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Efficient solution of a wave equation with fractional-order dissipative terms

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ABSTRACT

We consider a wave equation with fractional-order dissipative terms modeling visco-thermal losses on the lateral walls of a duct, namely the Webster–Lokshin model. Diffusive representations of fractional derivatives are used, first to prove existence and uniqueness results, then to design a numerical scheme which avoids the storage of the entire history of past data. Two schemes are proposed depending on the choice of a quadrature rule in the Laplace domain. The first one mimics the continuous energy balance but suffers from a loss of accuracy in long time simulation. The second one provides uniform control of the accuracy. However, even though the latter is more efficient and numerically stable under the standard CFL condition, no discrete energy balance has been yet found for it. Numerical results of comparisons with a closed-form solution are provided.

0. Introduction

The Webster–Lokshin system is a dissipative model that describes acoustic waves traveling in a duct with visco-thermal losses at the lateral walls. This system couples a wave equation with spatially-varying coefficients with absorbing terms involving fractional-order integrals and derivatives. The main goal of this article is to propose an efficient numerical discretization of this type of model that, in particular, would avoid storing the solution from all the past time steps, because that would be too computationally penalizing in long time simulations.

Our approach is based on the so-called diffusive representations of the fractional integral where, roughly speaking, the fractional-order time kernel in the integral is represented by its Laplace transform. This allows for efficient time domain discretization because the value of the integral at each time step can be updated from the value at the previous time step by operations which are local in time (contrary to a naive discretization of the fractional integral where global-in-time operations are required). However, these representations require the evaluation of an integral over the Laplace variable domain. We propose and analyze two schemes based on the choice of the quadrature rule associated with this integral.

The first one is inspired by the continuous stability analysis of the initial boundary value problem associated with the coupled system: a wave equation with a diffusive representation. The scheme is constructed so that it preserves the energy balance at the discrete level. This is done, however, at the expense of a loss of accuracy with the simulation time.

The second approach is numerically more efficient and provides uniform control of the accuracy with respect to the simulation time. The idea of this approach is inspired by the work in [1] and follows the detailed development in [2]: the convolution integral is split into a local part and a historical part, where for the latter one can exploit the exponential decay

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of the Laplace kernels to choose quadrature rules that provide uniform error bounds in time. Essentially, the number of quadrature points in the Laplace domain is $O(-\log(\Delta t))$ where Δt denotes the time step. Thus, if M is the number of time steps, the numerical scheme we propose requires $O(M\log(M))$ work and $O(\log(M))$ memory, compared to $O(M^2)$ work and $O(M)$ memory of a naive discretization. We note that even though the second scheme is more efficient and seems to be numerically stable under the standard CFL condition, no discrete energy balance has been yet found for it. Note that, for half order derivative, an error of $O(h^2 + \Delta t^{3/2})$ has been observed.

The paper is organized as follows. In Section 1, the model is briefly presented. We recall in Section 2 diffusive representations and obtain a semi-group formulation. In Section 3 the first numerical scheme is presented and the stability via the energy method is proved. In Section 4, an optimized quadrature is introduced in order to compute fractional derivatives with an *a priori* error estimate. The latter is used on the model in Section 1 to design a second numerical scheme, then we present numerical results showing the performance of the schemes.

1. Model under study

We focus on the one-dimensional Webster–Lokshin model in a simplified form

$$\partial_t^2 w + a(z)\partial_t^{2-\beta} w + b(z)\partial_t w - \frac{1}{r^2(z)}\partial_z(r^2(z)\partial_z w) = 0, \tag{1}$$

for $t > 0$ and $z \in [0, 1]$, where $\beta \in (0, 1)$, $r, a, b \in L^\infty([0, 1]; \mathbb{R}^+)$; the radius of the duct satisfies $r \geq r_0 > 0$. The Riemann–Liouville fractional integral operator $\partial_t^{-\beta}$ is defined by

$$(\partial_t^{-\beta} f)(t) := \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} \mathrm{d}\tau. \tag{2}$$

Working with $(p, v) := (\partial_t w, -r^2(z)\partial_z w)$ leads to the first order system:

$$\partial_t p = -r^{-2}\partial_z v - bp - a\partial_t^{-\beta}(\partial_t p), \tag{3}$$

$$\partial_t v = -r^2 \partial_z p, \tag{4}$$

which we supplement with the boundary conditions:

$$p_0(t) := p(z = 0, t) = u(t), \tag{5}$$

$$v_1(t) := v(z = 1, t) = 0. \tag{6}$$

We assume initial values and are interested in the relation between input $u(t) = p(z = 0, t)$ and output $y(t) = p(z = 1, t)$.

More general models allow for a $c(z)\partial_t^{1-\beta'} w$ term in (1), as well as boundary conditions of the impedance type instead of (5)–(6), both at $z = 0$ and $z = 1$. The additional term can be treated in a similar manner as $\partial_t^{2-\beta} w$.

2. Diffusive representation, semi-group formulation and well-posedness

This section recalls first known results on diffusive representations (we refer to [3] and [4, Section 5] for the treatment of *completely monotone kernels*, and [5] for links between diffusive representations and fractional integral and differential operators). We then present their application to our model problem and some theoretical results on the well-posedness of the obtained coupled system (proofs of these results are available in [6]).

2.1. Diffusive representations

The main idea is the representation of the convolution kernel as an integral of a family of decaying exponentials with respect to a positive measure. In other words, we utilize the following identity:

$$\frac{1}{\Gamma(\beta)} \frac{1}{(t-\tau)^{1-\beta}} = G_\beta \int_0^\infty \mathrm{e}^{-\xi(t-\tau)} \xi^{-\beta} \mathrm{d}\xi, \tag{7}$$

where $G_\beta := \frac{1}{\Gamma(\beta)\Gamma(1-\beta)} = \frac{\sin\beta\pi}{\pi}$, which in essence states that the convolution kernel in (2) is the Laplace transform of $G_\beta \xi^{-\beta}$.

One can easily check that the dynamical system with input $f \in L^2([0, T])$ and output $\theta^{[\beta]}(f) \in L^2([0, T])$:

$$\partial_t \varphi(\xi, t) = -\xi\varphi(\xi, t) + f(t), \quad \forall \xi \in \mathbb{R}^+, \tag{8}$$

$$\theta^{[\beta]}(f)(t) = G_\beta \int_0^\infty \varphi(\xi, t)\xi^{-\beta} \mathrm{d}\xi, \tag{9}$$

given $\varphi(\xi, 0) = 0$, provides a (diffusive) representation of the fractional integral $\partial_t^{-\beta}$; in other words, (8) and (9) realizes the input–output relation $\theta^{[\beta]}(f) = \partial_t^{-\beta} f$. It is clear that the state φ is such that $E_\varphi := \frac{G_\beta}{2} \int_0^\infty |\varphi|^2 \xi^{-\beta} \mathrm{d}\xi < \infty$.

Similarly, if $f \in H^1([0,T])$, the same dynamical system (8) with output:

$$\widetilde{\theta}^{[1-\beta]}(f)(t) := G_\beta \int_0^\infty [f(t) - \xi\varphi(\xi,t)]\,\xi^{-\beta}\mathrm{d}\xi, \tag{10}$$

given $\varphi(\xi,0) = 0$, provides a diffusive representation of the fractional derivative $\partial_t^{1-\beta}$. Here, the state φ is such that $\widetilde{E}_\varphi := \frac{G_\beta}{2}\int_0^\infty |\varphi|^2 \xi^\beta \mathrm{d}\xi$. Note that one has

$$\partial_t^{1-\beta} f = \widetilde{\theta}^{[1-\beta]}(f) = \theta^{[\beta]}(\partial_t f). \tag{11}$$

2.2. Rewriting the model as a coupled system

Using (8)–(10), the global system (3)–(6) can be put in the abstract form of a boundary control system (see e.g. [7]): $\partial_t X + \mathcal{A}X = 0$ and $\mathcal{B}X = u$, with $X = (p, v, \varphi)^{\mathrm{T}}$. The key point to the theoretical analysis of this boundary control system is the following energy balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^1 |p(z,t)|^2 r^2(z)\mathrm{d}z + \frac{1}{2}\int_0^1 |v(z,t)|^2 r^{-2}(z)\mathrm{d}z + \int_0^1 a(z)\widetilde{E}_\varphi(z,t) r^2(z)\mathrm{d}z\right)$$
$$= -\int_0^1 \|p - \xi\varphi\|^2_{H_\beta} a(z) r^2(z)\mathrm{d}z - \int_0^1 |p|^2 b(z) r^2(z)\mathrm{d}z + v(z=0,t)u(t), \tag{12}$$

with H_β defined in the Appendix. The natural *energy space* is the following Hilbert space:

$$\mathcal{H} = L_p^2 \times L_v^2 \times L^2(0,1;\widetilde{H}_\beta; ar^2\mathrm{d}z). \tag{13}$$

Then, the operator $\mathcal{A}$ is maximal monotone, and for the uncontrolled problem, we get the following result (see the Appendix).

Theorem 2.1. *$\forall X_0 \in D(\mathcal{A})$, $\exists! X(t) \in C^1([0,+\infty[;\mathcal{H}) \bigcap C^0([0,+\infty[; D(\mathcal{A}))$ such that $\partial_t X + \mathcal{A}X = 0$ on $[0,+\infty[$, and $X(0) = X_0$.*

Taking the control u into account, one can get the same regularity for the strong solutions under the hypothesis $u \in C^0([0,T])$. In the case $u \in L^2(0,T)$ and $X_0 \in \mathcal{H}$, only weak solutions can be found.

3. A stable numerical scheme

Let Δt and $h = 1/N$ be, respectively, the time and spatial step sizes, N is the number of discretization points of $[0, 1]$. We set $z_i = ih$, $z_{i+\frac{1}{2}} = (i+\frac{1}{2})h$ and denote

$$p_i^n \approx p(ih, n\Delta t); \qquad v_{i+\frac{1}{2}}^{n+\frac{1}{2}} \approx v\left(\left(i+\frac{1}{2}\right)h, (n+\frac{1}{2})\Delta t\right); \qquad \tilde{\theta}_i^n \approx \tilde{\theta}(ih, n\Delta t).$$

Then a second order centered explicit scheme associated with (3) and (4) can be written as

$$\frac{p_i^{n+1} - p_i^n}{\Delta t} = -\frac{1}{r^2(z_i)}\frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{h} - b(z_i)\frac{p_i^{n+1} + p_i^n}{2} - a(z_i)(\partial_t^{1-\beta} p_i)^{n+\frac{1}{2}}, \tag{14}$$

for $n > 0$ and $0 < i \le N$ and

$$\frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t} = -r^2\left(z_{i+\frac{1}{2}}\right)\frac{p_{i+1}^n - p_i^n}{h}, \tag{15}$$

where $(\partial_t^{1-\beta} p_i)^{n+\frac{1}{2}}$ is an approximation of $(\partial_t^{1-\beta} p_i)$ at $t = (n+\frac{1}{2})\Delta t$. Motivated by the energy balance of the continuous model (12), we choose here to use the identity, $\partial^{1-\beta}(p_i) = \widetilde{\theta}^{[1-\beta]}(p_i)$ and set

$$(\partial_t^{1-\beta} p_i)^{n+\frac{1}{2}} = \left(\tilde{\theta}_i^{n+1} + \tilde{\theta}_i^n\right)/2,$$

where $\tilde{\theta}_i^n$ is an approximation of $\widetilde{\theta}^{[1-\beta]}(p_i)(n\Delta t)$. This approximation requires the evaluation of the integral in (10). Let $(\xi_j, \rho_j(\beta))_{1\le j\le N_\xi}$ some quadrature rule associated with the evaluation of $G_\beta \int_0^\infty h(\xi)\xi^{-\beta}\mathrm{d}\xi$. Then we set

$$\tilde{\theta}_i^n = \sum_{j=1}^{N_\xi} \rho_j(\beta)(p_i^n - \xi_j \varphi_{i,j}^n), \tag{16}$$

where $\varphi_{i,j}^n \approx \varphi(\xi_j, z_i, n\Delta t)$ satisfies

$$\varphi_{i,j}^{n+1} = \mathrm{e}^{-\xi_j \Delta t}\varphi_{i,j}^n + \frac{1 - \mathrm{e}^{-\xi_j\Delta t}}{\xi_j}\frac{p_i^{n+1} + p_i^n}{2}, \tag{17}$$

which has been derived from the expression $\varphi(\xi, z, t) = \int_0^t e^{-\xi(t-s)} p(z, s) ds$ by exact integration of the right hand side between $n\Delta t$ and $(n+1)\Delta t$.

Based on Bode diagrams for fractional integrals, a possible and commonly used choice of the nodes ξ_j is a *geometric* grid of the ξ axis defined by a lower bound ξ_m, an upper bound ξ_M, and the number of points N_ξ; in this case, we define $\xi_j = \left(\frac{\xi_M}{\xi_m}\right)^{\frac{j-1}{N_\xi - 1}} \xi_m$,; $j = 1, \dots, N_\xi$. The weights $\rho_j(\beta)$ are obtained for instance by applying the trapezoidal rule. Choosing N_ξ, together with ξ_m and ξ_M is indeed an issue, which has been studied in e.g. [8].

The following stability lemma holds for *any* choice of quadrature of the form (16) with positive quadrature weights $\rho_j(\beta)$.

Let $\gamma = \max\left\{ r(z_{i+\frac{1}{2}})/r(z_{i+1}), r(z_{i+\frac{1}{2}})/r(z_i), i = 1, \dots, N-1 \right\}$ then we have:

Lemma 3.1. *The scheme defined by (14)–(17) is L^2-stable under the CFL condition: $\gamma \Delta t < h$.*

Proof. Without loss of generality we shall consider the case of $b = 0$ and study the scheme with $p_0 = 0$, i.e. with *no* input. Let us introduce a so-called discrete wave energy associated with our scheme as defined by

$$E^n = \frac{h}{2} \left(\sum_{i=0}^{N-1} \left| r(z_i) p_i^n \right|^2 + \frac{1}{r\left(z_{i+\frac{1}{2}}\right)^2} v_{i+\frac{1}{2}}^{n+\frac{1}{2}} v_{i+\frac{1}{2}}^{n-\frac{1}{2}} \right) + \frac{h}{4} \left| r(z_N) p_N^n \right|^2. \tag{18}$$

Then using (15), it is easy to see that there exists a constant $C > 0$, independent of n, such that

$$E^n \geq C \sum_{i=1}^{N} \left| r(z_i) p_i^n \right|^2 + \left| \left(v_{i-\frac{1}{2}}^{n+\frac{1}{2}} + v_{i-\frac{1}{2}}^{n-\frac{1}{2}} \right) \Big/ 2r\left(z_{i-\frac{1}{2}}\right) \right|^2.$$

It is then sufficient to prove that E^n remains bounded with respect to n. Denoting $\omega_j := \frac{\xi_j \Delta t}{2} \frac{1+e^{-\xi_j \Delta t}}{1-e^{-\xi_j \Delta t}}$, let us introduce now

$$\widetilde{E}_\varphi^n := \frac{1}{2} \sum_{j=1}^{N_\xi} \rho_j(\beta) \omega_j \xi_j \left(\sum_{i=1}^{N-1} a(z_i) |\varphi_{i,j}^n|^2 + \frac{1}{2} a(z_N) |\varphi_{N,j}^n|^2 \right) \tag{19}$$

as the energy associated with the derivative term $\partial_t^{1-\beta}$. After some careful manipulations, see [6, ch. 3], one obtains the following balance:

$$\begin{aligned} \frac{E^{n+1} - E^n}{\Delta t} &= -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} - \frac{\widetilde{E}_\varphi^{n+1} - \widetilde{E}_\varphi^n}{\Delta t} \\ &\quad - \sum_{j=1}^{N_\xi} \rho_j(\beta) \omega_j^2 \left(\sum_{i=1}^{N-1} a(z_i) \left| \dot{\varphi}_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2} a(z_N) \left| \dot{\varphi}_{N,j}^{n+\frac{1}{2}} \right|^2 \right), \end{aligned} \tag{20}$$

with the short-hand notation $\dot{\varphi}_{i,j}^{n+\frac{1}{2}} := \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^n}{\Delta t}$. Therefore, the total energy $\mathcal{E}^n := E^n + \widetilde{E}_\varphi^n$, is non-increasing with n. Hence E^n remains uniformly bounded with respect to n and the L^2 stability is proved under the CFL $\gamma \Delta t < h$. □

In summary, mimicking the energy balances of the continuous model, a numerical scheme has been designed, which proves to be stable under a standard CFL condition. One drawback of this scheme is that no error bounds are provided with respect to quadrature rules of the form (16), which makes it difficult to use. The following section is dedicated to the design of efficient and accurate computation of (9) or (10) independently from stability analysis of the coupled scheme.

4. Optimized weights for diffusive representations

We follow, with slight modifications, the approach of [1] where the special case $\beta = \frac{1}{2}$ was considered. Detailed exposition of this approach appears in [2].

Due to the fact that as τ approaches t, the support of $e^{-\xi(t-\tau)}$ becomes infinite, we write $\theta^{[\beta]}(f)(t)$ as the sum of a local (in time) part:

$$\theta^{[\beta,\mathrm{loc}]}(f)(t) = \frac{1}{\Gamma(\beta)} \int_{t-\Delta t}^{t} \frac{1}{(t-\tau)^{1-\beta}} f(\tau) d\tau, \tag{21}$$

and a historical part:

$$\theta^{[\beta,\mathrm{hist}]}(f)(t) = G_\beta \int_0^\infty \int_0^{t-\Delta t} e^{-\xi(t-\tau)} \xi^{-\beta} f(\tau) d\tau d\xi. \tag{22}$$

We use for $\theta^{[\beta,\mathrm{loc}]}(f)(t)$ the approximation :

$$\theta^{[\beta,\mathrm{loc}]}(f)(t) \approx \frac{1}{\Gamma(\beta)} f\left(t-\frac{\Delta t}{2}\right)\frac{\Delta t^{\beta}}{\beta}, \tag{23}$$

and build a quadrature in ξ for

$$\int_0^{\infty} \mathrm{e}^{-\xi(t-\tau)}\xi^{-\beta}\mathrm{d}\xi \approx \sum_{j=1}^{M} \mathrm{e}^{-\xi_j(t-\tau)}\xi_j^{-\beta} w_j, \tag{24}$$

where the w_j's are the weights of the quadrature rule, which will be accurate for all $\tau \in [0, t-\Delta t]$. Removing the integrable singularity $\xi^{-\beta}$ at the origin will make the design of the quadrature simpler, so we make the change of variables, $\gamma = \frac{1}{1-\beta}$, $\eta = \xi^{\frac{1}{\gamma}}$, to obtain

$$\int_0^{\infty} \mathrm{e}^{-\xi(t-\tau)}\xi^{-\beta}\mathrm{d}\xi = \int_0^{\infty} \gamma \mathrm{e}^{-(\eta^{\gamma}(t-\tau))}\mathrm{d}\eta.$$

In essence, we will construct a quadrature with an error tolerance of ϵ:

$$\left|\int_0^{\infty} \mathrm{e}^{-(\eta^{\gamma}\tau)}\mathrm{d}\eta - \sum_{j=1}^{L^{\Delta t,\epsilon}} \mathrm{e}^{-[(\eta_j^{\Delta t,\epsilon})^{\gamma}\tau]} w_j^{\Delta t,\epsilon},\right| \le \epsilon, \tag{25}$$

valid for all $\tau \in [\Delta t, t_f]$, where we have used the superscripts Δt and ϵ to indicate that the position, weight, and the number of quadrature nodes are dependent on these quantities.

First, we reduce the domain of integration to a finite interval. From the relation

$$\int_{\eta_f}^{\infty} \mathrm{e}^{-\eta^{\gamma}\tau}\mathrm{d}\eta \le \frac{\mathrm{e}^{-(\eta_f{}^{\gamma}\Delta t)}\Gamma(\frac{1}{\gamma})}{\gamma \Delta t^{(\frac{1}{\gamma})}}$$

we find that choosing $\eta_f = \left(\frac{-\log\frac{\gamma\epsilon/3\Delta t^{l}}{\Gamma(l)}}{\Delta t}\right)^{l}$, $l = \frac{1}{\gamma} = 1-\beta$, ensures that

$$\left|\int_0^{\infty} \mathrm{e}^{-\eta^{\gamma}\tau}\mathrm{d}\eta - \int_0^{\eta_f} \mathrm{e}^{-\eta^{\gamma}\tau}\mathrm{d}\eta\right| \le \frac{\epsilon}{3}. \tag{26}$$

The parameter τ varies from Δt to t_f and we need to construct a single quadrature which accurately approximates the integral for this one-parameter family of integrands. Over any fixed subinterval in $[a, b]$, the integrand $\mathrm{e}^{-\eta^{\gamma}\tau}$ varies from identically 1 to identically 0. A quadrature must approximate accurately this range of behavior. It is not difficult to see that the region of the most rapid range in the integrand $\mathrm{e}^{-\eta^{\gamma}\tau}$ occurs at the inflection point, $\eta_i = \left(\frac{\gamma-1}{\gamma\tau}\right)^{\frac{1}{\gamma}}$.

To capture the clustering of support of the integrands toward $\eta = 0$ as τ becomes larger, we follow the development in [1] and use Gauss–Legendre quadrature points on dyadic intervals.

On a dyadic interval $[a, b]$, we choose the smallest order Gauss–Legendre quadrature which accurately computes the integral to a tolerance $\frac{\epsilon}{3(b-a)}$. In other words, we choose the smallest L^a such that given the Gauss–Legendre quadrature on $[a, b]$, $\eta_1^a, \ldots, \eta_{L^a}^a$, $w_j^a, \ldots, w_{L^a}^a$, of order L^a, the following is satisfied:

$$\left|\int_a^{b} \mathrm{e}^{-\eta^{\gamma}\tau}\mathrm{d}\eta - \sum_{j=1}^{L^a} \mathrm{e}^{-(\eta_j^a)^{\gamma}\tau} w_j^a,\right| \le \frac{\epsilon}{3(b-a)} \tag{27}$$

for all $\tau \in [\tau_{\min}, \tau_{\max}]$, where $\tau_{\min}$ and $\tau_{\max}$ are chosen by solving the following equations:

$$\mathrm{e}^{-a^{\gamma}\tau_{\max}} = q,$$
$$\mathrm{e}^{-b^{\gamma}\tau_{\min}} = 1-q.$$

The number q is a small factor to indicate that the support of $\tau > \tau_{\max}$ is mostly outside of $[a, b]$ and that for $\tau < \tau_{\min}$ the integrand is almost identically 1. Hence, the only relevant τ for which the tolerance in (27) needs to be tested is between $\tau_{\min}$ and $\tau_{\max}$. The number of quadrature points needed on $[a, b]$ is determined numerically by testing (27) for a range of values of τ in $[\tau_{\min}, \tau_{\max}]$.

If a t_f is chosen, then we solve for the largest $a_{\min}$ of the form $a_{\min} = 2^{j_{\min}}$ satisfying

$$\mathrm{e}^{-a_{\min}^{\gamma}t_f} \le q, \tag{28}$$

i.e., $\mathrm{e}^{-\eta^{\gamma}t_f}$ is negligible outside of $[0, a_{\min}]$, and we treat the interval $[0, a_{\min}]$ like explained for the interval $[a, b]$, by numerically satisfying (27). But the number $\tau_{\max}$ is now simply t_f. Clearly, if $a_{\min} \le \frac{\epsilon}{3}$ this interval can be neglected. Thus, if

Table 1
Number of quadrature nodes $L^{\Delta t,\epsilon}$ for $\beta = \frac{1}{2}$. The first number is $L^{\Delta t,\epsilon}$ for $t_f = 10$, the second (in parenthesis) is $L^{\Delta t,\epsilon}$ for $t_f = \infty$.

Δt	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$
10^{-1}	9 (13)	12 (22)	22 (54)
10^{-2}	16 (18)	28 (48)	28 (60)

Table 2
Relative L^2-error of $t \to p(1,t)$ in terms of Δt. A linear regression of the logarithmic values indicates a slope of 1.6. The number of quadrature points corresponds to $\epsilon = 10^{-4}$.

Δt	10^{-2}	$10^{-2}/2$	$10^{-2}/4$	$10^{-2}/8$
$L^{\Delta t,\epsilon}(t_f = 20)$	30	52	58	60
Error	0.0716	0.0349	0.0138	0.0023

$t_f = \infty$ is chosen, the smallest dyadic interval to be treated is $[2^{j_{\min}}, 2^{j_{\min}+1}]$ where $j_{\min}$ is the largest integer satisfying $2^{j_{\min}} < \frac{\epsilon}{3}$.

The number of required quadrature nodes, $L^{\Delta t,\epsilon}$, for a given Δt and ϵ, is $O(-\log \Delta t, -\log \epsilon)$. In Table 1, $L^{\Delta t,\epsilon}$ for $\beta = \frac{1}{2}$ are given for two values of t_f: $t_f = 10$ and $t_f = \infty$.

5. Modified scheme with splitting between local and historical parts

Motivated by the analysis of the previous section, we choose here to use the identity, $\partial_t^{1-\beta}(p_i) = \theta^{[\beta]}(\partial_t p_i)$ and use the splitting of $\theta^{[\beta]}(f)$ in local and historical parts to compute an approximation of it. Hence we propose to couple scheme (14)-(15) with

$$(\partial_t^{1-\beta} p_i)^{n+\frac{1}{2}} = \frac{\Delta t^\beta}{\Gamma(\beta)\beta 2\Delta t}(p_i^{n+1} - p_i^{n-1}) + G_\beta \sum_{j=1}^{L^{\Delta t,\epsilon}} w_j \xi_j^{-\beta} \mathrm{e}^{-\xi_j \Delta t}(\tilde{\varphi}_{i,j}^n + \tilde{\varphi}_{i,j}^{n-1})/2,$$

where $(\xi_j, w_j)_{j=1,\ldots,L^{\Delta t,\epsilon}}$ is the quadrature associated with β and a given tolerance ϵ as derived in Section 4, and

$$\tilde{\varphi}_{i,j}^n = \mathrm{e}^{-\xi_j \Delta t}\tilde{\varphi}_{i,j}^{n-1} + \mathrm{e}^{-\xi_j \Delta t/2}(p_i^n - p_i^{n-1}), \tag{29}$$

which is the centered approximation associated with $\partial_t(\mathrm{e}^{\xi t}\tilde{\varphi}) = \mathrm{e}^{\xi t}\partial_t p$.

Unfortunately, the stability analysis derived in Section 3 does not apply to this scheme and this is mainly due to the shift introduced in the evaluation of the historical part. Let us notice however that numerical experiments suggest that the CFL is the same as for the first scheme.

6. Validation on a reference case

We conclude this work with a numerical validation in the case of the *Lokshin* model, i.e., when $r(z) = r_0$ (the duct is a cylinder) and $b = (a/2)^2$. In this case, a closed-form solution is available in both time and frequency domains (see [9]). In the frequency domain, the Laplace transforms satisfy $P(z = 1, s) = H(s)U(s)$, where,

$$H(s) \propto \frac{\mathrm{e}^{-\Upsilon(s)}}{1 + \mathrm{e}^{-2\Upsilon(s)}} = \sum_{k=0}^{\infty}(-1)^k \mathrm{e}^{-(2k+1)\Upsilon(s)}, \quad \text{for } \Re e(s) > 0,$$

with $\Upsilon(s) = s + \frac{a}{2}\sqrt{s}$. Hence, in the time domain, the output $p(z = 1, t) = h \star u$ can be decomposed into *wave-trains*, due to the fact that

$$h(t) = \sum_{k=0}^{\infty}(-1)^k \psi^{(2k+1)a}(t - (2k+1)),$$

where $\psi^a(t) = \frac{a}{2\sqrt{\pi}t^{3/2}}\exp(-\frac{a^2}{4t})$ for $t > 0$.

The following numerical experiments correspond to the use of the second scheme (Section 5) and to $a = 0.4$, $\beta = 1/2$, and $r_0 = 1$. We used a Gaussian incident pulse and applied a CFL coefficient $(\Delta t/h) = 0.99$.

Fig. 1 shows very good agreements between exact and numerical solutions in the Fourier domain: this experiment correspond to $\varepsilon = 10^{-3}$, $t_f = \infty$ and $\Delta t = 10^{-2}$ which requires the use of 60 quadrature nodes (see Table 1). The time domain response of exact and numerical solutions almost coincide and therefore only one plot is shown in Fig. 1 right. Table 2 indicates how the expected accuracy $O(\Delta t^{3/2})$ is achieved by our numerical scheme with a few quadrature points.

7. Conclusion and perspectives

Based on diffusive representations, two numerical schemes have been derived for a fractional wave equation, one is stable with no error bound, whereas the second is efficient with no proof of stability.

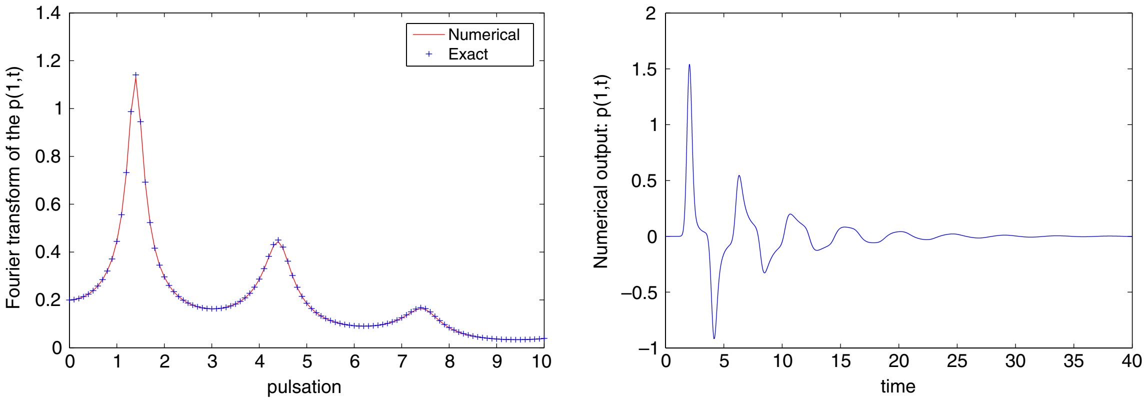

Fig. 1. Frequency domain response (left) and time domain response (right).

The theoretical and numerical work has been carried out here on a simplified model with a fractional derivative only, and Dirichlet and Neumann boundary conditions. It is also possible to incorporate both in the continuous model and its discretized versions a fractional integral for the internal damping, together with more realistic boundary conditions of the impedance type, thus introducing some external damping, as was done in [6].

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Appendix. Functional spaces

For the pressure variable, we denote $L^2_p := L^2(0, 1; r^2(z)\mathrm{d}z)$ and

$$H^1_p := \left\{ p \in L^2_p, \int_0^1 \left[p^2 + (\partial_z p)^2\right] r^2(z)\mathrm{d}z < +\infty \right\}.$$

For the velocity variable, we denote $L^2_v := L^2(0, 1; r^{-2}(z)\mathrm{d}z)$ and

$$H^1_v := \left\{ v \in L^2_v, \int_0^1 [v^2 + (\partial_z v)^2] r^{-2}(z)\mathrm{d}z < +\infty \right\}.$$

For the diffusive variable, we denote $H_\beta = L^2(\mathbb{R}^+, G_\beta \xi^{\beta-1}\mathrm{d}\xi)$, $\widetilde{H}_\beta = L^2(\mathbb{R}^+, G_\beta \xi^{\beta}\mathrm{d}\xi)$, and $V_\beta = L^2\left(\mathbb{R}^+, G_\beta (1+\xi)\xi^{-\beta}\mathrm{d}\xi\right)$.

For the state $X = (p, v, \varphi)^{\mathrm{T}}$, the natural *energy space* is: $\mathcal{H} = L^2_p \times L^2_v \times L^2(0, 1; \widetilde{H}_\beta; ar^2\mathrm{d}z)$, and with $\mathcal{V} = H^1_p \times H^1_v \times L^2(0, 1; \widetilde{H}_\beta; ar^2\mathrm{d}z) \subset \mathcal{H}$, the domain of $\mathcal{A}$ is:

$$D(\mathcal{A}) = \left\{ X \in \mathcal{V}, \left| p(z=0) = 0, v(z=1) = 0, (p - \xi\varphi) \in L^2(0, 1; V_\beta; ar^2\mathrm{d}z) \right.\right\}.$$

The *monotonicity* of $\mathcal{A}$ follows from the identity:

$$\forall X \in D(\mathcal{A}), \quad (\mathcal{A}X, X)_{\mathcal{H}} = \int_0^1 \|p - \xi\varphi\|^2_{H_\beta} ar^2\mathrm{d}z + \int_0^1 |p|^2 br^2 \mathrm{d}z \geq 0.$$

Moreover, from [10], with full proofs in [6, ch. 2], the operator $\mathcal{A}$ is maximal monotone; thus, due to the Hille–Yosida theorem, the original problem is well-posed, and we get Theorem 2.1.

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