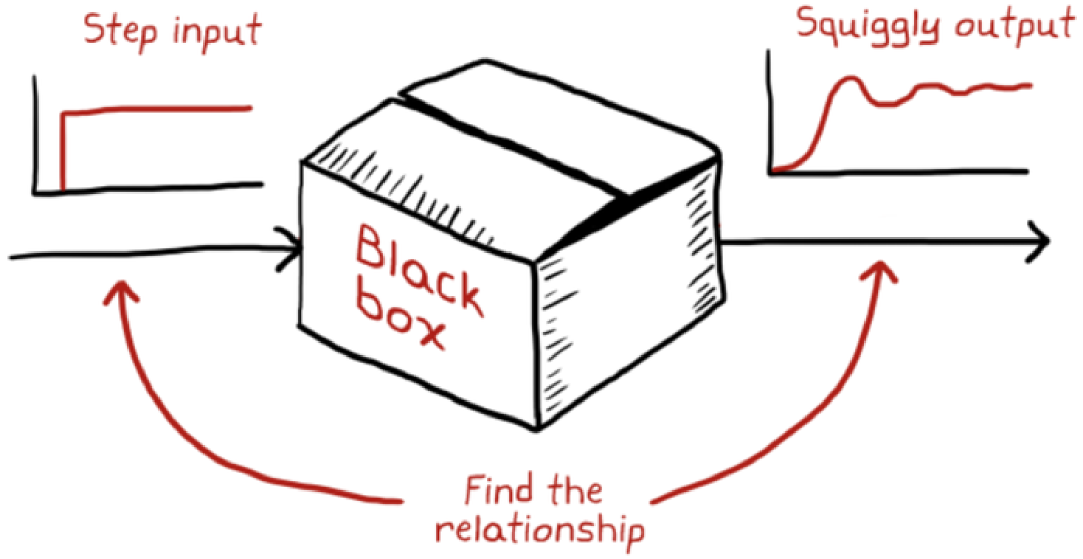


A **black box** model receives inputs and produces outputs but its workings are unknowable.
For example: neural networks

A **white box** model is a mathematical model of a physical process described by
ODE or PDE



Modeling



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For example: neural networks

Grey box models

A **white box** model is a mathematical model of a physical process described by
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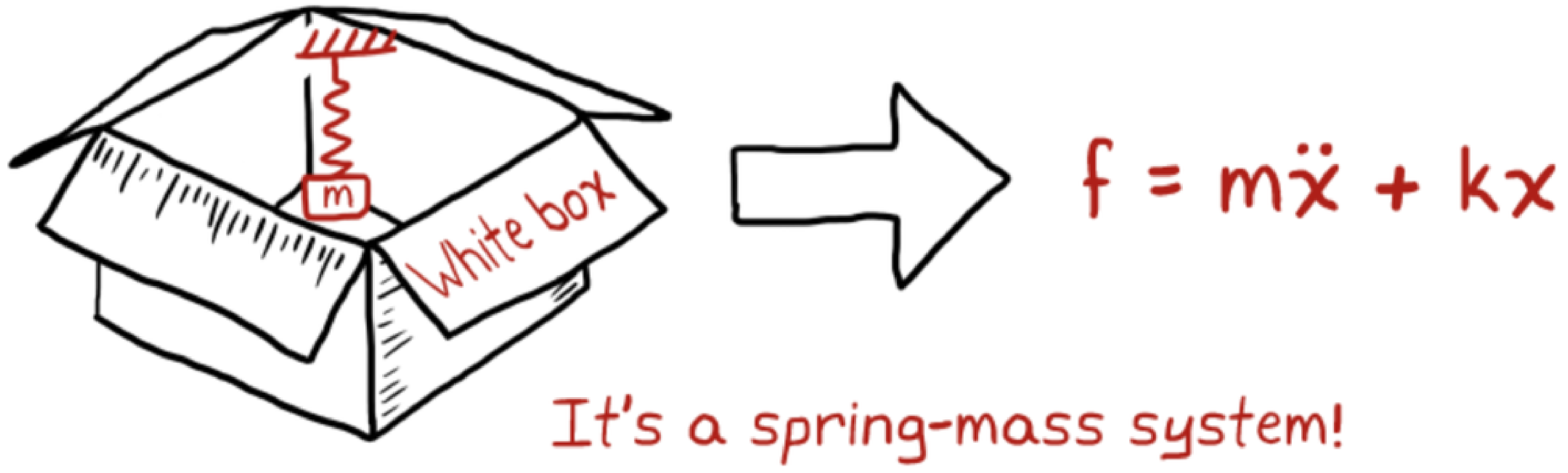


Modeling

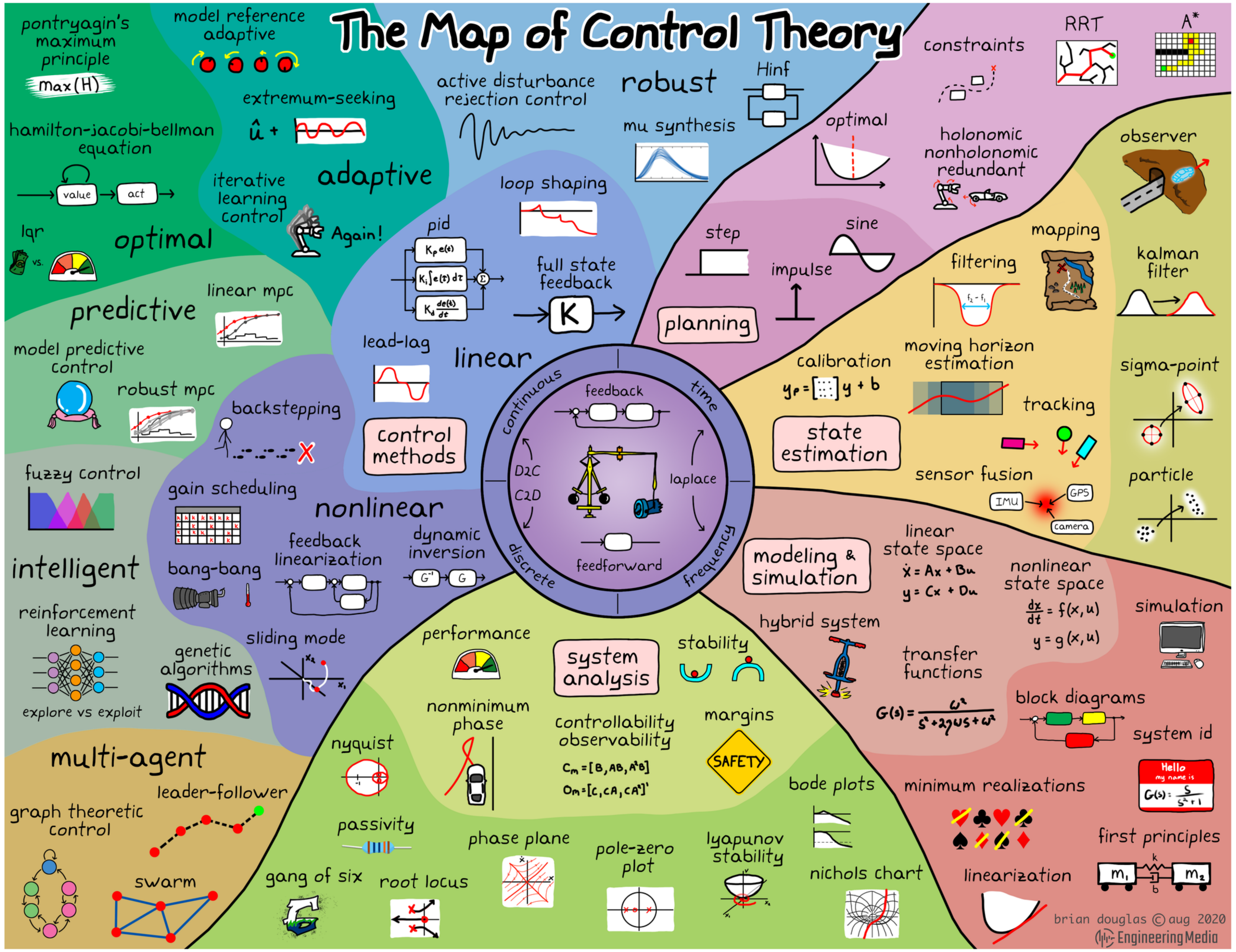
Models allow simulating and analyzing the system

Models are never exact

Modeling depends on your goal

A single system may have many models

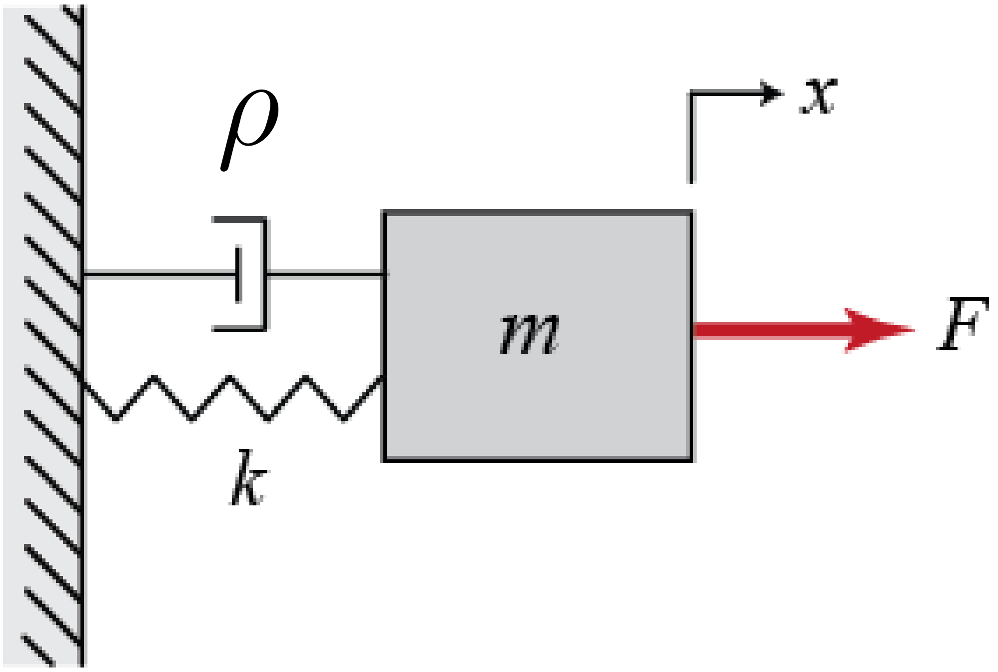
Lectures outline



by Brian Douglas

1. Modeling. LTI systems
2. Controllability and Observability
3. Stability and State Observer
4. Control design. PID controller
5. Optimal control design
6. Advanced control design
7. Project defense

Ex.1: Mass-Spring-Damper System



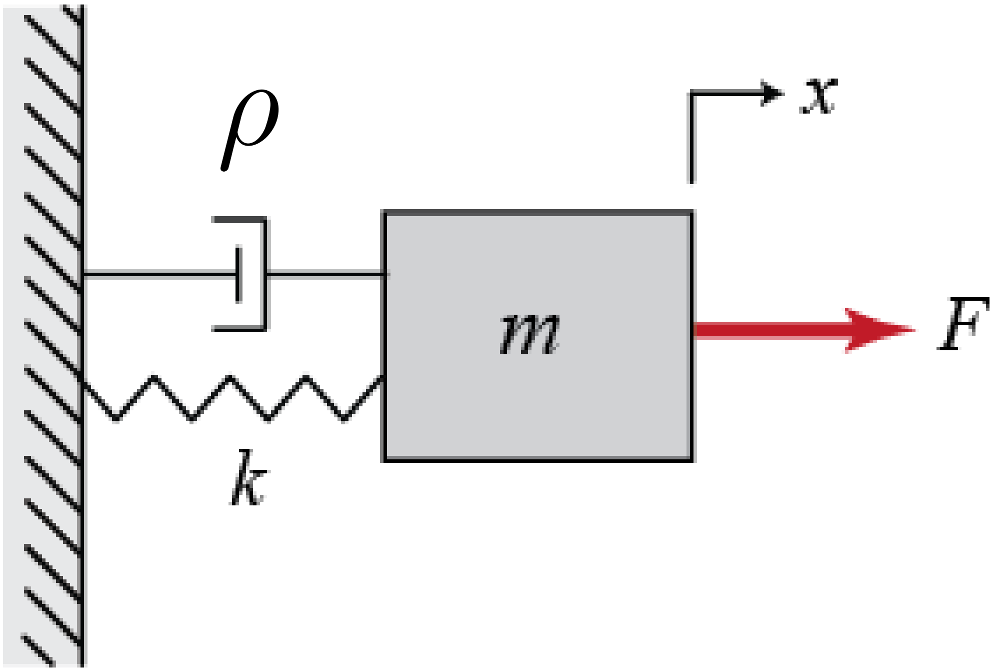
One common example of mass-spring system is the suspension of a car. The car itself is the mass; it is suspended by an elastic spring. A damper (the actual shock absorber) prevents oscillations.

Newton's second law (translational motion):

$$m\ddot{x} = F_{total} = \underset{\text{spring force}}{-kx} - \underset{\text{friction force}}{\rho\dot{x}} + \underset{\text{external force}}{u}$$

Hooke's law Stokes' law

Ex.1: Mass-Spring System



2nd-order linear ODE

$$\ddot{x} + \frac{k}{m}x + \frac{\rho}{m}\dot{x} = \frac{1}{m}u$$

Canonical form: 1st-order ODE

$$\dot{x} = v$$

definition of velocity

$$\dot{v} = -\frac{k}{m}x - \frac{\rho}{m}v + \frac{1}{m}u$$

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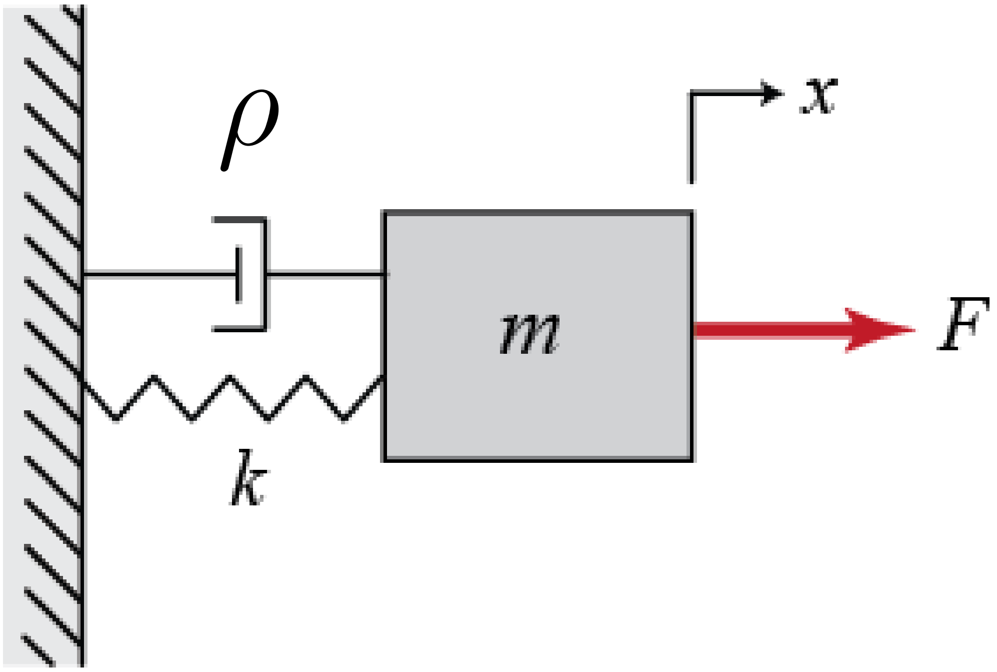

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Measurements

We are interested in controlling the position of the mass

output: $y = x$

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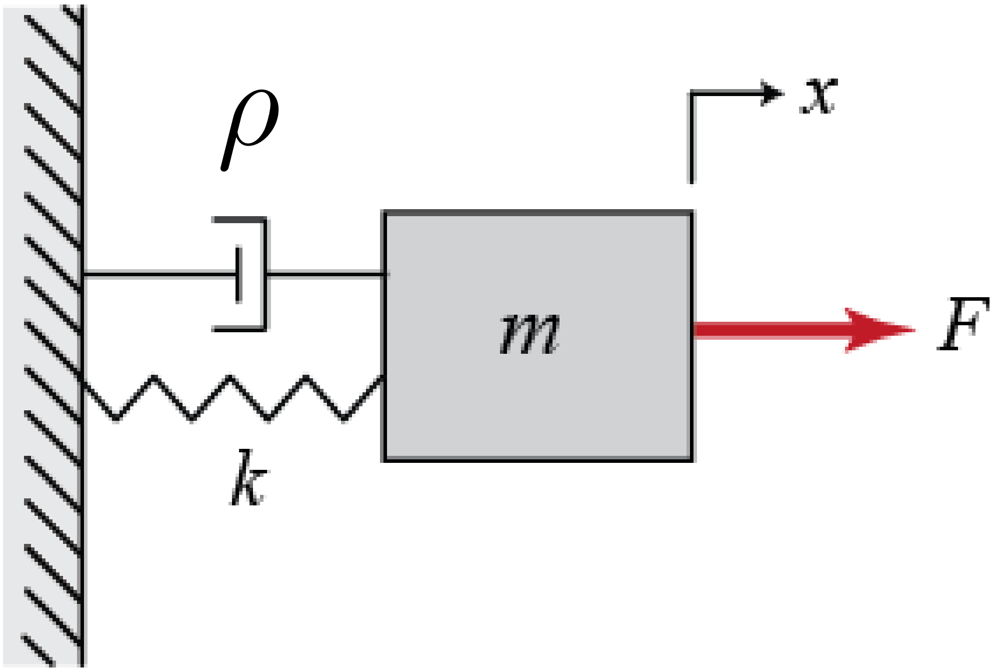

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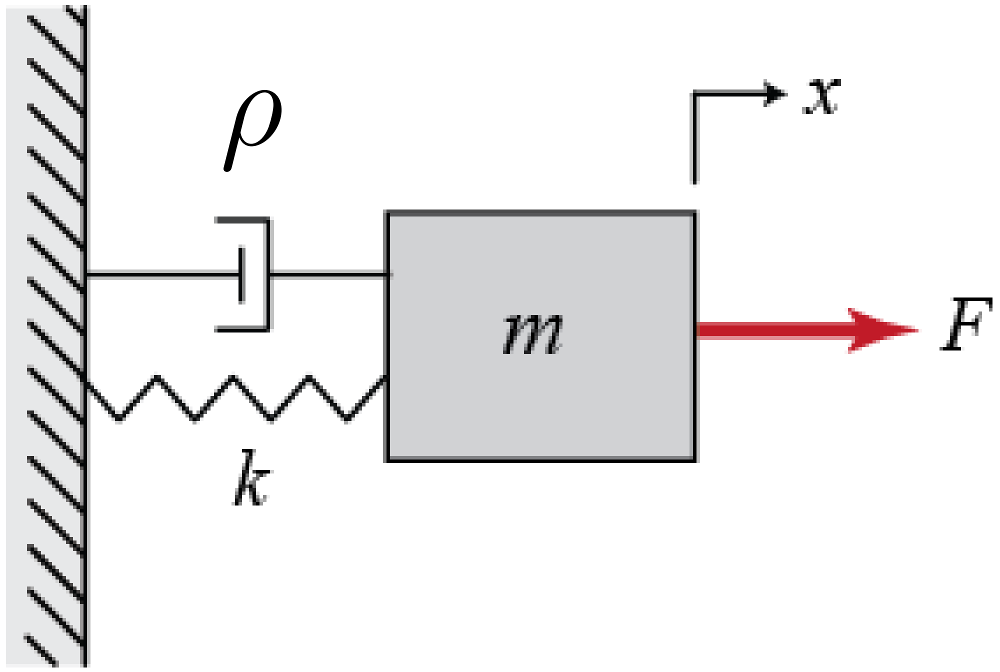

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Canonical form: 1st-order ODE

$$\begin{pmatrix}\dot{x}\\ \dot{v}\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{\rho}{m}\end{pmatrix}}_{\text{dynamic matrix}} \underbrace{\begin{pmatrix}x\\ v\end{pmatrix}}_{\text{state vector}} + \underbrace{\begin{pmatrix}0\\ \frac{1}{m}\end{pmatrix}}_{\text{control matrix}} \underbrace{u}_{\text{control vector}}$$

Measurements

$$y = \underbrace{(1 \quad 0)}_{\text{sensor matrix}} \begin{pmatrix}x\\ v\end{pmatrix} + \underbrace{0}_{\text{direct matrix}} u$$

State-space models of LTI systems

State equation

$$\dot{x} = \underset{\text{dynamic matrix}}{A}\overset{\text{state vector}}{x} + \underset{\text{control matrix}}{B}\overset{\text{control vector}}{u}$$

Output equation

$$y = \underset{\text{sensor matrix}}{C}\overset{\text{state vector}}{x} + \underset{\text{direct matrix}}{D}\overset{\text{control vector}}{u}$$

$$\underset{\text{n state variables}}{x \in \mathbb{R}^{n\times 1}},\ \underset{\text{p control inputs}}{u \in \mathbb{R}^{p\times 1}},\ \underset{\text{m output measurements}}{y \in \mathbb{R}^{m\times 1}}$$

$$A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times p},\ C \in \mathbb{R}^{m\times n},\ D \in \mathbb{R}^{m\times p}$$

State-space models of LTI systems

State equation

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Output equation

$$y = \underset{\text{sensor matrix}}{C}\overset{\text{state vector}}{x} + \underset{\text{direct matrix}}{D}\overset{\text{control vector}}{u}$$

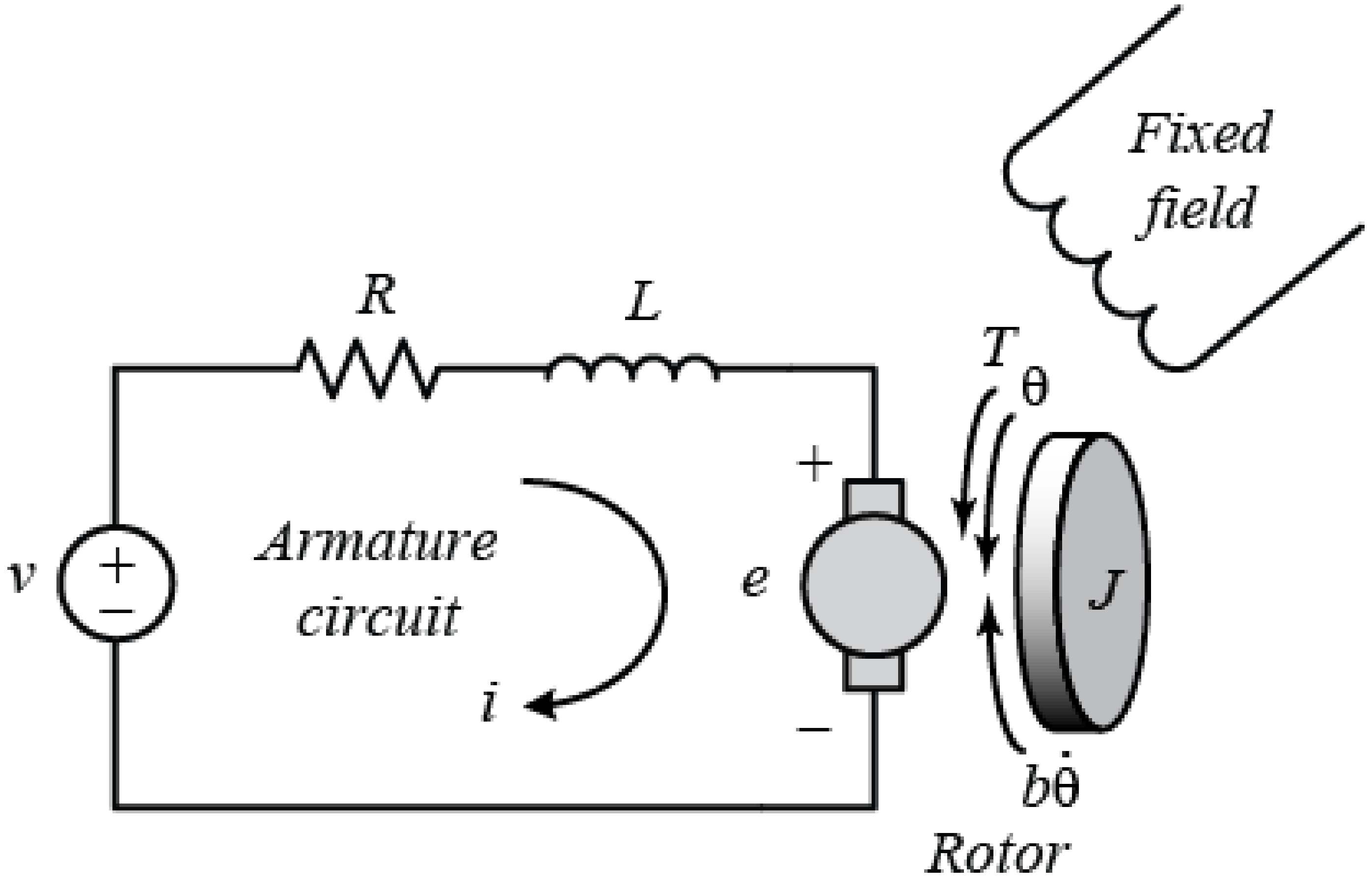
Linear time invariate systems

Linearity: functions are linear mappings

Time invariance: a certain input will always give the same output (up to timing), without regard to when the input was applied to the system.

Ex.2: DC motor

A common actuator in control systems is the DC motor. It directly provides rotary motion and, coupled with wheels or drums and cables, can provide translational motion.



Newton's 2nd law

$$J\ddot{\theta} + b\dot{\theta} = Ki$$

Kirchhoff's voltage law

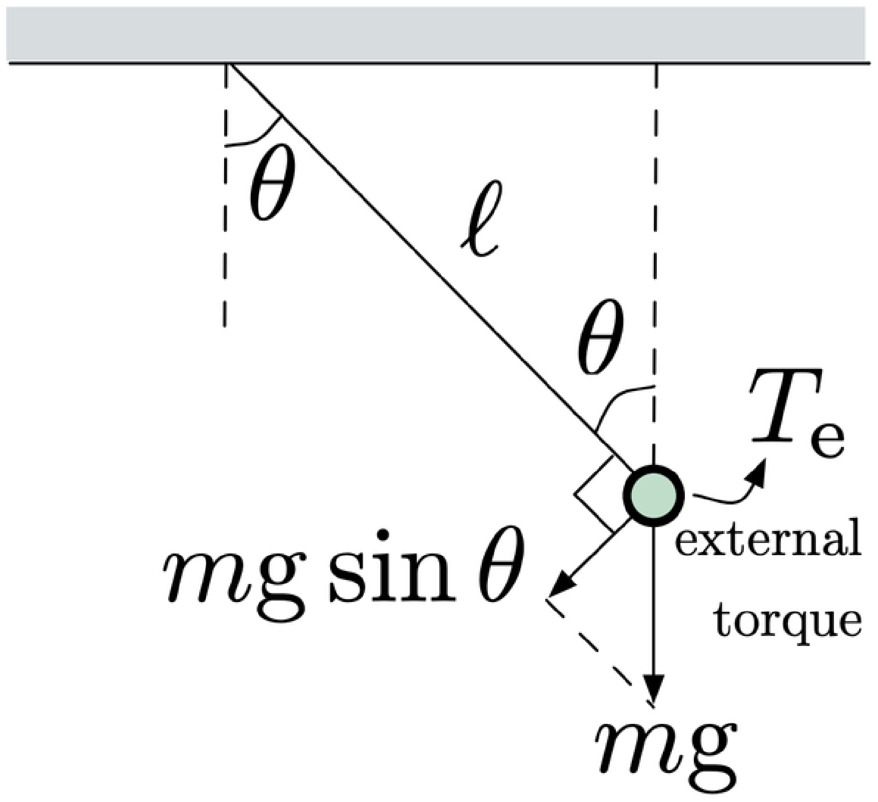
assuming that the magnetic field is constant

$$\underbrace{L\frac{di}{dt}}_{v_L} + \underbrace{Ri}_{v_R} = V - \underbrace{K\dot{\theta}}_{v_e}$$

Ex.3: Pendulum

Newton’s 2nd law (rotation motion):

$$\underbrace{ml^2}_{\text{moment of inertia}} \cdot \underbrace{\ddot{\theta}}_{\text{angular acceleration}} = \overbrace{\underbrace{-mg\sin(\theta)}_{\text{force}} \cdot \underbrace{l}_{\text{lever arm}}}^{\text{pendulum torque}} + \underbrace{T_e}_{\text{external torque}}$$



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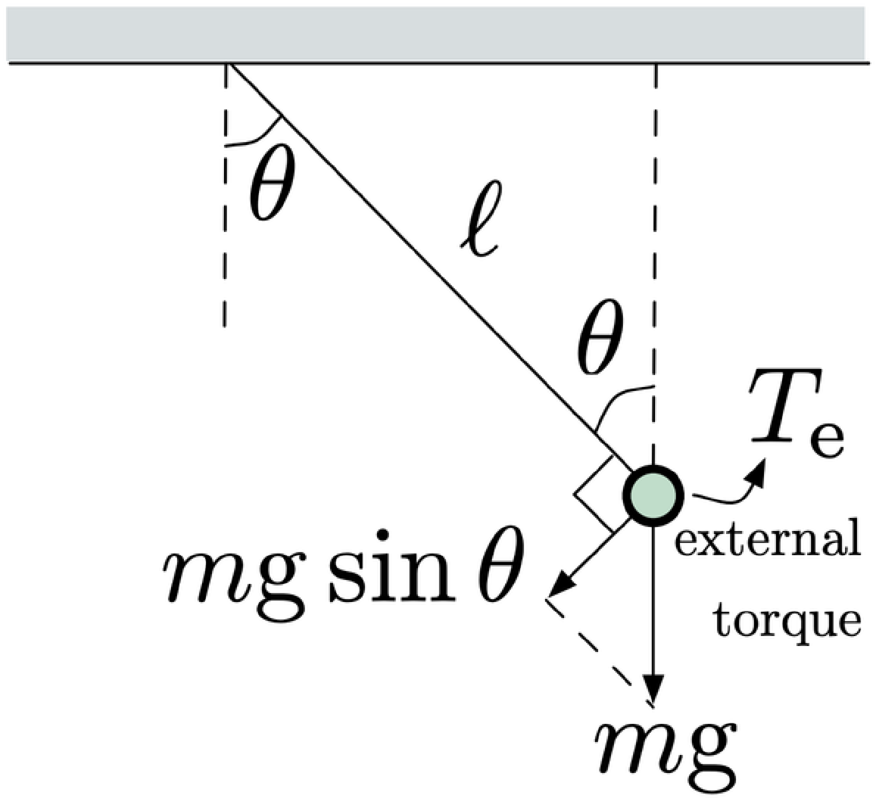

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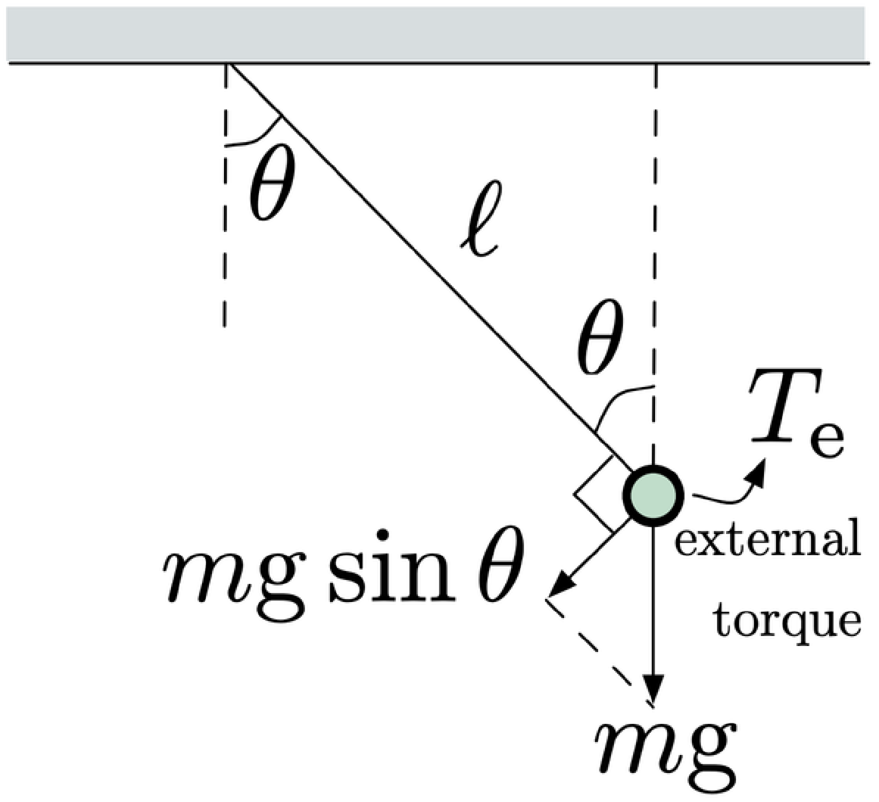
Nonlinear 2nd order equation

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{1}{ml^2}T_e$$

Ex.3: Pendulum



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Nonlinear canonical state-space

Let

$$\theta_1 = \theta,\ \theta_2 = \dot{\theta}$$

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$

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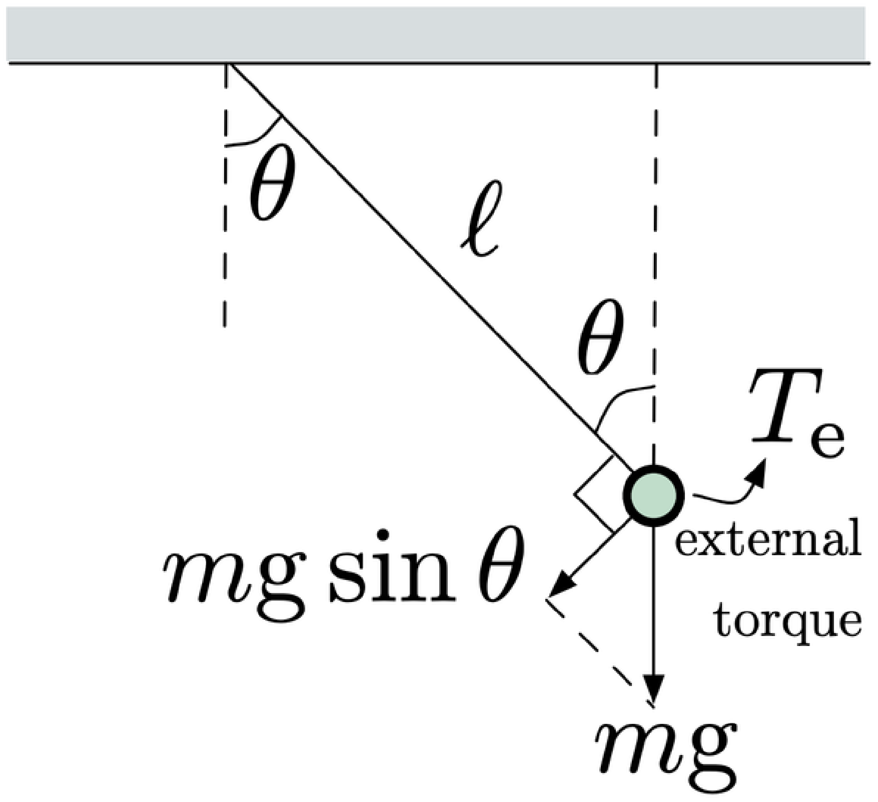

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$$\theta_1 = \theta,\ \ \theta_2 = \dot{\theta}$$

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$

$$\dot{x} = f(x, u)$$

$$x = \begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} \quad u = T_e$$

Ex.3: Pendulum

Newton’s 2nd law (rotation motion):

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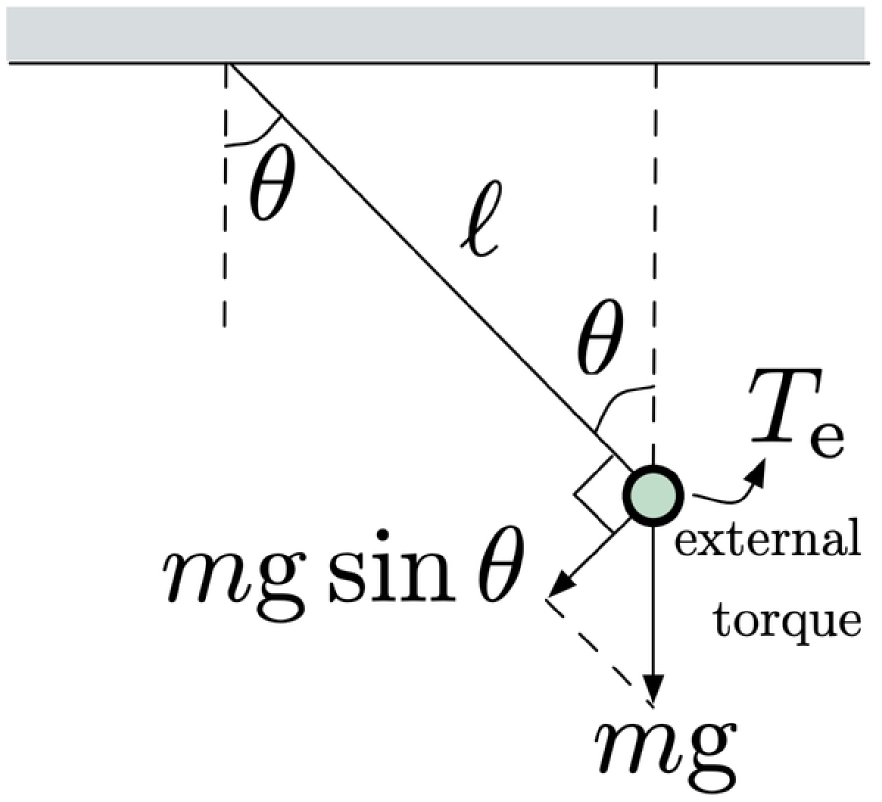

Nonlinear equation

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$

for small θ

$$\sin\theta \approx \theta$$

$$\dot{x} = f(x, u)$$

$$x = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \quad u = T_e$$

Ex.3: Pendulum

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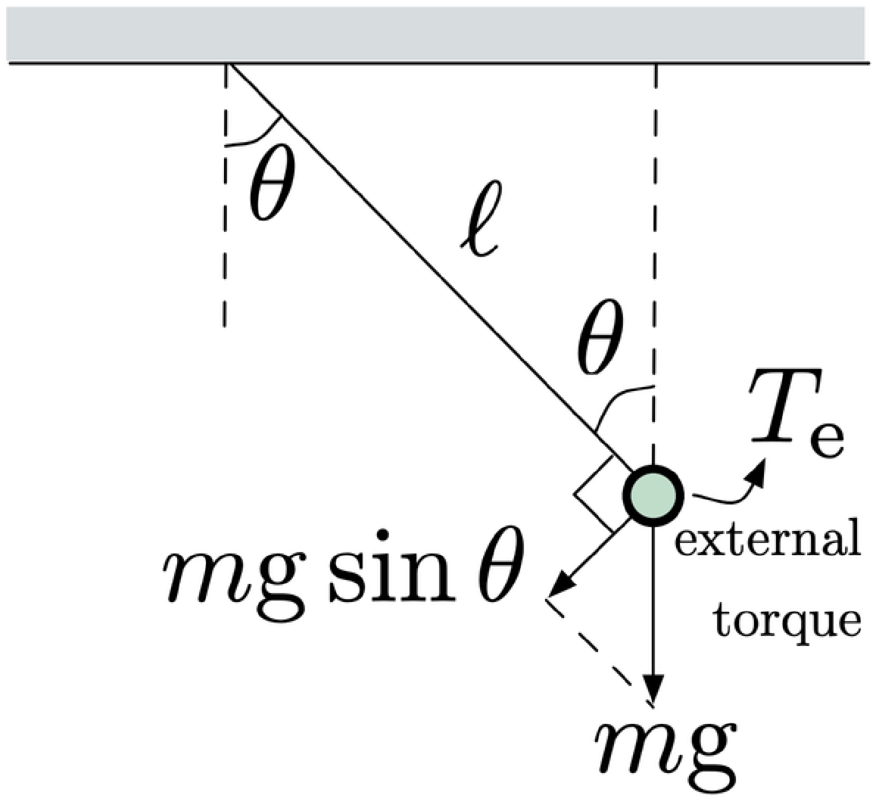

LTI model in canonical form

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} T_e$$

General linearisation procedure

- Start from nonlinear state-space model $\dot{x} = f(x, u)$
- Find equilibrium point (x_0, u_0) such that $f(x_0, u_0) = 0$

  Note: different systems may have different equilibria, not necessarily $(0, 0)$, so we need to shift variables:

$$\underline{x} = x - x_0 \qquad \underline{u} = u - u_0$$
$$\underline{f}(\underline{x}, \underline{u}) = f(\underline{x} + x_0, \underline{u} + u_0) = f(x, u)$$

Note that the transformation is *invertible*:

$$x = \underline{x} + x_0, \qquad u = \underline{u} + u_0$$

General linearisation procedure

- Pass to shifted variables $\underline{x} = x - x_0$, $\underline{u} = u - u_0$

$$\begin{aligned} \dot{\underline{x}} &= \dot{x} && (x_0 \text{ does not depend on } t) \\ &= f(x, u) \\ &= \underline{f}(\underline{x}, \underline{u}) \end{aligned}$$

— equivalent to original system

- The transformed system is in equilibrium at $(0, 0)$:

$$\underline{f}(0, 0) = f(x_0, u_0) = 0$$

General linearisation procedure

- Now linearize:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \qquad \text{where } A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{x=x_0\\u=u_0}}, \; B_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{x=x_0\\u=u_0}}$$

General linearisation procedure

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- Why do we require that $f(x_0, u_0) = 0$ in equilibrium?

General linearisation procedure

- This requires some thought. Indeed, we may talk about a *linear approximation* of any smooth function f at any point x_0:

  $$f(x) \approx f(x_0)+f'(x_0)(x-x_0) \qquad \text{— } f(x_0) \text{ does not have to be } 0$$

- The key is that we want to approximate a given nonlinear system $\dot{x} = f(x,u)$ by a *linear* system $\dot{x} = Ax + Bu$ (may have to shift coordinates: $x \mapsto x - x_0$, $u \mapsto u - u_0$)

  Any linear system *must* have an equilibrium point at $(x,u) = (0,0)$:

  $$f(x,u) = Ax + Bu \qquad f(0,0) = A0 + B0 = 0.$$

Ex. 4: Modeling a balance system

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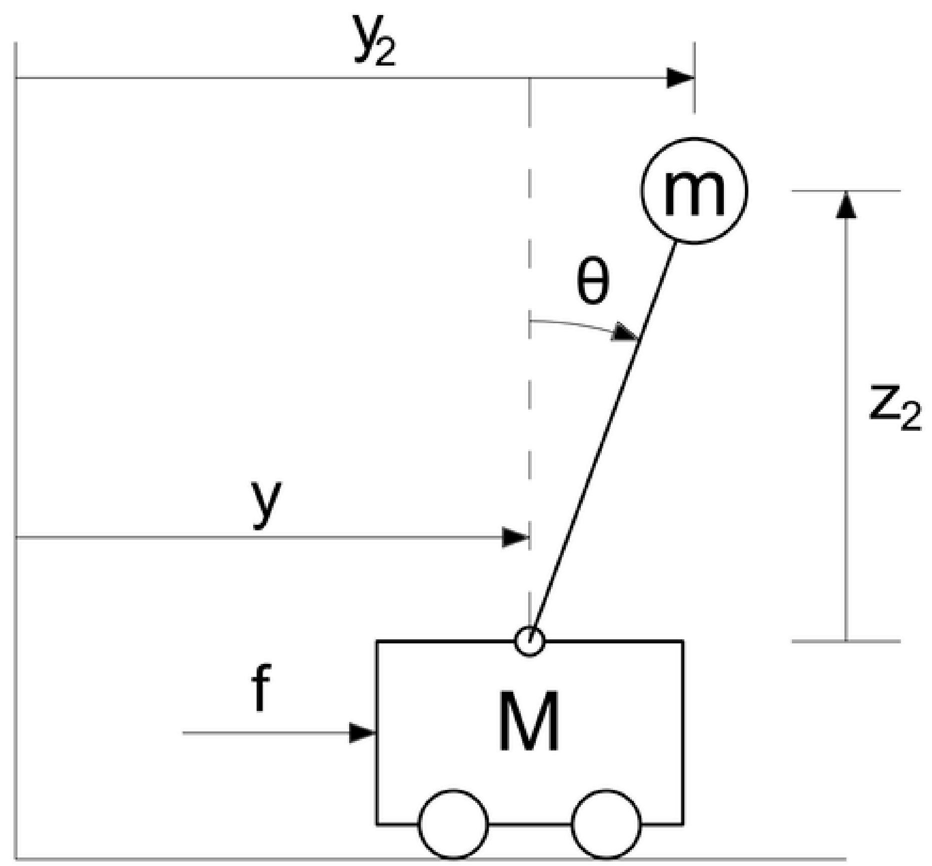
Real-world examples modeled as an inverted pendulum on the cart.

Ex. 4: Modeling a balance system



and other exercises for today session…

Please, install Jupyter Notebook

https://jupyter.org/install

and work on the notebook
I have sent to you by e-mail earlier today

The completed notebook should be sent to elena.vanneaux@ensta-paris.fr
before the beginning of the next session.

Please add [AUT202] to the topic of e-mail.