

Time harmonic wave propagation in one dimensional weakly randomly perturbed periodic media

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Abstract In this work we consider the solution of the time harmonic wave equation in a one dimensional periodic medium with weak random perturbations. More precisely, we study two types of weak perturbations: (1) the case of stationary, ergodic and oscillating coefficients, the typical size of the oscillations being small compared to the wavelength and (2) the case of rare random perturbations of the medium, where each period has a small probability to have its coefficients modified, independently of the other periods. Our goal is to derive an asymptotic approximation of the solution with respect to the small parameter. This can be used in order to construct absorbing boundary conditions for such media.

Keywords Wave equation · Random media · Periodic media

1 Introduction and model problem

The propagation of waves in periodic media has seen a regain of interest for many important applications, particularly in optics for micro and nano-technology. However, in real applications, the media are often not perfectly periodic and the perturbations can be partially known. The use of randomness to model this partial knowledge is particularly well suited.

We want to propose a numerical method for computing the propagation of waves in such media. More precisely, we want to reduce the pure numerical computation to a bounded region, typically a region where the medium is well-known (i.e. not random). It is then necessary to construct transparent or absorbing boundary conditions to impose at the boundary of the computational domain. These conditions should reflect the best possible the wave propagation in the exterior medium. They could then be used, for instance, to obtain the statistics or to quantify the uncertainty of the field in the computational domain (via the simulation of numerous realizations of the field). This paper is a first contribution to the construction of such boundary conditions in particular situations.

We are interested in this paper in the one-dimensional time harmonic wave equation in infinite media which are periodic with weak random perturbations. More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a random medium occupying \mathbb{R} , characterised by the following coefficients

$$\text{for a.e. } (x, \omega) \in \mathbb{R} \times \Omega, \quad \begin{aligned} \kappa_\varepsilon(x, \omega) &= \kappa_{\text{per}}(x) + \tilde{\kappa}_\varepsilon(x, \omega), \\ \rho_\varepsilon(x, \omega) &= \rho_{\text{per}}(x) + \tilde{\rho}_\varepsilon(x, \omega), \end{aligned} \quad (1)$$

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where

- κ_ε and ρ_ε are uniformly bounded from above and below, that is, that there exist strictly non negative constants $\kappa_+, \kappa_-, \rho_+, \rho_-$ such that almost everywhere (*a.e.*) and almost surely (*a.s.*)

$$0 < \kappa_- \leq \kappa_\varepsilon \leq \kappa_+, \quad 0 < \rho_- \leq \rho_\varepsilon \leq \rho_+. \quad (2)$$

- κ_{per} and ρ_{per} are L -periodic functions with $L > 0$ such that

$$0 < \kappa_- \leq \kappa_{\text{per}} \leq \kappa_+, \quad 0 < \rho_- \leq \rho_{\text{per}} \leq \rho_+.$$

- $\tilde{\kappa}_\varepsilon$ and $\tilde{\rho}_\varepsilon$ are random coefficients, depending on a small parameter ε . We consider in this paper two types of weak random perturbations that are specified later. We suppose that there exists $a > 0$ such that *a.s.*

$$\text{Supp } \tilde{\kappa}_\varepsilon \cap \text{Supp } \tilde{\rho}_\varepsilon \subset \mathbb{R} \setminus [-a, a]$$

We consider the time harmonic scalar wave equation (deduced from the wave equation assuming a time-dependence e^{-ikt} at a given frequency k), also known as the Helmholtz equation, in this medium,

$$-\partial_x[\kappa_\varepsilon(\cdot, \omega)\partial_x u_\varepsilon(\cdot, \omega)] - \rho_\varepsilon(\cdot, \omega)k^2 u_\varepsilon(\cdot, \omega) = f \quad \text{in } \mathbb{R}, \text{ for a.e. } \omega \in \Omega, \quad (3)$$

where $f \in L^2(\mathbb{R})$ has a compact support. We can suppose without loss of generality that the support of f is included in $(-a, a)$. Of course, if the coefficients are random, the solution, if any, will also be random.

It is well-known that the Helmholtz equation with real frequency is in general not well-posed in the classical L^2 setting. This is linked to the fact that the physical solution is in general not of finite energy (*i.e.* with a finite L^2 norm) since a propagation without attenuation is possible at infinity. On the other hand, in the L^2_{loc} framework, an infinity of solutions can be found. Usually, radiation conditions which characterize the behaviour at infinity of the physical solution, have to be determined and added to the problem in order to recover a well-posed problem. When the medium is perfectly periodic or if it is a local perturbation of a periodic medium, the authors in [15, 11] have derived radiation conditions and shown that the problem is well-posed. For random media and even for weakly random perturbations of periodic media, this question is still open. The classical framework to answer to this question is to use the limiting absorption principle. More precisely, we introduce a damping parameter η and we define the physical solution of (3) as the limit (in a sense to be determined), as η goes to 0, of the unique H^1 solution of the Helmholtz equation with damping

$$-\partial_x[\kappa_\varepsilon(\cdot, \omega)\partial_x u_\varepsilon^\eta(\cdot, \omega)] - \rho_\varepsilon(\cdot, \omega)(k^2 + i\eta)u_\varepsilon^\eta(\cdot, \omega) = f \quad \text{in } \mathbb{R}, \text{ for a.e. } \omega \in \Omega. \quad (4)$$

The proof of the limiting absorption principle requires a good knowledge of the solution of the Helmholtz equation with damping. This is what we investigate here. The subject of the limiting absorption principle will be the subject of a forthcoming paper. In the whole paper, we fix the damping parameter, so we suppress the subscript η . In other words, we consider (3) with $\text{Im}(k^2) > 0$ so that there exists *a.s.* a unique H^1 solution u_ε .

In order to restrict the computation to the bounded region $(-a, a)$, one would like to characterize the restriction of u_ε to $(-a, a)$ by the solution of (3) in $(-a, a)$ that satisfies boundary conditions of the form

$$\pm [\kappa_\varepsilon \frac{\partial u_\varepsilon}{\partial x}](\cdot, \omega) + \lambda_\varepsilon^\pm(\omega)u_\varepsilon(\cdot, \omega) = 0 \quad \text{on } \pm a, \text{ for a.e. } \omega \in \Omega \quad (5)$$

where $\lambda_\varepsilon^\pm(\omega)$ are the Dirichlet-to-Neumann (DtN) coefficients defined by

$$\lambda_\varepsilon^\pm(\omega) := \mp [\kappa_\varepsilon \frac{\partial u_\varepsilon}{\partial x}](\pm a, \omega), \text{ for a.e. } \omega \in \Omega$$

and $u_\varepsilon^\pm(\cdot, \omega)$ is the unique H^1 solution of the half-line problem,

$$\begin{cases} -\partial_x[\kappa_\varepsilon(\cdot, \omega)\partial_x u_\varepsilon^\pm(\cdot, \omega)] - \rho_\varepsilon(\cdot, \omega)k^2 u_\varepsilon^\pm(\cdot, \omega) = 0 \quad \text{on } (\pm a, \pm\infty), \text{ for a.e. } \omega \in \Omega \\ u_\varepsilon^\pm(\pm a, \omega) = 1, \text{ for a.e. } \omega \in \Omega. \end{cases} \quad (6)$$

For the periodic case, the construction of such DtN coefficients was proposed in [15,11]. A similar construction is not achievable in the present case, because of the partial knowledge of the exterior medium. Our goal here is to propose approximations of $\lambda_\varepsilon^\pm(\omega)$ at different orders in ε and quantify how these approximations approach the exact DtN coefficient. As you shall see, the zero-order approximation of $\lambda_\varepsilon^\pm(\omega)$ is a DtN coefficient associated to a periodic medium. We want to understand if and in which sense the higher order terms could correct this first deterministic order term theoretically. Moreover, we do not want to only give a theoretical definition of these terms, we also want to be able to compute them and use them from a numerical point of view.

Finally, let us describe now the two types of weak random perturbations that we study in this paper.

1. **the homogenization regime.** We suppose that for a.e. $x \in (\pm a, \pm\infty)$, $\omega \in \Omega$, κ_ε and ρ_ε can be written as

$$\kappa_\varepsilon(x) = \kappa_{\text{per}}(x)(1 + v_s^\kappa(\frac{x}{\varepsilon}, \omega))^{-1}, \quad \rho_\varepsilon(x, \omega) = \rho_{\text{per}}(x)(1 + v_s^\rho(\frac{x}{\varepsilon}, \omega)),$$

where v_s^κ and v_s^ρ are stationary and ergodic processes that are supposed to be with mean zero without loss of generality. When κ_{per} and ρ_{per} are constant (denoted resp. by κ_0 and ρ_0), this is the classical setting to apply stochastic homogenization theory (see for instance [18,22]). In that case, it is well known that when ε goes to 0, the family (u_ε^\pm) converges almost surely weakly in $H^1(\pm a, \pm\infty)$ to the solution of the Helmholtz equation with effective constant coefficients κ_0 and ρ_0 . Under additional assumptions on the coefficients (strongly mixing properties, short range of correlation,...) it is possible to provide error estimates in terms of ε . This is called the quantitative homogenization. This was investigated in lots of papers for any dimension (see for instance [14, 3,6]). A precise definition of the first corrector (*i.e.* the second term of the asymptotics in terms of ε) was given in 1D in [5,13] provided that the coefficients satisfy some mixing properties. Under the same properties for v_s^κ and v_s^ρ (stationarity and strongly mixing property), we propose here an extension of these results when the coefficient κ_{per} or ρ_{per} are general periodic functions. Let us emphasize that to derive an asymptotic expansion of the DtN coefficient, we need a precise description on the flux $\kappa_\varepsilon \partial_x u_\varepsilon^\pm$ and not only the solution. See Section 3.

2. **Rare random perturbations case.** We suppose here that for a.e. $x \in \pm a, \pm\infty$, $\omega \in \Omega$, the perturbations $\tilde{\kappa}_\varepsilon$ and $\tilde{\rho}_\varepsilon$ defined in (1) can be written as

$$\begin{aligned} \tilde{\kappa}_\varepsilon(x, \omega) &= b_\varepsilon(x, \omega) \tilde{\kappa}_{\text{per}}(x), \\ \tilde{\rho}_\varepsilon(x, \omega) &= b_\varepsilon(x, \omega) \tilde{\rho}_{\text{per}}(x), \end{aligned}$$

where $\tilde{\kappa}_{\text{per}}$ and $\tilde{\rho}_{\text{per}}$ are L -periodic functions and the random perturbation b_ε is given by

$$b_\varepsilon(x, \omega) = \sum_{j \in \mathbb{Z}} \mathbb{1}_{I_j}(x) B_\varepsilon^j(\omega), \quad I_j = [jL, (j+1)L]. \quad (7)$$

Here $(B_\varepsilon^j)_{j \in \mathbb{N}}$ are independent Bernoulli random variables with parameter $\varepsilon \in [0, 1]$. Intuitively, each period with parameters $\{\kappa_{\text{per}}, \rho_{\text{per}}\}$ is perturbed with probability ε , independently of the other periods, to have its coefficients modified to $\{\kappa_{\text{per}} + \tilde{\kappa}_{\text{per}}, \rho_{\text{per}} + \tilde{\rho}_{\text{per}}\}$. The idea of this problem originates from a paper [1] by C. Le Bris and A. Anantharaman who worked on this particular problem but in the context of homogenization to model composite materials with defects. [2] and [20] extend the framework to other random perturbations but are still considering the homogenized problem. We want here to investigate this case for the Helmholtz equation and in the general regime where homogenization techniques cannot necessarily be applied. See Section 4.

2 Preliminary results

In the whole paper, for any Banach space V , we define

$$L^2(\Omega, V) := \{ u, \text{ for a.e. } \omega \in \Omega, u(\cdot, \omega) \in V, \mathbb{E}[\|u(\cdot, \omega)\|_V^2] < +\infty \}$$

equipped with the norm

$$\forall u \in L^2(\Omega, V), \|u\|_{L^2(\Omega, V)} = \sqrt{\mathbb{E}[\|u(\cdot, \omega)\|_V^2]}.$$

The approximations of the DtN coefficients λ_ε^\pm at different orders in ε are obtained from similar approximations of the half-line solutions u_ε^\pm . The half-line problems (6) are similar but independent one from the other. We can then concentrate on one half-line problem to obtain approximations of the corresponding solution and the corresponding DtN coefficient. Without loss of generality, we suppose that $a = 0$. In the sequel, we thus study the solution $u_\varepsilon^+ \in L^2(\Omega, H^1((0, +\infty)))$ of

$$\begin{cases} -\partial_x[\kappa_\varepsilon \partial_x u_\varepsilon^+] - \rho_\varepsilon k^2 u_\varepsilon^+ = 0 \text{ on } (0, +\infty), \\ u_\varepsilon^+(0, \cdot) = 1. \end{cases} \quad (8)$$

and the corresponding DtN coefficient defined a.s. as

$$\lambda_\varepsilon^+ := -[\kappa_\varepsilon \frac{\partial u_\varepsilon^+}{\partial x}](0, \cdot). \quad (9)$$

The well-posedness of the half-line problem (8) is recalled in the following proposition.

Proposition 1 *There exists a unique solution in $L^2(\Omega, H^1((0, +\infty)))$ of (8).*

Proof The proof is really classical for deterministic problem, we only extend it for our random setting. It is sufficient to show that for any $f \in L^2(\Omega, H^{-1}((0, +\infty)))$, there exists a unique solution in $L^2(\Omega, H_0^1((0, +\infty)))$ of

$$-\partial_x[\kappa_\varepsilon \partial_x v] - \rho_\varepsilon k^2 v = f \text{ on } (0, +\infty).$$

Let $a_\varepsilon : L^2(\Omega, H_0^1((0, +\infty))) \times L^2(\Omega, H_0^1((0, +\infty))) \rightarrow \mathbb{C}$ be the sesquilinear form defined by

$$\forall v, w \in L^2(\Omega, H_0^1((0, +\infty))), \quad a_\varepsilon(v, w) = \mathbb{E} \left[\left\langle \kappa_\varepsilon \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x} \right\rangle - k^2 \langle \rho_\varepsilon v, w \rangle \right],$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2((0, +\infty))$. To apply Lax-Milgram theorem, the only difficulty comes from the proof of coercivity of a_ε . For all $v \in L^2(\Omega, H_0^1((0, +\infty)))$

$$\begin{aligned} \Im\left(\frac{1}{k} a(v, v)\right) &= \Im\left(\frac{1}{k} \mathbb{E} \left[\left\langle \kappa_\varepsilon \frac{\partial v}{\partial x}, \frac{\partial v}{\partial x} \right\rangle - k^2 \langle \rho_\eta v, v \rangle \right]\right), \\ &= -\Im(k) \mathbb{E} \left[\frac{1}{|k|^2} \left\langle \kappa_\varepsilon \frac{\partial v}{\partial x}, \frac{\partial v}{\partial x} \right\rangle + \langle \rho_\eta v, v \rangle \right]. \end{aligned}$$

Thus, using that for all $z \in \mathbb{C}$, $|z| \geq \Im z$, we obtain

$$|a(v, v)| \geq \Im(k) |k| \min\left(\frac{\kappa_-}{|k|^2}, \rho_-\right) \|v\|_{L^2(\Omega, H^1((0, +\infty)))}^2.$$

■

Since we are working in presence of dissipation $\Im(k^2) > 0$, we know that the solution's amplitude decreases exponentially to 0. The first step in our study is to take advantage of this decaying property and approximate the solution u_ε by the solution of the Helmholtz equation in a finite interval. So for any $N \in \mathbb{N}$, let us define u_ε^N the unique solution in $L^2(\Omega, H^1((0, NL)))$ of the following problem

$$\begin{cases} -\partial_x[\kappa_\varepsilon \partial_x u_\varepsilon^N] - \rho_\varepsilon k^2 u_\varepsilon^N = 0 \text{ on } \Omega \times (0, NL), \\ u_\varepsilon^N(0, \cdot) = 1, \quad u_\varepsilon^N(NL, \cdot) = 0. \end{cases} \quad (10)$$

This problem is well-posed in $L^2(\Omega, H^1((0, NL)))$ as a direct consequence of the following proposition.

Proposition 2 For any $f \in L^2(\Omega, H^{-1}((0, NL)))$, there exists a unique solution v^N in $L^2(\Omega, H_0^1((0, NL)))$ of

$$-\partial_x[\kappa_\varepsilon \partial_x v^N] - \rho_\varepsilon k^2 v^N = f \text{ on } (0, NL),$$

and

$$\|v^N\|_{L^2(\Omega, H^1((0, NL)))} \leq C \|f\|_{L^2(\Omega, H^{-1}((0, NL)))},$$

where $C = \left[\Im(k) |k| \min\left(\frac{\kappa_-}{|k|^2}, \rho_-\right) \right]^{-1}$.

The proof is the same than the one of Proposition 1.

Remark 1 The condition on $x = NL$ is a Dirichlet condition but it could be of course Neumann or any combination of the two.

We have the following estimate of the approximation of u_ε by u_ε^N .

Proposition 3 There exists a constant $C > 0$ depending on k, κ_+ and ρ_- such that for all $\alpha < \Im(k^2)\rho_-/\kappa_+$

$$\|u_\varepsilon^+ - u_\varepsilon^N\|_{L^2(\Omega, H^1((0, NL)))} \leq C e^{-\alpha NL}. \quad (11)$$

We can then choose N to obtain an error as small as we want to. Let us finally define the DtN coefficient associated to (10)

$$\lambda_\varepsilon^N := -[\kappa_\varepsilon \frac{\partial u_\varepsilon^N}{\partial x}](0, \cdot). \quad (12)$$

In the following sections, we propose an asymptotic expansion of u_ε^N and λ_ε^N in terms of ε for the two cases described at the end of the Introduction. In both cases, we expect that the limit behaviour of u_ε^N or in other words, the first term in the asymptotic expansion, is the solution of the problem with the periodic coefficients $(\kappa_{\text{per}}, \rho_{\text{per}})$. Let us define u_{per}^N the solution of the wave equation in the unperturbed domain $(0, NL)$. u_{per}^N is the unique solution in $H^1((0, NL))$ of

$$\begin{cases} -\partial_x[\kappa_{\text{per}} \partial_x u_{\text{per}}^N] - k^2 \rho_{\text{per}} u_{\text{per}}^N = 0 & \text{in } (0, NL), \\ u_{\text{per}}^N(0) = 1, \quad u_{\text{per}}^N(NL) = 0. \end{cases} \quad (13)$$

and λ_{per}^N is the associated DtN coefficient defined by

$$\lambda_{\text{per}}^N = -[\kappa_{\text{per}} \partial_x u_{\text{per}}^N](0). \quad (14)$$

3 Homogenization regime

We consider in this section the solution u_ε^N of (10) when the coefficients κ_ε and ρ_ε are such that for a.e. $x \in \mathbb{R}^+$ and $\omega \in \Omega$

$$\kappa_\varepsilon(x) = \kappa_{\text{per}}(x) \left(1 + v_s^\kappa\left(\frac{x}{\varepsilon}, \omega\right)\right)^{-1}, \quad \rho_\varepsilon(x, \omega) = \rho_{\text{per}}(x) \left(1 + v_s^\rho\left(\frac{x}{\varepsilon}, \omega\right)\right),$$

where we recall that κ_{per} and ρ_{per} are L -periodic functions and v_s^κ and v_s^ρ are jointly strictly stationary processes on $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfy strongly mixing properties. Without loss of generality, we suppose that v_s^κ and v_s^ρ have mean zero.

If v_s^κ and v_s^ρ are jointly strictly stationary ergodic processes and κ_{per} and ρ_{per} are constant (resp. κ_0 and ρ_0), we know from the stochastic homogenization theory that u_ε^N converges weakly a.s. in $H^1((0, NL))$ to u_0^N unique solution in $H^1((0, NL))$ of the following homogenized problem

$$\begin{cases} -\kappa_0 d_x^2 u_0^N - k^2 \rho_0 u_0^N = 0 & \text{in } (0, NL), \\ u_0^N(0) = 1, \quad u_0^N(NL) = 0. \end{cases}$$

We want to establish a similar result in the case of a perturbed periodic medium, quantify the rate of convergence and compute a corrector, *i.e.* the second term in the asymptotic expansion of u_ε^N in

terms of ε . As for the classical quantitative homogenization result, the mixing properties of v_s^κ and v_s^ρ are important to quantify the rate of convergence of u_ε^N and to exhibit a corrector. More precisely, we assume that v_s^ρ and v_s^κ verify jointly a specific type of strongly mixing assumption: they are jointly ρ -mixing processes with an integrable and decreasing coefficient φ . In the case of constant κ_{per} and ρ_{per} , extensive work has been done to quantify the convergence of u_ε^N to u_0^N (see for example [10]). To extend the setting to periodic κ_{per} and ρ_{per} we choose to follow the specific approach of [5].

Before doing so, we recall elementary definitions and properties of the stationary and strongly mixing processes.

3.1 Stationary, ergodic and strongly mixing processes : definitions and properties

Definition 1 A process $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^p$ is strictly stationary on $(\Omega, \mathcal{F}, \mathbb{P})$ if for all $n \in \mathbb{N}^*$, $x_1, \dots, x_n \in \mathbb{R}$, $I_1, I_2, \dots, I_n \in \mathcal{B}(\mathbb{R}^p)$ and all $y \in \mathbb{R}$

$$\mathbb{P}(f(x_1 + y) \in I_1, \dots, f(x_n + y) \in I_n) = \mathbb{P}(f(x_1) \in I_1, \dots, f(x_n) \in I_n)$$

In other words, the distribution of $f(x_1 + y), f(x_2 + y), \dots, f(x_n + y)$ under \mathbb{P} is independent of y .

Let X denote the set of bounded measurable functions g from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p))$ that verify a.e.

$$|g_i| \leq C < \min(\rho_-, \rho_+, \kappa_-, \kappa_+), \forall i \in [1, p].$$

$\mathcal{B}(\mathbb{R}^p)$ denotes here the Borel σ -algebra of \mathbb{R}^p . We endow X with the associated cylindrical σ -algebra \mathcal{F}_X .

For $y \in \mathbb{R}$, we introduce the shift operator on X defined by

$$\theta_y g = g(y + \cdot), \quad \forall g \in X.$$

If f is a strictly stationary process, we have that for all $y \in \mathbb{R}$, $A \in \mathcal{F}_X$

$$\mathbb{P}_f(A) = \mathbb{P}_f(\theta_y A),$$

where \mathbb{P}_f is the law of f defined by

$$\mathbb{P}_f(A) := \mathbb{P}(f \in A).$$

We say that the shifts θ_y determine a measure-preserving flow in $(X, \mathcal{F}_X, \mathbb{P}_f)$.

Definition 2 A set $A \in \mathcal{F}_X$ is called invariant if for all $y \in \mathbb{R}$, we have $A = \theta_y^{-1} A$.

Definition 3 We say that the process f is ergodic if for all $A \in \mathcal{F}_X$ invariant, $\mathbb{P}_f(A) \in \{0, 1\}$.

We recall now the following fundamental result on stationary ergodic processes, on which is based the demonstration of the stochastic homogenization result for one dimensional problems. The proof of this result can be found in [24, 19].

Theorem 1 (Ergodic theorem) We suppose that f is a strictly stationary ergodic process. Let $g : X \rightarrow \mathbb{R}^p$ be a measurable function in $L^1(\mathbb{P}_f)$. We have \mathbb{P} -a.s. and in $L^1(\mathbb{P})$

$$\frac{1}{2T} \int_{-T}^T g(f(y + \cdot)) dy \xrightarrow{T \rightarrow +\infty} \mathbb{E}[g(f)].$$

A sufficient condition for a stationary process to be ergodic is to be strongly mixing.

Definition 4 A stationary process f is strongly mixing if for all $A, B \in \mathcal{F}_X$

$$\mathbb{P}_f(A \cap \theta_y^{-1} B) \xrightarrow{y \rightarrow +\infty} \mathbb{P}_f(A) \mathbb{P}_f(B).$$

Proposition 4 A stationary strongly mixing process is ergodic.

Proof Let A be an invariant set of \mathcal{F}_X . We take $B = A$ in the definition of the strongly mixing process. Since for all $y \in \mathbb{R}$, $\theta_y^{-1}A = A$ we obtain $\mathbb{P}_f(A) = \mathbb{P}_f(A)^2$. Thus $\mathbb{P}_f(A) \in \{0, 1\}$. ■

Definition 5 The process $f := (f_1, \dots, f_p)$ is ρ -mixing with an integrable and decreasing coefficient φ if there exists a function $\varphi \in L^1((0, +\infty))$ such that for all Borel sets $A, B \in \mathcal{B}(\mathbb{R})$ and all $i, j \in \{1, \dots, p\}$ the following inequality holds for all $g_{A,i} \in L^2(\Omega, \mathcal{F}_{A,i}, \mathbb{P})$, $g_{B,j} \in L^2(\Omega, \mathcal{F}_{B,j}, \mathbb{P})$,

$$|\text{Corr}(g_{A,i}, g_{B,j})| := \left| \frac{\mathbb{E}[(g_{A,i} - \mathbb{E}[g_{A,i}])(g_{B,j} - \mathbb{E}[g_{B,j}])]}{\sqrt{\mathbb{E}[g_{A,i}^2]\mathbb{E}[g_{B,j}^2]}} \right| \leq \varphi(d(A, B)), \quad (15)$$

where $\mathcal{F}_{A,i}$ (resp. $\mathcal{F}_{B,j}$) denotes the sub- σ -algebra of \mathcal{F} generated by $f_i(x, \cdot)$ for $x \in A$ (resp. $f_j(x, \cdot)$ for $x \in B$). In this paper, we will also suppose that the corresponding function φ is such that $\varphi^{1/2} \in L^1((0, +\infty))$.

Let us define the cross-correlation functions associated to a stationary process $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^p$ defined by

$$R_{f_i, f_j}(x) := \mathbb{E}[f_i(y)f_j(y+x)], \forall i, j \in \{1, \dots, p\},$$

where $f = (f_1, \dots, f_p)$ and for all i , $f_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$.

Note that since f is strictly stationary, $(R_{f_i, f_j})_{i,j}$ as defined above are independent on y . Note also that if f is a strictly stationary and strongly mixing process then if f is ρ -mixing, its cross correlation functions are integrable on \mathbb{R} (take $A = \{y\}$ and $B = \{x+y\}$).

Finally we say that two processes ν^κ and ν^ρ are jointly strictly stationary and strongly mixing if (ν^κ, ν^ρ) is strictly stationary and strongly mixing.

Let us now state two technical results that will be useful for our study.

Lemma 1 Let $a^i, i \in \{1, \dots, 4\}$ be bounded zero-mean stationary jointly ρ -mixing processes with coefficient $\varphi \in L^1((0, +\infty))$ such that $\varphi^{1/2} \in L^1((0, +\infty))$. Let $a_\varepsilon^i := a^i\left(\frac{\cdot}{\varepsilon}\right)$. For a fixed $D > 0$, we have the following estimates

$$\int_0^D \int_0^D \left| \mathbb{E}[a_\varepsilon^1(t_1)a_\varepsilon^2(t_2)] \right| dt_1 dt_2 \leq C\varepsilon, \quad (16)$$

$$\int_0^D \int_0^D \int_0^D \int_0^D \left| \mathbb{E}[a_\varepsilon^1(t_1)a_\varepsilon^2(t_2)a_\varepsilon^3(t_3)a_\varepsilon^4(t_4)] \right| dt_1 dt_2 dt_3 dt_4 \leq C\varepsilon^2. \quad (17)$$

Proof Let $0 \leq x_1, x_2 \leq D$. By definition of the correlation function, we can express the left hand side of (16) as follows

$$\int_0^D \int_0^D \left| \mathbb{E}[a_\varepsilon^1(t_1)a_\varepsilon^2(t_2)] \right| dt_1 dt_2 = \int_0^D \int_0^D \left| R_{a_1, a_2}\left(\frac{t_1 - t_2}{\varepsilon}\right) \right| dt_1 dt_2.$$

By a change of variables and integrability of R_{a_1, a_2} , we get

$$\int_0^D \int_0^D \left| \mathbb{E}[a_\varepsilon^1(t_1)a_\varepsilon^2(t_2)] \right| dt_1 dt_2 \leq D \int_{\mathbb{R}} \left| R_{a_1, a_2}\left(\frac{x}{\varepsilon}\right) \right| dx \leq \varepsilon D \|R_{a_1, a_2}\|_{L^1(\mathbb{R})}.$$

Let Σ be the set of permutations of $\{1, \dots, 4\}$. From [5, Lemma 2.1] (which gives us the first inequality) and thanks to a change of variables, we can show that

$$\begin{aligned} & \int_0^D \int_0^D \int_0^D \int_0^D \left| \mathbb{E} [a_\varepsilon^1(t_1) a_\varepsilon^2(t_2) a_\varepsilon^3(t_3) a_\varepsilon^4(t_4)] \right| dt_1 dt_2 dt_3 dt_4, \\ & \leq C \sum_{\sigma \in \Sigma} \int_0^D \int_0^D \int_0^D \int_0^D \varphi^{\frac{1}{2}} \left(\frac{|t_{\sigma(1)} - t_{\sigma(2)}|}{2\varepsilon} \right) \varphi^{\frac{1}{2}} \left(\frac{|t_{\sigma(3)} - t_{\sigma(4)}|}{2\varepsilon} \right) dt_1 dt_2 dt_3 dt_4, \\ & \leq 24C D^2 \int_0^\infty \int_0^\infty \varphi^{\frac{1}{2}} \left(\frac{|t_1|}{2\varepsilon} \right) \varphi^{\frac{1}{2}} \left(\frac{|t_2|}{2\varepsilon} \right) dt_1 dt_2, \\ & \leq 24C \varepsilon^2 D^2 \left(\int_0^\infty \varphi^{\frac{1}{2}}(|s|) ds \right)^2. \end{aligned}$$

■

Lemma 2 Let $D > 0$, $\mathcal{H}^i, i \in \{1, 2\}$ be operators on $L^2(0, D)$ defined by

$$\mathcal{H}^i f(x) := \int_0^D H^i(x, y) f(y) dy$$

such that $H^i, i \in \{1, 2\}$ are bounded functions on $\{0 \leq x, y \leq D\}$. Let $a^i, i \in \{1, 2\}$ be bounded zero-mean stationary jointly ρ -mixing processes with coefficient $\varphi \in L^1((0, +\infty))$ such that $\varphi^{1/2} \in L^1((0, +\infty))$. Let $a_\varepsilon^i := a^i \left(\frac{\cdot}{\varepsilon} \right)$. Then there exists a positive constant C such that for all $f \in L^\infty([0, D])$,

$$\mathbb{E} [\|\mathcal{H}^1[a_\varepsilon^1 f]\|_{L^2}^2] \leq C\varepsilon \|f\|_{L^\infty}^2, \quad (18)$$

$$\mathbb{E} [\|\mathcal{H}^1[a_\varepsilon^1 \mathcal{H}^2[a_\varepsilon^2 f]]\|_{L^2}^2] \leq C\varepsilon^2 \|f\|_{L^\infty}^2. \quad (19)$$

Proof We have for all $f \in L^\infty([0, D])$

$$\begin{aligned} \mathbb{E} [\|\mathcal{H}^1[a_\varepsilon^1 f]\|_{L^2}^2] &= \mathbb{E} \left[\int_0^D \left| \int_0^D H^1(x, y) a_\varepsilon^1(y) f(y) dy \right|^2 dx \right], \\ &= \int_0^D \int_0^D \int_0^D H^1(x, t) \bar{H}^1(x, t') \mathbb{E} [a_\varepsilon^1(t) a_\varepsilon^1(t')] f(t) \bar{f}(t') dt dt' dx, \\ &\leq C \|f\|_\infty^2 \int_0^D \int_0^D \int_0^D |\mathbb{E} [a_\varepsilon^1(t) a_\varepsilon^1(t')] | dt dt' dx. \end{aligned}$$

The first inequality (16) in Lemma (1) allows finally to show that

$$\mathbb{E} [\|\mathcal{H}^1[a_\varepsilon^1 f]\|_{L^2}^2] \leq \varepsilon C \|f\|_\infty^2.$$

We proceed in the same manner to obtain the last estimate. Indeed, for all $f \in L^\infty([0, D])$

$$\begin{aligned} \mathbb{E} [\|\mathcal{H}^1[a_\varepsilon^1 \mathcal{H}^2[a_\varepsilon^2 f]]\|_{L^2}^2] &= \mathbb{E} \left[\int_0^D \left| \int_0^D H^1(x, y) a_\varepsilon^1(y) \int_0^D H^2(y, z) a_\varepsilon^2(z) f(z) dz dy \right|^2 dx \right], \\ &= \int_0^D \int_0^D \int_0^D \int_0^D \int_0^D H^1(x, y) \bar{H}^1(x, y') H^2(y, z) \bar{H}^2(y', z') \\ &\quad \mathbb{E} [a_\varepsilon^1(y) a_\varepsilon^1(y') a_\varepsilon^2(z) a_\varepsilon^2(z')] f(z) \bar{f}(z') dy dy' dz dz' dx, \\ &\leq C \|f\|_\infty^2 \int_0^D \int_0^D \int_0^D \int_0^D |\mathbb{E} [a_\varepsilon^1(y) a_\varepsilon^1(y') a_\varepsilon^2(z) a_\varepsilon^2(z')] | dy dy' dz dz'. \end{aligned}$$

The second inequality (17) in Lemma (1) gives us

$$\mathbb{E} [\|\mathcal{H}^1[a_\varepsilon^1 \mathcal{H}^2[a_\varepsilon^2 f]]\|_{L^2}^2] \leq \varepsilon^2 C \|f\|_\infty^2.$$

■

Remark 1 These estimates are central in the analysis of the following section. Similar results hold if f is in $L^2(0, D)$:

$$\mathbb{E} [\|\mathcal{H}^1[a_\varepsilon^1 \mathcal{H}^2[a_\varepsilon^2 f]]\|_{L^2}^2] \leq C\varepsilon \|f\|_{L^2}^2 \quad (20)$$

or if $H^i, i \in \{1, 2\}$ are supposed to be in $L^{2+\eta}(0, D)$ for some $\eta > 0$ (as it is done in [5]).

In the following, we suppose that v_s^κ and v_s^ρ are zero-mean stationary jointly ρ -mixing processes with coefficient $\varphi \in L^1((0, +\infty))$ such that $\varphi^{1/2} \in L^1((0, +\infty))$.

3.2 The limit behaviour

We expect that the solution u_ε^N of (10) tends, in a certain sense to be determined, to the solution u_{per}^N of the problem (13) with the periodic coefficients $(\kappa_{\text{per}}, \rho_{\text{per}})$.

Let us introduce the Green function G associated to the periodic problem as follows: for all $y \in (0, NL)$, $G(\cdot, y)$ is the unique solution in $H^1((0, NL))$ of

$$\begin{cases} -\partial_x [\kappa_{\text{per}} \partial_x G(\cdot, y)] - k^2 \rho_{\text{per}} G(\cdot, y) = \delta(\cdot - y) & \text{in } (0, NL), \\ G(0, y) = 0, \quad G(NL, y) = 0. \end{cases}$$

u_{per}^N and u_ε^N can then be expressed for all $y \in [0, NL]$ as

$$\begin{aligned} u_{\text{per}}^N(y) &= \kappa_{\text{per}}(0) \partial_x G(0, y), \\ u_\varepsilon^N(y) &= -\int_0^{NL} \partial_x G(x, y) [\tilde{\kappa}_\varepsilon \partial_x u_\varepsilon^N](x) dx + k^2 \int_0^{NL} G(x, y) [\tilde{\rho}_\varepsilon u_\varepsilon^N](x) dx + u_{\text{per}}^N(y), \end{aligned}$$

where we recall that $\tilde{\kappa}_\varepsilon := \kappa_\varepsilon - \kappa_{\text{per}}$ and $\tilde{\rho}_\varepsilon := \rho_\varepsilon - \rho_{\text{per}}$.

Let us introduce the volume integral operators \mathcal{G} and \mathcal{G}_1 on $L^2((0, NL))$ defined by

$$\forall f \in L^2(0, NL), \forall y \in (0, NL), \mathcal{G}f(y) = \int_0^{NL} G(x, y) f(x) dx \text{ and } \mathcal{G}_1 f(y) = \int_0^{NL} \partial_x G(x, y) f(x) dx,$$

u_ε^N reads then

$$u_\varepsilon^N = -\mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_\varepsilon^N \right] + k^2 \mathcal{G} [\tilde{\rho}_\varepsilon u_\varepsilon^N] + u_{\text{per}}^N, \quad (21)$$

where we have introduced $v_\varepsilon^N := \kappa_\varepsilon \partial_x u_\varepsilon^N$. Since u_ε^N is solution of (10), it is easy to see that v_ε^N , corresponding to a flux quantity, is solution of the following equation

$$\begin{cases} -\partial_x \left[\frac{1}{\rho_\varepsilon} \partial_x v_\varepsilon^N \right] - \frac{k^2}{\kappa_\varepsilon} v_\varepsilon^N = 0 & \text{in } (0, NL), \\ -\left[\frac{1}{\rho_\varepsilon} \partial_x v_\varepsilon^N \right](0) = k^2 \text{ and } \left[\frac{1}{\rho_\varepsilon} \partial_x v_\varepsilon^N \right](NL) = 0. \end{cases} \quad (22)$$

We expect that v_ε^N tends to $v_{\text{per}}^N := \kappa_{\text{per}} \partial_x u_{\text{per}}^N$, where v_{per}^N is solution of the same problem as v_ε^N but replacing κ_ε (resp. ρ_ε) by κ_{per} (resp. ρ_{per}). The associated Green function, denoted by G^* , is defined as follows: for all $y \in (0, NL)$, $G^*(\cdot, y)$ is the unique solution in $H^1((0, NL))$ of

$$\begin{cases} -\partial_x \left[\frac{1}{\rho_{\text{per}}} \partial_x G^*(\cdot, y) \right] - \frac{k^2}{\kappa_{\text{per}}} G^*(\cdot, y) = \delta(\cdot - y) & \text{in } (0, NL), \\ \left[\frac{1}{\rho_{\text{per}}} \partial_x G^* \right](0, y) = 0 \text{ and } \left[\frac{1}{\rho_{\text{per}}} \partial_x G^* \right](NL, y) = 0. \end{cases}$$

Correspondingly, v_{per}^N and v_ε^N can then be written for all $y \in (0, NL)$ as

$$v_{\text{per}}^N(y) = k^2 G^*(0, y),$$

$$v_\varepsilon^N(y) = \int_0^{NL} \partial_x G^*(x, y) \left[\frac{\tilde{\rho}_\varepsilon}{\rho_{\text{per}} \rho_\varepsilon} \partial_x v_\varepsilon^N \right](x) dx - k^2 \int_0^{NL} G^*(x, y) \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_{\text{per}} \kappa_\varepsilon} v_\varepsilon^N \right](x) dx + v_{\text{per}}^N(y).$$

By introducing the associated volume integral operators \mathcal{G}^* and \mathcal{G}_1^* defined by

$$\forall f \in L^2(0, NL), \forall y \in (0, NL), \mathcal{G}^* f(y) = \int_0^{NL} G^*(x, y) f(x) dx \text{ and } \mathcal{G}_1^* f(y) = \int_0^{NL} \partial_x G^*(x, y) f(x) dx,$$

v_ε^N reads then

$$v_\varepsilon^N = -k^2 \mathcal{G}_1^* \left[\frac{\tilde{\rho}_\varepsilon}{\rho_{\text{per}}} u_\varepsilon^N \right] - k^2 \mathcal{G}^* \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_{\text{per}} \kappa_\varepsilon} v_\varepsilon^N \right] + v_{\text{per}}^N, \quad (23)$$

where we have used that $u_\varepsilon^N = -\frac{1}{k^2 \rho_\varepsilon} d_x v_\varepsilon^N$.

Thanks to the two expressions (23) and (21), we can show the expected convergence result with a quantification of the convergence. The result is the same as the classical one of the stochastic homogenization theory, obtained in [5] and [13].

Theorem 2 *There exists a constant C such that*

$$\|u_\varepsilon^N - u_{\text{per}}^N\|_{L^2(\Omega, L^2((0, NL)))} \leq C\sqrt{\varepsilon}, \quad \|v_\varepsilon^N - v_{\text{per}}^N\|_{L^2(\Omega, L^2((0, NL)))} \leq C\sqrt{\varepsilon}$$

$$\text{and } \|\lambda_\varepsilon^N - \lambda_{\text{per}}^N\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}. \quad (24)$$

Proof First note that $G, \partial_x G, G^*$ and $\partial_x G^*$ are uniformly bounded in $\{0 \leq x, y \leq NL\}$.

Indeed, G is in $H^1((0, NL))$ and thus continuous on $(0, NL)$ with traces at 0 and NL . This ensures that G is bounded on $[0, NL]^2$. Similarly, one can write the equation verified by the flux $\kappa_{\text{per}} \partial_x G$ as we did for v_ε . $\kappa_{\text{per}} \partial_x G$ is H^1 in both intervals $\{0 < x < y < NL\}$ and $\{0 < y < x < NL\}$ and thus uniformly bounded on $[0, NL]$. Since $\kappa_{\text{per}} \in L^\infty(\mathbb{R})$, $\partial_x G$ is also bounded on $[0, NL]^2$. We can prove with similar arguments that G^* and $\partial_x G^*$ verify the same property.

Moreover, for a.e. $x \in (0, NL)$,

$$\frac{\tilde{\kappa}_\varepsilon(x, \cdot)}{\kappa_\varepsilon(x, \cdot)} = -v_s^\kappa\left(\frac{x}{\varepsilon}, \cdot\right) \quad \text{and} \quad \frac{\tilde{\rho}_\varepsilon(x, \cdot)}{\rho_{\text{per}}(x)} = v_s^\rho\left(\frac{x}{\varepsilon}, \cdot\right).$$

We have assumed that v_s^κ and v_s^ρ are zero-mean stationary jointly ρ -mixing processes with coefficient $\varphi \in L^1((0, +\infty))$ such that $\varphi^{1/2} \in L^1((0, +\infty))$. From the proof of proposition 2, we know that u_ε^N and v_ε^N are bounded in $H^1((0, NL))$ uniformly in ω and ε . The continuous embedding of $H^1((0, NL))$ in $L^\infty([0, NL])$ implies that u_ε^N and v_ε^N are bounded in $L^\infty([0, NL])$ uniformly in ω and ε . Moreover, κ_{per} and ρ_{per} are bounded from below by a strictly positive constant.

We can therefore apply the estimate 18 of Lemma (2) and get the existence of a constant $C > 0$ such that

$$\mathbb{E} \left[\left\| \mathcal{G}_1^* \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_\varepsilon^N \right] \right\|^2 \right] + \mathbb{E} \left[\left\| k^2 \mathcal{G}^* \left[\tilde{\rho}_\varepsilon u_\varepsilon^N \right] \right\|^2 \right] \leq C\varepsilon.$$

This allow us to get the estimate on $\mathbb{E} \left[\left\| u_\varepsilon^N - u_0^N \right\|^2 \right]$. Similar arguments hold to estimate $\mathbb{E} \left[\left\| v_\varepsilon^N - v_0^N \right\|^2 \right]$.

Finally, by definition (12) and (14), λ_ε^N and λ_{per}^N are nothing else but

$$\lambda_\varepsilon^N = v_\varepsilon^N(0) \quad \text{and} \quad \lambda_{\text{per}}^N = v_{\text{per}}^N(0).$$

Then using (23), we have

$$\lambda_\varepsilon^N - \lambda_{\text{per}}^N = -k^2 \int_0^{NL} \partial_x G^*(x, 0) \left[\frac{\tilde{\rho}_\varepsilon}{\rho_{\text{per}}} u_\varepsilon^N \right](x) dx - k^2 \int_0^{NL} G^*(x, 0) \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_{\text{per}} \kappa_\varepsilon} v_\varepsilon^N \right](x) dx.$$

By using that $G^*(\cdot, 0)$, $\partial_x G^*(\cdot, 0)$, u_ε^N and v_ε^N are uniformly bounded in $[0, NL]$ and that κ_{per} is bounded from below by a strictly positive constant, we show easily that there exists a constant $C > 0$ such that

$$\mathbb{E}[|\lambda_\varepsilon^N - \lambda_{\text{per}}^N|^2] \leq C \mathbb{E} \left[\int_0^{NL} \int_0^{NL} \nu_s^\rho \left(\frac{x}{\varepsilon} \right) \nu_s^\rho \left(\frac{y}{\varepsilon} \right) dx dy + \int_0^{NL} \int_0^{NL} \nu_s^\kappa \left(\frac{x}{\varepsilon} \right) \nu_s^\kappa \left(\frac{y}{\varepsilon} \right) dx dy \right].$$

It is sufficient to use the estimate (16) of Lemma 1 to conclude. \blacksquare

3.3 Study of the first order corrector

We need now to identify the term of order $\sqrt{\varepsilon}$ in the error terms. By looking at (21) and (23), since $u_\varepsilon^N \approx u_{\text{per}}^N$ and $v_\varepsilon^N \approx v_{\text{per}}^N$, a natural intuition is that

$$u_\varepsilon^N - u_{\text{per}}^N \approx -\mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_{\text{per}}^N \right] + k^2 \mathcal{G} [\tilde{\rho}_\varepsilon u_{\text{per}}^N] \quad \text{and} \quad v_\varepsilon^N - v_{\text{per}}^N \approx -k^2 \mathcal{G}_1^* \left[\frac{\tilde{\rho}_\varepsilon}{\rho_{\text{per}}} u_{\text{per}}^N \right] - k^2 \mathcal{G}^* \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_{\text{per}} \kappa_\varepsilon} v_{\text{per}}^N \right].$$

We prove and quantify this approximation in the following proposition.

Proposition 5 *The following estimate holds*

$$\begin{aligned} \left\| u_\varepsilon^N - u_{\text{per}}^N - \left(-\mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_{\text{per}}^N \right] + k^2 \mathcal{G} [\tilde{\rho}_\varepsilon u_{\text{per}}^N] \right) \right\|_{L^2(\Omega, L^2((0, NL)))} &\leq C \varepsilon, \\ \left\| v_\varepsilon^N - v_{\text{per}}^N - \left(-k^2 \mathcal{G}_1^* \left[\frac{\tilde{\rho}_\varepsilon}{\rho_{\text{per}}} u_{\text{per}}^N \right] - k^2 \mathcal{G}^* \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_{\text{per}} \kappa_\varepsilon} v_{\text{per}}^N \right] \right) \right\|_{L^2(\Omega, L^2((0, NL)))} &\leq C \varepsilon. \end{aligned}$$

Proof Let us denote

$$U_\varepsilon = u_\varepsilon^N - u_{\text{per}}^N - \left(-\mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_{\text{per}}^N \right] + k^2 \mathcal{G} [\tilde{\rho}_\varepsilon u_{\text{per}}^N] \right).$$

Using (21), we can show that

$$U_\varepsilon = \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} (v_\varepsilon^N - v_{\text{per}}^N) \right] - k^2 \mathcal{G} [\tilde{\rho}_\varepsilon (u_\varepsilon^N - u_{\text{per}}^N)].$$

We know that u_ε^N , u_{per}^N , v_ε^N and v_{per}^N are in $L^\infty([0, NL])$, so one could use (18). However, we would have in that case only $\|U_\varepsilon\|_{L^2(\Omega, L^2((0, NL)))} \sim \sqrt{\varepsilon}$. Instead, in order to have a better estimate, we have to use the same trick as in [5] and replace in the expression u_ε^N of (21), the expression (23) of v_ε^N and reuse the expression (21) of u_ε^N . We obtain

$$\begin{aligned} u_\varepsilon^N = u_{\text{per}}^N - \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_{\text{per}}^N \right] + k^2 \mathcal{G} [\tilde{\rho}_\varepsilon u_{\text{per}}^N] + k^2 \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} \mathcal{G}_1^* \left[\frac{\tilde{\rho}_\varepsilon}{\rho_{\text{per}}} u_{\text{per}}^N \right] \right] + k^2 \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} \mathcal{G}^* \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_{\text{per}} \kappa_\varepsilon} v_{\text{per}}^N \right] \right] \\ - k^2 \mathcal{G} [\tilde{\rho}_\varepsilon \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_{\text{per}}^N \right] + k^4 \mathcal{G} [\tilde{\rho}_\varepsilon \mathcal{G} [\tilde{\rho}_\varepsilon u_{\text{per}}^N]]], \quad (25) \end{aligned}$$

which yields directly

$$U_\varepsilon = k^2 \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} \mathcal{G}_1^* \left[\frac{\tilde{\rho}_\varepsilon}{\rho_{\text{per}}} u_{\text{per}}^N \right] \right] + k^2 \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} \mathcal{G}^* \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_{\text{per}} \kappa_\varepsilon} v_{\text{per}}^N \right] \right] - k^2 \mathcal{G} [\tilde{\rho}_\varepsilon \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_{\text{per}}^N \right] + k^4 \mathcal{G} [\tilde{\rho}_\varepsilon \mathcal{G} [\tilde{\rho}_\varepsilon u_{\text{per}}^N]]].$$

Since u_ε^N is bounded on $(0, NL)$ uniformly with respect to ω and ε , as in the proof of Theorem 2, the estimate (19) gives us

$$\begin{aligned} \mathbb{E} \left[\left\| \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} \mathcal{G}_1^* \left[\frac{\tilde{\rho}_\varepsilon}{\rho_{\text{per}}} u_{\text{per}}^N \right] \right] \right\|^2 \right] + \mathbb{E} \left[\left\| \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} \mathcal{G}^* \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_{\text{per}} \kappa_\varepsilon} v_{\text{per}}^N \right] \right] \right\|^2 \right] + \mathbb{E} \left[\left\| \mathcal{G} [\tilde{\rho}_\varepsilon \mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_{\text{per}}^N \right]] \right\|^2 \right] \\ + \mathbb{E} \left[\left\| \mathcal{G} [\tilde{\rho}_\varepsilon \mathcal{G} [\tilde{\rho}_\varepsilon u_{\text{per}}^N]] \right\|^2 \right] \leq C \varepsilon^2. \end{aligned}$$

This ends our proof for u_ε^N . The estimate for v_ε^N can be demonstrated in the same manner. \blacksquare

Let us introduce the so-called correctors

$$u_{1,\varepsilon} := \frac{1}{\sqrt{\varepsilon}} \left(-\mathcal{G}_1 \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_\varepsilon} v_{\text{per}}^N \right] + k^2 \mathcal{G} \left[\tilde{\rho}_\varepsilon u_{\text{per}}^N \right] \right) \quad \text{and} \quad v_{1,\varepsilon} := \frac{1}{\sqrt{\varepsilon}} \left(-k^2 \mathcal{G}_1^* \left[\frac{\tilde{\rho}_\varepsilon}{\rho_{\text{per}}} u_{\text{per}}^N \right] - k^2 \mathcal{G}^* \left[\frac{\tilde{\kappa}_\varepsilon}{\kappa_{\text{per}} \kappa_\varepsilon} v_{\text{per}}^N \right] \right). \quad (26)$$

and the associated DtN map

$$\lambda_{1,\varepsilon} = v_{1,\varepsilon}(0). \quad (27)$$

These correctors depend on ω , and thus they would be impossible to compute, if we have only a partial knowledge on the medium. The following theorem describes however the distribution of these terms.

Theorem 3 $u_{1,\varepsilon}$ and $v_{1,\varepsilon}$ converge in distribution in the space of continuous functions $\mathcal{C}([0, NL])$ to the Gaussian processes specified below

$$\begin{aligned} u_{1,\varepsilon} &\xrightarrow[\varepsilon \rightarrow 0]{\mathcal{L}} \int_0^{NL} \sigma_u(x, t) dW_t, \\ v_{1,\varepsilon} &\xrightarrow[\varepsilon \rightarrow 0]{\mathcal{L}} \int_0^{NL} \sigma_v(x, t) dW_t. \end{aligned}$$

where W_t is a standard Brownian motion and for all $\ell \in \{u, v\}$

$$\sigma_\ell^2(x, t) = 2 \int_0^\infty \mathbb{E}(F_\ell(x, t, 0) F_\ell(x, t, z)) dz,$$

with

$$\begin{aligned} F_u(x, t, z) &= k^2 \rho_s(z) G(x, t) \rho_{\text{per}} u_{\text{per}}^N(x) + \kappa_s(z) \partial_x G(x, t) v_{\text{per}}^N(x), \\ F_v(x, t, z) &= -k^2 \rho_s(z) \partial_x G^*(x, t) u_{\text{per}}^N(x) + k^2 \kappa_s(z) G^*(x, t) \frac{1}{\kappa_{\text{per}}} v_{\text{per}}^N(x). \end{aligned}$$

Moreover, $\frac{u_\varepsilon^N - u_{\text{per}}^N}{\sqrt{\varepsilon}}$ admits the same limit as $u_{1,\varepsilon}$ in distribution in the space of square integrable functions $L^2((0, NL))$.

Proof We follow the result from [7] that we give here for completeness.

Theorem 4 Let $(Y_n)_{n \in \mathbb{N}^*}$ be a random process with values into $\mathcal{C}([0, D])$. $(Y_n)_{n \in \mathbb{N}^*}$ converges in law to Y , a random variable with values in $\mathcal{C}([0, D])$ iff

- 1) $(Y_n)_{n \in \mathbb{N}^*}$ is tight.
- 2) the finite-dimensional distribution $(Y_n(y_1), \dots, Y_n(y_p))$ converges to the joint distribution $(Y(y_1), \dots, Y(y_p))$ for any $p \in \mathbb{N}^*$, $y_1, \dots, y_p \in [0, D]$.

We show in the sequel that $Y_n = u_{1,\varepsilon_n}$ and similarly $Z_n = v_{1,\varepsilon_n}$ satisfy 1) and 2) for any $(\varepsilon_n)_n$ such that $\varepsilon_n \rightarrow 0$.

Tightness First tightness is showed by proving that the following Kolmogorov criterion holds.

Lemma 1 Let $(Y_n)_{n \in \mathbb{N}^*}$ be a random process with values into $\mathcal{C}([0, D])$. If $(Y_n)_{n \in \mathbb{N}^*}$ verifies

- 1) there exists $y_0 \in [0, D]$, $\gamma > 0$ such that $\sup_{n \geq 1} \mathbb{E}[|Y_n(y_0)|^\gamma] < +\infty$,
 - and
 - 2) there exists $\alpha, \beta > 0$ such that $\forall y, y' \in [0, D]$, $\mathbb{E}[|Y_n(y) - Y_n(y')|^\alpha] \leq C|y - y'|^{1+\beta}$, where C does not depend on n ,
- then $(Y_n)_{n \in \mathbb{N}^*}$ is tight.

$u_{1,\varepsilon}(x)$ and $v_{1,\varepsilon}(x)$ are uniformly bounded in ε in $L^2(\Omega)$ for any $x \in [0, NL]$. Therefore, the first assumption is verified for $\gamma = 2$ and any $y_0 \in [0, NL]$. Next we show that for all $y, y' \in (0, NL)$ uniformly in ε

$$\mathbb{E}[|u_{1,\varepsilon}(y) - u_{1,\varepsilon}(y')|^4] + \mathbb{E}[|v_{1,\varepsilon}(y) - v_{1,\varepsilon}(y')|^4] \leq C|y - y'|^2.$$

Let $y, y' \in [0, NL]$. $\mathbb{E}[|u_{1,\varepsilon}(y) - u_{1,\varepsilon}(y')|^4]$ is a sum of terms of the form

$$\frac{1}{\varepsilon^2} \int_0^{NL} \int_0^{NL} \int_0^{NL} \int_0^{NL} \mathbb{E}[a^1(t_1)a^2(t_2)a^3(t_3)a^4(t_4)] \prod_{i=1}^4 [H^i(t_i, y) - H^i(t_i, y')] dt_1 dt_2 dt_3 dt_4,$$

where $(a^i, H^i) \in \{(\rho_s, G), (\kappa_s, \partial_x G), (\rho_s, \bar{G}), (\kappa_s, \partial_x \bar{G})\}$. We use once again Lemma (2.1) in [5] to bound the expectation and obtain

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_0^{NL} \int_0^{NL} \int_0^{NL} \int_0^{NL} \mathbb{E}(a^1(t_1)a^2(t_2)a^3(t_3)a^4(t_4)) \prod_{i=1}^4 (H^i(t_i, y) - H^i(t_i, y')) dt_1 dt_2 dt_3 dt_4, \\ & \leq \frac{1}{\varepsilon^2} \sum_{\sigma \in \Sigma} \int_0^{NL} \int_0^{NL} \int_0^{NL} \int_0^{NL} \varphi^{\frac{1}{2}} \left(\frac{|t_{\sigma(1)} - t_{\sigma(2)}|}{\varepsilon} \right) \varphi^{\frac{1}{2}} \left(\frac{|t_{\sigma(3)} - t_{\sigma(4)}|}{\varepsilon} \right) \\ & \quad \prod_{i=1}^4 |H^i(t_i, y) - H^i(t_i, y')| dt_1 dt_2 dt_3 dt_4, \end{aligned}$$

where Σ is the set of permutations of $\{1, \dots, 4\}$. This can be bounded in turn by terms of the form

$$\left| \frac{1}{\varepsilon} \int_0^{NL} \int_0^{NL} \varphi^{\frac{1}{2}} \left(\frac{|s_1 - s_2|}{\varepsilon} \right) \prod_{i=1}^2 |H^i(s_i, y) - H^i(s_i, y')| ds_1 ds_2 \right|^2. \quad (28)$$

If $H^i = G, \bar{G}$, we know directly that $|H^i(s_i, y) - H^i(s_i, y')| \leq C|y - y'|$, since G is Lipschitz continuous in y uniformly in s . Therefore if at least one H^i in the product is G or \bar{G} , we use that $\partial_x G$ is bounded on $[0, NL]$ and that $\varphi^{\frac{1}{2}}|\cdot|$ is integrable to get the result for the term.

We are thus left to deal with term of the form (28) where $H^1, H^2 \in \{\partial_x G, \partial_x \bar{G}\}$. In order to use that $\partial_x G$ is piecewise Lipschitz continuous and bounded, we divide our integration interval into 3 parts. We suppose without loss of generality that $y < y'$. On $(0, y)$ and (y', NL) , $\partial_x G$ is Lipschitz continuous uniformly in s and we can bound $|H^i(s_i, y) - H^i(s_i, y')|$ by $C|y - y'|$. Those terms can then be dealt as the previous ones. We are left with the term on (y, y') , on which $\partial_x G$ is not Lipschitz continuous. However we can use that $\partial_x G$ is bounded on $[0, NL]$ and that $\varphi^{1/2}|\cdot|$ is integrable.

$$\begin{aligned} & \frac{1}{\varepsilon} \int_y^{y'} \int_y^{y'} \varphi^{\frac{1}{2}} \left(\frac{|s_1 - s_2|}{\varepsilon} \right) |\partial_x G(s_1, y) - \partial_x G(s_1, y')| |\partial_x G(s_2, y) - \partial_x G(s_2, y')| ds_1 ds_2 \\ & \leq C \frac{y' - y}{\varepsilon} \int_{-\infty}^{+\infty} \varphi^{\frac{1}{2}} \left(\frac{|r|}{\varepsilon} \right) dr \leq C(y' - y) \int_{-\infty}^{+\infty} \varphi^{\frac{1}{2}} |r| dr. \end{aligned}$$

This concludes the proof of the tightness.

Finite-dimensional distribution Secondly, we need to prove that the finite-dimensional distributions converge to the finite-dimensional distributions of the wanted limit.

Let $n \in \mathbb{N}^*$. Let $y_1, \dots, y_n \in (0, NL)$.

The finite-dimensional distribution $(u_{1,\varepsilon}(y_1, \omega), \dots, u_{1,\varepsilon}(y_n, \omega))$ has the following characteristic function for all $t \in \mathbb{R}^n$

$$\begin{aligned} \Phi_\varepsilon(t) &= \mathbb{E} \left[\prod_{j=1}^n e^{it_j u_{1,\varepsilon}(y_j, \omega)} \right] \\ &= \mathbb{E} \left[\exp \left(\frac{i}{\sqrt{\varepsilon}} \left(\int_0^{NL} \rho_s \left(\frac{x}{\varepsilon} \right) f_\rho(x, y, t) dx + \int_0^{NL} \kappa_s \left(\frac{x}{\varepsilon} \right) f_\kappa(x, y, t) dx \right) \right) \right], \end{aligned}$$

where $f_\rho(x, y, t) = \sum_{j=1}^n t_j k^2 \rho_{\text{per}}(x) G(x, y_j) u_{\text{per}}^N(x)$ and $f_\kappa(x, y, t) = \sum_{j=1}^n t_j \partial_x G(x, y_j) v_{\text{per}}^N$.

The demonstration relies therefore on the convergence in distribution of oscillatory integrals of the type $\frac{1}{\sqrt{\varepsilon}} \int_0^L f\left(\frac{t}{\varepsilon}\right) m(t) dt$ where m is a continuous function and f a stationary ρ -mixing process with an integrable coefficient. Those integrals have been thoroughly studied, see for example [21, 16, 5] and are proved to converge in distribution to $\int_0^D m(x) \sigma dW_x$, where $\sigma^2 = 2 \int_0^\infty R_f(r) dr$. In particular, in Theorem (2.8) of [5], one can find a clear idea of the demonstration of this result.

The convergence of $\frac{u_\varepsilon^N - u_{\text{per}}^N}{\sqrt{\varepsilon}}$ comes consequently from the estimate in Proposition 5. ■

3.4 Numerical simulations

We present in this section the results of the numerical simulations that we conducted in the homogenization regime. Our goal is to illustrate and verify the theoretical results that we proved above and to show how our approach can be used to build approximated transparent boundary conditions.

3.4.1 Randomly perturbed periodic media in the homogenization regime

Our first step is to construct numerically the two zero-mean jointly stationary strongly mixing processes v_s^κ and v_s^ρ . Following [3], to obtain a stationary finite-range dependent or strongly mixing process, a scalar white noise W , i.e. a sequence of independent standard normal random variables, is generated and convolved with a bump function f . The desired process is then a smooth function of this convolution. We choose in our simulations to take the sin of the convolution to preserve the zero-mean property. The width and decreasing rate of the bump function gives the correlation length of the generated process and the strongly mixing coefficient φ . On Figure 1, we plotted an example of a scalar white noise, a bump function $f : x \rightarrow e^{-\frac{1}{1-x^2}} \mathbb{1}_{|x| < 1}$ and the zero-mean stationary strongly mixing process obtained through the sin of the convolution of the two former. Note that with this specific example of bump function, the process has a finite range of dependence of 2. The two processes v_s^κ and v_s^ρ are generated in the same way but from different white noises. This gives two independent processes that therefore verify the jointly mixing and stationary hypothesis.

Equipped with the two processes v_s^κ and v_s^ρ , we can now simulate periodic media perturbed by oscillating stationary processes. We choose a 1-periodic background medium characterized by the two parameters

$$\kappa_{\text{per}}^{-1} = 1.1 + \sin(2\pi \cdot) \quad \text{and} \quad \rho_{\text{per}} = 1.1 + \cos(2\pi \cdot),$$

that we perturb to obtain $\kappa_\varepsilon^{-1} = \kappa_{\text{per}}^{-1} (1 + v_s^\kappa(\frac{\cdot}{\varepsilon}))$ and $\rho_\varepsilon = \rho_{\text{per}} (1 + v_s^\rho(\frac{\cdot}{\varepsilon}))$. We represented on Figure 2 two realisations of the randomly perturbed periodic coefficient ρ_ε for two different ε . On the left in green $\varepsilon = 0.01$ and on the left in blue $\varepsilon = 0.2$. As ε goes to 0, the medium oscillates faster and faster. This oscillatory behavior hints at the difficulties of computing exact solutions in this type of media. Indeed, the faster the oscillation, the smaller the needed space discretization.

Different realizations of the medium lead to different solutions of (10). To give you an idea of the variety of solutions that we deal with, we plotted on Figure 3 three different solutions of (10) associated to three different media generated as described above for two different ε . On the left, $\varepsilon = 0.05$ and on the right, $\varepsilon = 0.2$.

3.4.2 Asymptotic behavior of u_ε^N

The first result that we want to corroborate is the convergence result of Theorem 2. We choose a frequency of 0.1 and a damping coefficient $\eta = 0.2$. Before solving the equation, we compute thanks

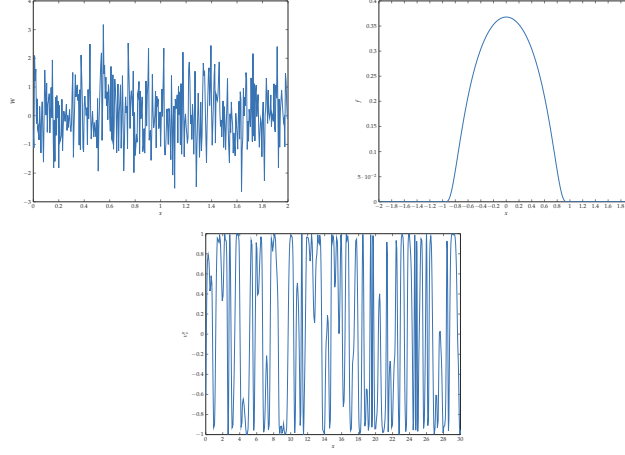


Fig. 1: White noise W , bump function f and zero-mean stationary strongly mixing process v_s^k obtained by convolution of W and f

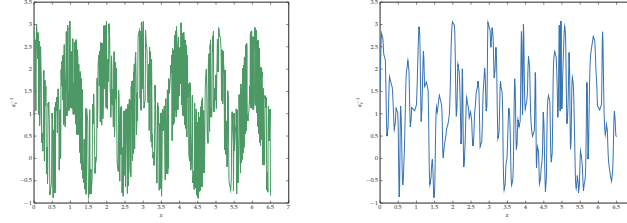


Fig. 2: ρ_ε with respect to x for two different ω and two different ε : $\varepsilon = 0.01$ (left) and $\varepsilon = 0.2$ (right)

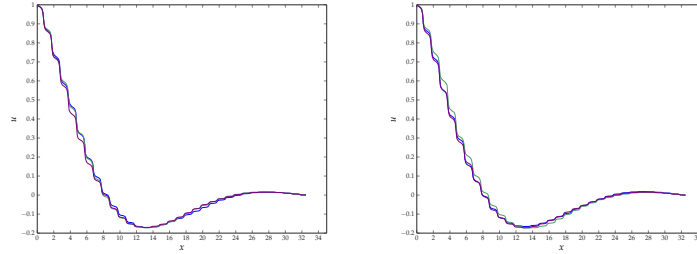


Fig. 3: u_ε^N for different ω and $\varepsilon = 0.05$ (left) and $\varepsilon = 0.2$ (right).

to Proposition 3 the number of periods N , that we need to consider our problem on, in order to obtain a good-enough approximation of our solution on the half-line, say for example an error of 10^{-5} . We then solve the equation (10) on $(0, NL)$, using a finite difference scheme with a discretization step $h = 0.001$, for 300 different randomly generated media over a range of ε from 0.05 to 0.2. By the Monte-Carlo method, we deduce, for each ε , the norm in $L^2(\Omega, L^2((0, NL)))$ of $u_\varepsilon^N - u_{per}^N$ and the norm in $L^2(\Omega)$ of the associated DtN coefficient error $\lambda_\varepsilon^N - \lambda_{per}^N$. In Figure 4, are represented in blue $\log \|u_\varepsilon^N - u_{per}^N\|_{L^2(\Omega, L^2((0, NL)))}$ and in green $\log \|\lambda_\varepsilon^N - \lambda_{per}^N\|_{L^2(\Omega)}$ with respect to $\log(\varepsilon)$. We recover the theoretical rate of convergence of $\sqrt{\varepsilon}$ in the slope of 0.5.

Next, we want to exhibit the convergence in law of the corrector stated in Theorem 3. We display this result for the DtN coefficient but note that we could have printed the exact same plots for the corrector of u_ε^N or the flux v_ε^N at any point $x \in (0, NL)$. We represented in Figure 5, on the left, the probability density of $\lambda_{1,\varepsilon} = \frac{\lambda_\varepsilon^N - \lambda_{per}^N}{\sqrt{\varepsilon}}$, for different ε ranging from 0.05 to 0.2, approximated over 300 simulations per ε . One can see that, as ε goes to 0, the probability density of $\lambda_{1,\varepsilon}$ converges to

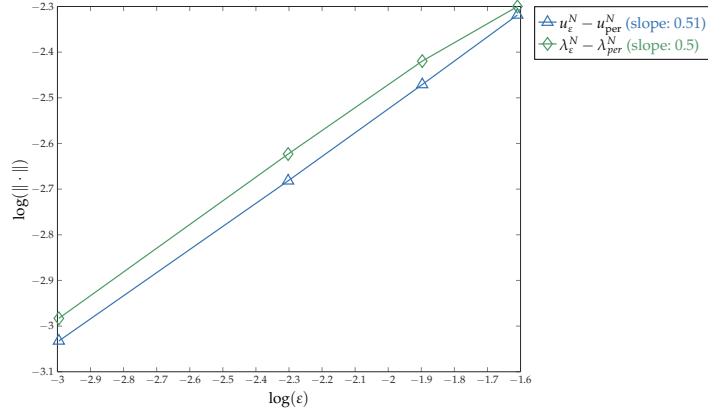


Fig. 4: Rate of convergence of u_ε^N in $L^2(\Omega, L^2((0, NL)))$ and λ_ε^N in $L^2(\Omega)$ to resp. u_{per}^N and λ_{per}^N .

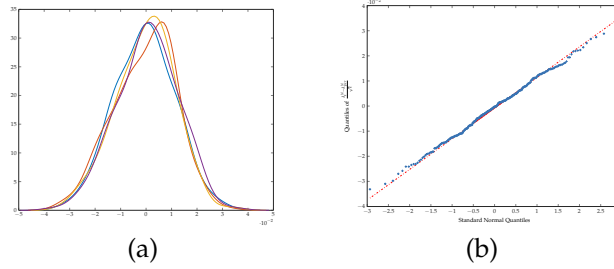


Fig. 5: (a) Probability density of $\frac{\lambda_\varepsilon^N - \lambda_{per}^N}{\sqrt{\varepsilon}}$ as ε goes to 0, (b) Normal quantile-quantile plot of the corrector $\frac{\lambda_\varepsilon^N - \lambda_{per}^N}{\sqrt{\varepsilon}}$ for $\varepsilon = 0.05$.

the density of a centered normal random variable. To quantify more accurately this limit behavior, we plotted on the right the normal quantile-quantile plot of the same quantity. On the x -axis are represented the quantiles of a normal random variable and on the y -axis the quantiles of our data. If the corrector follows as ε goes to 0 a normal distribution, its quantiles should accumulate on the red line. We can see that it is here the case for $\varepsilon = 0.05$.

3.4.3 Transparent boundary conditions

In this last section, we expose how our approach can be of use in the construction of transparent boundary conditions for randomly perturbed periodic media. We want to solve the wave equation on the half-line $[-1, +\infty)$, where $[-1, 0]$ is a known homogeneous medium with $\kappa_{hom}^{-1} = \rho_{hom} = 1$ and $[0, +\infty)$ is a randomly perturbed periodic medium modeled as above. u_ε is thus the unique solution in $H^1(-1, +\infty)$ of

$$\begin{cases} -\partial_x \kappa_\varepsilon \partial_x u_\varepsilon - k^2 \rho_\varepsilon u_\varepsilon = 0 & \text{in } (-1, +\infty), \\ u_\varepsilon(-1) = 0. \end{cases}$$

We know that the above problem is equivalent as solving

$$\begin{cases} -\partial_x \kappa_{hom} \partial_x u_\varepsilon - \rho_{hom} k^2 u_\varepsilon = 0 & \text{in } (-1, 0), \\ u_\varepsilon(-1) = 1, \quad \partial_x u_\varepsilon(0) = \lambda_\varepsilon^+ u_\varepsilon(0), \end{cases}$$

where λ_ε^+ is the DtN coefficient associated to the problem of $[0, +\infty)$. We approximate λ_ε^+ by λ_{per}^N and construct an approximation of u_ε on $[-1, 0]$ that is cheap to compute. We solve on $[0, 1]$ the

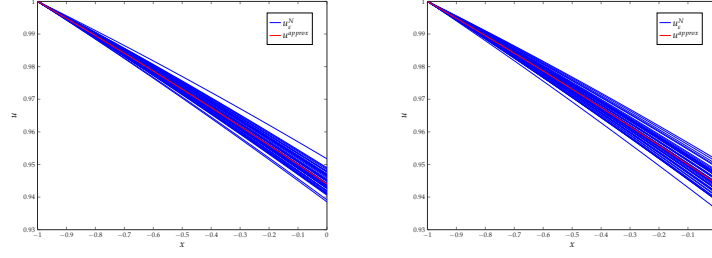


Fig. 6: Solution u in $[-1, 0]$ with exact and approximated DtN conditions for $\varepsilon = 0.05$ (left) and $\varepsilon = 0.1$ (right).

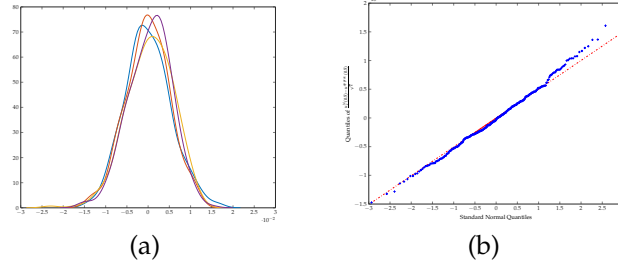


Fig. 7: (a) Probability density of $\frac{u_\varepsilon^N(-0.5) - u^{approx}(-0.5)}{\sqrt{\varepsilon}}$ as ε goes to 0, (b) Normal quantile-quantile plot of the corrector $\frac{u_\varepsilon^N(-0.5) - u^{approx}(-0.5)}{\sqrt{\varepsilon}}$ for $\varepsilon = 0.05$.

homogeneous problem

$$\begin{cases} -\partial_x \kappa_{hom} \partial_x u^{approx} - \rho_{hom} k^2 u^{approx} = 0 & \text{in } (-1, 0), \\ u^{approx}(-1) = 1, \quad \partial_x u^{approx}(0) = \lambda_{per}^N u^{approx}(0). \end{cases}$$

In Figure 6 are represented in blue different solutions u_ε^N corresponding to exact DtN boundary conditions associated to different random media occupying $(0, NL)$ and in red the approximated solution u^{approx} corresponding to the approximated DtN boundary condition. On the left, $\varepsilon = 0.05$ and on the right, $\varepsilon = 0.1$. We can notice that as ε goes to 0 the width of the different realizations of u_ε^N diminishes.

As proven in Theorem 3, we know nonetheless the limit distribution of the corrector $u_{1,\varepsilon} := \frac{u_\varepsilon^N - u^{approx}}{\sqrt{\varepsilon}}$ as ε goes to 0. To verify this result, we plotted in Figure 7, on the left, the probability density of $u_{1,\varepsilon}(-0.5)$ for different ε ranging from 0.05 to 0.2 approximated over 300 simulations per ε . The distribution of $u_{1,\varepsilon}$ seems to converge to a Gaussian as ε goes to 0. This is confirmed by the normal quantile-quantile plot of the same quantity on the right for $\varepsilon = 0.05$. The quantiles of the data samples are well aligned on the red line representing the quantile of a normal law. We have thus access to an approximation of order $\sqrt{\varepsilon}$ in law of the exact solution u_ε . This gives us among others all its statistics and its cumulative distribution function at an order of $\sqrt{\varepsilon}$ for small ε .

3.5 Conclusions and extensions

We provided in this section quantitative estimates of the convergence of the solution of the perturbed equation to the solution of the non-perturbed problem and exhibited how we can build a better approximation with a faster rate of convergence by considering the first-order corrector. Moreover, we displayed an analytical expression for the limit in distribution of this rescaled corrector.

A natural question concerns the possible extensions of this study to higher dimension problems. This is not straightforward since our proofs really use 1D arguments (for instance the definition (22) of v_ε^N).

In higher dimensions extensive work has been done to establish quantitative estimates of the convergence of the solution of elliptic equations with random oscillating coefficients. This corresponds to the case where κ_{per} and ρ_{per} are constant functions. A. Gloria and F. Otto in [14] and S. Armstrong, T. Kuusi and J.-C. Mourrat in [3] proved optimal error estimates with close-to-optimal stochastic integrability in the case of coefficients with a finite-range of dependance. In the latter, the convergence in law of the corrector to a variant of a Gaussian free field is demonstrated. In [6], P. Bella, B. Fehrman, J. Fischer and F. Otto obtained higher order approximations in weak spatial norms also in higher dimensions and for coefficients with finite-range of dependance.

In the case where κ_ε is unperturbed (that is $v_s^\kappa = 0$), similar results as ours are established in [5].

4 Rare perturbations case

In this section, let $(B_\varepsilon^j)_{j \in \mathbb{N}}$ be independent Bernoulli random variables with parameter $\varepsilon \in [0, 1]$, we consider a random medium occupying \mathbb{R}^+ , characterized by the following coefficients

$$\begin{aligned} \text{for a.e. } (x, \omega) \in \mathbb{R}^+ \times \Omega, \quad \kappa_\varepsilon(x, \omega) &= \kappa_{\text{per}}(x) + b_\varepsilon(x, \omega) \tilde{\kappa}_{\text{per}}(x), \\ \rho_\varepsilon(x, \omega) &= \rho_{\text{per}}(x) + b_\varepsilon(x, \omega) \tilde{\rho}_{\text{per}}(x), \end{aligned}$$

where $\kappa_{\text{per}}, \tilde{\kappa}_{\text{per}}, \rho_{\text{per}}, \tilde{\rho}_{\text{per}}$ are L -periodic functions with $L > 0$ and the random perturbation b_ε is given by

$$b_\varepsilon(x, \omega) = \sum_{j \in \mathbb{N}} \mathbb{1}_{I_j}(x) B_\varepsilon^j(\omega), \quad I_j = [jL, (j+1)L]. \quad (29)$$

We recall that we investigate for this case the asymptotic expansion of the solution u_ε^N of (10) in terms of ε and deduce an expansion for the associated DtN coefficient defined in (12).

4.1 The limit behaviour

As the probability ε for each period to be altered goes to 0, it is natural to think that the solution u_ε^N converges, in a sense to be determined, to the solution u_{per}^N of the problem (13) with the periodic coefficients $(\kappa_{\text{per}}, \rho_{\text{per}})$. We prove in this section that indeed u_ε^N converges to u_{per}^N as $\varepsilon \rightarrow 0$ in $L^2(\Omega, H^1((0, NL)))$ but only at a rate of $\sqrt{\varepsilon}$ as in the homogenization case. The weak convergence rate in $L^2(\Omega \times (0, NL))$ is slightly better.

Theorem 5 (i) *There exists a constant $C > 0$ such that*

$$\|u_\varepsilon^N - u_{\text{per}}^N\|_{L^2(\Omega, H^1((0, NL)))} \leq C \sqrt{\varepsilon} \|u_{\text{per}}^N\|_{H^1((0, NL))}, \quad (30)$$

and

$$|\lambda_\varepsilon^N - \lambda_{\text{per}}^N| \leq C \sqrt{\varepsilon}, \quad (31)$$

where $C = \max(\kappa_+, \rho_+ |k|^2) [\Im(k) |k| \min\left(\frac{\kappa_-}{|k|^2}, \rho_-\right)]^{-1}$.

(ii) *Weakly in $L^2(\Omega \times (0, NL))$, u_ε^N admits the 0-order asymptotic expansion*

$$u_\varepsilon^N = u_{\text{per}}^N + o(\sqrt{\varepsilon}).$$

Proof (i) Let $r_\varepsilon^0 = u_\varepsilon^N - u_{\text{per}}^N$. It is easy to see that r_ε^0 is the unique solution in $L^2(\Omega, H^1((0, NL)))$ of

$$\begin{cases} -\partial_x[\kappa_\varepsilon \partial_x r_\varepsilon^0] - k^2 \rho_\varepsilon r_\varepsilon^0 = \partial_x[b_\varepsilon \tilde{\kappa}_{\text{per}} \partial_x u_{\text{per}}^N] + b_\varepsilon \tilde{\rho}_{\text{per}} k^2 u_{\text{per}}^N, & \text{in } (0, NL), \\ r_\varepsilon^0(0) = 0, \quad r_\varepsilon^0(NL) = 0. \end{cases} \quad (32)$$

Using Proposition 2, we can show that

$$\|r_\varepsilon^0\|_{L^2(\Omega, H^1((0, NL)))} \leq C \left(\|b_\varepsilon \partial_x u_{\text{per}}^N\|_{L^2(\Omega, L^2((0, NL)))} + \|b_\varepsilon u_{\text{per}}^N\|_{L^2(\Omega, L^2((0, NL)))} \right), \quad (33)$$

where $C = \max(\kappa_+, \rho_+ |k|^2) [\Im(k) |k| \min\left(\frac{\kappa_-}{|k|^2}, \rho_-\right)]^{-1}$.

We can compute similarly each term appearing in the right hand side of the previous inequality. For instance,

$$\begin{aligned} \|b_\varepsilon u_{\text{per}}\|_{L^2(\Omega, L^2((0, NL)))}^2 &= \mathbb{E} \sum_{j=0}^{N-1} \int_{jL}^{(j+1)L} (B_\varepsilon^j(\omega))^2 |u_{\text{per}}^N(x)|^2 dx, \\ &= \sum_{j=0}^{N-1} \mathbb{E}[B_\varepsilon^j(\omega)] \int_{jL}^{(j+1)L} |u_{\text{per}}^N(x)|^2 dx = \varepsilon \|u_{\text{per}}^N\|_{L^2((0, NL))}^2. \end{aligned}$$

which, combined with (33), enables to obtain (30).

(ii) Let $w_\varepsilon^0 := \frac{r_\varepsilon^0}{\sqrt{\varepsilon}}$. w_ε^0 is bounded in $L^2(\Omega, H^1((0, NL)))$ thanks to (i). Thus w_ε^0 and ∂w_ε^0 are bounded in $L^2(\Omega \times (0, NL))$. As κ_ε is uniformly bounded, we know that $\kappa_\varepsilon \partial_x w_\varepsilon^0$ is also bounded in $L^2(\Omega \times (0, NL))$. Up to extraction, w_ε^0 and $\kappa_\varepsilon \partial_x w_\varepsilon^0$ converge weakly in $L^2(\Omega \times (0, NL))$. Moreover, since $\|b_\varepsilon\|_{L^2(\Omega \times (0, NL))} = \sqrt{\varepsilon NL}$, $b_\varepsilon / \sqrt{\varepsilon}$ converges weakly in $L^2(\Omega \times (0, NL))$, up to extraction, to a limit b_0 . Since it converges to 0 in probability, we know that $b_0 = 0$. We divide (32) by $\sqrt{\varepsilon}$, take the limit of the variational formulation as ε to 0 and obtain that $r_\varepsilon^0 = o(\sqrt{\varepsilon})$. ■

4.2 First-order corrections

We want to refine the approximation obtained in the previous section. We know that at a fixed period $I_j = (jL, (j+1)L)$, the periodic coefficients are perturbed with a probability ε . A natural intuition is that a more accurate approximation of u_ε could be obtained by considering the solutions of the problem where the periodic coefficients are perturbed in one and only one period. Let us then introduce for all $j \in \{0, \dots, N-1\}$, u_j^N the unique solution in $H^1((0, NL))$ of

$$\begin{cases} -\partial_x \left[(\kappa_{\text{per}} + \mathbb{1}_{I_j} \tilde{\kappa}_{\text{per}}) \partial_x u_j^N \right] - k^2 (\rho_{\text{per}} + \mathbb{1}_{I_{jm}} \tilde{\rho}_{\text{per}}) u_j^N = 0 \\ \hspace{15em} \text{in } (0, NL), \\ u_j^N(0) = 1, \quad u_j^N(NL) = 0. \end{cases} \quad (34)$$

and λ_j^N the associated DtN coefficient

$$\lambda_j^N := -[(\kappa_{\text{per}} + \mathbb{1}_{I_j}) \partial_x u_j^N](0).$$

More generally, let p be an integer in $\{1, \dots, N\}$ and $j_1 < \dots < j_p$ be p indices in $\{0, \dots, N-1\}$, we consider the periodic medium where exactly only the periods I_{j_1}, \dots, I_{j_p} are perturbed. This medium is characterized by the coefficients

$$\kappa_{\text{per}} + \sum_{m=1}^p \mathbb{1}_{I_{j_m}} \tilde{\kappa}_{\text{per}} \quad \text{and} \quad \rho_{\text{per}} + \sum_{m=1}^p \mathbb{1}_{I_{j_m}} \tilde{\rho}_{\text{per}}$$

We denote u_{j_1, \dots, j_p}^N the unique solution in $H^1((0, NL))$ of

$$\begin{cases} -\partial_x \left[(\kappa_{\text{per}} + \sum_{m=1}^p \mathbb{1}_{I_{j_m}} \tilde{\kappa}_{\text{per}}) \partial_x u_{j_1, \dots, j_p}^N \right] - k^2 (\rho_{\text{per}} + \sum_{m=1}^p \mathbb{1}_{I_{j_m}} \tilde{\rho}_{\text{per}}) u_{j_1, \dots, j_p}^N = 0 \\ \hspace{15em} \text{in } (0, NL), \\ u_{j_1, \dots, j_p}^N(0) = 1, \quad u_{j_1, \dots, j_p}^N(NL) = 0. \end{cases} \quad (35)$$

and λ_j^N the associated DtN coefficient

$$\lambda_{j_1, \dots, j_p}^N := -[\kappa_{\text{per}} + \sum_{m=1}^p \mathbb{1}_{I_{j_m}} \tilde{\kappa}_{\text{per}}] \partial_x u_{j_1, \dots, j_p}^N(0).$$

For a.e. $\omega \in \Omega$, we denote \mathcal{P}_ω the set defined by

$$\mathcal{P}_\omega := \{j \in \{0, \dots, N-1\}, B_\varepsilon^j(\omega) = 1\}.$$

It is easy to see that u_ε^N can be written for a.e. $(x, \omega) \in (0, NL) \times \Omega$

$$u_\varepsilon^N(x, \omega) = \mathbb{1}_{\mathcal{P}_\omega = \emptyset}(\omega) u_{\text{per}}^N(x) + \sum_{p=1}^N \sum_{j_1 < \dots < j_p} \mathbb{1}_{\mathcal{P}_\omega = \{j_1, \dots, j_p\}}(\omega) u_{j_1, \dots, j_p}^N(x). \quad (36)$$

The expectation of u_ε^N has therefore the following expression

$$\mathbb{E}(u_\varepsilon^N) = (1 - \varepsilon)^N u_{\text{per}}^N + \varepsilon^p (1 - \varepsilon)^{N-p} \sum_{p=1}^N \sum_{j_1 < \dots < j_p} u_{j_1, \dots, j_p}^N, \quad (37)$$

which yields directly the following result.

Theorem 6 *In mean, u_ε and λ_ε verify the following first-order asymptotic expansions*

$$\begin{aligned} \|\mathbb{E}[u_\varepsilon^N] - u_{\text{per}}^N - \varepsilon \sum_{j=0}^{N-1} (u_j^N - u_{\text{per}}^N)\|_{H^1((0, NL))} &= \mathcal{O}(\varepsilon^2), \\ |\mathbb{E}[\lambda_\varepsilon^N] - \lambda_{\text{per}}^N - \varepsilon \sum_{j=0}^{N-1} (\lambda_j^N - \lambda_{\text{per}}^N)| &= \mathcal{O}(\varepsilon^2). \end{aligned}$$

Remark 2 *Higher order approximations can also be derived from (37). For example, the second order corrector reads*

$$\|\mathbb{E}[u_\varepsilon^N] - u_{\text{per}}^N - \varepsilon \sum_{j=0}^{N-1} (u_j^N - u_{\text{per}}^N) - \varepsilon^2 \sum_{i \neq j} u_{i,j}^N - 2u_i^N + u_{\text{per}}^N\|_{H^1((0, NL))} = \mathcal{O}(\varepsilon^3).$$

However, from a numerical point of view, we did not feel the need of taking into account higher order approximations.

The first-order correction for $\mathbb{E}(u_\varepsilon^N)$ is indeed linked to the solutions in a medium with one and only one defect. The advantage of such expansion is that each term is solution of a deterministic problem set on a locally perturbed periodic medium. Using [15], these problems can be solved considering only cell problems with $(\kappa_{\text{per}}, \rho_{\text{per}})$ or $(\kappa_{\text{per}} + \tilde{\kappa}_{\text{per}}, \rho_{\text{per}} + \tilde{\rho}_{\text{per}})$. Computationally, the resolution is then really cheap.

Unfortunately, adding this corrector to u_ε^N does not change the rate of convergence found in Theorem 5 in $L^2(\Omega, H^1((0, NL)))$, as stated in the following proposition. Figure 10 illustrates the different rates of convergence that we exhibited so far.

Proposition 6 *There exists a constant C such that*

$$\|u_\varepsilon^N - u_{\text{per}}^N - \varepsilon \sum_{j=0}^{N-1} (u_j^N - u_{\text{per}}^N)\|_{L^2(\Omega, H^1((0, NL)))} \leq C \sqrt{\varepsilon}, \quad (38)$$

and

$$\left| \lambda_\varepsilon^N - \lambda_{\text{per}}^N - \varepsilon \sum_{j=0}^{N-1} (\lambda_j^N - \lambda_{\text{per}}^N) \right| \leq C \sqrt{\varepsilon}. \quad (39)$$

Proof For all $j \in \{0, \dots, N-1\}$, let consider $v_j^N = u_j^N - u_{\text{per}}^N$. It is the unique solution in $H^1((0, NL))$ of

$$\begin{cases} -\partial_x \left[(\kappa_{\text{per}} + \mathbb{1}_{I_j} \tilde{\kappa}_{\text{per}}) \right] \partial_x v_j^N - k^2 (\rho_{\text{per}} + \mathbb{1}_{I_j} \tilde{\rho}_{\text{per}}) v_j^N \\ \quad = \partial_x [\mathbb{1}_{I_j} \tilde{\kappa}_{\text{per}} \partial_x u_{\text{per}}^N] + k^2 \mathbb{1}_{I_j} \tilde{\rho}_{\text{per}} u_{\text{per}}^N, \quad \text{in } (0, NL), \\ v_j^N(0) = 0, \quad v_j^N(NL) = 0. \end{cases}$$

For all $j \in \{0, \dots, N-1\}$, we have the estimate

$$\|v_j^N\|_{H^1((0, NL))} \leq C \|u_{\text{per}}^N\|_{H^1((jL, (j+1)L))}$$

for a positive constant C .

Let us introduce $r_\varepsilon^1 = u_\varepsilon^N - u_{\text{per}}^N - \varepsilon \sum_{j=0}^{N-1} v_j^N$. It is the unique solution in $L^2(\Omega, H^2((0, NL)))$ of

$$\begin{cases} -\partial_x [\kappa_\varepsilon \partial_x r_\varepsilon^1] - k^2 \rho_\varepsilon r_\varepsilon^1 = \partial_x [\tilde{\kappa}_{\text{per}} (b_\varepsilon - \varepsilon) \partial_x u_{\text{per}}^N] + k^2 \tilde{\rho}_{\text{per}} (b_\varepsilon - \varepsilon) u_{\text{per}}^N \\ \quad + \varepsilon \left[\partial_x [\tilde{\kappa}_{\text{per}} b_\varepsilon \partial_x \sum_{j=0}^{N-1} v_j^N] + k^2 \tilde{\rho}_{\text{per}} b_\varepsilon \sum_{j=0}^{N-1} v_j^N \right] \\ \quad - \varepsilon \left[\partial_x [\tilde{\kappa}_{\text{per}} \sum_{j=0}^{N-1} \mathbb{1}_{I_j} \partial_x v_j^N] + k^2 \tilde{\rho}_{\text{per}} \sum_{j=0}^{N-1} \mathbb{1}_{I_j} v_j^N \right], \\ \quad \text{in } (0, NL), \\ r_\varepsilon^1(0) = 0, \quad r_\varepsilon^1(NL) = 0. \end{cases}$$

As in the proof of Theorem 5 and using Proposition 2 and we have that

$$\begin{aligned} \|r_\varepsilon^1\|_{L^2(\Omega, H^1((0, NL)))} &\leq C \left[\|(b_\varepsilon - \varepsilon) \partial_x u_{\text{per}}^N\|_{L^2(\Omega, L^2((0, NL)))} + \|(b_\varepsilon - \varepsilon) u_{\text{per}}^N\|_{L^2(\Omega, L^2((0, NL)))} \right. \\ &\quad \left. + \varepsilon [\|b_\varepsilon \partial_x \sum_{j=0}^{N-1} v_j^N\|_{L^2(\Omega, L^2((0, NL)))} + \|b_\varepsilon \sum_{j=0}^{N-1} v_j^N\|_{L^2(\Omega, L^2((0, NL)))}] + \varepsilon \left\| \sum_{j=0}^{N-1} \mathbb{1}_{I_j} v_j^N \right\|_{H^1} \right], \end{aligned}$$

and then

$$\|r_\varepsilon^1\|_{L^2(\Omega, H^1((0, NL)))} \leq C \left[\sqrt{\varepsilon(1-\varepsilon)} \|u_{\text{per}}^N\|_{H^1} + \varepsilon^{3/2} \left\| \sum_{j=0}^{N-1} v_j^N \right\|_{H^1} + \varepsilon \left\| \sum_{j=0}^{N-1} \mathbb{1}_{I_j} v_j^N \right\|_{H^1} \right].$$

■

However, similarly as (37), one can obtain from (36) asymptotic expansions for the expectation of any smooth function of u_ε^N . Higher order approximations can also be derived.

Theorem 7 For any $\varphi \in C_c^\infty(\mathbb{R})$

$$\|\mathbb{E}[\varphi(u_\varepsilon^N)] - \varphi(u_{\text{per}}^N) - \varepsilon \sum_{j=0}^{N-1} \varphi(u_j^N) - \varphi(u_{\text{per}}^N)\|_{H^1((0, NL))} = \mathcal{O}(\varepsilon^2), \quad (40)$$

$$|\mathbb{E}[\varphi(u_\varepsilon^N)] - \varphi(\lambda_{\text{per}}^N) - \varepsilon \sum_{j=0}^{N-1} \varphi(\lambda_j^N) - \varphi(\lambda_{\text{per}}^N)| = \mathcal{O}(\varepsilon^2), \quad (41)$$

This can be seen as an approximation in law in the first order in ε of u_ε^N . Indeed, let $P_{u_\varepsilon^N}$ be the distribution of u_ε^N . Let $\mathcal{P}(\mathbb{R})$ be the set of probability measures with compact support on \mathbb{R} . In $\mathcal{P}(\mathbb{R})$, the following asymptotic expansion holds weakly-*

$$P_{\lambda_\varepsilon^N} = \delta_{u_{\text{per}}^N} + \varepsilon \sum_{j=0}^{N-1} \delta_{u_j^N} - \delta_{u_{\text{per}}^N} + \mathcal{O}(\varepsilon^2).$$

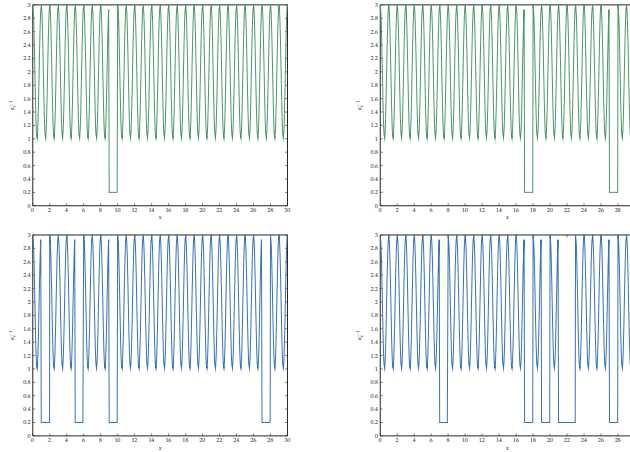


Fig. 8: κ_ε^{-1} with respect to x for two different ω and two different ε : $\varepsilon = 0.05$ (top) and $\varepsilon = 0.2$ (bottom)

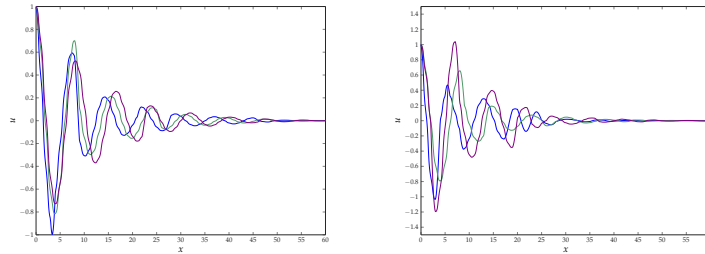


Fig. 9: u_ε^N for different ω and $\varepsilon = 0.05$ (left) and $\varepsilon = 0.2$ (right).

4.3 Numerical simulations

We present, in this section, a few simulation results to illustrate our theorems and exhibit how our approach can provide a tool to build transparent boundary conditions for the studied randomly perturbed periodic media.

We start off by exposing in Figure 8 two random realizations of the perturbed periodic media that we consider for two different ε . We take our two parameters κ_{per}^{-1} and ρ_{per} of the periodic medium to be

$$\cos(2\pi \cdot) + 2,$$

and perturb each period with a probability ε by $\tilde{\kappa}_{per}$ and $\tilde{\rho}_{per}$ such that

$$\kappa_{per} + \tilde{\kappa}_{per} = \rho_{per} + \tilde{\rho}_{per} = 0.2.$$

The parameters of the medium stay this way bounded above and below. The top row represents two different random realizations of κ_ε^{-1} for $\varepsilon = 0.05$, that is a medium where each period has a probability of $1/20$ to be altered, and the bottom row represents two different random realizations of κ_ε^{-1} for $\varepsilon = 0.2$, that is a medium where each period has a probability of $1/5$ to be altered.

For the two same ε , we plot in Figure 9 three different solutions u_ε^N obtained by solving the equation in three random media generated as described above. We chose a frequency of 1 and a damping coefficient of 0.2. Before solving the system, we compute, using Proposition 3, the appropriate number N of periods to write our equation on, such that the error between the truncated problem and the problem on the positive half-line is as small as desired, for example 10^{-5} .

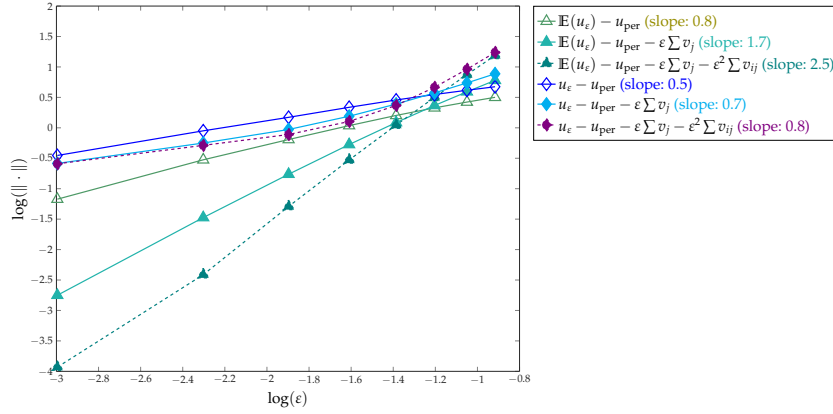


Fig. 10: Rates of convergence of $\mathbb{E}(u_\varepsilon^N)$ and u_ε^N and their first-order corrections

4.3.1 Different modes of convergence and their orders

The first results that we verify numerically are the rates of convergence proved in the two previous sections. We chose 8 values of ε between 0.05 and 0.4 and solved the equations for 2000 different realizations of the medium for each value of ε . Note that we need more simulations than in the homogenization regime since we are working here with rare perturbations.

We computed by the Monte-Carlo method $\mathbb{E}[u_\varepsilon^N]$ and plotted the log of the norm of the error in $H^1((0, NL))$ of the 0-, 1st- and 2nd- order asymptotic expansion, with respect to the log of ε . One can see in Figure 10 that the green lines with triangle markers have a slope of respectively 0.8, 1.7 and 2.5, for the 0-, 1st- and 2nd- order approximations, close to what we expected from Theorem 6, i.e. resp. 1, 2 and 3.

Similarly, as Theorem 5, and Proposition 6 predicted, when we plot the log of the error in $L^2(\Omega, H^1((0, NL)))$ of the 0-, 1st- and 2nd- order asymptotic expansion of u_ε^N , with respect to the log of ε , we find that the slopes do not change and stay around 0.5. On Figure 10, those lines correspond to the blue, cyan and magenta lines with diamond markers. We find slopes of 0.5, 0.7 and 0.8 respectively. Once again, to compute the expectation that appear in the norm, we used the Monte-Carlo method.

Note that we computed for this figure the 2nd order approximation. Here $v_{i,j} = u_{i,j}^N - 2u_i^N + u_{per}$ for all $i \neq j \in [1, N]$. Recall that $u_{i,j}^N$ denotes the solution of the equation in a medium where only the i th and j th periods are perturbed.

Finally, we want to illustrate the last convergence result stated Theorem 7, i.e. the convergence in law of the 1st order asymptotic expansion on u_ε^N with a rate of ε^2 . Since a similar result holds for the Dirichlet-to-Neumann coefficient, we chose to represent this result on it. We thus computed the DtN coefficients associated to the solutions in the 2000 realizations of our medium over the range of ε . We approximated over this sample the probability density of $\varepsilon^{-2}(\lambda_\varepsilon - \lambda_{per} - \varepsilon \sum \lambda_j - \lambda_{per})$ for each ε . We then plotted those density functions in Figure 11. You can see that as ε goes to 0 the density function converges to a dirac as expected.

4.3.2 Homogenization regime

A second verification that we pursued is to compare our results with those of A. Anantharaman and C. Le Bris. They obtain in [2] an asymptotic expansion for the homogenized coefficient of the elliptic pde in the same media that we studied but in any dimension $d \geq 1$ and of course in the homogenization regime. For $d = 1$, they derive an exact expression for the first order expression of the homogenized coefficient a_ε . We used this expression to compute an approximation of the solution u_ε^N in the low frequency regime (frequency of 0.1) and compared it to our first order approximation. In Figures 12, we represented in green with diamond markers our approximation and in blue with triangle markers their approximations for two different ε , $\varepsilon = 0.05$ on the left and $\varepsilon = 0.2$ on the right. One can see that they match in both cases.

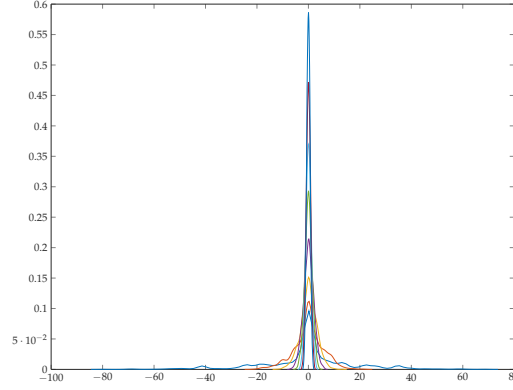


Fig. 11: Probability density of $\epsilon^{-2} (\lambda_\epsilon - \lambda_{per} - \epsilon \sum \lambda_j - \lambda_{per})$ as ϵ goes to 0

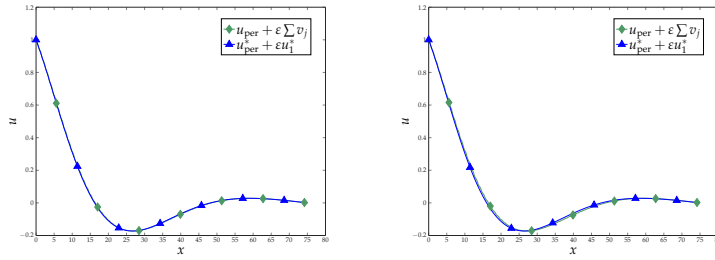


Fig. 12: Comparison between our asymptotic expansion and [A. Anantharaman and C. Le Bris]'s expansion for $\epsilon = 0.05$ (left) and $\epsilon = 0.2$ (right).

4.3.3 Transparent conditions and DtN

We consider a homogeneous medium on $[-1, 0]$, where $\kappa_{hom} = \rho_{hom} = 1$, and we want to solve the wave equation with a Dirichlet condition at -1

$$\begin{cases} -\partial_x \kappa_\epsilon \partial_x u_\epsilon - k^2 \rho_\epsilon u_\epsilon = 0 & \text{in } (-1, +\infty), \\ u_\epsilon(-1) = 1. \end{cases}$$

The randomly perturbed periodic medium lies in $(0, +\infty)$ as before. As explained in the introduction, the restriction of this solution to $(-1, 0)$ is the unique solution of the following equation with a boundary condition of the form (5) at 0

$$\begin{cases} -\partial_x \kappa_{hom} \partial_x u_\epsilon - \rho_{hom} k^2 u_\epsilon = 0 & \text{in } (-1, 0), \\ u_\epsilon(-1) = 1, \quad \partial_x u_\epsilon(0) = \lambda_\epsilon^+ u_\epsilon(0). \end{cases}$$

We use Theorem 7 to approximate at a first order in law λ_ϵ and construct an approximation of u_ϵ on $[-1, 0]$ that is cheap to compute. We solve

$$\begin{cases} -\partial_x \kappa_{hom} \partial_x u_\epsilon^{approx} - \rho_{hom} k^2 u_\epsilon^{approx} = 0 & \text{in } (-1, 0), \\ u_\epsilon^{approx}(-1) = 1, \quad \partial_x u_\epsilon^{approx}(0) = \left(\lambda_{per}^N + \epsilon \sum_{j=0}^{N-1} \lambda_j^N - \lambda_{per}^N \right) u_\epsilon^{approx}(0), \end{cases}$$

which is equivalent to imposing $u_\epsilon^{approx} = u_{per}^N + \epsilon \sum_{j=0}^{N-1} v_j^N$ on $(0, NL)$. u_ϵ^{approx} provides therefore a first order approximation in law of u_ϵ in our bounded known domain $(-1, 0)$ bordering the unknown random medium.

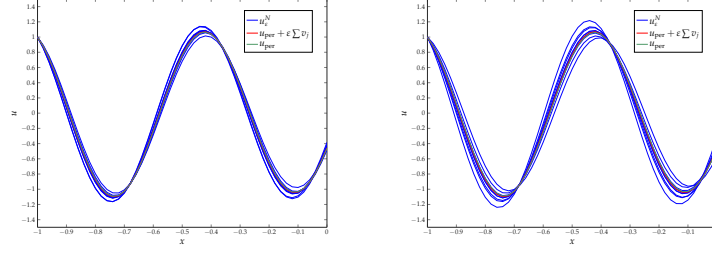


Fig. 13: Solution u_ε in $[-1, 0]$ with exact and approximated DtN conditions for $\varepsilon = 0.05$ (left) and $\varepsilon = 0.1$ (right).

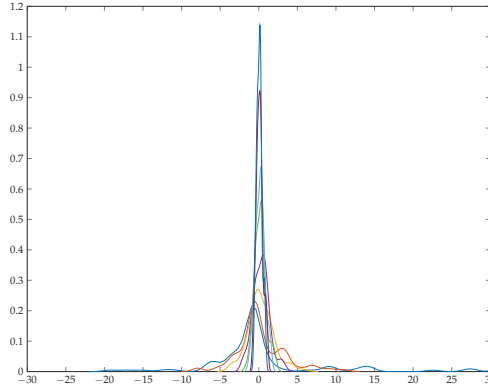


Fig. 14: Probability density of $\varepsilon^{-2} \left(u_\varepsilon(-0.5) - u_\varepsilon^{approx}(-0.5) \right)$ as ε goes to 0

We plotted in Figure 13 a few random realisations of the true solution u_ε on $[-1, 0]$ for two different ε , on the left $\varepsilon = 0.05$ and on the right $\varepsilon = 0.1$. We represented as well the two approximated solutions obtained by taking the 0th order (in green) and 1st order (in red) approximation of the DtN coefficient in our boundary condition. Here the frequency is high and taken to be 20. Even though our approximated solutions seem to provide a good estimate for u_ε , it seems that the 1st order approximation does not make at first sight a significant difference in bettering the estimate compared to the 0th order. This is not surprising however, because we know that the convergence of u_ε^{approx} to u_ε does not hold almost surely but in law.

We thus exhibited the convergence in law of u_ε^{approx} to u_ε by representing the probability density of $\varepsilon^{-2} \left(u_\varepsilon(-0.5) - u_\varepsilon^{approx}(-0.5) \right)$ in Figure 14 for different ε . Like before, the probability density is computed over 2000 simulations per ε . A similar figure could be constructed for each point $x \in [-1, 0]$. This result might be a justification for using the 1st order approximation against the 0th-order since the convergence in law to a Dirac is faster for the former.

4.4 Other randomly perturbed periodic media

We consider a more general setting, where the random term b_ε in the perturbation follows on each period, independently of the others, a distribution $\mathbb{P}_{B_\varepsilon}$ which is not a Bernoulli distribution anymore but any distribution that is rare in the sense that it verifies the properties enounced below. We express thus b_ε as follows

$$\forall (x, \omega) \in \mathbb{R}_+^* \times \Omega, \quad \tilde{b}_\varepsilon(x, \omega) = \sum_{j \in \mathbb{N}} \mathbb{1}_{I_j}(x) \tilde{B}_\varepsilon^j(\omega),$$

where $(\tilde{B}_\varepsilon^j)_{j \in \mathbb{N}}$ are i.i.d. random variables in $L^2(\Omega)$. We suppose that \tilde{B}_ε is uniformly bounded in ω and ε by a constant $K < M$ and that its second moment $\mathbb{E}((\tilde{B}_\varepsilon)^2)$ converges to 0 as $\varepsilon \rightarrow 0$. M is chosen

such that

$$\kappa_{\text{per}} \pm M\tilde{\kappa}_{\text{per}} \text{ and } \rho_{\text{per}} \pm M\tilde{\rho}_{\text{per}}$$

still verify (2). This allows us to write the equivalent of theorem 5 in this setting

Theorem 8 u_ε^N converges to u_0^N at the following converging rate

$$\|u_\varepsilon^N - u_{\text{per}}^N\|_{L^2(\Omega, H^1((0, NL)))} \leq C \|\tilde{B}_\varepsilon\|_{L^2(\Omega)} \|u_{\text{per}}^N\|_{H^1((0, NL))}. \quad (42)$$

To obtain the first order asymptotic expansion in law, we need to impose an asymptotic assumption on $P_{\tilde{B}_\varepsilon}$. Note that the Bernoulli law of parameter ε verifies

$$P_{B_\varepsilon} = \delta_0 + \varepsilon(\delta_1 - \delta_0) + \mathcal{O}(\varepsilon^2), \quad \text{weakly-}^* \text{ in } \mathcal{P}((0, 1)).$$

We follow then the first section and assume that

$$P_{\tilde{B}_\varepsilon} = \delta_0 + \varepsilon P^1 + \mathcal{O}(\varepsilon^2), \quad \text{weakly-}^* \text{ in } \mathcal{P}((-M, M)),$$

i.e. for all $\varphi \in C_c^\infty((-M, M))$

$$\left| \mathbb{E}(\varphi(\tilde{B}_\varepsilon)) - \varphi(0) - \varepsilon \int \varphi dP^1 \right| = \mathcal{O}(\varepsilon^2).$$

In order to exhibit the first order term in our asymptotic expansion, we introduce $u_{s_0, \dots, s_{N-1}}^N$ for $s_1, \dots, s_N \in (-M, M)$, the solution of the equation in a medium where the j th period is perturbed at a level s_j , i.e. $b_\varepsilon^j(\omega) = s_j$ for $j \in \llbracket 1, n \rrbracket$.

$u_{s_0, \dots, s_{N-1}}^N$ is the solution in $H^1((0, NL))$ of the following equation

$$\begin{cases} -\partial_x \left(\kappa_{\text{per}} + \sum_{j=0}^{N-1} s_j \mathbb{1}_{I_j} \tilde{\kappa}_{\text{per}} \right) \partial_x u_{s_0, \dots, s_{N-1}}^N - k^2 \left(\rho_{\text{per}} + \sum_{j=0}^{N-1} s_j \mathbb{1}_{I_j} \tilde{\rho}_{\text{per}} \right) u_{s_0, \dots, s_{N-1}}^N = 0 \\ u_{s_0, \dots, s_{N-1}}^N(0) = 1, \quad u_{s_0, \dots, s_{N-1}}^N(NL) = 0. \end{cases} \quad \text{in } (0, NL),$$

We can show by induction that

$$\begin{aligned} F : \quad (-M, M)^N &\rightarrow H^1((0, NL)) \\ (s_0, \dots, s_{N-1}) &\mapsto u_{s_0, \dots, s_{N-1}}^N \end{aligned}$$

is in $C^\infty((-M, M)^N)$.

We can then express $\mathbb{E}(\varphi(u_\varepsilon^N))$ for any $\varphi \in C_c^\infty(\mathbb{R})$

$$\forall \varphi \in C_c(\mathbb{R}), \quad \mathbb{E}(\varphi(u_\varepsilon^N)) = \int_{(-M, M)^N} \varphi(u_{s_0, \dots, s_{N-1}}^N) \prod_{j=0}^{N-1} P_{B_\varepsilon}(ds_j). \quad (43)$$

We use our asymptotic expansion on P_{B_ε} to obtain the expansion for the product. Weakly- * in $\mathcal{P}(-M, M)$

$$\prod_{j=0}^{N-1} P_{B_\varepsilon}(ds_j) = \prod_{j=0}^{N-1} \delta_0(ds_j) + \varepsilon \sum_{j=0}^{N-1} P^1(ds_j) \prod_{l \neq j} \delta_0(ds_l) + \mathcal{O}(\varepsilon^2).$$

We plug this expansion into (43) and obtain the following theorem.

Theorem 9 In $H^1((0, NL))$, the following expansion holds for all $\varphi \in C_c^\infty(\mathbb{R})$

$$\mathbb{E}(\varphi(u_\varepsilon^N)) = \varphi(u_{\text{per}}^N) + \varepsilon \sum_{j=0}^{N-1} \int_{-M}^M \varphi(u_{0, \dots, s_j, \dots, 0}^N) P^1(ds_j) + \mathcal{O}(\varepsilon^2),$$

i.e. weakly- * in $\mathcal{P}(\mathbb{R})$, $P_{u_\varepsilon^N}$ verifies

$$P_{u_\varepsilon^N} = \delta_{u_{\text{per}}^N} + \varepsilon \sum_{j=0}^{N-1} \int_{-M}^M \delta_{u_{0, \dots, s_j, \dots, 0}^N} P^1(ds_j) + \mathcal{O}(\varepsilon^2).$$

4.5 Conclusions and extensions

In this section we proved in the case of rare random perturbations of a periodic medium the convergence of the solution of the perturbed problem to the solution of the periodic equation. We exhibited asymptotic expansions at any order in $L^2(\Omega, H^1((0, NL)))$ for the mean of the perturbed solution with respect to the rarity of the perturbation and a first order corrector in law for the perturbed solution. This study can be easily extended to higher dimensions problems.

5 Conclusion

We have proposed approximations of the DtN coefficient for two types of weakly randomly perturbed periodic media. This analysis is for now based on the exponential decay of the solution in presence of dissipation. Indeed this allows us to approximate the solution by the solution of a problem set on a truncated domain. We think that this analysis could be extended in absence of dissipation when the frequency lies in the so-called spectral gaps of the underlying periodic operator. This will be the subject of a forthcoming paper. Another extension that we are also looking at, is to consider other weakly random perturbations of periodic media where the wavelength is still large with respect to the correlation length of the random processes but is small compared to the distance propagation of the wave.

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