2.2. Nonhomogeneous Wave Equation $\dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial^2 w}{\partial x^2} + \Phi(x,t)$

2.2-1. Solutions of boundary value problems in terms of the Green's function.

We consider boundary value problems for the nonhomogeneous wave equation on a finite interval $0 \le x \le l$ with the general initial conditions

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w(x,t) = \frac{\partial}{\partial t} \int_0^l f(\xi) G(x,\xi,t)\,d\xi + \int_0^l g(\xi) G(x,\xi,t)\,d\xi + \int_0^t \int_0^l \Phi(\xi,\tau) G(x,\xi,t-\tau)\,d\xi\,d\tau.$$

2.2-2. Domain: $0 \le x \le l$. First boundary value problem for the wave equation.

Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x,\xi,t) = \frac{2}{a\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi \xi}{l}\right) \sin\left(\frac{n\pi a t}{l}\right).$$

2.2-3. Domain: $0 \le x \le l$. Second boundary value problem for the wave equation.

Boundary conditions are prescribed:

$$\frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0, \qquad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x,\xi,t) = \frac{t}{l} + \frac{2}{a\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi \xi}{l}\right) \sin\left(\frac{n\pi a t}{l}\right).$$

2.2-4. Domain: $0 \le x \le l$. Third boundary value problem ($k_1 > 0$, $k_2 > 0$).

Boundary conditions are prescribed:

$$\frac{\partial w}{\partial x} - k_1 w = 0 \quad \text{at} \quad x = 0, \qquad \frac{\partial w}{\partial x} + k_2 w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x,\xi,t) = \frac{1}{a} \sum_{n=1}^{\infty} \frac{1}{\lambda_n \|u_n\|^2} \sin(\lambda_n x + \varphi_n) \sin(\lambda_n \xi + \varphi_n) \sin(\lambda_n a t),$$

$$\varphi_n = \arctan\frac{\lambda_n}{k_1}, \qquad \|u_n\|^2 = \frac{l}{2} + \frac{(\lambda_n^2 + k_1 k_2)(k_1 + k_2)}{2(\lambda_n^2 + k_1^2)(\lambda_n^2 + k_2^2)};$$

the λ_n are positive roots of the transcendental equation $\cot(\lambda l) = \dfrac{\lambda^2 - k_1 k_2}{\lambda(k_1 + k_2)}$.

References

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