Chapter 3

Multi-Dimensional Variational Principles

3.1 Galerkin's Method and Extremal Principles

The construction of Galerkin formulations presented in Chapters 1 and 2 for one-dimensional problems readily extends to higher dimensions. Following our prior developments, we'll focus on the model two-dimensional self-adjoint diffusion problem

$$\mathcal{L}[u] = -(p(x,y)u_x)_x - (p(x,y)u_y)_y + q(x,y)u = f(x,y), \qquad (x,y) \in \Omega, \tag{3.1.1a}$$

where $\Omega \subset \Re^2$ with boundary $\partial\Omega$ (Figure 3.1.1) and $p(x,y) > 0$, $q(x,y) \geq 0$, $(x,y) \in \Omega$. Essential boundary conditions

$$u(x,y) = \alpha(x,y), \qquad (x,y) \in \partial\Omega_E, \tag{3.1.1b}$$

are prescribed on the portion $\partial\Omega_E$ of $\partial\Omega$ and natural boundary conditions

$$p(x,y)\frac{\partial u(x,y)}{\partial \mathbf{n}} = p\nabla u \cdot \mathbf{n} := p(u_x \cos\theta + u_y \sin\theta) = \beta(x,y), \qquad (x,y) \in \partial\Omega_N, \tag{3.1.1c}$$

are prescribed on the remaining portion $\partial\Omega_N$ of $\partial\Omega$. The angle θ is the angle between the x-axis and the outward normal $\mathbf{n}$ to $\partial\Omega$ (Figure 3.1.1).

The Galerkin form of (3.1.1) is obtained by multiplying (3.1.1a) by a test function v and integrating over Ω to obtain

$$\iint_{\Omega} v[-(pu_x)_x - (pu_y)_y + qu - f]dxdy = 0. \tag{3.1.2}$$

In order to integrate the second derivative terms by parts in two and three dimensions, we use Green's theorem or the divergence theorem

$$\iint_{\Omega} \nabla \cdot \mathbf{a} dxdy = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} ds \tag{3.1.3a}$$

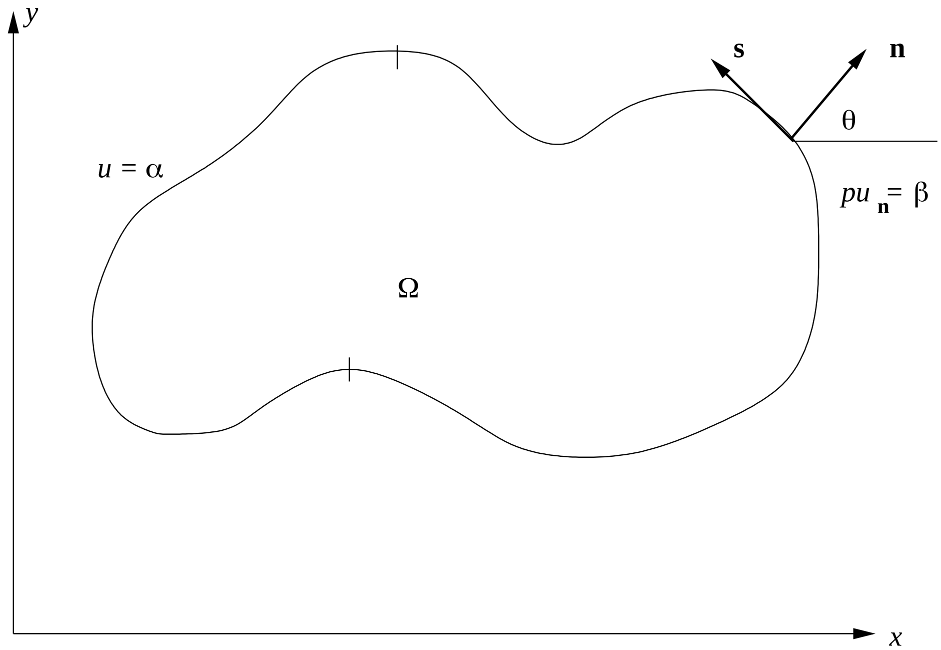

Figure 3.1.1: Two-dimensional region Ω with boundary $\partial\Omega$ and normal vector $\mathbf{n}$ to $\partial\Omega$.

where s is a coordinate on $\partial\Omega$, $\mathbf{a} = [a_1, a_2]^T$, and

$$\nabla \cdot \mathbf{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y}. \tag{3.1.3b}$$

In order to use this result in the present circumstances, let us introduce vector notation

$$(pu_x)_x + (pu_y)_y := \nabla \cdot (p\nabla u)$$

and use the "product rule" for the divergence and gradient operators

$$\nabla \cdot (vp\nabla u) = (\nabla v) \cdot (p\nabla u) + v\nabla \cdot (p\nabla u). \tag{3.1.3c}$$

Thus,

$$\iint_\Omega -v\nabla \cdot (p\nabla u)dxdy = \iint_\Omega [(\nabla v) \cdot (p\nabla u) - \nabla \cdot (vp\nabla u)]dxdy.$$

Now apply the divergence theorem (3.1.3) to the second term to obtain

$$\iint_\Omega -v\nabla \cdot (p\nabla u)dxdy = \iint_\Omega \nabla v \cdot p\nabla u dxdy - \int_{\partial\Omega} vp\nabla u \cdot \mathbf{n} ds.$$

Thus, (3.1.2) becomes

$$\iint_\Omega [\nabla v \cdot p\nabla u + v(qu - f)]dxdy - \int_{\partial\Omega} vpu_{\mathbf{n}} ds = 0 \tag{3.1.4}$$

where (3.1.1c) was used to simplify the surface integral.

The integrals in (3.1.4) must exist and, with u and v of the same class and p and q smooth, this implies

$$\iint_{\Omega} (u_x^2 + u_y^2 + u^2) dxdy$$

exists. This is the two-dimensional Sobolev space H^1. Drawing upon our experiences in one dimension, we expect $u \in H_E^1$, where functions in H_E^1 are in H^1 and satisfy the Dirichlet boundary conditions (3.1.1b) on Ω_E. Likewise, we expect $v \in H_0^1$, which denotes that $v = 0$ on $\partial\Omega_E$. Thus, the variation v should vanish where the trial function u is prescribed.

Let us extend the one-dimensional notation as well. Thus, the L^2 inner product is

$$(v, f) := \iint_{\Omega} v f dxdy \tag{3.1.5a}$$

and the strain energy is

$$A(v, u) := (\nabla v, p\nabla u) + (v, qu) = \iint_{\Omega} [p(v_x u_x + v_y u_y) + qvu] dxdy. \tag{3.1.5b}$$

We also introduce a boundary L^2 inner product as

$$< v, w >= \int_{\partial\Omega_N} vwds. \tag{3.1.5c}$$

The boundary integral may be restricted to $\partial\Omega_N$ since $v = 0$ on $\partial\Omega_E$. With this nomenclature, the variational problem (3.1.4) may be stated as: find $u \in H_E^1$ satisfying

$$A(v, u) = (v, f)+ < v, \beta >, \qquad \forall v \in H_0^1. \tag{3.1.6}$$

The Neumann boundary condition (3.1.1c) was used to replace $pu_{\mathbf{n}}$ in the boundary inner product. The variational problem (3.1.6) has the same form as the one-dimensional problem (2.3.3). Indeed, the theory and extremal principles developed in Chapter 2 apply to multi-dimensional problems of this form.

Theorem 3.1.1. *The function $w \in H_E^1$ that minimizes*

$$I[w] = A(w, w) - 2(w, f) - 2 < w, \beta > . \tag{3.1.7}$$

is the one that satisfies (3.1.6), and conversely.

Proof. The proof is similar to that of Theorem 2.2.1 and appears as Problem 1 at the end of this section. □

Corollary 3.1.1. *Smooth functions $u \in H_E^1$ satisfying (3.1.6) or minimizing (3.1.7) also satisfy (3.1.1).*

Proof. Again, the proof is left as an exercise. □

Example 3.1.1. Suppose that the Neumann boundary conditions (3.1.1c) are changed to Robin boundary conditions

$$pu_{\mathbf{n}} + \gamma u = \beta, \qquad (x, y) \in \partial\Omega_N. \tag{3.1.8a}$$

Very little changes in the variational statement of the problem (3.1.1a,b), (3.1.8). Instead of replacing $pu_{\mathbf{n}}$ by β in the boundary inner product (3.1.5c), we replace it by $\beta - \gamma u$. Thus, the Galerkin form of the problem is: find $u \in H_E^1$ satisfying

$$A(v, u) = (v, f) + < v, \beta - \gamma u >, \qquad \forall v \in H_0^1. \tag{3.1.8b}$$

Example 3.1.2. Variational principles for nonlinear problems and vector systems of partial differential equations are constructed in the same manner as for the linear scalar problems (3.1.1). As an example, consider a thin elastic sheet occupying a two-dimensional region Ω. As shown in Figure 3.1.2, the Cartesian components (u_1, u_2) of the displacement vector vanish on the portion $\partial\Omega_E$ of of the boundary $\partial\Omega$ and the components of the traction are prescribed as (S_1, S_2) on the remaining portion $\partial\Omega_N$ of $\partial\Omega$.

The equations of equilibrium for such a problem are (*cf.*, *e.g.*, [6], Chapter 4)

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} = 0, \tag{3.1.9a}$$

$$\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} = 0, \qquad (x, y) \in \Omega, \tag{3.1.9b}$$

where σ_{ij}, $i, j = 1, 2$, are the components of the two-dimensional symmetric stress tensor (matrix). The stress components are related to the displacement components by Hooke's law

$$\sigma_{11} = \frac{E}{1 - \nu^2}\left(\frac{\partial u_1}{\partial x} + \nu \frac{\partial u_2}{\partial y}\right), \tag{3.1.10a}$$

$$\sigma_{22} = \frac{E}{1 - \nu^2}\left(\nu \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right), \tag{3.1.10b}$$

$$\sigma_{12} = \frac{E}{2(1 + \nu)}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right), \tag{3.1.10c}$$

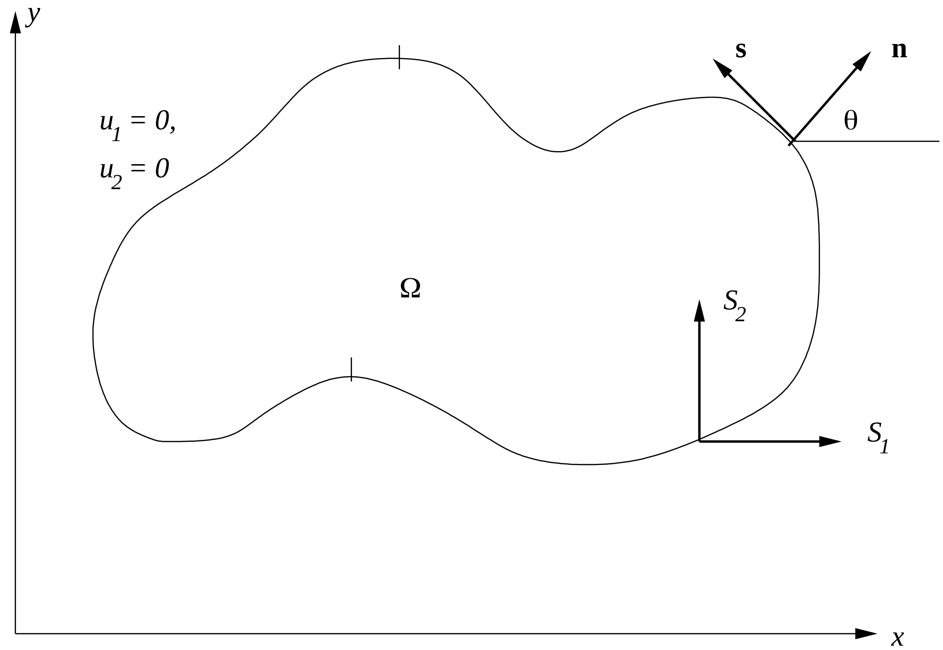

Figure 3.1.2: Two-dimensional elastic sheet occupying the region Ω. Displacement components (u_1, u_2) vanish on $\partial\Omega_E$ and traction components (S_1, S_2) are prescribed on $\partial\Omega_N$.

where E and ν are constants called Young's modulus and Poisson's ratio, respectively.

The displacement and traction boundary conditions are

$$u_1(x,y) = 0, \qquad u_2(x,y) = 0, \qquad (x,y) \in \partial\Omega_E, \tag{3.1.11a}$$

$$n_1\sigma_{11} + n_2\sigma_{12} = S_1, \qquad n_1\sigma_{12} + n_2\sigma_{22} = S_2, \qquad (x,y) \in \partial\Omega_N, \tag{3.1.11b}$$

where $\mathbf{n} = [n_1, n_2]^T = [\cos\theta, \sin\theta]^T$ is the unit outward normal vector to $\partial\Omega$ (Figure 3.1.2).

Following the one-dimensional formulations, the Galerkin form of this problem is obtained by multiplying (3.1.9a) and (3.1.9b) by test functions v_1 and v_2, respectively, integrated over Ω, and using the divergence theorem. With u_1 and u_2 being components of a displacement field, the functions v_1 and v_2 are referred to as components of the *virtual displacement field.*

We use (3.1.9a) to illustrate the process; thus, multiplying by v_1 and integrating over Ω, we find

$$\iint_\Omega v_1\left[\frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\sigma_{12}}{\partial y}\right]dxdy = 0.$$

The three stress components are dependent on the two displacement components and are typically replaced by these using (3.1.10). Were this done, the variational principle would involve second derivatives of u_1 and u_2. Hence, we would want to use the divergence theorem to obtain a symmetric variational form and reduce the continuity requirements on u_1 and u_2. We'll do this, but omit the explicit substitution of (3.1.10) to simplify the presentation. Thus, we regard σ_{11} and σ_{12} as components of a two-vector, we use the divergence theorem (3.1.3) to obrain

$$\iint_{\Omega} [\frac{\partial v_1}{\partial x}\sigma_{11} + \frac{\partial v_1}{\partial y}\sigma_{12}]dxdy = \int_{\partial\Omega} v_1[n_1\sigma_{11} + n_2\sigma_{12}]ds.$$

Selecting $v_1 \in H_0^1$ implies that the boundary integral vanishes on $\partial\Omega_E$. This and the subsequent use of the natural boundary condition (3.1.11b) give

$$\iint_{\Omega} [\frac{\partial v_1}{\partial x}\sigma_{11} + \frac{\partial v_1}{\partial y}\sigma_{12}]dxdy = \int_{\partial\Omega_N} v_1 S_1 ds, \qquad \forall v_1 \in H_0^1. \tag{3.1.12a}$$

Similar treatment of (3.1.9b) gives

$$\iint_{\Omega} [\frac{\partial v_2}{\partial x}\sigma_{12} + \frac{\partial v_2}{\partial y}\sigma_{22}]dxdy = \int_{\partial\Omega_N} v_2 S_2 ds, \qquad \forall v_2 \in H_0^1. \tag{3.1.12b}$$

Equations (3.1.12a) and (3.1.12b) may be combined and written in a vector form. Letting $\mathbf{u} = [u_1, u_2]^T$, *etc.*, we add (3.1.12a) and (3.1.12b) to obtain the Galerkin problem: find $\mathbf{u} \in H_0^1$ such that

$$A(\mathbf{v}, \mathbf{u}) =< \mathbf{v}, \mathbf{S} >, \qquad \forall \mathbf{v} \in H_0^1, \tag{3.1.13a}$$

where

$$A(\mathbf{v}, \mathbf{u}) = \iint_{\Omega} [\frac{\partial v_1}{\partial x}\sigma_{11} + \frac{\partial v_2}{\partial y}\sigma_{22} + (\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x})\sigma_{12}]dxdy, \tag{3.1.13b}$$

$$< \mathbf{v}, \mathbf{S} >= \int_{\partial\Omega_N} (v_1 S_1 + v_2 S_2)ds. \tag{3.1.13c}$$

When a vector function belongs to H^1, we mean that each of its components is in H^1. The spaces H_E^1 and H_0^1 are identical since the displacement is trivial on $\partial\Omega_E$.

The solution of (3.1.13) also satisfies the following minimum problem.

Theorem 3.1.2. *Among all functions* $\mathbf{w} = [w_1, w_2]^T \in H_E^1$ *the solution* $\mathbf{u} = [u_1, u_2]^T$ *of (3.1.13) is the one that minimizes*

$$I[\mathbf{w}] = \frac{E}{2(1-\nu^2)} \iint_{\Omega} \{(1-\nu)[(\frac{\partial w_1}{\partial x})^2 + (\frac{\partial w_2}{\partial y})^2] + \nu(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y})^2$$

$$+\frac{(1-\nu)}{2}(\frac{\partial w_1}{\partial y}+\frac{\partial w_2}{\partial x})^2\}dxdy-\int\limits_{\partial\Omega_N}(w_1S_1+w_2S_2)ds,$$

and conversely.

Proof. The proof is similar to that of Theorem 2.2.1. The stress components σ_{ij}, $i,j=1,2$, have been eliminated in favor of the displacements using (3.1.10). □

Let us conclude this section with a brief summary.

- A solution of the differential problem, *e.g.*, (3.1.1), is called a "classical" or "strong" solution. The function $u \in H^2_B$, where functions in H^2 have finite values of

$$\iint\limits_{\Omega}[(u_{xx})^2+(u_{xy})^2+(u_{yy})^2+(u_x)^2+(u_y)^2+u^2]dxdy$$

  and functions in H^2_B also satisfy *all* prescribed boundary conditions, *e.g.*, (3.1.1b,c).

- Solutions of a Galerkin problem such as (3.1.6) are called "weak" solutions. They may be elements of a larger class of functions than strong solutions since the high-order derivatives are missing from the variational statement of the problem. For the second-order differential equations that we have been studying, the variational form (*e.g.*, (3.1.6)) only contains first derivatives and $u \in H^1_E$. Functions in H^1 have finite values of

$$\iint\limits_{\Omega}[(u_x)^2+(u_y)^2+u^2]dxdy.$$

  and functions in H^1_E also satisfy the prescribed essential (Dirichlet) boundary condition (3.1.1b). Test functions v are not varied where essential data is prescribed and are elements of H^1_0. They satisfy trivial versions of the essential boundary conditions.

- While essential boundary conditions constrain the trial and test spaces, natural (Neumann or Robin) boundary conditions alter the variational statement of the problem. As with (3.1.6) and (3.1.13), inhomogeneous conditions add boundary inner product terms to the variational statement.

- Smooth solutions of the Galerkin problem satisfy the original partial differential equation(s) and natural boundary conditions, and conversely.

- Galerkin problems arising from self-adjoint differential equations also satisfy extremal problems. In this case, approximate solutions found by Galerkin's method are *best* in the sense of (2.6.5), *i.e.*, in the sense of minimizing the strain energy of the error.

Problems

1. Prove Theorem 3.1.1 and its Corollary.

2. Prove Theorem 3.1.2 and aslo show that smooth solutions of (3.1.13) satisfy the differential system (3.1.9) - (3.1.11).

3. Consider an infinite solid medium of material M containing an infinite number of periodically spaced circular cylindrical fibers made of material F. The fibers are arranged in a square array with centers two units apart in the x and y directions (Figure 3.1.3). The radius of each fiber is a (< 1). The aim of this problem is to find a Galerkin problem that can be used to determine the effective conductivity of the composite medium. Because of embedded symmetries, it suffices to solve a

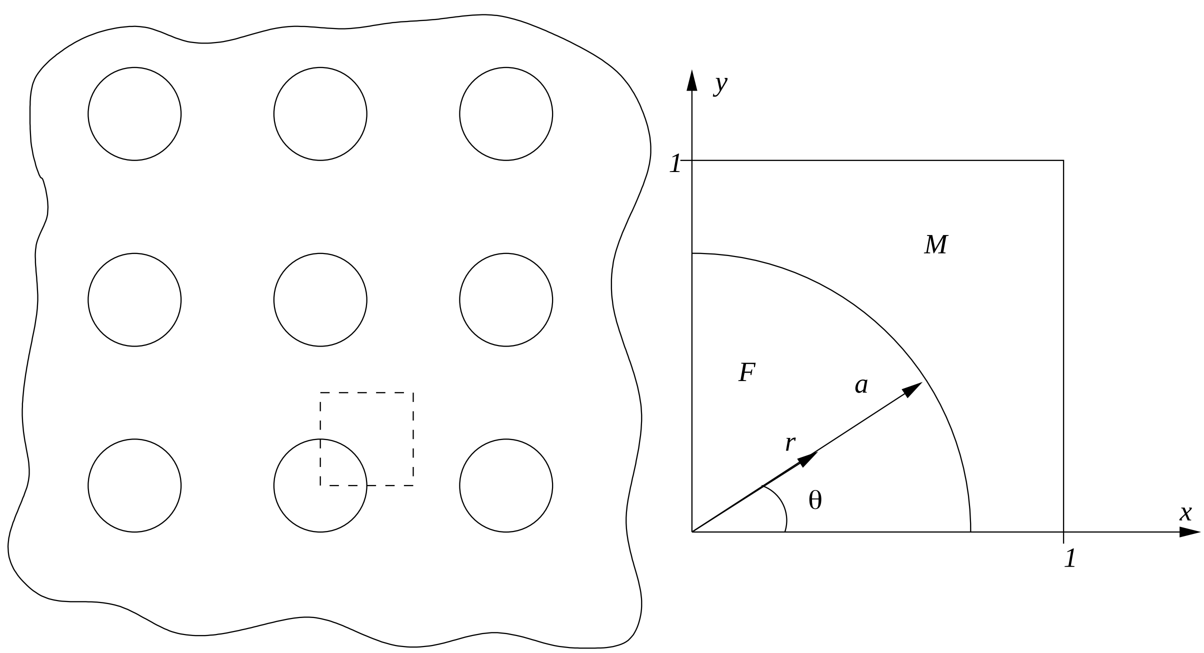

Figure 3.1.3: Composite medium consisting of a regular array of circular cylindrical fibers embedded in in a matrix (left). Quadrant of a Periodicity cell used to solve this problem (right).

problem on one quarter of a periodicity cell as shown on the right of Figure 3.1.3. The governing differential equations and boundary conditions for the temperature (or potential, *etc.*) $u(x,y)$ within this quadrant are

$$\begin{aligned} \nabla\cdot(p\nabla u) &= 0, && (x,y)\in\Omega_F\cup\Omega_M,\\ u_x(0,y) = u_x(1,y) &= 0, && 0\le y\le 1,\\ u(x,0)=0, \qquad u(x,1) &= 1, && 0\le x\le 1,\\ u\in C^0, \qquad pu_r &\in C^0, && (x,y)\in x^2+y^2=a^2. \end{aligned} \tag{3.1.14}$$

The subscripts F and M are used to indicate the regions and properties of the fiber and matrix, respectively. Thus, letting

$$\Omega := \{(x,y)|\ 0\le x\le 1,\ 0\le y\le 1\},$$

we have

$$\Omega_F := \{(r,\theta)|\ 0\le r\le a,\ 0\le\theta\le\pi/2\},$$

and

$$\Omega_M := \Omega - \Omega_F.$$

The conductivity p of the fiber and matrix will generally be different and, hence, p will jump at $r=a$. If necessary, we can write

$$p(x,y) = \begin{cases} p_F, & \text{if } x^2+y^2<a^2 \\ p_M, & \text{if } x^2+y^2>a^2 \end{cases}.$$

Although the conductivities are discontinuous, the last boundary condition confirms that the temperature u and flux pu_r are continuous at $r=a$.

3.1. Following the steps leading to (3.1.6), show that the Galerkin form of this problem consists of determining $u\in H_E^1$ as the solution of

$$\iint_{\Omega_F\cup\Omega_M} p(u_xv_x+u_yv_y)dxdy = 0, \qquad \forall v\in H_0^1.$$

Define the spaces H_E^1 and H_0^1 for this problem. The Galerkin problem appears to be the same as it would for a homogeneous medium. There is no indication of the continuity conditions at $r=a$.

3.2. Show that the function $w\in H_E^1$ that minimizes

$$I[w] = \iint_{\Omega_F\cup\Omega_M} p(w_x^2+w_y^2)dxdy$$

is the solution u of the Galerkin problem, and conversely. Again, there is little evidence that the problem involves an inhomogeneous medium.

3.2 Function Spaces and Approximation

Let us try to formalize some of the considerations that were raised about the properties of function spaces and their smoothness requirements. Consider a Galerkin problem in the form of (3.1.6). Using Galerkin's method, we find approximate solutions by solving (3.1.6) in a finite-dimensional subspace S^N of H^1. Selecting a basis $\{\phi_j\}_{j=1}^N$ for S^N, we consider approximations $U \in S_E^N$ of u in the form

$$U(x,y) = \sum_{j=1}^{N} c_j \phi_j(x,y). \tag{3.2.1}$$

With approximations $V \in S_0^N$ of v having a similar form, we determine U as the solution of

$$A(V,U) = (V,f) + < V, \beta >, \qquad \forall V \in S_0^N. \tag{3.2.2}$$

(Nontrivial essential boundary conditions introduce differences between S_E^N and S_0^N and we have not explicitly identified these differences in (3.2.2).)

We've mentioned the criticality of knowing the minimum smoothness requirements of an approximating space S^N. Smooth (*e.g.* C^1) approximations are difficult to construct on nonuniform two- and three-dimensional meshes. We have already seen that smoothness requirements of the solutions of partial differential equations are usually expressed in terms of Sobolev spaces, so let us define these spaces and examine some of their properties. First, let's review some preliminaries from linear algebra and functional analysis.

Definition 3.2.1. $\mathcal{V}$ is a *linear space* if

1. $u, v \in \mathcal{V}$ then $u + v \in \mathcal{V}$,
2. $u \in \mathcal{V}$ then $\alpha u \in \mathcal{V}$, for all constants α, and
3. $u, v \in \mathcal{V}$ then $\alpha u + \beta v \in \mathcal{V}$, for all constants α, β.

Definition 3.2.2. $A(u,v)$ is a *bilinear form* on $\mathcal{V} \times \mathcal{V}$ if, for $u, v, w \in \mathcal{V}$ and all constants α and β,

1. $A(u,v) \in \Re$, and
2. $A(u,v)$ is linear in each argument; thus,

$$A(u, \alpha v + \beta w) = \alpha A(u,v) + \beta A(u,w),$$

$$A(\alpha u + \beta v, w) = \alpha A(u,w) + \beta A(v,w).$$

Definition 3.2.3. An *inner product* $A(u, v)$ is a bilinear form on $\mathcal{V} \times \mathcal{V}$ that

1. is symmetric in the sense that $A(u, v) = A(v, u)$, $\forall u, v \in \mathcal{V}$, and

2. $A(u, u) > 0$, $u \neq 0$ and $A(0, 0) = 0$, $\forall u \in \mathcal{V}$.

Definition 3.2.4. The *norm* $\|\cdot\|_A$ associated with the inner product $A(u, v)$ is

$$\|u\|_A = \sqrt{A(u, u)} \tag{3.2.3}$$

and it satisfies

1. $\|u\|_A > 0$, $u \neq 0$, $\|0\|_A = 0$,

2. $\|u + v\|_A \leq \|u\|_A + \|v\|_A$, and

3. $\|\alpha u\|_A = |\alpha|\|u\|_A$, for all constants α.

The integrals involved in the norms and inner products are Lebesgue integrals rather than the customary Riemann integrals. Functions that are Riemann integrable are also Lebesgue integrable but not conversely. We have neither time nor space to delve into Lebesgue integration nor will it be necessary for most of our discussions. It is, however, helpful when seeking understanding of the continuity requirements of the various function spaces. So, we'll make a few brief remarks and refer those seeking more information to texts on functional analysis [3, 4, 5].

With Lebesgue integration, the concept of the length of a subinterval is replaced by the *measure of an arbitrary point set*. Certain sets are so sparse as to have *measure zero*. An example is the set of rational numbers on $[0, 1]$. Indeed, all countably infinite sets have measure zero. If a function $u \in \mathcal{V}$ possesses a given property except on a set of measure zero then it is said to have that property almost everywhere. A relevant property is the notion of an *equivalence class*. Two functions $u, v \in \mathcal{V}$ belong to the same *equivalence class* if

$$\|u - v\|_A = 0.$$

With Lebesgue integration, two functions in the same equivalence class are equal almost everywhere. Thus, if we are given a function $u \in \mathcal{V}$ and change its values on a set of measure zero to obtain a function v, then u and v belong to the same equivalence class.

We need one more concept, the notion of *completeness*. A *Cauchy sequence* $\{u_n\}_{n=1}^{\infty} \in \mathcal{V}$ is one where

$$\lim_{m,n\to\infty} \|u_m - u_n\|_A = 0.$$

If $\{u_n\}_{n=1}^{\infty}$ converges in $\|\cdot\|_A$ to a function $u \in \mathcal{V}$ then it is a Cauchy sequence. Thus, using the triangular inequality,

$$\lim_{m,n\to\infty} \|u_m - u_n\|_A \leq \lim_{m,n\to\infty} \{\|u_m - u\|_A + \|u - u_n\|_A\} = 0.$$

A space $\mathcal{V}$ where the converse is true, *i.e.*, where all Cauchy sequences $\{u_n\}_{n=1}^{\infty}$ converge in $\|\cdot\|_A$ to functions $u \in \mathcal{V}$, is said to be *complete.*

Definition 3.2.5. A complete linear space $\mathcal{V}$ with inner product $A(u, v)$ and corresponding norm $\|u\|_A$, $u, v \in \mathcal{V}$ is called a *Hilbert space.*

Let's list some relevant Hilbert spaces for use with variational formulations of boundary value problems. We'll present their definitions in two space dimensions. Their extension to one and three dimensions is obvious.

Definition 3.2.6. The space $L^2(\Omega)$ consists of functions satisfying

$$L^2(\Omega) := \{u| \iint_\Omega u^2 dxdy < \infty\}. \tag{3.2.4a}$$

It has the inner product

$$(u, v) = \iint_\Omega uvdxdy \tag{3.2.4b}$$

and norm

$$\|u\|_0 = \sqrt{(u, u)}. \tag{3.2.4c}$$

Definition 3.2.7. The *Sobolev space* H^k consists of functions u which belong to L^2 with their first $k \geq 0$ derivatives. The space has the inner product and norm

$$(u, v)_k := \sum_{|\boldsymbol{\kappa}|\leq k} (D^{\boldsymbol{\kappa}}u, D^{\boldsymbol{\kappa}}v), \tag{3.2.5a}$$

$$\|u\|_k = \sqrt{(u, u)_k}, \tag{3.2.5b}$$

where

$$\boldsymbol{\kappa} = [\kappa_1, \kappa_2]^T, \qquad |\boldsymbol{\kappa}| = \kappa_1 + \kappa_2, \tag{3.2.5c}$$

with κ_1 and κ_2 non-negative integers, and

$$D^{\boldsymbol{\kappa}}u := \frac{\partial^{\kappa_1+\kappa_2}u}{\partial x^{\kappa_1}\partial y^{\kappa_2}}. \tag{3.2.5d}$$

In particular, the space H^1 has the inner product and norm

$$(u,v)_1 = (u,v) + (u_x,v_x) + (u_y,v_y) = \iint_\Omega (uv + u_x v_x + u_y v_y)dxdy \tag{3.2.6a}$$

$$\|u\|_1 = \left[\iint_\Omega (u^2 + u_x^2 + u_y^2)dxdy\right]^{1/2}. \tag{3.2.6b}$$

Likewise, functions $u \in H^2$ have finite values of

$$\|u\|_2^2 = \iint_\Omega [u_{xx}^2 + u_{xy}^2 + u_{yy}^2 + u_x^2 + u_y^2 + u^2]dxdy.$$

Example 3.2.1. We have been studying second-order differential equations of the form (3.1.1) and seeking weak solutions $u \in H^1$ and $U \in S^N \subset H^1$ of (3.1.6) and (3.2.2), respectively. Let us verify that H^1 is the correct space, at least in one dimension. Thus, consider a basis of the familiar piecewise-linear hat functions on a uniform mesh with spacing $h = 1/N$

$$\phi_j(x) = \begin{cases} (x - x_{j-1})/h, & \text{if } x_{j-1} \le x < x_j \\ (x_{j+1} - x)/h, & \text{if } x_j \le x < x_{j+1} \\ 0, & \text{otherwise} \end{cases}. \tag{3.2.7}$$

Since $S^N \subset H^1$, ϕ_j and ϕ_j' must be in L^2, $j = 1, 2, ..., N$. Consider C^∞ approximations of $\phi_j(x)$ and $\phi_j'(x)$ obtained by "rounding corners" in $O(h/n)$-neighborhoods of the nodes x_{j-1}, x_j, x_{j+1} as shown in Figure3.2.1. A possible smooth approximation of $\phi_j'(x)$ is

$$\phi_j'(x) \approx \phi_{j,n}'(x) = \frac{1}{2h}[\tanh \frac{n(x - x_{j+1})}{h} + \tanh \frac{n(x - x_{j-1})}{h} - 2\tanh \frac{n(x - x_j)}{h}].$$

A smooth approximation $\phi_{j,n}$ of ϕ_j is obtained by integration as

$$\phi_{j,n}(x) = \frac{h}{2n} \ln \left[\frac{\cosh n((x - x_{j+1})/h) \cosh n((x - x_{j-1})/h)}{\cosh^2 n((x - x_j)/h)}\right].$$

Clearly, $\phi_{j,n}$ and $\phi_{j,n}'$ are elements of L^2. The "rounding" disappears as $n \to \infty$ and

$$\lim_{n\to\infty} \int_0^1 [\phi_{j,n}'(x)]^2 dx \approx 2h(1/h)^2 = 2/h.$$

The explicit calculations are somewhat involved and will not be shown. However, it seems clear that the limiting function $\phi_j' \in L^2$ and, hence, $\phi_j \in S^N$ for fixed h.

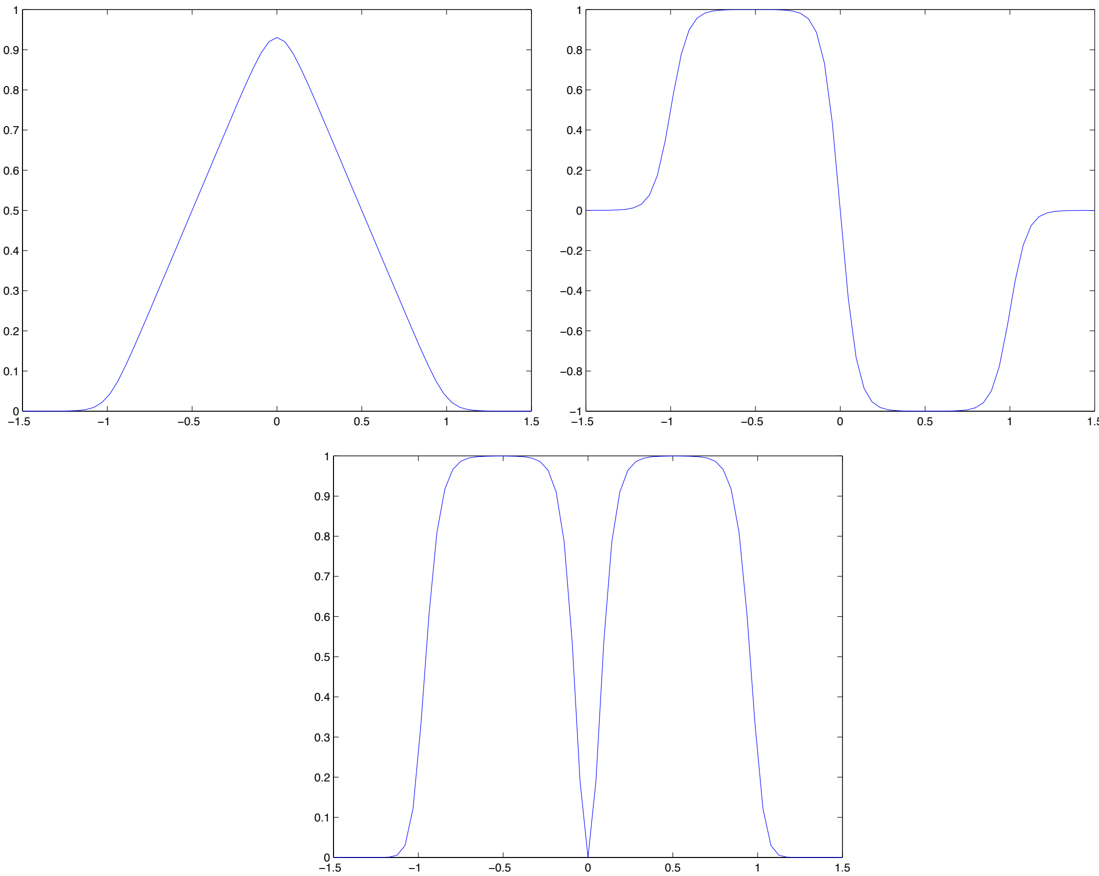

Figure 3.2.1: Smooth version of a piecewise linear hat function (3.2.7) (top), its first derivative (center), and the square of its first derivative (bottom). Results are shown with $x_{j-1} = -1$, $x_j = 0$, $x_{j+1} = 1$ ($h = 1$), and $n = 10$.

Example 3.2.2. Consider the piecewise-constant basis function on a uniform mesh

$$\phi_j(x) = \begin{cases} 1, & \text{if } x_{j-1} \le x < x_j \\ 0, & \text{otherwise} \end{cases} . \tag{3.2.8}$$

A smooth version of this function and its first derivative are shown in Figure 3.2.2 and may be written as

$$\phi_{j,n}(x) = \frac{1}{2}[\tanh \frac{n(x - x_{j-1})}{h} - \tanh \frac{n(x - x_j)}{h}]$$

$$\phi'_{j,n}(x) = \frac{n}{2h}[\operatorname{sech}^2 \frac{n(x - x_{j-1})}{h} - \operatorname{sech}^2 \frac{n(x - x_j)}{h}].$$

As $n \to \infty$, $\phi_{j,n}$ approaches a square pulse; however, $\phi'_{j,n}$ is proportional to the combination of delta functions

$$\phi'_{j,n}(x) \propto \delta(x - x_{j-1}) - \delta(x - x_j).$$

Thus, we anticipate problems since delta functions are not elements of L^2. Squaring $\phi'_{j,n}(x)$

$$[\phi'_{j,n}(x)]^2 = (\frac{n}{2h})^2[\text{sech}^4\frac{n(x-x_{j-1})}{h} - 2\text{sech}^2\frac{n(x-x_{j-1})}{h}\text{sech}^2\frac{n(x-x_j)}{h} + \text{sech}^4\frac{n(x-x_j)}{h}].$$

As shown in Figure 3.2.2, the function $\text{sech}\, n(x-x_j)/h$ is largest at x_j and decays exponentially fast from x_j; thus, the center term in the above expression is exponentially small relative to the first and third terms. Neglecting it yields

$$[\phi'_{j,n}(x)]^2 \approx (\frac{n}{2h})^2[\text{sech}^4\frac{n(x-x_{j-1})}{h} + \text{sech}^4\frac{n(x-x_j)}{h}].$$

Thus,

$$\int_0^1 [\phi'_{j,n}(x)]^2 dx \approx \frac{n}{12h}[\tanh\frac{n(x-x_{j-1})}{h}(2+\text{sech}^2\frac{n(x-x_{j-1})}{h}) + \tanh\frac{n(x-x_j)}{h}(2+\text{sech}^2\frac{n(x-x_j)}{h})]_0^1.$$

This is unbounded as $n \to \infty$; hence, $\phi'_j(x) \notin L^2$ and $\phi_j(x) \notin H^1$.

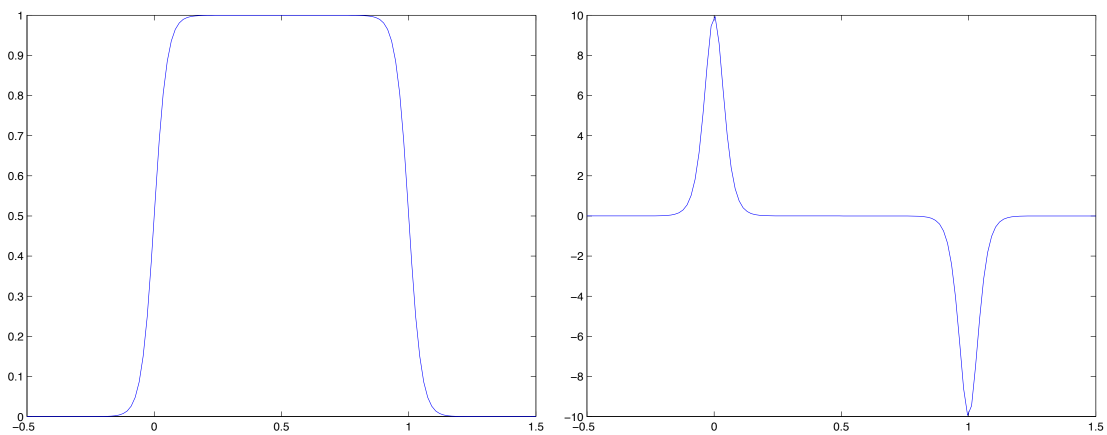

Figure 3.2.2: Smooth version of a piecewise constant function (3.2.8) (left) and its first derivative (right). Results are shown with $x_{j-1} = 0$, $x_j = 1$ ($h = 1$), and $n = 20$.

Although the previous examples lack rigor, we may conclude that a basis of continuous functions will belong to H^1 in one dimension. More generally, $u \in H^k$ implies that $u \in C^{k-1}$ in one dimension. The situation is not as simple in two and three dimensions. The Sobolev space H^k is the completion with respect to the norm (3.2.5) of C^k functions whose first k partial derivatives are elements of L^2. Thus, for example, $u \in H^1$ implies that u, u_x, and u_y are all elements of L^2. This is not sufficient to ensure that u is continuous in two and three dimensions. Typically, if $\partial\Omega$ is smooth then $u \in H^k$ implies that $u \in C^s(\Omega \cup \partial\Omega)$ where s is the largest integer *less than* $(k - d/2)$ in d dimensions [1, 2]. In two and three dimensions, this condition implies that $u \in C^{k-2}$.

Problems

1. Assuming that $p(x, y) > 0$ and $q(x, y) \geq 0$, $(x, y) \in \Omega$, find any other conditions that must be satisfied for the strain energy

$$A(v, u) = \iint_{\Omega} [p(v_x u_x + v_y u_y) + qvu] dxdy$$

to be an inner product and norm, *i.e.*, to satisfy Definitions 3.2.3 and 3.2.4.

2. Construct a variational problem for the fourth-order biharmonic equation

$$\Delta(p\Delta u) = f(x, y), \qquad (x, y) \in \Omega,$$

where

$$\Delta u = u_{xx} + u_{yy}$$

and $p(x, y) > 0$ is smooth. Assume that u satisfies the essential boundary conditions

$$u(x, y) = 0, \qquad u_{\mathbf{n}}(x, y) = 0, \qquad (x, y) \in \partial\Omega,$$

where $\mathbf{n}$ is a unit outward normal vector to $\partial\Omega$. To what function space should the weak solution of the variational problem belong?

3.3 Overview of the Finite Element Method

Let us conclude this chapter with a brief summary of the key steps in constructing a finite-element solution in two or three dimensions. Although not necessary, we will continue to focus on (3.1.1) as a model.

1. Construct a variational form of the problem. Generally, we will use Galerkin's method to construct a variational problem. As described, this involves multiplying the differential equation be a suitable test function and using the divergence theorem to get a symmetric formulation. The trial function $u \in H_E^1$ and, hence, satisfies any prescribed essential boundary conditions. The test function $v \in H_0^1$ and, hence, vanishes where essential boundary conditions are prescribed. Any prescribed Neumann or Robin boundary conditions are used to alter the variational problem as, *e.g.*, with (3.1.6) or (3.1.8b), respectively.

Nontrivial essential boundary conditions introduce differences in the spaces H_E^1 and H_0^1. Furthermore, the finite element subspace S_E^N cannot satisfy non-polynomial boundary conditions. One way of overcoming this is to transform the differential equation to one having trivial essential boundary conditions (*cf.* Problem 1 at the end of this section). This approach is difficult to use when the boundary data is discontinuous or when the problem is nonlinear. It is more important for theoretical than for practical reasons.

The usual approach for handling nontrivial Dirichlet data is to interpolate it by the finite element trial function. Thus, consider approximations in the usual form

$$U(x,y) = \sum_{j=1}^{N} c_j \phi_j(x,y); \tag{3.3.1}$$

however, we include basis functions ϕ_k for mesh entities (vertices, edges) k that are on $\partial\Omega_E$. The coefficients c_k associated with these nodes are not varied during the solution process but, rather, are selected to interpolate the boundary data. Thus, with a Lagrangian basis where $\phi_k(x_j, y_j) = \delta_{k,j}$, we have

$$U(x_k, y_k) = \alpha(x_k, y_k) = c_k, \qquad (x_k, y_k) \in \partial\Omega_E.$$

The interpolation is more difficult with hierarchical functions, but it is manageable (*cf.* Section 4.4). We will have to appraise the effect of this interpolation on solution accuracy. Although the spaces S_E^N and S_0^N differ, the stiffness and mass matrices can be made symmetric for self-adjoint linear problems (*cf.* Section 5.5).

A third method of satisfying essential boundary conditions is given as Problem 2 at the end of this section.

2. Discretize the domain. Divide Ω into *finite elements* having simple shapes, such as triangles or quadrilaterals in two dimensions and tetrahedra and hexahedra in three dimensions. This nontrivial task generally introduces errors near $\partial\Omega$. Thus, the problem is typically solved on a polygonal region $\tilde{\Omega}$ defined by the finite element mesh (Figure 3.3.1) rather than on Ω. Such errors may be reduced by using finite elements with curved sides and/or faces near $\partial\Omega$ (*cf.* Chapter 4). The relative advantages of using fewer curved elements or a larger number of smaller straight-sided or planar-faced elements will have to be determined.

3. Generate the element stiffness and mass matrices and element load vector. Piecewise polynomial approximations $U \in S_E^N$ of u and $V \in S_0^N$ of v are chosen. The approximating spaces S_E^N and S_0^N are supposed to be subspaces of H_E^1 and H_0^1, respectively; however, this may not be the case because of errors introduced in approximating the essential boundary conditions and/or the domain Ω. These effects will also have to be appraised (*cf.* Section 7.3). Choosing a basis for S^N, we write U and V in the form of (3.3.1).

The variational problem is written as a sum of contributions over the elements and the element stiffness and mass matrices and load vectors are generated. For the model problem (3.1.1) this would involve solving

$$\sum_{e=1}^{N_\Delta} [A_e(V,U) - (V,f)_e - < V, \beta >_e] = 0, \qquad \forall V \in S_0^N, \tag{3.3.2a}$$

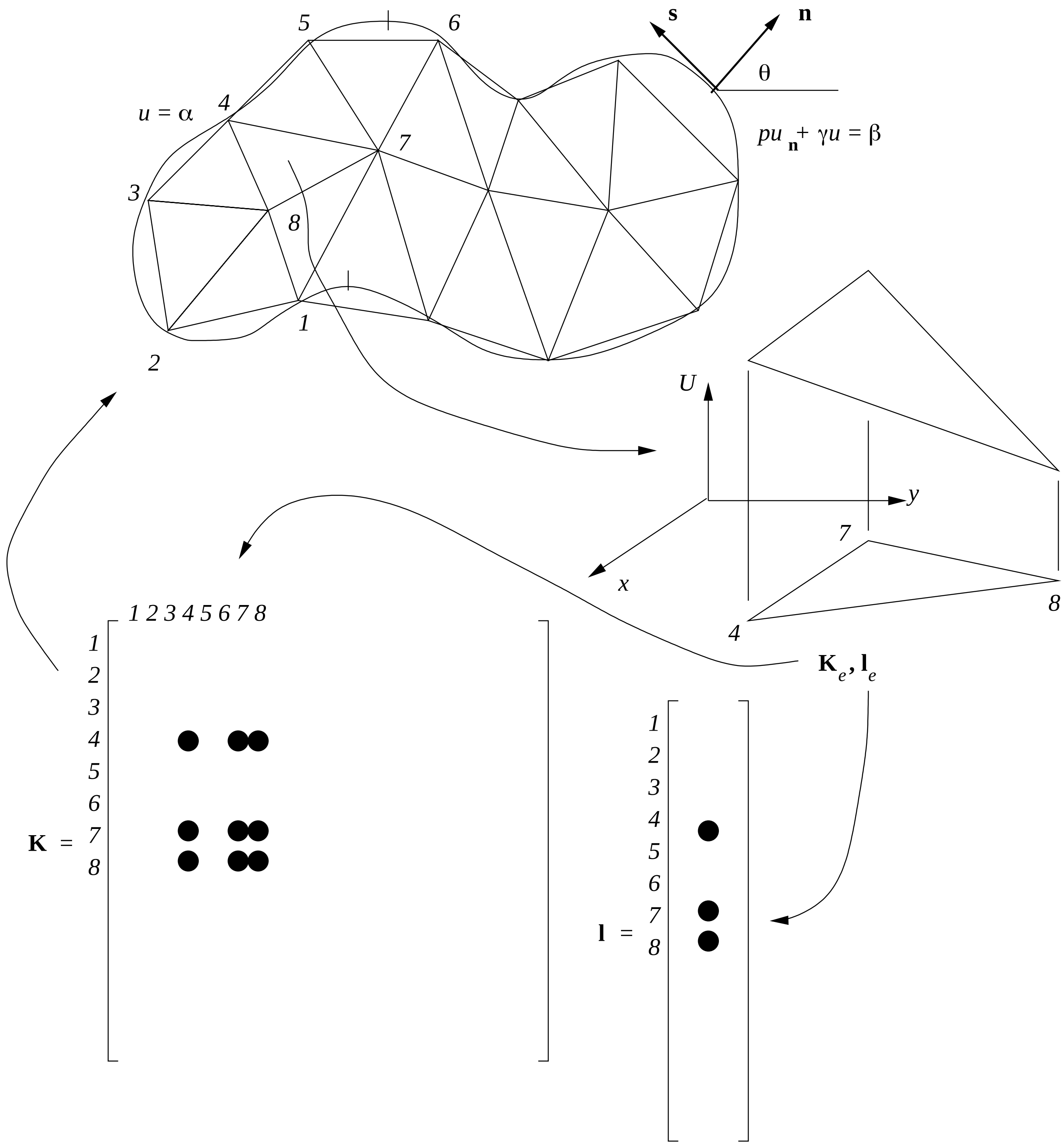

Figure 3.3.1: Two-dimensional domain Ω having boundary $\partial\Omega = \partial\Omega_E \cup \partial\Omega_N$ with unit normal $\mathbf{n}$ discretized by triangular finite elements. Schematic representation of the assembly of the element stiffness matrix $\mathbf{K}_e$ and element load vector $\mathbf{l}_e$ into the global stiffness matrix $\mathbf{K}$ and load vector $\mathbf{l}$.

where

$$A_e(V,U) = \iint_{\Omega_e} (V_x p U_x + V_y p U_y + VqU)\,dxdy, \tag{3.3.2b}$$

$$(V, f)_e = \iint\limits_{\Omega_e} V f dx dy, \tag{3.3.2c}$$

$$< V, \beta >_e = \int\limits_{\partial\Omega_e \cap \partial\tilde{\Omega}_N} V \beta ds, \tag{3.3.2d}$$

Ω_e is the domain occupied by element e, and N_Δ is the number of elements in the mesh. The boundary integral (3.3.2d) is zero unless a portion of $\partial\Omega_e$ coincides with the boundary of the finite element domain $\partial\tilde{\Omega}$.

Galerkin formulations for self-adjoint problems such as (3.1.6) lead to minimum problems in the sense of Theorem 3.1.1. Thus, the finite element solution is the best solution in S^N in the sense of minimizing the strain energy of the error $A(u-U, u-U)$. The strain energy of the error is orthogonal to all functions V in S_E^N as illustrated in Figure 3.3.2 for three-vectors.

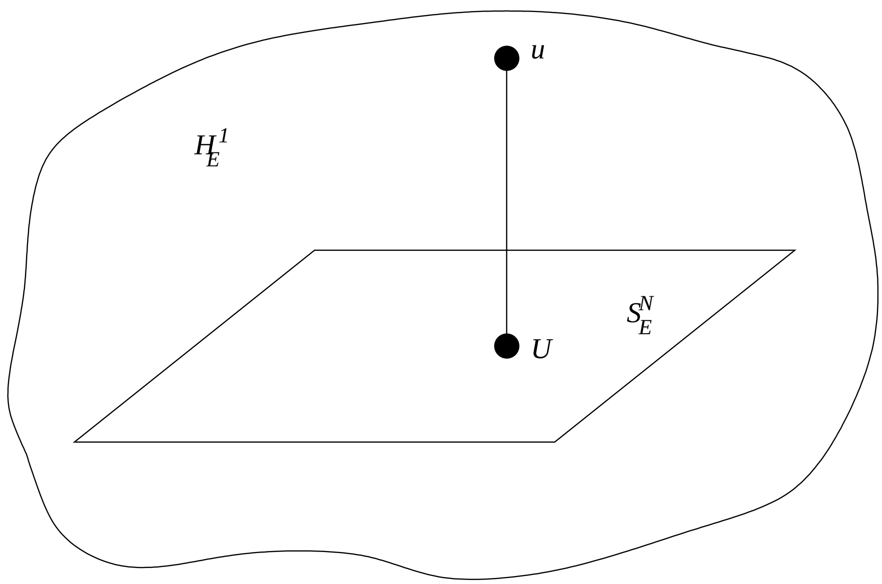

Figure 3.3.2: Subspace S_E^N of H_E^1 illustrating the "best" approximation property of the solution of Galerkin's method.

4. Assemble the global stiffness and mass matrices and load vector. The element stiffness and mass matrices and load vectors that result from evaluating (3.3.2b-d) are added directly into global stiffness and mass matrices and a load vector. As depicted in Figure 3.3.1, the indices assigned to unknowns associated with mesh entities (vertices as shown) determine the correct positions of the elemental matrices and vectors in the global stiffness and mass matrices and load vector.

5. Solve the algebraic system. For linear problems, the assembly of (3.3.2) gives rise to a system of the form

$$\mathbf{d}^T[(\mathbf{K}+\mathbf{M})\mathbf{c}-\mathbf{l}]=\mathbf{0}, \tag{3.3.3a}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the global stiffness and mass matrices, $\mathbf{l}$ is the global load vector,

$$\mathbf{c}^T=[c_1,c_2,...,c_N]^T, \tag{3.3.3b}$$

and

$$\mathbf{d}^T=[d_1,d_2,...,d_N]^T. \tag{3.3.3c}$$

Since (3.3.3a) must be satisfied for *all* choices of $\mathbf{d}$, we must have

$$(\mathbf{K}+\mathbf{M})\mathbf{c}=\mathbf{l}. \tag{3.3.4}$$

For the model problem (3.1.1), $\mathbf{K}+\mathbf{M}$ will be sparse and positive definite. With proper treatment of the boundary conditions, it will also be symmetric (*cf.* Chapter 5).

Each step in the finite element solution will be examined in greater detail. Basis construction is described in Chapter 4, mesh generation and assembly appear in Chapter 5, error analysis is discussed in Chapter 7, and linear algebraic solution strategies are presented in Chapter 11.

Problems

1. By introducing the transformation

$$\hat{u}=u-\alpha$$

show that (3.1.1) can be changed to a problem with homogeneous essential boundary conditions. Thus, we can seek $\hat{u}\in H_0^1$.

2. Another method of treating essential boundary conditions is to remove them by using a "penalty function." Penalty methods are rarely used for this purpose, but they are important for other reasons. This problem will introduce the concept and reinforce the material of Section 3.1. Consider the variational statement (3.1.6) as an example, and modify it by including the essential boundary conditions

$$A(v,u)=(v,f)+<v,\beta>_{\partial\Omega_N}+\lambda<v,\alpha-u>_{\partial\Omega_E}, \qquad \forall v\in H^1.$$

Here λ is a penalty parameter and subscripts on the boundary integral indicate their domain. No boundary conditions are applied and the problem is solved for u and v ranging over the whole of H^1.

Show that smooth solutions of this variational problem satisfy the differential equation (3.1.1a) as well as the natural boundary conditions (3.1.1c) and

$$u + \frac{p}{\lambda}\frac{\partial u}{\partial \mathbf{n}} = \alpha, \qquad (x, y) \in \Omega_E.$$

The penalty parameter λ must be selected large enough for this natural boundary condition to approximate the prescribed essential condition (3.1.1b). This can be tricky. If selected too large, it will introduce ill-conditioning into the resulting algebraic system.

Bibliography

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] O. Axelsson and V.A. Barker. *Finite Element Solution of Boundary Value Problems.* Academic Press, Orlando, 1984.

[3] C. Geoffman and G. Pedrick. *First Course in Functional Analysis.* Prentice-Hall, Englewood Cliffs, 1965.

[4] P.R. Halmos. *Measure Theory.* Springer-Verlag, New York, 1991.

[5] J.T. Oden and L.F. Demkowicz. *Applied Functional Analysis.* CRC Press, Boca Raton, 1996.

[6] R. Wait and A.R. Mitchell. *The Finite Element Analysis and Applications.* John Wiley and Sons, Chichester, 1985.