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# Types of PDEs
# and
# Galerkin discretization

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## Classification of Second-Order Equations in $n$ Variables

For more than three independent variables it is convenient to write the above PDE in the following form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0 \tag{42}$$

where the coefficients $a_{ij}$, $b_i$, $c$, $d$ are functions of $x = (x_1, x_2, \ldots, x_n)$, $u = u(x_1, x_2, \ldots, x_n)$, and $n$ is the number of independent variables. Equation (42) can be written in matrix form as

$$\begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} + \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} + cu + d = 0$$

We assume that the coefficient matrix $A = (a_{ij})$ to be symmetric. If $A$ is not symmetric, we can always find a symmetric matrix $\overline{a}_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ such that (42) can be rewritten as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{a}_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0$$

## Classification of Second-Order Equations in $n$ Variables

- Equation is called elliptic if all eigenvalues $\lambda_i$ of $A$ are non-zero and have the same sign.
- Equation is called hyperbolic if all eigenvalues $\lambda_i$ of $A$ are non-zero and have the same sign except for one of the eigenvalues.
- Equation is called parabolic if any of the eigenvalues $\lambda_i$ of $A$ is zero. This means that the coefficient matrix $A$ is singular.

## Classification of Second-Order Equations in $n$ Variables

**Example 7**

Classify the three-dimensional Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$

**Solution** The coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As the coefficient matrix is already in diagonalized form it can be seen immediately that it has three non-zero eigenvalues which are all positive. Hence, according to the classification rule the given PDE is elliptic.

## Classification of Second-Order Equations in $n$ Variables

**Example 8**

Classify the two-dimensional wave equation

$$u_{tt} - c^2 \left( u_{xx} + u_{yy} \right) = 0$$

**Solution** The coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -c^2 & 0 \\ 0 & 0 & -c^2 \end{bmatrix}$$

As the coefficient matrix is already in diagonalized form it can be seen immediately that it has three non-zero eigenvalues which are all negative except one. Hence, according to the classification rule the given PDE is hyperbolic.

## Classification of Second-Order Equations in $n$ Variables

**Example 9**

Classify the two-dimensional heat equation

$$u_t - \alpha\,(u_{xx} + u_{yy}) = 0$$

**Solution** The coefficient matrix is given by

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\alpha \end{bmatrix}$$

As the coefficient matrix is already in diagonalized form it can be seen immediately that it has a zero eigenvalue. Hence, according to the classification rule the given PDE is parabolic.

Partial Differential Equation Toolbox solves equations of the form

$$m\frac{\partial^2 u}{\partial t^2}+d\frac{\partial u}{\partial t}-\nabla\cdot(c\nabla u)+au=f.$$

## Green's first identity

This identity is derived from the divergence theorem applied to the vector field $\mathbf{F} = \psi\,\nabla\varphi$: Let $\varphi$ and $\psi$ be scalar functions defined on some region $U \subset \mathbf{R}^d$, and suppose that $\varphi$ is twice continuously differentiable, and $\psi$ is once continuously differentiable. Then[1]

$$\int_U (\psi\,\Delta\varphi + \nabla\psi\cdot\nabla\varphi)\,dV = \oint_{\partial U} \psi\,(\nabla\varphi\cdot\mathbf{n})\,dS = \oint_{\partial U} \psi\,\nabla\varphi\cdot d\mathbf{S}$$

where $\Delta$ is the Laplace operator, $\partial U$ is the boundary of region $U$, $\mathbf{n}$ is the outward pointing unit normal of surface element $dS$ and $d\mathbf{S}$ is the oriented surface element.

## Green's second identity

If $\varphi$ and $\psi$ are both twice continuously differentiable on $U \subset \mathbf{R}^3$, and $\varepsilon$ is once continuously differentiable, one may choose $\mathbf{F} = \psi\varepsilon\,\nabla\varphi - \varphi\varepsilon\,\nabla\psi$ to obtain

$$\int_U [\psi\,\nabla\cdot(\varepsilon\,\nabla\varphi) - \varphi\,\nabla\cdot(\varepsilon\,\nabla\psi)]\,dV = \oint_{\partial U} \varepsilon\left(\psi\frac{\partial\varphi}{\partial\mathbf{n}} - \varphi\frac{\partial\psi}{\partial\mathbf{n}}\right)dS\,.$$

When the $m$ and $d$ coefficients are 0, this reduces to

$$-\nabla \cdot (c\nabla u) + au = f,$$

which the documentation calls an *elliptic* equation, whether or not the equation is elliptic in the mathematical sense. The equation holds in $\Omega$, where $\Omega$ is a bounded domain in two or three dimensions. $c$, $a$, $f$, and the unknown solution $u$ are complex functions defined on $\Omega$. $c$ can also be a 2-by-2 matrix function on $\Omega$. The boundary conditions specify a combination of $u$ and its normal derivative on the boundary:

- *Dirichlet:* $hu = r$ on the boundary $\partial\Omega$.
- *Generalized Neumann:* $\vec{n} \cdot (c\nabla u) + qu = g$ on $\partial\Omega$.
- *Mixed:* Only applicable to *systems*. A combination of Dirichlet and generalized Neumann.

$\vec{n}$ is the outward unit normal. $g$, $q$, $h$, and $r$ are functions defined on $\partial\Omega$.

Assume that $u$ is a solution of the differential equation. Multiply the equation with an arbitrary *test function* $v$ and integrate on $\Omega$:

$$\int_\Omega \left(-(\nabla \cdot c\nabla u)v + auv\right)dx = \int_\Omega fv\,dx.$$

Integrate by parts (i.e., use Green's formula) to obtain

$$\int_\Omega \left((c\nabla u)\cdot\nabla v + auv\right)dx - \int_{\partial\Omega} \vec{n}\cdot(c\nabla u)v\;ds = \int_\Omega fv\;dx.$$

The boundary integral can be replaced by the boundary condition:

$$\int_{\Omega}\big((c\nabla u)\cdot\nabla v + auv\big)\,dx - \int_{\partial\Omega}(-qu+g)v\;ds = \int_{\Omega} fv\;dx.$$

Replace the original problem with *Find u such that*

$$\int_{\Omega}\big((c\nabla u)\cdot\nabla v + auv - fv\big)\,dx - \int_{\partial\Omega}(-qu+g)v\;ds = 0 \quad \forall v.$$

This equation is called the variational, or weak, form of the differential equation. Obviously, any solution of the differential equation is also a solution of the variational problem. The reverse is true under some restrictions on the domain and on the coefficient functions. The solution of the variational problem is also called the weak solution of the differential equation.

The solution $u$ and the test functions $v$ belong to some function space $V$. The next step is to choose an Np-dimensional subspace $V_{N_p} \subset V$. *Project the weak form of the differential equation onto a finite-dimensional function space* simply means requesting $u$ and $v$ to lie in $V_{N_p}$ rather than $V$. The solution of the finite dimensional problem turns out to be the element of $V_{N_p}$ that lies closest to the weak solution when measured in the energy norm. Convergence is guaranteed if the space $V_{N_p}$ tends to $V$ as $N_p \to \infty$. Since the differential operator is linear, we demand that the variational equation is satisfied for $N_p$ test-functions $\Phi_i \in V_{N_p}$ that form a basis, i.e.,

$$\int_\Omega \left( (c\nabla u) \cdot \nabla \phi_i + au\phi_i - f\phi_i \right) dx - \int_{\partial\Omega} (-qu + g)\phi_i \, ds = 0, \;\; i = 1, \ldots, N_p .$$

Expand $u$ in the same basis of $V_{N_p}$ elements

$$u(x) = \sum_{j=1}^{N_p} U_j \phi_j(x),$$

and obtain the system of equations

$$\sum_{j=1}^{N_p} \left( \int_{\Omega} \left( \left( c \nabla \phi_j \right) \cdot \nabla \phi_i + a \phi_j \phi_i \right) dx + \int_{\partial\Omega} q \phi_j \phi_i \, ds \right) U_j = \int_{\Omega} f \phi_i \, dx + \int_{\partial\Omega} g \phi_i \, ds, \; i = 1, \ldots, N_p.$$

Use the following notations:

$$K_{i,j} = \int_{\Omega} \left( c\nabla \phi_j \right) \cdot \nabla \phi_i \; dx \quad \text{(stiffness matrix)}$$

$$M_{i,j} = \int_{\Omega} a\phi_j \phi_i \; dx \quad \text{(mass matrix)}$$

$$Q_{i,j} = \int_{\partial\Omega} q\phi_j \phi_i \; ds$$

$$F_i = \int_{\Omega} f\phi_i \; dx$$

$$G_i = \int_{\partial\Omega} g\phi_i \; ds$$

and rewrite the system in the form
$(K + M + Q)U = F + G.$

$K$, $M$, and $Q$ are $N_p$-by-$N_p$ matrices, and $F$ and $G$ are $N_p$-vectors. $K$, $M$, and $F$ are produced by `assema`, while $Q$, $G$ are produced by `assemb`. When it is not necessary to distinguish $K$, $M$, and $Q$ or $F$ and $G$, we collapse the notations to $KU = F$, which form the output of `assempde`.

When the problem is *self-adjoint* and *elliptic* in the usual mathematical sense, the matrix $K + M + Q$ becomes symmetric and positive definite. Many common problems have these characteristics, most notably those that can also be formulated as minimization problems. For the case of a scalar equation, $K$, $M$, and $Q$ are obviously symmetric. If $c(x) \geq \delta > 0$, $a(x) \geq 0$ and $q(x) \geq 0$ with $q(x) > 0$ on some part of $\partial\Omega$, then, if $U \neq 0$.

$$U^T (K + M + Q)U = \int_\Omega \left(c|u|^2 + au^2\right) dx + \int_{\partial\Omega} qu^2 \; ds > 0, \;\; \text{if } U \neq 0.$$

$U^T(K + M + Q)U$ is the *energy norm*.

spaces. The software uses continuous functions that are linear on each element of a 2-D mesh, and are linear or quadratic on elements of a 3-D mesh. Piecewise linearity guarantees that the integrals defining the stiffness matrix $K$ exist. Projection onto $V_{N_p}$ is nothing more than linear interpolation, and the evaluation of the solution inside an element is done just in terms of the nodal values. If the mesh is uniformly refined, $V_{N_p}$ approximates the set of smooth functions on $\Omega$.

A suitable basis for $V_{N_p}$ in 2-D is the set of "tent" or "hat" functions $\phi_i$. These are linear on each element and take the value 0 at all nodes $x_j$ except for $x_i$. For the definition of basis functions for 3-D geometry, see "Finite Element Basis for 3-D" on page 5-10. Requesting $\phi_i(x_i) = 1$ yields the very pleasant property

$$u(x_i) = \sum_{j=1}^{N_p} U_j \phi_j(x_i) = U_i.$$

That is, by solving the FEM system we obtain the nodal values of the approximate solution. The basis function $\phi_i$ vanishes on all the elements that do not contain the node $x_i$. The immediate consequence is that the integrals appearing in $K_{i,j}$, $M_{i,j}$, $Q_{i,j}$, $F_i$ and $G_i$ only need to be computed on the elements that contain the node $x_i$. Secondly, it means that $K_{i,j}$ and$M_{i,j}$ are zero unless $x_i$ and $x_j$ are vertices of the same element and thus $K$ and $M$ are very sparse matrices. Their sparse structure depends on the ordering of the indices of the mesh points.

The integrals in the FEM matrices are computed by adding the contributions from each element to the corresponding entries (i.e., only if the corresponding mesh point is a vertex of the element). This process is commonly called *assembling*, hence the name of the function `assempde`.

The assembling routines scan the elements of the mesh. For each element they compute the so-called local matrices and add their components to the correct positions in the sparse matrices or vectors.

## Triangular finite elements in 2 dimensions

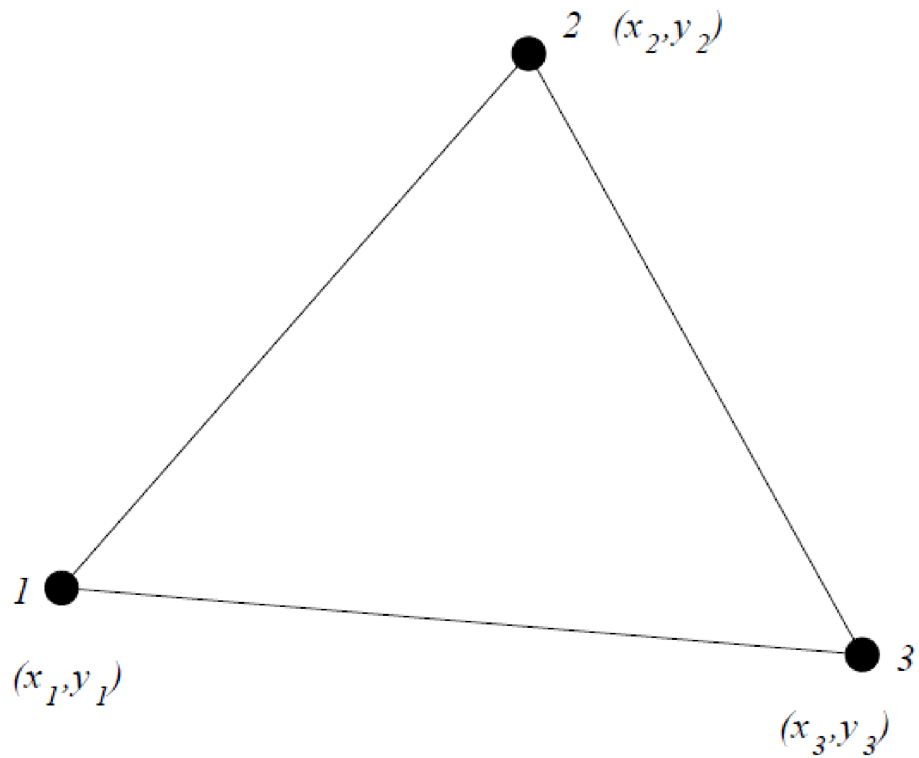



Figure 4.2.1: Triangular element with vertices 1, 2, 3 having coordinates $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$.

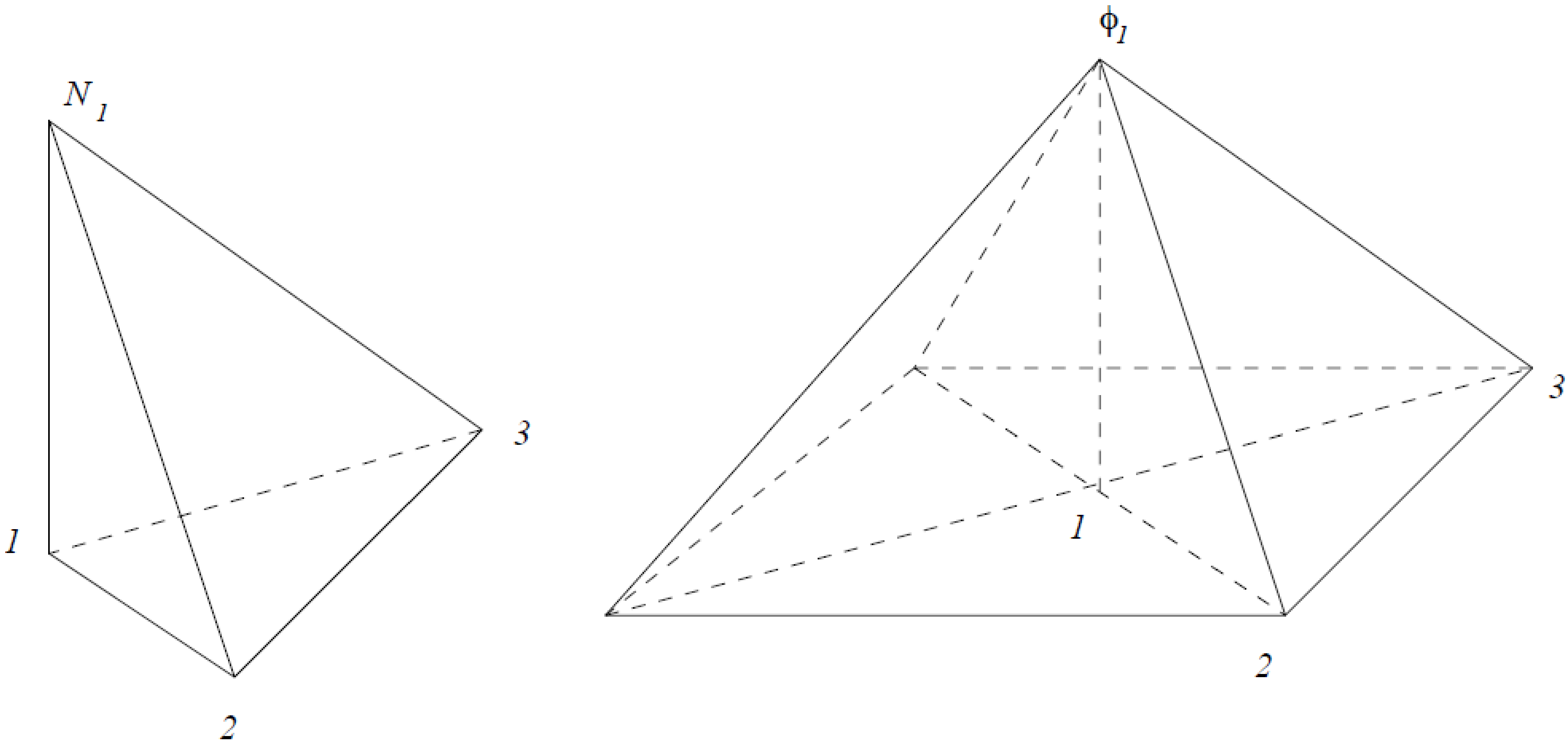
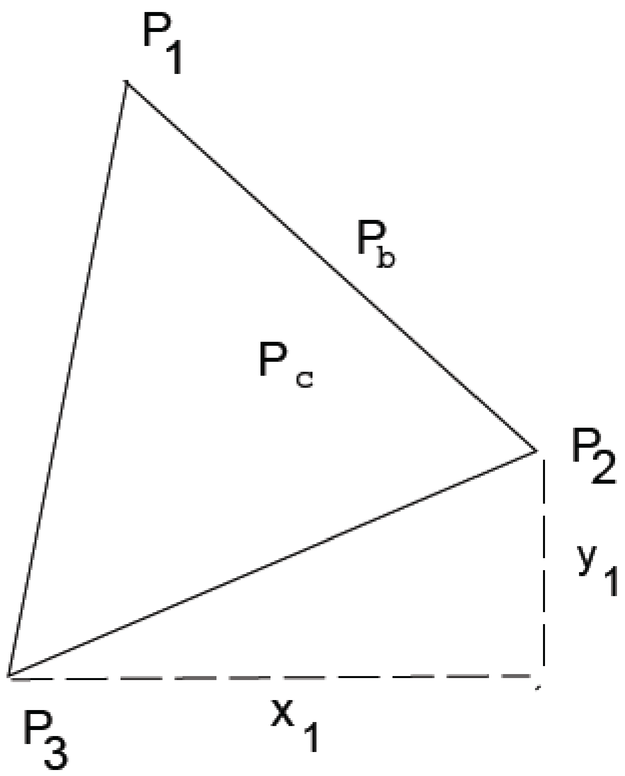



Figure 4.2.2: Shape function $N_1$ for Node 1 of element $e$ (left) and basis function $\phi_1$ for a cluster of four finite elements at Node 1.

Consider a triangle given by the nodes $P_1$, $P_2$, and $P_3$ as in the following figure.




**The Local Triangle P1P2P3**

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**Note:** The local 3-by-3 matrices contain the integrals evaluated only on the current triangle. The coefficients are assumed constant on the triangle and they are evaluated only in the triangle barycenter.

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The simplest computations are for the local mass matrix $m$:

$$m_{i,j} = \int_{\Delta P_1 P_2 P_3} a(P_c)\phi_i(x)\phi_j(x)\,dx = a(P_c)\frac{\text{area}(\Delta P_1 P_2 P_3)}{12}(1+\delta_{i,j}),$$

where $P_c$ is the center of mass of $\Delta\, P_1 P_2 P_3$, i.e.,

$$P_c = \frac{P_1 + P_2 + P_3}{3}.$$

$$f_i = f(P_c)\frac{\text{area}(\Delta P_1 P_2 P_3)}{3}.$$

For the local stiffness matrix we have to evaluate the gradients of the basis functions that do not vanish on $P_1P_2P_3$. Since the basis functions are linear on the triangle $P_1P_2P_3$, the gradients are constants. Denote the basis functions $\phi_1$, $\phi_2$, and $\phi_3$ such that $\phi(P_i) = 1$. If $P_2 - P_3 = [x_1, y_1]^T$ then we have that

$$\nabla \phi_1 = \frac{1}{2\text{area}(\Delta P_1 P_2 P_3)} \begin{bmatrix} y_1 \\ -x_1 \end{bmatrix}$$

and after integration (taking $c$ as a constant matrix on the triangle)

$$k_{i,j} = \frac{1}{4\ \text{area}(\Delta P_1 P_2 P_3)} \left[ y_j, -x_j \right] c(P_c) \begin{bmatrix} y_1 \\ -x_1 \end{bmatrix}.$$

If two vertices of the triangle lie on the boundary $\partial\Omega$, they contribute to the line integrals associated to the boundary conditions. If the two boundary points are $P_1$ and $P_2$, then we have

$$Q_{i,j} = q(P_b)\frac{\|P_1 - P_2\|}{6}\left(1+\delta_{i,j}\right), \quad i,j = 1,2$$

and

$$G_i = g(P_b)\frac{\|P_1 - P_2\|}{2}, \quad i = 1,2$$

where $P_b$ is the midpoint of $P_1P_2$.

For each triangle the vertices $P_m$ of the local triangle correspond to the indices $i_m$ of the mesh points. The contributions of the individual triangle are added to the matrices such that, e.g.,

$$K_{i_m,i_n}t \leftarrow K_{i_m,i_n} + k_{m,n}, \quad m,n = 1,2,3.$$

PAUSE

Time for a break