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Analytical solutions

Green's functions/fundamental solutions

Jing-Rebecca Li
Equipe DEFI CMAP, Ecole Polytechnique
Institut national de recherche en informatique et en automatique (INRIA) Saclay

Many exact solutions

http://eqworld.ipmnet.ru/en/solutions/lpde.htm

Linear Partial Differential Equations of Mathematical Physics

Second-Order Parabolic Partial Differential Equations
Second-Order Hyperbolic Partial Differential Equations
Second-Order Elliptic Partial Differential Equations
Other Second-Order Partial Differential Equations
Higher-Order Partial Differential Equations

Heat equation

1. The Fundamental solution

As we will see, in the case $\Omega = \mathbb{R}^n$, we will be able to represent general solutions the inhomogeneous heat equation

$$u_t - D\Delta u = f, \qquad \Delta \stackrel{\text{def}}{=} \sum_{i=1}^{n} \partial_i^2 \tag{1.0.1}$$

in terms of f, the initial data, and a single solution that has very special properties. This special solution is called the *fundamental solution.*

Remark 1.0.1. Note that when $\Omega = \mathbb{R}^n$, there are no finite boundary conditions to worry about. However, we do have to worry about "boundary conditions at ∞." Roughly speaking, this means that we have to assume something about the growth rate of the solution as $|x| \to \infty$.

Definition 1.0.1. The fundamental solution $\Gamma_D(t,x)$ to (1.0.1) is defined to be

$$\Gamma_D(t,x) \stackrel{\text{def}}{=} \frac{1}{(4\pi Dt)^{n/2}} e^{-\frac{|x|^2}{4Dt}}, \qquad t > 0, x \in \mathbb{R}^n, \tag{1.0.2}$$

where $x \stackrel{\text{def}}{=} (x^1, \cdots, x^n)$, $|x|^2 \stackrel{\text{def}}{=} \sum_{i=1}^n (x^i)^2$.

Let's check that $\Gamma_D(t,x)$ solves (1.0.1) when $f = 0$ in the next lemma.

Heat equation

Lemma 1.0.1. *$\Gamma_D(t,x)$ is a solution to the heat equation (1.0.1) when $f=0$ for $x \in \mathbb{R}^n, t>0$.*

Proof. We compute that $\partial_t \Gamma_D(t,x) = \left(-\frac{2\pi D n}{(4\pi D t)^{n/2+1}} + \frac{1}{(4\pi D t)^{n/2}} \frac{|x|^2}{4Dt^2}\right) e^{-\frac{|x|^2}{4Dt}}$. Also, we compute $\partial_i \Gamma_D(t,x) = -\frac{2\pi x^i}{(4\pi D t)^{n/2+1}} e^{-\frac{|x|^2}{4Dt}}$ and $\partial_i^2 \Gamma_D(t,x) = \left(-\frac{2\pi}{(4\pi D t)^{n/2+1}} + \frac{1}{4Dt} \frac{2\pi (x^i)^2}{(4\pi D t)^{n/2+1}}\right) e^{-\frac{|x|^2}{4Dt}}$,
$D\Delta \Gamma_D(t,x) = \left(-\frac{2\pi D n}{(4\pi D t)^{n/2+1}} + \frac{1}{4Dt} \frac{2\pi D |x|^2}{(4\pi D t)^{n/2+1}}\right) e^{-\frac{|x|^2}{4Dt}}$. Lemma 1.0.1 now easily follows. □

Here are a few very important properties of $\Gamma_D(t,x)$.

Lemma 1.0.2. *$\Gamma_D(t,x)$ has the following properties:*

(1) *If $x \neq 0$, then* $\lim_{t\to 0^+} \Gamma_D(t,x) = 0$
(2) $\lim_{t\to 0^+} \Gamma_D(t,0) = \infty$
(3) $\int_{\mathbb{R}^n} \Gamma_D(t,x)\, d^n x = 1$ *for all $t>0$*

Proof. This is a good exercise for you to do on your own. □

As we will see, (1) - (3) suggest that at $t=0$, $\Gamma_D(0,x)$ behaves like the "delta distribution centered at 0." We'll make sense of this in the next lemma.

Heat equation

Definition 1.0.2. The delta distribution δ is an example of a mathematical object called a *distribution*. It acts on suitable functions $\phi(x)$ as follows:

$$\langle \delta, \phi \rangle \stackrel{\text{def}}{=} \phi(0). \tag{1.0.3}$$

Remark 1.0.3. The notation $\langle \cdot, \cdot \rangle$ is meant to remind you of the L^2 inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g(x)\, d^n x. \tag{1.0.4}$$

The next lemma shows that $\Gamma_D(t,x)$ behaves like the delta distribution as $t \to 0^+$.

Lemma 1.0.3. *Suppose that $\phi(x)$ is a continuous function on $\mathbb{R}^n$ and that there exist constants $a, b \geq 0$ such that*

$$|\phi(x)| \leq a e^{b|x|^2}. \tag{1.0.5}$$

Then

$$\lim_{t \to 0^+} \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)\, d^n x = \phi(0). \tag{1.0.6}$$

Heat equation

Definition 1.1.1. If f and g are two functions on $\mathbb{R}^n$, then we define their convolution $f * g$ to be the following function on $\mathbb{R}^n$:

$$(f * g)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(y)g(x-y)\, d^n y. \tag{1.1.2}$$

Convolution is an averaging process, in which the function $f(x)$ is replaced by the "average value" of $f(x)$ relative to the "profile" function $g(x)$.

The convolution operator plays a very important role in many areas of mathematics. Here are two key properties. First, by making the change of variables $z = x - y$, $d^n z = d^n y$ in (1.1.2), we see that

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\, d^n y = \int_{\mathbb{R}^n} f(x-z)g(z)\, d^n z = (g * f)(x), \tag{1.1.3}$$

which implies that convolution is a *commutative* operation. Next, Fubini's theorem can be used to show that

$$f * (g * h) = (f * g) * h, \tag{1.1.4}$$

so that $*$ is also associative.

Heat equation

Theorem 1.1 (Solving the global Cauchy problem via the fundamental solution). *Assume that $g(x)$ is a continuous function on $\mathbb{R}^n$ that verifies the bounds $|g(x)| \le ae^{b|x|^2}$, where $a, b > 0$ are constants. Then there exists a solution $u(t,x)$ to the homogeneous heat equation*

(1.1.10)
$$u_t - D\Delta u = 0, \qquad (t > 0, x \in \mathbb{R}^n),$$
$$u(0,x) = g(x), \qquad x \in \mathbb{R}^n$$

existing for $(t,x) \in [0,T) \times \mathbb{R}^n$, where

(1.1.11)
$$T \stackrel{def}{=} \frac{1}{4Db}.$$

Furthermore, $u(t,x)$ can be represented as

(1.1.12)
$$u(t,x) = [g(\cdot) * \Gamma_D(t,\cdot)](x) = \int_{\mathbb{R}^n} g(y)\Gamma_D(t, x-y)\, d^n y$$
$$= \frac{1}{(4\pi Dt)^{n/2}} \int_{\mathbb{R}^n} g(y) e^{-\frac{|x-y|^2}{4Dt}}\, d^n y.$$

Heat equation

The solution $u(t,x)$ *is of regularity* $C^\infty((0,\frac{1}{4Db})\times\mathbb{R}^n)$ *(i.e., it is infinitely differentiable). Finally, for each compact subinterval* $[0,T'] \subset [0,T)$, *there exist constants* $A, B > 0$ *(depending on the compact subinterval) such that*

$$|u(t,x)| \le Ae^{B|x|^2} \qquad (1.1.13)$$

for all $(t,x) \in [0,T'] \times \mathbb{R}^n$. *The solution* $u(t,x)$ *is the unique solution in the class of functions verifying a bound of the form* (1.1.13).

Remark 1.1.3. Note the very important **smoothing property** of diffusion: the solution to the heat equation on all of $\mathbb{R}^n$ is *smooth* even if the data are merely *continuous*.

Remark 1.1.4. The formula (1.1.12) shows that solutions to (1.1.10) propagate with **infinite speed**: even if the initial data $g(x)$ have support that is contained within some compact region, (1.1.12) shows that at any time $t > 0$, the solution $u(t,x)$ has "spread out over the entire space $\mathbb{R}^n$." In contrast, as we will see later in the course, some important PDEs have finite speeds of propagation (for example, the wave equation).

Heat equation

In the next theorem, we extend the results of Theorem 1.1 to allow for an inhomogeneous term $f(t,x)$.

Theorem 1.2 (Duhamel's principle). *Let $g(x)$ and $T \stackrel{def}{=} \frac{1}{4Db}$ be as in Theorem 1.1. Also assume that $f(t,x)$, $\partial_i f(t,x)$, and $\partial_i \partial_j f(t,x)$ are continuous, bounded functions on $[0,T) \times \mathbb{R}^n$ for $1 \le i,j \le n$. Then there exists a unique solution $u(t,x)$ to the inhomogeneous heat equation*

$$
\begin{aligned}
u_t - D\Delta u &= f(t,x), \qquad (t,x) \in (0,\infty) \times \mathbb{R}, \\
u(0,x) &= g(x), \qquad x \in \mathbb{R}
\end{aligned}
\tag{1.1.19}
$$

existing for $(t,x) \in [0,T) \times \mathbb{R}$. Furthermore, $u(t,x)$ can be represented as

$$
u(t,x) = (\Gamma_D(t,\cdot) * g)(x) + \int_0^t (\Gamma_D(t-s,\cdot) * f(s,\cdot))(x)\, ds. \tag{1.1.20}
$$

The solution has the following regularity properties: $u \in C^0([0,T) \times \mathbb{R}) \cap C^{1,2}((0,T) \times \mathbb{R})$.

Proof. A slightly less technical version of this theorem is one of your homework exercises. □

Laplace equation

Definition 1.0.1. The fundamental solution Φ corresponding to the operator Δ is

$$\Phi(x) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2\pi} \ln |x| & n = 2, \\ -\frac{1}{\omega_n |x|^{n-2}} & n \geq 3, \end{cases} \tag{1.0.3}$$

where as usual $|x| \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} (x^i)^2}$ and ω_n is the surface area of a unit ball in $\mathbb{R}^n$ (e.g. $\omega_3 = 4\pi$).

In mathematics, a **fundamental solution** for a linear partial differential operator L is a formulation in the language of distribution theory of the older idea of a **Green's function**, which normally further addresses boundary conditions.

Analytical solutions/Green's functions

1.2. Nonhomogeneous Heat Equation $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \Phi(x,t)$

1.2-2. Solutions of boundary value problems in terms of the Green's function.

We consider boundary value problems for the heat equation* on an interval $0 \le x \le l$ with the general initial condition

$$w = f(x) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w(x,t) = \int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^t \int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau.$$

1.2. Nonhomogeneous Heat Equation $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \Phi(x,t)$

1.2-6. Domain: $0 \le x \le l$. First boundary value problem for the heat equation.

Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l.$$

Two forms of representation of the Green's function:

$$G(x,\xi,t) = \frac{2}{l}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right)$$
$$= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(x-\xi+2nl)^2}{4at}\right] - \exp\left[-\frac{(x+\xi+2nl)^2}{4at}\right]\right\}.$$

The first series converges rapidly at large t and the second series at small t.

1.2. Nonhomogeneous Heat Equation $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \Phi(x,t)$

1.2-7. Domain: $0 \le x \le l$. Second boundary value problem for the heat equation.

Boundary conditions are prescribed:

$$\frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0, \qquad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = l.$$

Two forms of representation of the Green's function:

$$G(x,\xi,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right)$$
$$= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(x-\xi+2nl)^2}{4at}\right] + \exp\left[-\frac{(x+\xi+2nl)^2}{4at}\right]\right\}.$$

The first series converges rapidly at large t and the second series at small t.

1.5. Heat Equation of the Form $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}\right) + \Phi(r,t)$

Nonhomogeneous heat (diffusion) equation with axial symmetry.

The two-dimensional Laplace equation has the following form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{in the Cartesian coordinate system,}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} = 0 \quad \text{in the polar coordinate system,}$$

where $x = r\cos\varphi$, $y = r\sin\varphi$, and $r = \sqrt{x^2 + y^2}$.

1.5. Heat Equation of the Form $\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \Phi(r,t)$

Nonhomogeneous heat (diffusion) equation with axial symmetry.

1.5-1. Solutions of boundary value problems in terms of the Green's function.

We consider boundary value problems for the nonhomogeneous heat equation with axial symmetry in domain $0 \le r \le R$ with the general initial condition

$$w = f(r) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions (the solutions bounded at $r = 0$ are sought). The solution can be represented in terms of the Green's function as

$$w(x,t) = \int_0^R f(\xi)G(r,\xi,t)\,d\xi + \int_0^t \int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau.$$

1.5. Heat Equation of the Form $\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \Phi(r,t)$

Nonhomogeneous heat (diffusion) equation with axial symmetry.

1.5-2. Domain: $0 \le r \le R$. First boundary value problem for the heat equation.

A boundary condition is prescribed:

$$w = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \sum_{n=1}^{\infty} \frac{2\xi}{R^2 J_1^2(\mu_n)} J_0\left(\mu_n \frac{r}{R}\right) J_0\left(\mu_n \frac{\xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$. Below are the numerical values of the first ten roots:

$$\mu_1 = 2.4048, \quad \mu_2 = 5.5201, \quad \mu_3 = 8.6537, \quad \mu_4 = 11.7915, \quad \mu_5 = 14.9309,$$
$$\mu_6 = 18.0711, \quad \mu_7 = 21.2116, \quad \mu_8 = 24.3525, \quad \mu_9 = 27.4935, \quad \mu_{10} = 30.6346.$$

The zeroes of the Bessel function $J_0(\mu)$ may be approximated by the formula

$$\mu_n = 2.4 + 3.13(n-1) \qquad (n = 1,\ 2,\ 3,\ \dots),$$

which is accurate within 0.3%. As $n \to \infty$, we have $\mu_{n+1} - \mu_n \to \pi$.

1.5. Heat Equation of the Form $\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \Phi(r,t)$

Nonhomogeneous heat (diffusion) equation with axial symmetry.

1.5-3. Domain: $0 \le r \le R$. Second boundary value problem for the heat equation.

A boundary condition is prescribed:

$$\frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \frac{2}{R^2}\xi + \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{\xi}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu)=0$. Below are the numerical values of the first ten roots:

$$\mu_1 = 3.8317, \quad \mu_2 = 7.0156, \quad \mu_3 = 10.1735, \quad \mu_4 = 13.3237, \quad \mu_5 = 16.4706,$$
$$\mu_6 = 19.6159, \quad \mu_7 = 22.7601, \quad \mu_8 = 25.9037, \quad \mu_9 = 29.0468, \quad \mu_{10} = 32.1897.$$

As $n \to \infty$, we have $\mu_{n+1} - \mu_n \to \pi$.

PAUSE

Time for a break