Spectroscopic imaging of a dilute cell suspension*

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Abstract

A rigorous homogenization theory is derived to describe the effective admittivity of cell suspensions. A new formula is reported for dilute cases that gives the frequency-dependent effective admittivity with respect to the membrane polarization. Different microstructures are shown to be distinguishable via spectroscopic measurements of the overall admittivity using the spectral properties of the membrane polarization. The Debye relaxation times associated with the membrane polarization tensor are shown to be able to give the microscopic structure of the medium. A natural measure of the admittivity anisotropy is introduced and its dependence on the frequency of applied current is derived. A Maxwell-Wagner-Fricke formula is given for concentric circular cells, and the results can be extended to the random cases. A randomly deformed periodic medium is also considered and a new formula is derived for the overall admittivity of a dilute suspension of randomly deformed cells.

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Contents

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1 Introduction

The electric behavior of biological tissue under the influence of an electric field at frequency ω can be characterized by its frequency-dependent effective admittivity $k_{ef} := \sigma_{ef}(\omega) + i\omega\epsilon_{ef}(\omega)$, where σ_{ef} and ε_{ef} are respectively its effective conductivity and permittivity. Electrical impedance spectroscopy assesses the frequency dependence of the effective admittivity by measuring it across a range of frequencies from a few Hz to hundreds of MHz. Effective admittivity of biological tissues and its frequency dependence vary with tissue composition, membrane characteristics, intra-and extra-cellular fluids and other factors. Hence, the admittance spectroscopy provides information about the microscopic structure of the medium and physiological and pathological conditions of the tissue.

The determination of the effective, or macroscopic, property of a suspension is an enduring problem in physics [42]. It has been studied by many distinguished scientists, including Maxwell, Poisson [49], Faraday, Rayleigh [51], Fricke [31], Lorentz, Debye, and Einstein [26]. Many studies have been conducted on approximate analytic expressions for overall admittivity of a cell suspension from the knowledge of pointwise conductivity distribution, and these studies were mostly restricted to the simplified model of a strongly dilute suspension of spherical or ellipsoidal cells.

In this paper, we consider a periodic suspension of identical cells of arbitrary shape. We apply at the boundary of the medium an electric field of frequency ω. The medium outside the cells has an admittivity of $k_0 := \sigma_0 + i\omega\epsilon_0$. Each cell is composed of an isotropic homogeneous core of admittivity k_0 and a thin membrane of constant thickness δ and admittivity $k_m := \sigma_m + i\omega\epsilon_m$. The thickness δ is considered to be very small relative to the typical cell size and the membrane is considered very resistive, *i.e.*, $\sigma_m \ll \sigma_0$. In this context, the potential in the medium passes an effective discontinuity over the cell boundary; the jump is proportional to its normal derivative with a coefficient of the effective thickness, given by $\delta k_0 / k_m$. The normal derivative of the potential is continuous across the cell boundaries.

We use homogenization techniques with asymptotic expansions to derive a homogenized problem and to define an effective admittivity of the medium. We prove a rigorous convergence of the initial problem to the homogenized problem via two-scale convergence. For dilute cell suspensions, we use layer potential techniques to expand the effective admittivity in terms of cell volume fraction. Through the effective thickness, $\delta k_0/k_m$, the first-order term in this expansion can be expressed in terms of a membrane polarization tensor, M, that depends on the operating frequency ω. We retrieve the Maxwell-Wagner-Fricke formula for concentric circular-shaped cells. This explicit formula has been generalized in many directions: in three dimension for concentric spherical cells; to include higher power terms of the volume fraction for concentric circular and spherical cells; and to include various shapes such as concentric, confocal ellipses and ellipsoids; see [14, 15, 28, 29, 30, 41, 52, 53, 54].

The imaginary part of M is positive for δ small enough. Its two eigenvalues are maximal for frequencies $1/\tau_i, i = 1, 2$, of order of a few MHz with physically plausible parameters values. This dispersion phenomenon well known by the biologists is referred to as the β-dispersion. The associated characteristic times τ_i correspond to Debye relaxation times. Given this, we show that different microscopic organizations of the medium can be distinguished via $\tau_i, i = 1, 2$, alone. The relaxation times τ_i are computed numerically for different configurations: one circular or elliptic cell, two or three cells in close proximity. The obtained results illustrate the viability of imaging cell suspensions using the spectral properties of the membrane polarization. The Debye relaxation times are shown to be able to give the microscopic structure of the medium.

In the second part of this paper, we show that our results can be extended to the random case by considering a randomly deformed periodic medium. We also derive a rigorous homogenization theory for cells (and hence interfaces) that are randomly deformed from a periodic structure by random, ergodic, and stationary deformations. We prove a new formula for the overall conductivity of a dilute suspension of randomly deformed cells. Again, the spectral properties of the membrane polarization can be used to classify different microscopic structures of the medium through their Debye relaxation times. For recent works on effective properties of dilute random media, we refer to [7, 17].

Our results in this paper have potential applicability in cancer imaging, food sciences and biotechnology [39, 40], and applied and environmental geophysics. They can be used to model and improve the MarginProbe system for breast cancer [58], which emits an electric field and senses the returning signal from tissue under evaluation. The greater vascularization, differently polarized cell membranes, and other anatomical differences of tumors compared with healthy tissue cause them to show different electromagnetic signatures. The ability of the probe to detect signals characteristic of cancer helps surgeons ensure the removal of all unwanted tissue around tumor margins.

Another commercial medical system to which our results can be applied is ZedScan [59]. ZedScan is based on electrical impedance spectroscopy for detecting neoplasias in cervical disease [1, 20]. Malignant white blood cells can be also detected using induced membrane polarization [50]. In food quality inspection, spectroscopic conductivity imaging can be used to detect bacterial cells [12, 56]. In applied and environmental geophysics, induced membrane polarization can be used to probe up to subsurface depths of thousands of meters [55, 57].

The structure of the rest of this paper is as follows. Section 2 introduces the problem settings and state the main results of this work. Section 3 is devoted to the analysis of the problem. We prove existence and uniqueness results and establish useful *a priori* estimates. In section 4 we consider a periodic cell suspension and derive spectral properties of the overall conductivity. In section 5 we consider the problem of determining the effective property of a suspension of cells when the volume fraction goes to zero. Section 6 is devoted to spectroscopic imaging of a dilute suspension. We make use of the asymptotic expansion of the effective admittivity in terms of the volume fraction to image a permittivity inclusion. We also discuss selective spectroscopic imaging using a pulsed approach. Finally, we introduce a natural measure of the conductivity anisotropy and derive its dependence on the frequency of applied current. In section 7 we extend our results to the case of randomly deformed periodic media. In section 8 we provide numerical examples that support our findings. A few concluding remarks are given in the last section.

2 Problem settings and main results

The aim of this section is to introduce the problem settings and state the main results of this paper.

2.1 Periodic domain

We consider the probe domain Ω to be a bounded open set of $\mathbb{R}^2$ of class $\mathcal{C}^2$. The domain contains a periodic array of cells whose size is controlled by ε. Let C be a $\mathcal{C}^{2,\eta}$ domain being contained in the unit square $Y = [0,1]^2$, see Figure 2.1. Here, $0 < \eta < 1$ and C represents a reference cell. We divide the domain Ω periodically in each direction in identical squares $(Y_{\varepsilon,n})_n$ of size ε, where

$$Y_{\varepsilon,n} = \varepsilon n + \varepsilon Y.$$

Here, $n \in N_\varepsilon := \left\{ n \in \mathbb{Z}^2 | Y_{\varepsilon,n} \cap \Omega \neq \emptyset \right\}$.

We consider that a cell $C_{\varepsilon,n}$ lives in each small square $Y_{\varepsilon,n}$. As shown in Figure 2.2, all cells are identical, up to a translation and scaling of size ε, to the reference cell C:

$$\forall n \in N_\varepsilon, \quad C_{\varepsilon,n} = \varepsilon n + \varepsilon C.$$

So are their boundaries $(\Gamma_{\varepsilon,n})_{n \in N_\varepsilon}$ to the boundary Γ of C:

$$\forall n \in N_\varepsilon, \quad \Gamma_{\varepsilon,n} = \varepsilon n + \varepsilon \Gamma.$$

Let us also assume that all the cells are strictly contained in Ω, that is for every $n \in N_\varepsilon$, the boundary $\Gamma_{\varepsilon,n}$ of the cell $C_{\varepsilon,n}$ does not intersect the boundary $\partial\Omega$:

$$\partial\Omega \cap \Big(\bigcup_{n \in N_\varepsilon} \Gamma_{\varepsilon,n} \Big) = \emptyset.$$

2.2 Electrical model of the cell

Set for any open set D of $\mathbb{R}^2$:

$$L^2_0(D) := \left\{ f \in L^2(D) \Big| \int_{\partial D} f(x) ds(x) = 0 \right\}$$

and

$$H^1(D) := \left\{ f \in L^2(D) \Big| |\nabla f| \in L^2(D) \right\}.$$

We consider in this section the reference cell C immersed in a domain D. We apply a sinusoidal electrical current $g \in L^2_0(\partial D)$ with angular frequency ω at the boundary of D.

The medium outside the cell, $D \setminus \overline{C}$, is a homogeneous isotropic medium with admittivity $k_0 := \sigma_0 + i\omega\epsilon_0$. The cell C is composed of an isotropic homogeneous core of admittivity k_0 and a thin membrane of constant thickness δ with admittivity $k_m := \sigma_m + i\omega\epsilon_m$. We make the following assumptions :

$$\sigma_0 > 0, \sigma_m > 0, \epsilon_0 > 0, \epsilon_m \geq 0.$$

If we apply a sinusoidal current $g(x)\sin(\omega t)$ on the boundary ∂D in the low frequency range below 10 MHz, the resulting time harmonic potential $\breve{u}$ is governed approximately by

$$\begin{cases} \nabla \cdot (k_0 + (k_m - k_0)\chi_{\Gamma^\delta})\nabla \breve{u}) = 0 & \text{in } D \\ k_0 \dfrac{\partial \breve{u}}{\partial n}\Big|_{\partial D} = g, \end{cases}$$

where $\Gamma^\delta := \{x \in C \ : \ \text{dist}(x, \Gamma) < \delta\}$ and χ_{Γ^δ} is the characteristic function of the set Γ^δ.

The membrane thickness δ is considered to be very small compared to the typical size ρ of the cell *i.e.* $\delta/\rho \ll 1$. According to the transmission condition, the normal component of the current density $k_0 \dfrac{\partial u}{\partial n}$ can be regarded as continuous across the thin membrane Γ.

We set $\beta := \dfrac{\delta}{k_m}$. Since the membrane is very resistive, *i.e.* $\sigma_m/\sigma_0 \ll 1$, the potential u in D undergoes a jump across the cell membrane Γ, which can be approximated at first order by $\beta k_0 \dfrac{\partial u}{\partial n}$. A rigorous proof of this result, based on asymptotic expansions of layer potentials, can be found in [36].

More precisely, u is the solution of the following equations:

$$\begin{cases} \nabla \cdot k_0 \nabla u = 0 & \text{in } D \setminus \overline{C}, \\ \nabla \cdot k_0 \nabla u = 0 & \text{in } C, \\ k_0 \dfrac{\partial u}{\partial n}\Big|_+ = k_0 \dfrac{\partial u}{\partial n}\Big|_- & \text{on } \Gamma, \\ u|_+ - u|_- - \beta k_0 \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma, \\ k_0 \dfrac{\partial u}{\partial n}\Big|_{\partial D} = g, \quad \displaystyle\int_{\partial D} g(x)ds(x) = 0, \quad \int_{D\setminus\overline{C}} u(x)dx = 0. \end{cases} \tag{2.1}$$

Here n is the outward unit normal vector and $u|_\pm(x)$ denotes $\lim_{t\to 0^+} u(x \pm tn(x))$ for x on the concerned boundary. Likewise, $\dfrac{\partial u}{\partial n}\Big|_\pm := \lim_{t\to 0^+} \nabla u(x \pm tn(x)) \cdot n(x)$.

2.3 Governing equation

We denote by Ω_ε^+ the medium outside the cells and Ω_ε^- the medium inside the cells:

$$\Omega_\varepsilon^+ = \Omega \cap \Big(\bigcup_{n\in N_\varepsilon} Y_{\varepsilon,n} \setminus \overline{C_{\varepsilon,n}} \Big), \quad \Omega_\varepsilon^- = \bigcup_{n\in N_\varepsilon} C_{\varepsilon,n}.$$

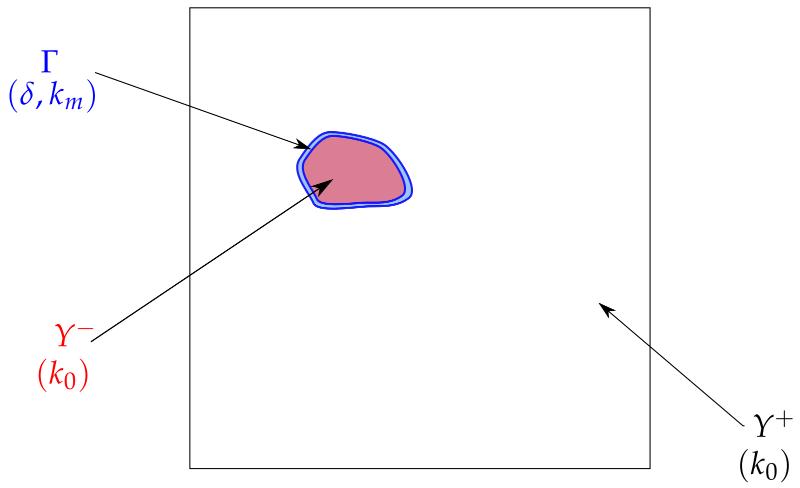

Figure 2.1: *Schematic illustration of a unit period Y.*

Set $\Gamma_\varepsilon := \bigcup_{n \in N_\varepsilon} \Gamma_{\varepsilon,n}$. By definition, the boundaries $\partial\Omega_\varepsilon^+$ and $\partial\Omega_\varepsilon^-$ of respectively Ω_ε^+ and Ω_ε^- satisfy:

$$\partial\Omega_\varepsilon^+ = \partial\Omega \cup \Gamma_\varepsilon, \quad \partial\Omega_\varepsilon^- = \Gamma_\varepsilon.$$

We apply a sinusoidal current $g(x)\sin(\omega t)$ at $x \in \partial\Omega$, where $g \in L_0^2(\partial\Omega)$. The induced time-harmonic potential u_ε in Ω satisfies [?, 47, 48]:

$$\begin{cases} \nabla \cdot k_0 \nabla u_\varepsilon^+ = 0 & \text{in } \Omega_\varepsilon^+, \\ \nabla \cdot k_0 \nabla u_\varepsilon^- = 0 & \text{in } \Omega_\varepsilon^-, \\ k_0 \dfrac{\partial u_\varepsilon^+}{\partial n} = k_0 \dfrac{\partial u_\varepsilon^-}{\partial n} & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon^+ - u_\varepsilon^- - \varepsilon \beta k_0 \dfrac{\partial u_\varepsilon^+}{\partial n} = 0 & \text{on } \Gamma_\varepsilon, \\ k_0 \dfrac{\partial u_\varepsilon^+}{\partial n}\Big|_{\partial\Omega} = g, \quad \displaystyle\int_{\partial\Omega} g(x)ds(x) = 0, \quad \int_{\Omega_\varepsilon^+} u_\varepsilon^+(x)dx = 0, \end{cases} \tag{2.2}$$

where $u_\varepsilon = \begin{cases} u_\varepsilon^+ & \text{in } \Omega_\varepsilon^+, \\ u_\varepsilon^- & \text{in } \Omega_\varepsilon^-. \end{cases}$

Note that the previously introduced constant β, *i.e.*, the ratio between the thickness of the membrane of C and its admittivity, becomes $\varepsilon\beta$. Because the cells $(C_{\varepsilon,n})_{n \in N_\varepsilon}$ are in squares of size ε, the thickness of their membranes is given by $\varepsilon\delta$ and consequently, a factor ε appears.

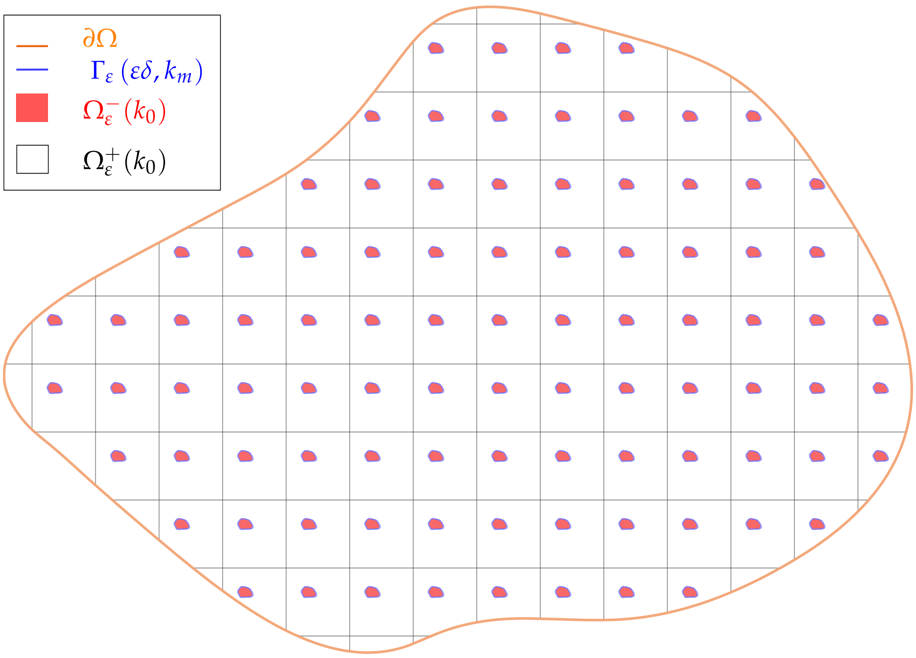

Figure 2.2: *Schematic illustration of the periodic medium Ω.*

2.4 Main results in the periodic case

We set $Y^+ := Y \setminus \overline{C}$ and $Y^- := C$. For any open set D in $\mathbb{R}^2$, we denote $H^1_{\mathbb{C}}(D)$ the Sobolev space $H^1(D)/\mathbb{C}$ which can be represented as :

$$H^1_{\mathbb{C}}(D) = \left\{ u \in H^1(D) \mid \int_D u(x)dx = 0 \right\}.$$

Throughout this paper, we assume that $\operatorname{dist}(Y^-, \partial Y) = O(1)$. We write the solution u_ε as

$$\forall x \in \Omega \;\; u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + o(\varepsilon), \tag{2.3}$$

with

$$y \longmapsto u_1(x,y) \; Y\text{-periodic and } u_1(x,y) = \begin{cases} u_1^+(x,y) \text{ in } \Omega \times Y^+, \\ u_1^-(x,y) \text{ in } \Omega \times Y^-. \end{cases}$$

The following theorem holds.

Theorem 2.1. (i) *The solution u_ε to* (2.2) *two-scale converges to u_0 and $\nabla u_\varepsilon(x)$ two-scale converges to $\nabla u_0(x) + \chi_{Y^+}(y)\nabla_y u_1^+(x,y) + \chi_{Y^-}(y)\nabla_y u_1^-(x,y)$, where $\chi_{Y^\pm}$ are the characteristic functions of $Y^\pm$.*

(ii) *The function u_0 in* (2.3) *is the solution in $H^1_{\mathbb{C}}(\Omega)$ to the following homogenized problem:*

$$\begin{cases} \nabla \cdot K^* \nabla u_0(x) = 0 & \text{in } \Omega, \\ n \cdot K^* \nabla u_0 = g & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where K^*, the effective admittivity of the medium, is given by

$$\forall (i,j) \in \{1,2\}^2, \qquad K^*_{i,j} = k_0 \left(\delta_{ij} + \int_Y (\chi_{Y^+} \nabla w_i^+ + \chi_{Y^-} \nabla w_i^-) \cdot e_j \right), \tag{2.5}$$

and the function $(w_i)_{i=1,2}$ are the solutions of the following cell problems:

$$\begin{cases} \nabla \cdot k_0 \nabla (w_i^+(y) + y_i) = 0 & \text{in } Y^+, \\ \nabla \cdot k_0 \nabla (w_i^-(y) + y_i) = 0 & \text{in } Y^-, \\ k_0 \dfrac{\partial}{\partial n}(w_i^+(y) + y_i) = k_0 \dfrac{\partial}{\partial n}(w_i^-(y) + y_i) & \text{on } \Gamma, \\ w_i^+ - w_i^- - \beta k_0 \dfrac{\partial}{\partial n}(w_i^+(y) + y_i) = 0 & \text{on } \Gamma, \\ y \longmapsto w_i(y) \ Y\text{-periodic}. \end{cases} \tag{2.6}$$

(iii) *Moreover, u_1 can be written as*

$$\forall (x,y) \in \Omega \times Y, \ \ u_1(x,y) = \sum_{i=1}^{2} \frac{\partial u_0}{\partial x_i}(x) w_i(y). \tag{2.7}$$

We define the integral operator $\mathcal{L}_\Gamma : \mathcal{C}^{2,\eta}(\Gamma) \to \mathcal{C}^{1,\eta}(\Gamma)$, with $0 < \eta < 1$ by

$$\mathcal{L}_\Gamma[\varphi](x) = \frac{1}{2\pi} \int_\Gamma \frac{\partial^2 \ln |x-y|}{\partial n(x) \partial n(y)} \varphi(y) ds(y), \quad x \in \Gamma. \tag{2.8}$$

$\mathcal{L}_\Gamma$ is the normal derivative of the double layer potential $\mathcal{D}_\Gamma$.

Since $\mathcal{L}_\Gamma$ is positive, one can prove that the operator $I + \alpha \mathcal{L}_\Gamma : \mathcal{C}^{2,\eta}(\Gamma) \to \mathcal{C}^{1,\eta}(\Gamma)$ is a bounded operator and has a bounded inverse provided that $\Re \alpha > 0$ [23, 45].

As the fraction f of the volume occupied by the cells goes to zero, we derive an expansion of the effective admittivity for arbitrary shaped cells in terms of the volume fraction. We refer to the suspension, as periodic dilute. The following theorem holds.

Theorem 2.2. *The effective admittivity of a periodic dilute suspension admits the following asymptotic expansion:*

$$K^* = k_0 \left(I + f M \left(I - \frac{f}{2} M \right)^{-1} \right) + o(f^2), \tag{2.9}$$

where $\rho = \sqrt{|Y^-|}$, $f = \rho^2$,

$$M = \left(M_{ij} = \beta k_0 \int_{\rho^{-1}\Gamma} n_j \psi_i^*(y) ds(y) \right)_{(i,j) \in \{1,2\}^2}, \tag{2.10}$$

and ψ_i^* is defined by

$$\psi_i^* = - \left(I + \beta k_0 \mathcal{L}_{\rho^{-1}\Gamma} \right)^{-1} [n_i]. \tag{2.11}$$

2.5 Description of the random cells and interfaces

We describe the domains occupied by the cells. As mentioned earlier, they are formed by randomly deforming a periodic structure. We transform the aforementioned periodic structure by a random diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. Let

$$\mathbb{R}_2^+ := \bigcup_{n \in \mathbb{Z}^2} (n + Y^+), \quad \mathbb{R}_2^- := \bigcup_{n \in \mathbb{Z}^2} (n + Y^-), \quad \mathbf{\Gamma}_2 := \bigcup_{n \in \mathbb{Z}^2} (n + \Gamma). \tag{2.12}$$

The cells, the environment and the interfaces are hence deformed to $\Phi(\mathbb{R}_2^-)$, $\Phi(\mathbb{R}_2^+)$ and $\Phi(\mathbf{\Gamma}_2)$. We emphasize that the topology of these sets are the same as before. Finally, the deformed structure is scaled to size ε, where $0 < \varepsilon \ll 1$, by the dilation operator $\varepsilon \mathbf{I}$ where $\mathbf{I}$ is the identity operator. The final sets $\varepsilon\Phi(\mathbb{R}_2^-)$, $\varepsilon\Phi(\mathbf{\Gamma}_2)$ and $\varepsilon\Phi(\mathbb{R}_2^+)$ thus are realistic models for the random cells, membranes and the environment for the biological problem at hand.

To model the cells inside an arbitrary bounded domain Ω as in (2.2), we would like to set $\Omega_\varepsilon^+ := \Omega \cap \varepsilon\Phi(\mathbb{R}_2^-)$ and $\Gamma_\varepsilon := \Omega \cap \varepsilon\Phi(\mathbf{\Gamma}_2)$. However, a technicality is encountered, precisely, the intersection of $\varepsilon\Phi(\mathbf{\Gamma}_2)$ with the boundary $\partial\Omega$ may not be empty. In this case, some cells are cut by the boundary of the body, which is not physically admissible. Moreover, an arbitrary diffeomorphism Φ may allow some deformed cells in $\varepsilon\Phi(\mathbb{R}_2^-)$ to get arbitrarily close to each other. This imposes difficulties for rigorous mathematical analysis. In order to resolve these issues, we will impose a few conditions on Φ and refine the above construction in the next subsection.

2.6 Stationary ergodic setting

Let $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ be some probability space on which $\Phi(x, \gamma) : \mathbb{R}^2 \times \mathcal{O} \to \mathbb{R}^2$ is defined. For a random variable $X \in L^1(\mathcal{O}, d\mathbb{P})$, we will denote its expectation by

$$\mathbb{E}X = \int_{\mathcal{O}} X(\gamma) d\mathbb{P}(\gamma).$$

Throughout this paper, we assume that the group $(\mathbb{Z}^2, +)$ acts on $\mathcal{O}$ by some action $\{\tau_n : \mathcal{O} \to \mathcal{O}\}_{n \in \mathbb{Z}^2}$, and that for all $n \in \mathbb{Z}^2$, τ_n is $\mathbb{P}$-preserving, that is,

$$\mathbb{P}(A) = \mathbb{P}(\tau_n A), \quad \text{for all } A \in \mathcal{F}.$$

We assume further that the action is *ergodic*, which means that for any $A \in \mathcal{F}$, if $\tau_n A = A$ for all $n \in \mathbb{Z}^2$, then necessarily $\mathbb{P}(A) \in \{0, 1\}$.

Following [19], we say that a random process $F \in L^1_{\text{loc}}(\mathbb{R}^2, L^1(\mathcal{O}))$ is (discrete) *stationary* if

$$\forall n \in \mathbb{Z}^2, \quad F(x + n, \gamma) = F(x, \tau_n \gamma) \quad \text{for almost every } x \text{ and } \gamma. \tag{2.13}$$

Clearly, a deterministic periodic function is a special case of stationary process. However, we precise that the above notion of stationarity is different from the classical one, see for instance [46] and [38]. Throughout this paper, we presume stationarity in the sense of (2.13) if not stated otherwise. What makes this notion useful is the following version of ergodic theorem [25, 34].

Proposition 2.1. *Let $F \in L^\infty(\mathbb{R}^2, L^1(\mathcal{O}))$ be a stationary random process. Equip $\mathbb{Z}^2$ with the norm $|n|_\infty = \max_{1\le i\le 2}|n_i|$ for all $n \in \mathbb{Z}^2$. Then*

$$\frac{1}{(2N+1)^2}\sum_{|n|_\infty \le N} F(x,\tau_n\gamma) \xrightarrow[N\to\infty]{L^\infty} \mathbb{E}F(x,\cdot) \quad \textit{for a.e. } \gamma\in\mathcal{O}. \tag{2.14}$$

This implies in particular that if the family $\{F(\frac{\cdot}{\varepsilon},\gamma)\}$ is bounded in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$, for some $p \in [1,\infty)$, then

$$F\left(\frac{x}{\varepsilon},\gamma\right) \underset{\varepsilon\to 0}{\rightharpoonup} \mathbb{E}\left(\int_Y F(x,\cdot)dx\right) \textit{ in } L^p_{\mathrm{loc}}(\mathbb{R}^2) \textit{ for a.e. } \gamma\in\mathcal{O}. \tag{2.15}$$

The convergence holds also in the weak-$*$ sense for $p=\infty$.

We assume that for every $\gamma\in\mathcal{O}$, $\Phi(\cdot,\gamma)$ is a diffeomorphsim from $\mathbb{R}^2$ to $\mathbb{R}^2$ and that it satisfies

$$\nabla\Phi(x,\gamma) \text{ is stationary.} \tag{2.16}$$

$$\operatorname*{ess\,inf}_{\gamma\in\mathcal{O},x\in\mathbb{R}^2} \det(\nabla\Phi(x,\gamma)) = \kappa > 0, \tag{2.17}$$

$$\operatorname*{ess\,sup}_{\gamma\in\mathcal{O},x\in\mathbb{R}^2} |\nabla\Phi(x,\gamma)|_F = \kappa' > 0, \tag{2.18}$$

where $|\cdot|_F$ is the Frobenius norm and ess inf and ess sup are the essential infimum and the essentiel supremum, respectively. To avoid the intersection of $\partial\Omega$ and the random cells $\varepsilon\Phi(\mathbb{R}^-_2)$ and the collision of cells, that is when two connected components of $\varepsilon\Phi(\mathbb{R}^-_2)$ get as close as $o(\varepsilon)$, we need the further modification in the construction of cells. To this end, we assume further that

$$\|\Phi(\cdot,\gamma) - \mathbf{I}(\cdot)\|_{L^\infty(\mathbb{R}^2)} \le \frac{\operatorname{dist}(Y^-,\partial Y)}{2} \text{ for a.e. } \gamma\in\mathcal{O}. \tag{2.19}$$

Note that this implies also that $\|\Phi^{-1}-\mathbf{I}\|_{L^\infty} \le \operatorname{dist}(Y^-,\partial Y)/2$ a.s. in $\mathcal{O}$. Now, given a bounded and simply connected open set Ω with smooth boundary and a small number $\varepsilon \ll 1$, we denote by $\Omega_{1/\varepsilon}$ the scaled set $\{x\in\mathbb{R}^2 \mid \varepsilon x\in\Omega\}$. Let $\widetilde{\Omega_{1/\varepsilon}}$ be the shrunk set

$$\widetilde{\Omega_{1/\varepsilon}} := \{x\in\Omega_{1/\varepsilon} \mid \operatorname{dist}(x,\partial\Omega_{1/\varepsilon}) \ge \operatorname{dist}(Y^-,\partial Y)\}.$$

We introduce for $n\in\mathbb{Z}^2$, Y_n and $Y_n^\pm$ the translated cubes, reference cells and reference environments: $Y_n := n+Y$, $Y_n^\pm := n+Y^\pm$. Let $\mathcal{I}_\varepsilon \subset \mathbb{Z}^2$ be the indices of cubes Y_n such that $Y_n \in \widetilde{\Omega_{1/\varepsilon}}$. Note that $\mathcal{I}_\varepsilon$ corresponds to N_ε in the periodic case. We set Ω^-_ε to be

$$\Omega^-_\varepsilon := \sum_{n\in\mathcal{I}_\varepsilon} \varepsilon\Phi(Y^-_n) \tag{2.20}$$

and then $\Omega^+_\varepsilon = \Omega\setminus\overline{\Omega^-_\varepsilon}$. We also define the following two notations:

$$E_\varepsilon := \sum_{n\in\mathcal{I}_\varepsilon}\varepsilon\Phi(Y_n) \quad\text{and}\quad K_\varepsilon := \Omega\setminus\overline{E_\varepsilon}. \tag{2.21}$$

Clearly, E_ε encloses all the cells in $\varepsilon\Phi(Y_n^-), n \in \mathcal{I}_\varepsilon$ and their immediate surroundings $\varepsilon\Phi(Y_n^+)$; K_ε is a cushion layer near the boundary that prevents the cells from touching the boundary. From the construction we see that

$$\inf_{x\in\Omega_\varepsilon^-} \mathrm{dist}(x,\partial\Omega) \geq \varepsilon \mathrm{dist}(Y^-,\partial Y) \quad \text{and} \quad \sup_{x\in K_\varepsilon} \mathrm{dist}(x,\partial\Omega) \leq (3\mathrm{dist}(Y^-,\partial Y)+\sqrt{2})\varepsilon. \tag{2.22}$$

Furthermore, we can check that

$$\sup_{n,j\in\mathcal{I}_\varepsilon, n\neq j} \;\inf_{x\in\varepsilon\Phi(Y_n^-), y\in\varepsilon\Phi(Y_j^-)} |x-y| \geq \mathrm{dist}(Y^-,\partial Y)\varepsilon. \tag{2.23}$$

This shows that the cells in Ω are well separated, *i.e.*, with a distance comparable to (if not much larger than) the size of the cells, see Figure 2.3.

2.7 Main results in the random case

The first important result in the random case concerns an auxiliary problem which produces oscillating test functions that are used in the stochastic homogenization procedure. In the following theorem, a function f^{ext} in $W^{1,s}_{\mathrm{loc}}(\mathbb{R}^2)$ is said to be an extension of $f \in W^{1,s}_{\mathrm{loc}}(\mathbb{R}^+_2)$ if $f^{\mathrm{ext}} = f$ on $\mathbb{R}^+_2$ and $\|f^{\mathrm{ext}}\|_{W^{1,s}(K)} \leq C(K,\mathbb{R}^+_2)\|f\|_{W^{1,s}(\mathbb{R}^+_2\cap K)}$, for any compact subset K.

The following theorem holds.

Theorem 2.3. *Let $\Phi(\cdot,\gamma)$ be a random process defined on the probability space $(\mathcal{O},\mathcal{F},\mathbb{P})$, valued in the space of diffeomorphisms from $\mathbb{R}^2$ to $\mathbb{R}^2$ satisfying* (2.16)-(2.18)*. Then for a.e. $\gamma \in \mathcal{O}$ and for an arbitrary fixed $p \in \mathbb{R}^2$, the system*

$$\begin{cases} \nabla\cdot k_0(\nabla w_p^+(y)+p) = 0 & \text{in } \Phi(\mathbb{R}^\pm_2,\gamma),\\ \nabla\cdot k_0(\nabla w_p^-(y)+p) = 0 & \text{in } \Phi(\mathbb{R}^\pm_2,\gamma),\\ k_0\dfrac{\partial w_p^+}{\partial n}(y) - \dfrac{\partial w_p^-}{\partial n}(y) = 0 & \text{on } \Phi(\Gamma_2,\gamma),\\ w_p^+ - w_p^- - \beta k_0 \dfrac{\partial w_p^+}{\partial n}(y) = 0 & \text{on } \Phi(\Gamma_2,\gamma),\\ w_p^\pm(y,\gamma) = \tilde{w}_p^\pm(\Phi^{-1}(y,\gamma),\gamma), & \\ \nabla\tilde{w}_p^\pm \text{ are stationary}, & \\ \exists \tilde{w}_p^{\mathrm{ext}} \in H^1_{\mathrm{loc}}(\mathbb{R}^2) \text{ extends } \tilde{w}_p^+ \quad \text{s.t. } \mathbb{E}\left(\int_Y \nabla\tilde{w}_p^{\mathrm{ext}}(\tilde{y},\cdot)d\tilde{y}\right) = 0, & \end{cases} \tag{2.24}$$

admits a unique solution (up to an additive constant) $w_p = w_p^+\chi_{\Phi(\mathbb{R}^+_2)} + w_p^-\chi_{\Phi(\mathbb{R}^-_2)}$, where $w_p^\pm \in H^1_{\mathrm{loc}}(\Phi(\mathbb{R}^\pm_2))$.

Clearly, one could add another condition to the above problem, namely that the integral of $\tilde{w}_p^+$ in Y^+ vanishes, to fix the additional constant. The second main result in the random case is the following homogenization theorem.

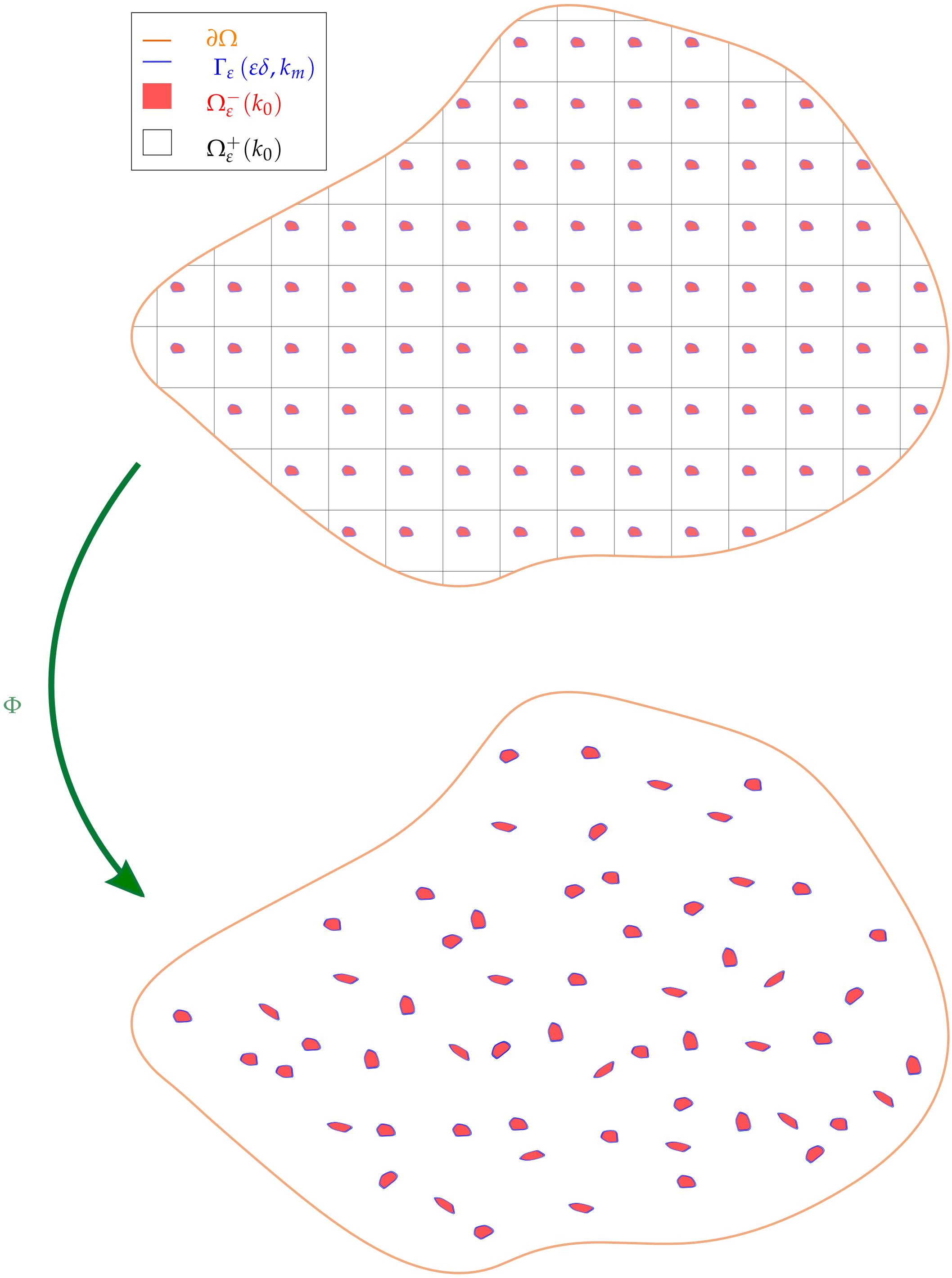

Figure 2.3: *Schematic illustration of the randomly deformed periodic medium* Ω.

Theorem 2.4. *Let Ω be a bounded and connected open subset of $\mathbb{R}^2$ with regular boundary. Let Φ be a random diffeomorphism on $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ satisfying* (2.16)-(2.19). *Assume that the cells Ω_ε^- are constructed as in the previous section. Then for a.e. $\gamma \in \mathcal{O}$, the solution $u_\varepsilon(\cdot,\gamma) = (u_\varepsilon^+, u_\varepsilon^-)$ of* (2.2) *satisfies the following properties:*

(i) *We can extend $u_\varepsilon^+(\cdot,\gamma)$ to $u_\varepsilon^{\text{ext}}(\cdot,\gamma) \in H^1(\Omega)$, where $u_\varepsilon^{\text{ext}}(\cdot,\gamma)$ converges weakly, as $\varepsilon \to 0$, to some deterministic function $u_0 \in H^1(\Omega)$.*

(ii) *The function $u_\varepsilon(\cdot,\gamma)$ converges strongly in L^2 to u_0 above. Further, let Q be the trivial extension operator setting $Qf = 0$ outside the domain of f, and define*

$$\varrho := \det\left(\mathbb{E}\int_Y \nabla\Phi(z,\cdot)dz\right)^{-1}, \quad \theta := \varrho\, \mathbb{E}\int_{Y^-} \det\nabla\Phi(z,\cdot)dz,$$

where det *denotes the determinant. Then, Qu_ε^- converges weakly to θu_0 in $L^2(\Omega)$ with $\theta < 1$.*

(iii) *The function u_0 is the unique weak solution in $H^1_{\mathbb{C}}(\Omega)$ to the homogenized equation*

$$\begin{cases} \nabla \cdot K^*\nabla u_0(x) = 0, & x \in \Omega, \\ n(x)\cdot K^*\nabla u_0(x) = g, & x \in \partial\Omega, \end{cases} \tag{2.25}$$

The homogenized admittivity coefficient K^* is given by

$$\forall (i,j) \in \{1,2\}^2, \quad K^*_{ij} = k_0\left(\delta_{ij} + \varrho\mathbb{E}\int_{\Phi(Y)} e_j\cdot(\chi_{Y^+}\nabla w^+_{e_i} + \chi_{Y^-}\nabla w^-_{e_i})(y,\cdot)\,dy\right), \tag{2.26}$$

where $\{e_i\}_{i=1}^2$ is the Euclidean basis of $\mathbb{R}^2$ and for each $p \in \mathbb{R}^2$, the pair of functions (w_p^+, w_p^-) is the unique solution to the auxiliary system (2.24).

In the dilute limit, we obtain the following approximation of the effective permittivity for the dilute suspension:

$$K^*_{ij} = k_0(I + \varrho f \mathbb{E} M_{ij}) + o(f), \tag{2.27}$$

where ϱ accounts for the averaged change of volume due to the random diffeomorphism and f is the volume fraction occupied by the cells ; the polarization matrix M is defined by

$$M_{ij} = \beta k_0 \int_{\rho^{-1}\Phi(\Gamma)} \tilde{\psi}_i n_j \, ds(\tilde{y}), \tag{2.28}$$

and is associated to the deformed inclusion scaled to the unit length scale.

3 Analysis of the problem

For a fixed ε, recall that $H^1_{\mathbb{C}}(\Omega_\varepsilon^+)$ denotes the Sobolev space $H^1(\Omega_\varepsilon^+)/\mathbb{C}$, which can be represented as

$$H^1_{\mathbb{C}}(\Omega_\varepsilon^+) = \left\{ u \in H^1(\Omega_\varepsilon^+) \mid \int_{\Omega_\varepsilon^+} u(x)dx = 0 \right\}. \tag{3.1}$$

The natural functional space for (2.2) is

$$W_\varepsilon := \left\{ u = u^+\chi_\varepsilon^+ + u^-\chi_\varepsilon^- \mid u^+ \in H^1_{\mathbb{C}}(\Omega_\varepsilon^+), u^- \in H^1(\Omega_\varepsilon^-) \right\}, \tag{3.2}$$

where $\chi_\varepsilon^\pm$ are the characteristic functions of the sets $\Omega_\varepsilon^\pm$. We can verify that

$$\|u\|_{W_\varepsilon} = \left(\|\nabla u^+\|^2_{L^2(\Omega_\varepsilon^+)} + \|\nabla u^-\|^2_{L^2(\Omega_\varepsilon^-)} + \varepsilon \|u^+ - u^-\|^2_{L^2(\Gamma_\varepsilon)} \right)^{\frac{1}{2}} \tag{3.3}$$

defines a norm on W_ε. In fact, as it will be seen in Proposition 3.2, this norm is equivalent to the standard norm on W_ε which is

$$\|u\|_{H^1_{\mathbb{C}}(\Omega_\varepsilon^+)\times H^1(\Omega_\varepsilon^-)} = \left(\|\nabla u^+\|^2_{L^2(\Omega_\varepsilon^+)} + \|\nabla u^-\|^2_{L^2(\Omega_\varepsilon^-)} + \|u^-\|^2_{L^2(\Omega_\varepsilon^-)} \right)^{\frac{1}{2}}. \tag{3.4}$$

3.1 Existence and uniqueness of a solution

Problem (2.2) should be understood through its weak formulation as follows: For a fixed $\varepsilon > 0$, find $u_\varepsilon \in W_\varepsilon$ such that

$$\int_{\Omega_\varepsilon^+} k_0 \nabla u_\varepsilon^+(x) \cdot \nabla \overline{v^+}(x) dx + \int_{\Omega_\varepsilon^-} k_0 \nabla u_\varepsilon^-(x) \cdot \nabla \overline{v^-}(x) ds(x) \\ + \frac{1}{\varepsilon\beta} \int_{\Gamma_\varepsilon} (u_\varepsilon^+ - u_\varepsilon^-)(x) \overline{(v^+ - v^-)}(x) ds(x) = \int_{\partial\Omega} g(x) \overline{v^+}(x) ds(x), \tag{3.5}$$

for any function $v \in W_\varepsilon$.

Define the sesquilinear form $a_\varepsilon(\cdot,\cdot)$ on $W_\varepsilon \times W_\varepsilon$ by

$$a_\varepsilon(u,v) := \int_{\Omega_\varepsilon^+} k_0 \nabla u^+ \cdot \nabla \overline{v^+} dx + \int_{\Omega_\varepsilon^-} k_0 \nabla u^- \cdot \nabla \overline{v^-} dx + \frac{1}{\varepsilon\beta} \int_{\Gamma_\varepsilon} (u^+ - u^-) \overline{(v^+ - v^-)} ds. \tag{3.6}$$

Associate the following anti-linear form on W_ε to the boundary data g:

$$\ell(u) := \int_{\partial\Omega} g \overline{u^+} ds. \tag{3.7}$$

The forms a_ε and ℓ are bounded. Moreover, a_ε is coercive in the following sense

$$\Re\, k_0^{-1} a_\varepsilon(u,u) = \left(\int_{\Omega_\varepsilon^+} |\nabla u^+|^2 dx + \int_{\Omega_\varepsilon^-} |\nabla u^-|^2 dx \right) + \frac{1}{\varepsilon\beta'} \int_{\Gamma_\varepsilon} |u^+ - u^-|^2 ds \geq C \|u\|^2_{W_\varepsilon}, \tag{3.8}$$

where $\beta' := \delta(\sigma_0\sigma_m + \omega^2\epsilon_0\varepsilon_m)/(\sigma_m^2 + \omega^2\epsilon_m^2)$. Consequently, due to the Lax–Milgram theorem we have existence and uniqueness for (2.2) for each fixed ε and for every $\gamma \in \mathcal{O}$. Note that C can be chosen independent of ε.

Proposition 3.1. *Let $g \in H^{-1/2}(\partial\Omega)$. There exists a unique $u_\varepsilon \in W_\varepsilon$ so that*

$$a_\varepsilon(u_\varepsilon, \varphi) = \ell(\varphi), \quad \forall \varphi \in W_\varepsilon. \tag{3.9}$$

To end this subsection we remark that the two norms on W_ε are equivalent.

Proposition 3.2. *The norm $\|\cdot\|_{W_\varepsilon}$ is equivalent with the standard norm on $H^1_{\mathbb{C}}(\Omega^+_\varepsilon) \times H^1(\Omega^-_\varepsilon)$. Moreover, we can find two positive constants $C_1 < C_2$, independent of ε, so that*

$$C_1\|u\|_{W_\varepsilon} \leq \|u\|_{H^1_{\mathbb{C}}\times H^1} \leq C_2\|u\|_{W_\varepsilon}, \tag{3.10}$$

for any $u \in H^1_{\mathbb{C}}(\Omega^+_\varepsilon) \times H^1(\Omega^-_\varepsilon)$.

Similar equivalence relation was established by Monsurrò [43], whose method can be adapted easily to the current case. For the sake of completeness, we present the details in Appendix C.

3.2 Energy estimate

For any fixed $\gamma \in \mathcal{O}$ and a sequence of $\varepsilon \to 0$, by solving (2.2) we obtain the sequence $u_\varepsilon = u^+_\varepsilon\chi^+_\varepsilon + u^-_\varepsilon\chi^-_\varepsilon$. We obtain some *a priori* estimates for u_ε.

We first recall that the extension theorem (Theorem A.2) yields a Poincaré–Wirtinger inequality in $H^1_{\mathbb{C}}(\Omega^+_\varepsilon)$ with a constant independent of ε. Indeed, Corollary B.1 shows that for all $v^+ \in H^1_{\mathbb{C}}(\Omega^+_\varepsilon)$, there exists a constant C, independent of ε, such that

$$\|v^+\|_{L^2(\Omega^+_\varepsilon)} \leq C\|\nabla v^+\|_{L^2(\Omega^+_\varepsilon)}.$$

Similarly, we can find a constant, independent of ε, by applying the trace theorem in $H^1(\Omega^+_\varepsilon)$. Using Corollary B.2, the following result holds.

Proposition 3.3. *Let $g \in H^{-\frac{1}{2}}(\partial\Omega)$. For any $\gamma \in \mathcal{O}$, let $\Omega = \Omega^+_\varepsilon \cup \Gamma_\varepsilon \cup \Omega^-_\varepsilon$. Then there exist constants C's, independent of ε and γ, such that the solution u_ε to* (2.2) *satisfies the following estimates:*

$$\|\nabla u^+_\varepsilon\|_{L^2(\Omega^+_\varepsilon)} + \|\nabla u^-_\varepsilon\|_{L^2(\Omega^-_\varepsilon)} \leq C|k_0|^{-1}\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}, \tag{3.11}$$

$$\|u^+_\varepsilon - u^-_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \leq C|k_0|^{-1}\sqrt{\varepsilon\beta'}\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{3.12}$$

Proof. By taking $\varphi = u_\varepsilon$ in (3.9), and taking the real part of resultant equality, we get

$$\|\nabla u^+_\varepsilon\|^2_{L^2(\Omega^+_\varepsilon)} + \|\nabla u^-_\varepsilon\|^2_{L^2(\Omega^-_\varepsilon)} + (\varepsilon\beta')^{-1}\|u^+_\varepsilon - u^-_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} = \Re k_0^{-1}\langle g, u^+_\varepsilon\rangle. \tag{3.13}$$

Here $\langle g, u^+_\varepsilon\rangle = \int_{\partial\Omega} g\overline{u^+_\varepsilon}ds$ is the pairing on $H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, for which we have the estimate

$$|\langle g, u^+_\varepsilon\rangle| \leq \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\|u^+_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_1\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\|u^+_\varepsilon\|_{H^1(\Omega^+_\varepsilon)}.$$

thanks to the Cauchy - Schwartz inequality and Corollary (B.2). C_1 is here a constant which does not depend on ε.

Applying Proposition (3.2) yields

$$|\langle g, u^+_\varepsilon\rangle| \leq C_2\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\|u_\varepsilon\|_{W_\varepsilon},$$

with a constant C_2 independent of ε.

Using this in (3.13) along with the coercivity of a we get

$$\|u_\varepsilon\|_{W_\varepsilon} \leq C_3 |k_0|^{-1} \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)},$$

where C_3 is still independent of ε.

It follows also that

$$|\langle g, u_\varepsilon^+ \rangle| \leq C_2 C_3 |k_0|^{-1} \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}.$$

Substitute this estimate into the right-hand side of (3.13), we get the desired estimates. □

Next, we apply the extension theorem (Theorem A.2) to obtain a bounded sequence in $H^1(\Omega)$ for which we can extract a converging subsequence.

Proposition 3.4. *Suppose that the same conditions of the previous proposition hold. Let $P_\gamma^\varepsilon : H^1(\Omega_\varepsilon^+) \to H^1(\Omega)$ be the extension operator of Theorem A.2. Then we have*

$$\|P_\gamma^\varepsilon u_\varepsilon^+\|_{H^1(\Omega)} \leq C |k_0|^{-1} \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}, \tag{3.14}$$

and

$$\|P_\gamma^\varepsilon u_\varepsilon^+ - u_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon |k_0|^{-1} (1 + \sqrt{\beta'}) \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{3.15}$$

Proof. The first inequality is a direct result of (A.11), (A.11), (B.2) and (3.11). For the second inequality, we have

$$\begin{aligned}\|P_\gamma^\varepsilon u_\varepsilon^+ - u_\varepsilon\|_{L^2(\Omega)} = \|P_\gamma^\varepsilon u_\varepsilon^+ - u_\varepsilon^-\|_{L^2(\Omega_\varepsilon^-)} \leq C\sqrt{\varepsilon} \|P_\gamma^\varepsilon u_\varepsilon^+ - u_\varepsilon^-\|_{L^2(\Gamma_\varepsilon)} \\ + C\varepsilon \|\nabla (P_\gamma^\varepsilon u_\varepsilon^+ - u_\varepsilon^-)\|_{L^2(\Omega_\varepsilon^-)}.\end{aligned}$$

Here, we have used estimate (C.3). Now, $\|P_\gamma^\varepsilon u_\varepsilon^+ - u_\varepsilon^-\|_{L^2(\Gamma_\varepsilon)} = \|u_\varepsilon^+ - u_\varepsilon^-\|_{L^2(\Gamma_\varepsilon)}$ is bounded in (3.12). The second term is bounded from above by

$$C\varepsilon \|\nabla P_\gamma^\varepsilon u_\varepsilon^+\|_{L^2(\Omega_\varepsilon^-)} + C\varepsilon \|\nabla u_\varepsilon^-\|_{L^2(\Omega_\varepsilon^-)} \leq C\varepsilon (\|\nabla u_\varepsilon^+\|_{L^2(\Omega_\varepsilon^+)} + \|\nabla u_\varepsilon^-\|_{L^2(\Omega_\varepsilon^-)}),$$

where we have used again (A.11). This gives the desired estimates. □

Remark 3.1. *As a consequence of the previous proposition, we get a sequence in $H^1(\Omega)$, namely $P_\gamma^\varepsilon u_\varepsilon^+$, which is a good estimate of u_ε in $L^2(\Omega)$ and from which we can extract a subsequence weakly converging in $H^1(\Omega)$ and strongly in $L^2(\Omega)$.*

4 Homogenization

We follow [5, 6] to derive a homogenized problem for the model with two-scale asymptotic expansions and to prove a rigorous two-scale convergence. In [43], the homogenization of an analogue problem is developed and proved with another method.

4.1 Two-scale asymptotic expansions

We assume that the solution u_ε admits the following two-scale asymptotic expansion

$$\forall x \in \Omega \;\; u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + o(\varepsilon),$$

with

$$y \longmapsto u_1(x,y)\, Y\text{-periodic and } u_1(x,y) = \begin{cases} u_1^+(x,y) \text{ in } \Omega \times Y^+, \\ u_1^-(x,y) \text{ in } \Omega \times Y^-. \end{cases}$$

We choose a test function φ_ε of the same form as u_ε:

$$\forall x \in \Omega, \;\; \varphi_\varepsilon(x) = \varphi_0(x) + \varepsilon \varphi_1(x, \frac{x}{\varepsilon}),$$

with φ_0 smooth in Ω, $\varphi_1(x,.)$ Y-periodic,

$$\varphi_1(x,y) = \begin{cases} \varphi_1^+(x,y) \text{ in } \Omega \times Y^+, \\ \varphi_1^-(x,y) \text{ in } \Omega \times Y^-, \end{cases}$$

and $\varphi_1^\pm$ smooth in $\Omega \times Y^\pm$.

In order to prove items (ii) and (iii) in Theorem 2.1, we perform an asymptotic expansion of the variational formulation (3.9). We thus inject these ansatz in the variational formulation and only consider the order 0 of the different integrals.

At order 0,

$$\nabla u_\varepsilon(x) = \nabla u_0(x) + \nabla_y u_1(x, \frac{x}{\varepsilon}) + o(\varepsilon).$$

Thanks to Lemma 4.1, we then have for the two first integrals:

$$\int_{\Omega_\varepsilon^+} k_0 \left(\nabla u_0(x) + \nabla_y u_1^+(x, \frac{x}{\varepsilon})\right) \cdot \left(\nabla \overline{\varphi}_0(x) + \nabla_y \overline{\varphi}_1^+(x, \frac{x}{\varepsilon})\right) dx$$
$$= \int_\Omega \int_{Y^+} k_0 \left(\nabla u_0(x) + \nabla_y u_1^+(x,y)\right) \cdot \left(\nabla \overline{\varphi}_0(x) + \nabla_y \overline{\varphi}_1^+(x,y)\right) dxdy + o(\varepsilon)$$

and

$$\int_{\Omega_\varepsilon^-} k_0 \left(\nabla u_0(x) + \nabla_y u_1^-(x, \frac{x}{\varepsilon})\right) \cdot \left(\nabla \overline{\varphi}_0(x) + \nabla_y \overline{\varphi}_1^-(x, \frac{x}{\varepsilon})\right) dx$$
$$= \int_\Omega \int_{Y^-} k_0 \left(\nabla u_0(x) + \nabla_y u_1^-(x,y)\right) \cdot \left(\nabla \overline{\varphi}_0(x) + \nabla_y \overline{\varphi}_1^-(x,y)\right) dxdy + o(\varepsilon).$$

We write the third integral in (3.6) as the sum, over all squares $Y_{\varepsilon,n}$, of integrals on the boundaries $\Gamma_{\varepsilon,n}$. We have

$$\frac{1}{\beta\varepsilon} \int_{\Gamma_\varepsilon} \left(u_\varepsilon^+(x, \frac{x}{\varepsilon}) - u_\varepsilon^-(x, \frac{x}{\varepsilon})\right) \left(\overline{\varphi}_\varepsilon^+(x, \frac{x}{\varepsilon}) - \overline{\varphi}_\varepsilon^-(x, \frac{x}{\varepsilon})\right) ds(x)$$
$$= \frac{1}{\beta\varepsilon} \sum_{n \in N_\varepsilon} \int_{\Gamma_{\varepsilon,n}} \left(u_\varepsilon^+(x, \frac{x}{\varepsilon}) - u_\varepsilon^-(x, \frac{x}{\varepsilon})\right) \left(\overline{\varphi}_\varepsilon^+(x, \frac{x}{\varepsilon}) - \overline{\varphi}_\varepsilon^-(x, \frac{x}{\varepsilon})\right) ds(x).$$

Let $x_{0,n}$ be the center of $Y_{\varepsilon,n}$ for each $n \in N_\varepsilon$. We perform Taylor expansions with respect to the variable x around $x_{0,n}$ for all functions $(u_i)_{i\in\{1,2\}}$ and $(\varphi_i)_{i\in\{1,2\}}$ on $Y_{\varepsilon,n}$. After the change of variables $\varepsilon(y - y_{0,n}) = x - x_{0,n}$, we obtain that

$$u_\varepsilon(x) = u_0(x_{0,n}) + \varepsilon u_1(x,y) + \varepsilon(y - y_{0,n}) \cdot \nabla u_0(x_{0,n}) + o(\varepsilon),$$
$$\varphi_\varepsilon(x) = \varphi_0(x_{0,n}) + \varepsilon\varphi_1(x,y) + \varepsilon(y - y_{0,n}) \cdot \nabla\varphi_0(x_{0,n}) + o(\varepsilon).$$

Consequently, the third integral in the variational formulation (3.9) becomes

$$\frac{\varepsilon^2}{\beta} \sum_{n\in N_\varepsilon} \int_{\Gamma_n} \left(u_1^+(x_{0,n},y) - u_1^-(x_{0,n},y)\right) \left(\overline{\varphi}_1^+(x_{0,n},y) - \overline{\varphi}_1^-(x_{0,n},y)\right) ds(y).$$

Finally, Lemma 4.1 gives us that

$$\frac{1}{\varepsilon\beta} \int_{\Gamma_\varepsilon} \left(u_\varepsilon^+ - u_\varepsilon^-\right) \left(\overline{\varphi}_\varepsilon^+ - \overline{\varphi}_\varepsilon^-\right) ds$$
$$= \frac{1}{\beta} \int_\Omega \int_\Gamma \left(u_1^+(x,y) - u_1^-(x,y)\right) \left(\overline{\varphi}_1^+(x,y) - \overline{\varphi}_1^-(x,y)\right) dxds(y) + o(\varepsilon).$$

Moreover, we can easily see that

$$\int_{\partial\Omega} g\overline{\varphi}_\varepsilon^+ ds = \int_{\partial\Omega} g\overline{\varphi}_0 ds + o(\varepsilon).$$

The order 0 of the variational formula is thus given by

$$\begin{aligned}
& \int_\Omega \int_{Y^+} k_0 \left(\nabla u_0(x) + \nabla_y u_1^+(x,y)\right) \cdot \left(\nabla\overline{\varphi}_0(x) + \nabla_y \overline{\varphi}_1^+(x,y)\right) dxdy \\
+ & \int_\Omega \int_{Y^-} k_0 \left(\nabla u_0(x) + \nabla_y u_1^-(x,y)\right) \cdot \left(\nabla\overline{\varphi}_0(x) + \nabla_y \overline{\varphi}_1^-(x,y)\right) dxdy \\
+ & \frac{1}{\beta} \int_\Omega \int_\Gamma \left(u_1^+(x,y) - u_1^-(x,y)\right) \left(\overline{\varphi}_1^+(x,y) - \overline{\varphi}_1^-(x,y)\right) dxds(y) \\
- & \int_{\partial\Omega} g(x)\overline{\varphi}_0(x)ds(x) = 0.
\end{aligned}$$

By taking $\varphi_0 = 0$, it follows that

$$\begin{aligned}
& \int_\Omega \int_{Y^+} k_0 \left(\nabla u_0(x) + \nabla_y u_1^+(x,y)\right) \cdot \nabla_y \overline{\varphi}_1^+(x,y)dxdy \\
+ & \int_\Omega \int_{Y^-} k_0 \left(\nabla u_0(x) + \nabla_y u_1^-(x,y)\right) \cdot \nabla_y \overline{\varphi}_1^-(x,y)dxdy \\
+ & \frac{1}{\beta} \int_\Omega \int_\Gamma \left(u_1^+(x,y) - u_1^-(x,y)\right) \left(\overline{\varphi}_1^+(x,y) - \overline{\varphi}_1^-(x,y)\right) dxds(y) = 0,
\end{aligned}$$

which is exactly the variational formulation of the cell problem (2.6) and definition (2.7) of u_1.

By taking $\varphi_1 = 0$, we recover the variational formulation of the homogenized problem (2.4):

$$\begin{aligned} & \int_\Omega \int_{Y^+} k_0 \left(\nabla u_0(x) + \nabla_y u_1^+(x,y)\right) \cdot \nabla \overline{\varphi}_0(x) dx dy \\ + & \int_\Omega \int_{Y^-} k_0 \left(\nabla u_0(x) + \nabla_y u_1^-(x,y)\right) \cdot \nabla \overline{\varphi}_0(x) dx dy \\ - & \int_{\partial\Omega} g(x) \overline{\varphi}_0(x) ds(x) = 0. \end{aligned}$$

We introduce some function spaces, which will be very useful in the following:

- $C^\infty_\sharp(D)$ is the space of functions, which are Y - periodic and in $C^\infty(D)$,
- $L^2_\sharp(D)$ is the completion of $C^\infty_\sharp(D)$ in the L^2-norm,
- $H^1_\sharp(D)$ is the completion of $C^\infty_\sharp(D)$ in the H^1-norm,
- $L^2(\Omega, H^1_\sharp(D))$ is the space of square integrable functions on Ω with values in the space $H^1_\sharp(D)$,
- $\mathcal{D}(\Omega)$ is the space of infinitely smooth functions with compact support in Ω,
- $\mathcal{D}(\Omega, C^\infty_\sharp(D))$ is the space of infinitely smooth functions with compact support in Ω and with values in the space $C^\infty_\sharp$,

where D is Y, Y^+, Y^- or Γ.

The following lemma was used in the preceding proof. It follows from [5, Lemma 3.1].

Lemma 4.1. *Let f be a smooth function. We have*

$$(i) \quad \varepsilon^2 \sum_{n \in N_\varepsilon} \int_{\Gamma_{\varepsilon,n}} f(x_{0,n}, y) ds(y) = \int_\Omega \int_\Gamma f(x,y) dx ds(y) + o(\varepsilon);$$

$$(ii) \quad \int_{\Omega_\varepsilon^+} f(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_{Y^+} f(x,y)\, dx dy + o(\varepsilon)$$

$$\text{and} \quad \int_{\Omega_\varepsilon^-} f(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_{Y^-} f(x,y)\, dx dy + o(\varepsilon).$$

We prove that the following lemmas hold.

Lemma 4.2. *The homogenized problem admits a unique solution in $H^1_{\mathbb{C}}(\Omega)$.*

Proof. The effective admittivity can be rewritten as

$$\begin{aligned} K^*_{i,j} &= k_0 \int_{Y^+} (\nabla w_i^+ + e_i) \cdot (\nabla \overline{w_j^+} + e_j) dx + k_0 \int_{Y^-} (\nabla w_i^- + e_i) \cdot (\nabla \overline{w_j^-} + e_j) dx \\ &\quad + \frac{1}{\beta} \int_\Gamma (w_i^+ - w_i^-)(\overline{w_j^+} - \overline{w_j^-}) ds, \quad i,j = 1,2. \end{aligned}$$

Therefore, it follows that, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$,

$$K^*\xi \cdot \xi = k_0 \int_{Y^+} |\nabla w^+ + \xi|^2 dx + k_0 \int_{Y^-} |\nabla w^- + \xi|^2 dx + \frac{1}{\beta} \int_\Gamma |w^+ - w^-|^2 ds,$$

where $w = \sum_i \xi_i w_i$. Since $\Re e \beta \geq 0$,

$$K^*\xi \cdot \xi \geq k_0 \int_{Y^+} |\nabla w^+ + \xi|^2 dx + k_0 \int_{Y^-} |\nabla w^- + \xi|^2 dx.$$

Consequently, it follows from [3] that there exist two positive constants C_1 and C_2 such that

$$C_1 |\xi|^2 \leq \Re e K^* \xi \cdot \xi \leq C_2 |\xi|^2.$$

Standard elliptic theory yields existence and uniqueness of a solution to the homogenized problem in $H^1_{\mathbb{C}}(\Omega)$. □

Lemma 4.3. *The cell problem (2.6) admits a unique solution in $H^1_\sharp(Y^+)/\mathbb{C} \times H^1_\sharp(Y^-)$.*

Proof. Let us introduce the Hilbert space

$$W := \left\{ v := v^+ \chi_{Y^+} + v^- \chi_{Y^-} | (v^+, v^-) \in H^1_{\mathbb{C}}(Y^+) \times H^1(Y^-) \right\},$$

equipped with the norm

$$\|v\|^2_W = \|\nabla v^+\|^2_{L^2(Y^+)} + \|\nabla v^-\|^2_{L^2(Y^-)} + \|v^+ - v^-\|^2_{L^2(\Gamma)}.$$

We consider the following problem:

$$\begin{cases} \text{Find } w_i \in W_\sharp \text{ such that for all } \varphi \in W_\sharp \\ \displaystyle\int_{Y^+} k_0 \nabla w_i^+(y) \cdot \nabla \overline{\varphi}^+(y) dy + \int_{Y^-} k_0 \nabla w_i^-(y) \cdot \nabla \overline{\varphi}^-(y) dy \\ \qquad \displaystyle + \frac{1}{\beta} \int_\Gamma \left(w_i^+ - w_i^-\right)(y) \left(\overline{\varphi}^+ - \overline{\varphi}^-\right)(y) ds(y) = \\ \qquad \displaystyle - \int_{Y^+} k_0 \nabla y_i \cdot \nabla \overline{\varphi}^+(y) dy - \int_{Y^-} k_0 \nabla y_i \cdot \nabla \overline{\varphi}^-(y) dy. \end{cases} \tag{4.1}$$

Lax–Milgram theorem gives us existence and uniqueness of a solution. Moreover, one can show that this ensures the existence of a unique solution in $H^1_\sharp(Y^+)/\mathbb{C} \times H^1_\sharp(Y^-)$ for the cell problem (2.6). □

4.2 Convergence

We present in this section a rigorous proof of the convergence of the initial problem to the homogenized one. We use for this purpose the two-scale convergence technique and hence need first of all some bounds on u_ε to ensure the convergence.

4.2.1 A priori estimates

Theorem 4.1. *The function u_ε^+ is uniformly bounded with respect to ε in $H^1(\Omega_\varepsilon^+)$, i.e., there exists a constant C, independent of ε, such that*

$$\|u_\varepsilon^+\|_{H^1(\Omega_\varepsilon^+)} \leq C.$$

Proof. Combining (3.11) and Poincaré - Wirtinger inequality, we obtain immediately the wanted result. □

The proof of the following result follows the one of Lemma 2.8 in [43].

Lemma 4.4. *There exists a constant C, which does not depend on ε, such that for all $v \in W_\varepsilon$:*

$$\|v^-\|_{L^2(\Omega_\varepsilon^-)} \leq C\|v\|_{W_\varepsilon}.$$

Proof. We write the norm $\|v^-\|_{L^2(\Omega_\varepsilon^-)}$ as a sum over all the cells.

$$\|v^-\|^2_{L^2(\Omega_\varepsilon^-)} = \sum_{n\in N_\varepsilon} \|v^-\|^2_{L^2(Y^-_{\varepsilon,n})} = \sum_{n\in N_\varepsilon} \int_{Y^-_{\varepsilon,n}} |v^-(x)|^2 dx.$$

We perform the change of variable $y = \dfrac{x}{\varepsilon}$ and get

$$\|v^-\|^2_{L^2(\Omega_\varepsilon^-)} = \varepsilon^2 \sum_{n\in N_\varepsilon} \int_{Y_n^-} |v_\varepsilon^-(y)|^2 dy, \tag{4.2}$$

where $v_\varepsilon^-(y) := v^-(\varepsilon y)$ for all $y \in Y^-$ and $Y_n^- = n + Y^-$ with $n \in N_\varepsilon$.

Recall that W denotes the following Hilbert space:

$$W := \left\{ v := v^+\chi_{Y^+} + v^-\chi_{Y^-} | (v^+, v^-) \in H^1_{\mathbb{C}}(Y^+) \times H^1(Y^-) \right\},$$

equipped with the norm:

$$\|v\|_W^2 = \|\nabla v^+\|^2_{L^2(Y^+)} + \|\nabla v^-\|^2_{L^2(Y^-)} + \|v^+ - v^-\|^2_{L^2(\Gamma)}.$$

We first prove that there exists a constant C_1, independent of ε, such that for every $v \in W$:

$$\|v^-\|_{L^2(Y^-)} \leq C_1 \|v\|_W. \tag{4.3}$$

We proceed by contradiction. Suppose that for any $n \in \mathbb{N}^*$, there exists $v_n \in W_\varepsilon$ such that

$$\|v_n^-\|_{L^2(Y^-)} = 1 \quad \text{and} \quad \|v_n\|_W \leq \frac{1}{n}.$$

Since $\|v_n^-\|_{L^2(Y^-)} = 1$ and $\|\nabla v_n^-\|_{L^2(Y^-)} \leq \|v_n\|_W \leq \dfrac{1}{n}$, v_n^- is bounded in $H^1(Y^-)$. Therefore it converges weakly in $H^1(Y^-)$. By compactness, we can extract a subsequence, still denoted v_n^-, such that v_n^- converges strongly in $L^2(Y^-)$. We denote by l its limit.

Besides, ∇v_n^- converges strongly to 0 in $L^2(Y^-)$. We thus have $\nabla l = 0$ and l constant in Y^-.

By applying in Y^+ the trace theorem and Poincaré–Wirtinger inequality to v_n^+, one also gets that

$$\|v_n^-\|_{L^2(\Gamma)} \leq \|v_n^+ - v_n^-\|_{L^2(\Gamma)} + \|v_n^+\|_{L^2(\Gamma)} \leq \|v_n^+ - v_n^-\|_{L^2(\Gamma)} + C\|v_n^+\|_{H^1(Y^+)} \leq \frac{C'}{n}.$$

Consequently, v_n^- converges strongly to 0 in $L^2(\Gamma)$ and $l = 0$ on Γ.
We have then $l = 0$ in Y^-, which leads to a contradiction. This proves (4.3).
We can now find an upper bound to (4.2):

$$\|v^-\|^2_{L^2(\Omega_\varepsilon^-)} \leq \varepsilon^2 C_1 \sum_{n\in N_\varepsilon} \int_{Y_n^+} |\nabla v_\varepsilon^+(y)|^2 dy + \int_{Y_n^-} |\nabla v_\varepsilon^-(y)|^2 dy + \int_{\Gamma_n} |v_\varepsilon^+(y) - v_\varepsilon^-(y)|^2 ds(y).$$

After the change of variable $x = \varepsilon y$, one gets

$$\|v^-\|^2_{L^2(\Omega_\varepsilon^-)} \leq \varepsilon C_1 \left(\|\nabla v^+\|^2_{L^2(\Omega_\varepsilon^+)} + \|\nabla v^-\|^2_{L^2(\Omega_\varepsilon^-)} + \varepsilon \|v^+ - v^-\|^2_{L^2(\Gamma_\varepsilon)} \right).$$

Since $\varepsilon < 1$, there exists a constant C_2, which does not depend on ε such that for every $v \in W_\varepsilon$,

$$\|v^-\|_{L^2(\Omega_\varepsilon^-)} \leq C_2 \|v\|_{W_\varepsilon},$$

which completes the proof. □

Theorem 4.2. *u_ε^- is uniformly bounded in ε in $H^1(\Omega_\varepsilon^-)$, i.e., there exists a constant C independent of ε, such that*

$$\|u_\varepsilon^-\|_{H^1(\Omega_\varepsilon^-)} \leq C.$$

Proof. By definition of the norm on W_ε , $\|\nabla u_\varepsilon^-\|^2_{L^2(\Omega_\varepsilon^-)} \leq \|u_\varepsilon\|^2_{W_\varepsilon}$.
We thus have with the result of Lemma 4.4:

$$\|u_\varepsilon^-\|^2_{H^1(\Omega_\varepsilon^-)} \leq C_1 \|u_\varepsilon\|^2_{W_\varepsilon}, \tag{4.4}$$

with a constant C_1 which does not depend on ε.

Furthermore, by combining (**??**) and the result of Theorem 4.1, there exists a constant C_2 independent of ε such that

$$|a(u_\varepsilon, u_\varepsilon)| \leq C_2.$$

We use the coercivity of a and get a uniform bound in ε of u_ε in W_ε. This bound and (4.4) complete the proof. □

4.2.2 Two-scale convergence

We first recall the definition of two-scale convergence and a few results of this theory [2].

Definition 4.1. *A sequence of functions u_ε in $L^2(\Omega)$ is said to* two-scale converge *to a limit u_0 belonging to $L^2(\Omega \times Y)$ if, for any function ψ in $L^2(\Omega, C_\sharp(Y))$, we have*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi(x, \frac{x}{\varepsilon})dx = \int_\Omega \int_Y u_0(x,y)\psi(x,y)dxdy.$$

This notion of two-scale convergence makes sense because of the next compactness theorem.

Theorem 4.3. *From each bounded sequence u_ε in $L^2(\Omega)$, we can extract a subsequence, and there exists a limit $u_0 \in L^2(\Omega \times Y)$ such that this subsequence two-scale converges to u_0.*

Two-scale convergence can be extended to sequences defined on periodic surfaces.

Proposition 4.1. *For any sequence u_ε in $L^2(\Gamma_\varepsilon)$ such that*

$$\varepsilon \int_{\Gamma_\varepsilon} |u_\varepsilon|^2 dx \leq C, \tag{4.5}$$

there exists a subsequence, still denoted u_ε, and a limit function $u_0 \in L^2(\Omega, L^2(\Gamma))$ such that u_ε two-scale converges to u_0 in the sense

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma_\varepsilon} u_\varepsilon(x)\psi(x, \frac{x}{\varepsilon}) ds(x) = \int_\Omega \int_\Gamma u_0(x,y)\psi(x,y) dx ds(y),$$

for any function $\psi \in L^2(\Omega, C_\sharp(Y))$.

Remark 4.1. *If u_ε and ∇u_ε are bounded in $L^2(\Omega)$, one can prove by using for example [4, Lemma 2.4.9] that u_ε verifies the uniform bound (4.5). The two-scale limit on the surface is then the trace on Γ of the two-scale limit of u_ε in Ω.*

In order to prove item (i) in Theorem 2.1, we need the following results.

Lemma 4.5. *Let the functions $(u_\varepsilon)_\varepsilon$ be the sequence of solutions of (2.2). There exist functions $u(x) \in H^1(\Omega)$, $v^+(x,y) \in L^2(\Omega, H^1_\sharp(Y^+))$ and $v^-(x,y) \in L^2(\Omega, H^1_\sharp(Y^-))$ such that, up to a subsequence,*

$$\begin{pmatrix} u_\varepsilon \\ \chi^+_\varepsilon(\frac{x}{\varepsilon})\nabla u^+_\varepsilon \\ \chi^-_\varepsilon(\frac{x}{\varepsilon})\nabla u^-_\varepsilon \end{pmatrix} \quad \textit{two-scale converge to} \quad \begin{pmatrix} u(x) \\ \chi_{Y^+}(y)\Big(\nabla u(x) + \nabla_y v^+(x,y)\Big) \\ \chi_{Y^-}(y)\Big(\nabla u(x) + \nabla_y v^-(x,y)\Big) \end{pmatrix}.$$

Proof. We denote by $\tilde{\cdot}$ the extension by zero of functions on Ω^+_ε and Ω^-_ε in the respective domains Ω^-_ε and Ω^+_ε.

From the previous estimates, $\widetilde{u}^\pm_\varepsilon$ and $\widetilde{\nabla u_\varepsilon}^\pm$ are bounded sequences in $L^2(\Omega)$. Up to a subsequence, they two-scale converge to $\tau^\pm(x,y)$ and $\xi^\pm(x,y)$. Since $\widetilde{u}^\pm_\varepsilon$ and $\widetilde{\nabla u_\varepsilon}^\pm$ vanish in $\Omega^\mp_\varepsilon$, so do $\tau^\pm$ and $\xi^\pm$.

Consider $\varphi \in \mathcal{D}(\Omega, C^\infty_\sharp(Y))^2$ such that $\varphi = 0$ for $y \in \overline{Y^-}$. By integrating by parts, it follows that

$$\varepsilon \int_{\Omega^+_\varepsilon} \nabla u^+_\varepsilon(x) \cdot \overline{\varphi}(x, \frac{x}{\varepsilon}) dx = -\int_{\Omega^+_\varepsilon} u^+_\varepsilon(x) \left(\operatorname{div}_y \overline{\varphi}(x, \frac{x}{\varepsilon}) + \varepsilon \operatorname{div}_x \overline{\varphi}(x, \frac{x}{\varepsilon}) \right) dx.$$

We take the limit of this equality as $\varepsilon \to 0$:

$$0 = -\int_\Omega \int_{Y^+} \tau^+(x,y) \operatorname{div}_y \overline{\varphi}(x,y) dx dy.$$

Therefore, τ^+ does not depend on y in Y^+, *i.e.*, there exists a function $u^+ \in L^2(\Omega)$ such that $\tau^+(x,y) = \chi_{Y^+}(y)u^+(x)$ for all $(x,y) \in \Omega \times Y$.

Take now $\varphi \in \mathcal{D}(\Omega, C^\infty_\sharp(Y))^2$ such that $\varphi = 0$ for $y \in \overline{Y^-}$ and $\mathrm{div}_y\varphi = 0$. Similarly, we have

$$\int_{\Omega_\varepsilon^+} \nabla u_\varepsilon^+(x) \cdot \overline{\varphi}(x, \frac{x}{\varepsilon})dx = -\int_{\Omega_\varepsilon^+} u_\varepsilon^+(x)\mathrm{div}_x\,\overline{\varphi}(x, \frac{x}{\varepsilon})dx,$$

and thus

$$\int_\Omega \int_{Y^+} \xi^+(x,y) \cdot \overline{\varphi}(x,y)dxdy = -\int_\Omega \int_{Y^+} u^+(x)\mathrm{div}_x\,\overline{\varphi}(x,y)dxdy. \tag{4.6}$$

For φ independent of y, this implies that $u^+ \in H^1(\Omega)$. Furthermore, if we integrate by parts the right-hand side of (4.6), we get

$$\int_\Omega \int_{Y^+} \xi^+(x,y) \cdot \overline{\varphi}(x,y)dxdy = \int_\Omega \int_{Y^+} \nabla u^+(x) \cdot \overline{\varphi}(x,y)dxdy,$$

for all $\varphi \in \mathcal{D}(\Omega, C^\infty_\sharp(Y^+))^2$ such that $\mathrm{div}_y\varphi = 0$ and $\varphi(x,y)\cdot n(y) = 0$ for y on Γ.

Since the orthogonal of the divergence-free functions are exactly the gradients, there exists a function $v^+ \in L^2(\Omega, H^1_\sharp(Y^+))$ such that

$$\xi^+(x,y) = \chi_{Y^+}(y)\left(\nabla u^+(x) + \nabla_y v^+(x,y)\right),$$

for all $(x,y) \in \Omega \times Y$.

Likewise, there exist functions $u^- \in H^1(\Omega)$ and $v^- \in L^2(\Omega, H^1_\sharp(Y^-))$ such that

$$\tau^-(x,y) = \chi_{Y^-}(y)u^-(x), \text{ and } \xi^-(x,y) = \chi_{Y^-}(y)\left(\nabla u^-(x) + \nabla_y v^-(x,y)\right),$$

for all $(x,y) \in \Omega \times Y$.

Furthermore, thanks to Remark 4.1, we have also

$$\varepsilon \int_{\Gamma_\varepsilon} u_\varepsilon^\pm(x)\overline{\varphi}(x, \frac{x}{\varepsilon})dx \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \int_\Gamma u^\pm(x,y)\overline{\varphi}(x,y)dxdy,$$

for all $\varphi \in L^2(\Omega, C^\infty_\sharp(\Gamma))$.

Recall that u_ε is a solution to the following variational form:

$$\int_{\Omega_\varepsilon^+} k_0 \nabla u_\varepsilon^+(x) \cdot \nabla\overline{\varphi}_\varepsilon^+(x)dx + \int_{\Omega_\varepsilon^-} k_0 \nabla u_\varepsilon^-(x) \cdot \nabla\overline{\varphi}_\varepsilon^-(x)dx$$
$$+\frac{1}{\varepsilon\beta}\int_{\Gamma_\varepsilon}\left(u_\varepsilon^+ - u_\varepsilon^-\right)\left(\overline{\varphi}_\varepsilon^+ - \overline{\varphi}_\varepsilon^-\right)ds - k_0\int_{\partial\Omega} g\overline{\varphi}_\varepsilon^+ ds = 0,$$

for all $(\varphi_\varepsilon^+, \varphi_\varepsilon^-) \in (H^1(\Omega_\varepsilon^+), H^1(\Omega_\varepsilon^-))$.

We multiply this equality by ε^2 and take the limit when ε goes to 0. The first two terms disappear and we obtain, for all $(\varphi^+, \varphi^-) \in \mathcal{D}(\Omega, C^\infty_\sharp(Y^+)) \times \mathcal{D}(\Omega, C^\infty_\sharp(Y^-))$:

$$\frac{1}{\beta}\int_\Omega \int_\Gamma (u^+(x) - u^-(x))(\overline{\varphi}^+(x,y) - \overline{\varphi}^-(x,y))dxdy = 0.$$

Thus $u^+(x) = u^-(x)$ for all $x \in \Omega$, and u_ε two-scale converges to $u = u^+ = u^- \in H^1(\Omega)$. This completes the proof. □

Now, we are ready to prove Theorem 2.1. For this, we need to show that u, v^+ and v^- are respectively u_0, solution of the homogenized problem (up to a constant), u_1^+ defined in (2.7) (up to a constant) and u_1^- defined in (2.7). The uniqueness of a solution for the homogenized problem and the cell problems will then allow us to conclude the convergence, not only up to a subsequence.

Proof. We first want to retrieve the expression of u_1 as a test function of the derivatives of u_0 and the cell problem solutions w_i.

We choose in the variational formulation (3.5) a function φ_ε of the form

$$\varphi_\varepsilon(x) = \varepsilon\varphi_1(x, \frac{x}{\varepsilon}),$$

where $\varphi_1 \in \mathcal{D}(\Omega, C_\sharp^\infty(Y^+)) \times \mathcal{D}(\Omega, C_\sharp^\infty(Y^-))$.

Lemma 4.5 shows the two-scale convergence of the following three terms:

$$\int_{\Omega_\varepsilon^+} k_0 \nabla u_\varepsilon^+(x) \cdot \nabla \overline{\varphi}_\varepsilon^+(x) dx \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \int_{Y^+} k_0 \left(\nabla u(x) + \nabla_y v^+(x,y)\right) \cdot \nabla_y \overline{\varphi}_1^+(x,y) dx dy$$

$$\int_{\Omega_\varepsilon^-} k_0 \nabla u_\varepsilon^-(x) \cdot \nabla \overline{\varphi}_\varepsilon^-(x) dx \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \int_{Y^-} k_0 \left(\nabla u(x) + \nabla_y v^-(x,y)\right) \cdot \nabla_y \overline{\varphi}_1^-(x,y) dx dy$$

$$\int_{\partial\Omega} g(x) \overline{\varphi}_\varepsilon^+(x) ds(x) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

We can not take directly the limit as $\varepsilon \to 0$ in the last term:

$$\frac{1}{\varepsilon\beta} \int_{\Gamma_\varepsilon} (u_\varepsilon^+(x) - u_\varepsilon^-(x))(\overline{\varphi}_\varepsilon^+(x) - \overline{\phi}_\varepsilon^-(x)) ds(x)$$
$$= \frac{1}{\beta} \int_{\Gamma_\varepsilon} (u_\varepsilon^+(x) - u_\varepsilon^-(x)) \left(\overline{\varphi}_1^+(x, \frac{x}{\varepsilon}) - \overline{\varphi}_1^-(x, \frac{x}{\varepsilon})\right) ds(x).$$

Lemma D.1 ensures the existence of a function $\theta \in (\mathcal{D}(\Omega, H_\sharp^1(Y^+)) \times \mathcal{D}(\Omega, H_\sharp^1(Y^-)))^2$ such that for all $\psi \in H_\sharp^1(Y^+)/\mathbb{C} \times H_\sharp^1(Y^-)$:

$$\int_{Y^+} \nabla \psi^+(y) \cdot \overline{\theta}^+(x,y) dy + \int_{Y^-} \nabla \psi^-(y) \cdot \overline{\theta}^-(x,y) dy + \int_\Gamma \left(\psi^+(y) - \psi^-(y)\right) \left(\overline{\varphi}_1^+(x,y) - \overline{\varphi}_1^-(x,y)\right) ds(y) = 0. \tag{4.7}$$

We make the change of variables $y = \dfrac{x}{\varepsilon}$, sum over all $(Y_{\varepsilon,n})_{n\in N_\varepsilon}$, and choose $\psi = u_\varepsilon$ to get

$$\int_{\Gamma_\varepsilon} (u_\varepsilon^+(x) - u_\varepsilon^-(x)) \left(\overline{\varphi}_1^+(x, \frac{x}{\varepsilon}) - \overline{\varphi}_1^-(x, \frac{x}{\varepsilon})\right) ds(x) =$$
$$- \int_{\Omega_\varepsilon^+} \nabla u_\varepsilon^+(x, \frac{x}{\varepsilon}) \cdot \theta^+(x, \frac{x}{\varepsilon}) dx - \int_{\Omega_\varepsilon^-} \nabla u_\varepsilon^-(x, \frac{x}{\varepsilon}) \cdot \theta^-(x, \frac{x}{\varepsilon}) dx.$$

We can now take the limit as ε goes to 0:

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} (u_\varepsilon^+(x) - u_\varepsilon^-(x)) \left(\overline{\varphi}_1^+(x, \frac{x}{\varepsilon}) - \overline{\varphi}_1^-(x, \frac{x}{\varepsilon})\right) ds(x) =$$
$$- \int_{Y^+} \left(\nabla u(x) + \nabla_y v^+(x,y)\right) \cdot \overline{\theta}^+(x,y) dx dy - \int_{Y^-} \left(\nabla u(x) + \nabla_y v^-(x,y)\right) \cdot \overline{\theta}^-(x,y) dx dy.$$

Finally, the variational formula (4.7) gives us

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}(u_\varepsilon^+(x)-u_\varepsilon^-(x))\left(\overline{\varphi}_1^+(x,\frac{x}{\varepsilon})-\overline{\varphi}_1^-(x,\frac{x}{\varepsilon})\right)ds(x)=$$
$$\int_\Omega\int_\Gamma(v^+(y)-v^-(y))\left(\overline{\varphi}_1^+(x,y)-\overline{\varphi}_1^-(x,y)\right)ds(y).$$

For $\varphi_\varepsilon(x)=\varepsilon\varphi_1(x,\frac{x}{\varepsilon})$, with $\varphi_1\in\mathcal{D}(\Omega,C_\sharp^\infty(Y^+))\times\mathcal{D}(\Omega,C_\sharp^\infty(Y^-))$, the two-scale limit of the variational formula is

$$\int_\Omega\int_{Y^+}k_0\left(\nabla u(x)+\nabla_y v^+(x,y)\right)\cdot\nabla_y\overline{\varphi}_1^+(x,y)dxdy$$
$$+\int_\Omega\int_{Y^-}k_0\left(\nabla u(x)+\nabla_y v^-(x,y)\right)\cdot\nabla_y\overline{\varphi}_1^-(x,y)dxdy$$
$$+\frac{1}{\beta}\int_\Omega\int_\Gamma(v^+(y)-v^-(y))\left(\overline{\varphi}_1^+(x,y)-\overline{\varphi}_1^-(x,y)\right)ds(y)=0.$$

By density, this formula hold true for $\varphi_1\in L^2(\Omega,H_\sharp^1(Y^+))\times L^2(\Omega,H_\sharp^1(Y^-))$. One can recognize the formula verified by $u_1^\pm$ and the definition of the cell problems. Hence, separation of variables and uniqueness of the solutions of the cell problems in W give

$$v^-(x,y)=u_1^-=\sum_{i=1,2}\frac{\partial u_0}{\partial x_i}(x)w_i^-(y)$$

and, up to a constant:

$$v^+(x,y)=u_1^+=\sum_{i=1,2}\frac{\partial u_0}{\partial x_i}(x)w_i^+(y).$$

We now choose in the variational formula verified by u_ε a test function $\varphi_\varepsilon(x)=\varphi(x)$, with $\varphi\in C_c^\infty(\overline{\Omega})$.

The limit of (3.5) as ε goes to 0 is then given by

$$\int_\Omega\int_{Y^+}k_0\left(\nabla u(x)+\nabla_y v^+(x,y)\right)\cdot\nabla\overline{\varphi}(x)dxdy$$
$$+\int_\Omega\int_{Y^-}k_0\left(\nabla u(x)+\nabla_y v^-(x,y)\right)\cdot\nabla\overline{\varphi}(x)dxdy$$
$$+\int_{\partial\Omega}g(x)\overline{\phi}(x)ds(x)=0.$$

By density, this formula hold true for $\varphi\in H^1(\Omega)$, which leads exactly to the variational formula of the homogenized problem (2.4). Since the solution of this problem is unique in $H^1_{\mathbb{C}}(\Omega)$, u_ε converges to u_0, not only up to a subsequence. Likewise, ∇u_ε two-scale converges to $\nabla u_0+\chi_{Y^+}\nabla_y u_1^++\chi_{Y^-}\nabla_y u_1^-$. □

5 Effective admittivity for a dilute suspension

In general, the effective admittivity given by formula (2.5) can not be computed exactly except for a few configurations. In this section, we consider the problem of determining the effective property of a suspension of cells when the volume fraction $|Y^-|$ goes to zero. In other words, the cells have much less volume than the medium surrounding them. This kind of suspension is called dilute. Many approximations for the effective properties of composites are based on the solution for dilute suspension.

5.1 Computation of the effective admittivity

We investigate the periodic double-layer potential used in calculating effective permittivity of a suspension of cells. We introduce the periodic Green function $G_\sharp$, for the Laplace equation in Y, given by

$$\forall x \in Y, \qquad G_\sharp(x) = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{e^{i2\pi n \cdot x}}{4\pi^2 |n|^2}.$$

The following lemma from [11, 9] plays an essential role in deriving the effective properties of a suspension in the dilute limit.

Lemma 5.1. *The periodic Green function $G_\sharp$ admits the following decomposition:*

$$\forall x \in Y, \qquad G_\sharp(x) = \frac{1}{2\pi} \ln |x| + R_2(x), \tag{5.1}$$

where R_2 is a smooth function with the following Taylor expansion at 0*:*

$$R_2(x) = R_2(0) - \frac{1}{4}(x_1^2 + x_2^2) + O(|x|^4). \tag{5.2}$$

Let $L^2_0(\Gamma) := \left\{ \varphi \in L^2(\Gamma) \Big| \int_\Gamma \varphi(x) ds(x) = 0 \right\}$.

We define the periodic double-layer potential $\widetilde{\mathcal{D}}_\Gamma$ of the density function $\varphi \in L^2_0(\Gamma)$:

$$\widetilde{\mathcal{D}}_\Gamma[\varphi](x) = \int_\Gamma \frac{\partial}{\partial n_y} G_\sharp(x-y) \varphi(y) ds(y).$$

The double-layer potential has the following properties [9].

Lemma 5.2. *Let $\varphi \in L^2_0(\Gamma)$. $\widetilde{\mathcal{D}}_\Gamma[\varphi]$ verifies:*

$$\begin{aligned}
&\textit{(i)} && \Delta \widetilde{\mathcal{D}}_\Gamma[\varphi] = 0 && \textit{in } Y^+, \\
& && \Delta \widetilde{\mathcal{D}}_\Gamma[\varphi] = 0 && \textit{in } Y^-, \\
&\textit{(ii)} && \frac{\partial}{\partial n} \widetilde{\mathcal{D}}_\Gamma[\varphi]\Big|_+ = \frac{\partial}{\partial n} \widetilde{\mathcal{D}}_\Gamma[\varphi]\Big|_- && \textit{on } \Gamma, \\
&\textit{(iii)} && \widetilde{\mathcal{D}}_\Gamma[\varphi]\Big|_\pm = \left(\mp \frac{1}{2} I + \widetilde{\mathcal{K}}_\Gamma \right)[\varphi] && \textit{on } \Gamma,
\end{aligned}$$

where $\widetilde{\mathcal{K}}_\Gamma : L^2_0(\Gamma) \mapsto L^2_0(\Gamma)$ *is the Neumann–Poincaré operator defined by*

$$\forall x \in \Gamma, \qquad \widetilde{\mathcal{K}}_\Gamma[\varphi](x) = \int_\Gamma \frac{\partial}{\partial n_y} G_\sharp(x-y)\varphi(y)ds(y).$$

The following integral representation formula holds.

Theorem 5.1. *Let* w_i *be the unique solution in* W *of (2.6) for* $i = 1, 2$. w_i *admits the following integral representation in* Y*:*

$$w_i = -\beta k_0\, \widetilde{\mathcal{D}}_\Gamma \left(I + \beta k_0 \widetilde{\mathcal{L}}_\Gamma \right)^{-1} [n_i], \tag{5.3}$$

where $\widetilde{\mathcal{L}}_\Gamma = \dfrac{\partial \widetilde{\mathcal{D}}_\Gamma}{\partial n}$ *and* $n = (n_i)_{i=1,2}$ *is the outward unit normal to* Γ.

Proof. Let $\varphi := -\beta k_0 \left(I + \beta k_0 \widetilde{\mathcal{L}} \right)^{-1} [n_i]$. φ verifies :

$$\int_\Gamma \varphi(y)ds(y) = -\beta k_0 \int_\Gamma \frac{\partial}{\partial n}(\widetilde{\mathcal{D}}_\Gamma[\varphi](y) + y_i)ds(y) = 0.$$

The first equality comes from the definition of φ and the second from an integration by parts and the fact that $\widetilde{\mathcal{D}}_\Gamma[\varphi]$ and I are harmonic. Consequently, $\varphi \in L^2_0(\Gamma)$.

We now introduce $V_i := \widetilde{\mathcal{D}}_\Gamma[\varphi]$. V_i is solution to the following problem:

$$\begin{cases} \nabla \cdot k_0 \nabla V_i = 0 & \text{in } Y^+, \\ \nabla \cdot k_0 \nabla V_i = 0 & \text{in } Y^-, \\ k_0 \dfrac{\partial V_i}{\partial n}\Big|_+ = k_0 \dfrac{\partial V_i}{\partial n}\Big|_- & \text{on } \Gamma, \\ V_i|_+ - V_i|_- = \varphi & \text{on } \Gamma, \\ y \longmapsto V_i(y) \ Y\text{-periodic.} \end{cases}$$

We use the definitions of φ and V_i and recognize that the last problem is exactly problem (2.6). The uniqueness of the solution in W gives us the wanted result. □

From Theorem 2.1, the effective admittivity of the medium K^* is given by

$$\forall (i,j) \in \{1,2\}^2, \qquad K^*_{i,j} = k_0 \left(\delta_{ij} + \int_Y \nabla w_i \cdot e_j \right).$$

After an integration by parts, we get

$$\forall (i,j) \in \{1,2\}^2, \qquad K^*_{i,j} = k_0 \left(\delta_{ij} + \int_{\partial Y} w_i(y) n_j(y)\, ds(y) - \int_\Gamma \left(w_i^+ - w_i^- \right) n_j(y)\, ds(y) \right).$$

Because of the Y-periodicity of w_i, we have: $\displaystyle\int_{\partial Y} w_i(y) n_j\, ds(y) = 0.$

Finally, the integral representation 5.3 gives us that

$$\forall (i,j) \in \{1,2\}^2, \qquad K^*_{i,j} = k_0 \left(\delta_{ij} - (\beta k_0) \int_\Gamma \left(I + \beta k_0 \widetilde{\mathcal{L}}_\Gamma \right)^{-1} [n_i] n_j \, ds(y) \right).$$

We consider that we are in the context of a dilute suspension, *i.e.*, the size of the cell is small compared to the square: $\left|Y^-\right| \ll |Y| = 1$. We perform the change of variable: $z = \rho^{-1} y$ with $\rho = |Y^-|^{\frac{1}{2}}$ and obtain that

$$\forall (i,j) \in \{1,2\}^2, \qquad K^*_{i,j} = k_0 \left(\delta_{ij} - \rho^2 (\beta k_0) \int_{\rho^{-1}\Gamma} \left(I + \rho\beta k_0 \widetilde{\mathcal{L}}_\Gamma \right)^{-1} [n_i](\rho z) n_j(z) \, ds(z) \right),$$

where n is the outward unit normal to Γ. Note that, in the same way as before, β becomes $\rho\beta$ when we rescale the cell.

Let us introduce $\varphi_i = -\left(I + \rho\beta k_0 \widetilde{\mathcal{L}}_\Gamma \right)^{-1} [n_i]$ and $\psi_i(z) = \varphi_i(\rho z)$ for all $z \in \rho^{-1}\Gamma$. From (5.1), we get, for any $z \in \rho^{-1}\Gamma$, after changes of variable in the integrals:

$$\widetilde{\mathcal{L}}_\Gamma[\varphi_i](\rho z) = \frac{\partial}{\partial n} \widetilde{\mathcal{D}}_\Gamma[\varphi_i](\rho z) = \rho^{-1} \frac{\partial}{\partial n} \mathcal{D}_{\rho^{-1}\Gamma}[\psi_i](z) + \frac{\partial}{\partial n(z)} \int_{\rho^{-1}\Gamma} \frac{\partial}{\partial n(y)} R_2(\rho z - \rho y) \varphi(\rho y) ds(y).$$

Besides, the expansion (5.2) gives us that the estimate

$$\nabla R_2(\rho(z-y)) \cdot n(y) = -\frac{\rho}{2}(z-y) \cdot n(y) + O(\rho^3),$$

holds uniformly in $z, y \in \rho^{-1}\Gamma$.

We thus get the following expansion:

$$\widetilde{\mathcal{L}}_\Gamma[\varphi_i](\rho z) = \rho^{-1} \mathcal{L}_{\rho^{-1}\Gamma}[\psi_i](z) - \frac{\rho}{2} \sum_{j=1,2} n_j \int_{\rho^{-1}\Gamma} n_j \psi_i(y) ds(y) + O(\rho^4).$$

Using ψ^*_i defined by (2.11) we get on $\rho^{-1}\Gamma$:

$$\psi_i = \psi^*_i + \beta k_0 \frac{\rho^2}{2} \sum_{j=1,2} \psi^*_j \int_{\rho^{-1}\Gamma} n_j(y) \psi_i(y) ds(y) + O(\rho^4). \tag{5.4}$$

By iterating the formula (5.4), we obtain on $\rho^{-1}\Gamma$ that

$$\psi_i = \psi^*_i + \beta k_0 \frac{\rho^2}{2} \sum_{j=1,2} \psi^*_j \int_{\rho^{-1}\Gamma} n_j(y) \psi^*_i(y) ds(y) + O(\rho^4).$$

Therefore, one can easily see that Theorem 2.2 holds.

5.2 Case of concentric circular-shaped cells: the Maxwell-Wagner-Fricke formula

We consider in this section that the cells are disks of radius r_0. $\rho^{-1}\Gamma$ becomes a circle of radius r_0.

For all $g \in L^2(]0, 2\pi[)$, we introduce the Fourier coefficients:

$$\forall m \in \mathbb{Z}, \quad \hat{g}(m) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) e^{-im\varphi} d\varphi,$$

and have then for all $\varphi \in]0, 2\pi[$:

$$g(\varphi) = \sum_{m=-\infty}^{\infty} \hat{g}(m) e^{im\varphi}.$$

For $f \in \mathcal{C}^{2,\eta}(\rho^{-1}\Gamma)$, we obtain after a few computations:

$$\forall \theta \in]0, 2\pi[, \quad (I + \beta k_0 \mathcal{L}_{\rho^{-1}\Gamma})^{-1}[f](\theta) = \sum_{n \in \mathbb{Z}^*} \left(1 + \beta k_0 \frac{|n|}{2r_0}\right)^{-1} \hat{f}(n)\, e^{in\theta}.$$

For $p = 1, 2$, $\psi_p^* = -(I + \beta k_0 \mathcal{L}_{\rho^{-1}\Gamma})^{-1}[n_p]$ then have the following expression:

$$\forall \theta \in]0, 2\pi[, \quad \psi_p^* = -\left(1 + \frac{\beta k_0}{2r_0}\right)^{-1} n_p.$$

Consequently, we get for $(p, q) \in \{1, 2\}^2$:

$$M_{p,q} = -\delta_{pq} \frac{\beta k_0 \pi r_0}{1 + \dfrac{\beta k_0}{2r_0}},$$

and hence,

$$\Im M_{p,q} = \delta_{p,q} \frac{\pi r_0 \delta \omega (\epsilon_m \sigma_0 - \epsilon_0 \sigma_m)}{(\sigma_m + \sigma_0 \dfrac{\delta}{2r_0})^2 + \omega^2 (\epsilon_m + \epsilon_0 \dfrac{\delta}{2r_0})^2}. \tag{5.5}$$

Formula (5.5) is the two-dimensional version of the Maxwell-Wagner-Fricke formula, which gives the effective admittivity of a dilute suspension of spherical cells covered by a thin membrane.

An explicit formula for the case of elliptic cells can be derived by using the spectrum of the integral operator $\mathcal{L}_{\rho^{-1}\Gamma}$, which can be identified by standard Fourier methods [35].

5.3 Debye relaxation times

From (5.5), it follows that the imaginary part of the membrane polarization attains its maximum with respect to the frequency at

$$\frac{1}{\tau} = \frac{\sigma_m + \sigma_0 \dfrac{\delta}{2r_0}}{\epsilon_m + \epsilon_0 \dfrac{\delta}{2r_0}}.$$

This dispersion phenomenon due to the membrane polarization is well known and referred to as the β-dispersion. The associated characteristic time τ corresponds to a Debye relaxation time.

For arbitrary-shaped cells, we define the first and second Debye relaxation times, $\tau_i, i = 1, 2$, by

$$\frac{1}{\tau_i} := \arg\max_{\omega} |\lambda_i(\omega)|, \tag{5.6}$$

where $\lambda_1 \leq \lambda_2$ are the eigenvalues of the imaginary part of the membrane polarization tensor $M(\omega)$. Note that if the cell is of circular shape, $\lambda_1 = \lambda_2$.

As it will be shown later, the Debye relaxation times can be used for identifying the microstructure.

5.4 Properties of the membrane polarization tensor and the Debye relaxation times

In this subsection, we derive important properties of the membrane polarization tensor and the Debye relaxation times defined respectively by (2.10) and (5.6). In particular, we prove that the Debye relaxation times are invariant with respect to translation, scaling, and rotation of the cell.

First, since the kernel of $\mathcal{L}_{\rho^{-1}\Gamma}$ is invariant with respect to translation, it follows that $M(C, \beta k_0)$ is invariant with respect to translation of the cell C.

Next, from the scaling properties of the kernel of $\mathcal{L}_{\rho^{-1}\Gamma}$ we have

$$M(sC, \beta k_0) = s^2 M(C, \frac{\beta k_0}{s})$$

for any scaling parameter $s > 0$.

Finally, we have

$$M(\mathcal{R}C, \beta k_0) = \mathcal{R} M(C, \beta k_0) \mathcal{R}^t \quad \text{for any rotation } \mathcal{R},$$

where t denotes the transpose.

Therefore, the Debye relaxation times are translation and rotation invariant. Moreover, for scaling, we have

$$\tau_i(hC, \beta k_0) = \tau_i(C, \frac{\beta k_0}{h}), \qquad i = 1, 2, \quad h > 0.$$

Since β is proportional to the thickness of the cell membrane, β/h is nothing else than the real rescaled coefficient β for the cell C. The Debye relaxation times (τ_i) are therefore invariant by scaling.

Since $\mathcal{L}_{\rho^{-1}\Gamma}$ is self-adjoint, it follows that M is symmetric. Finally, we show positivity of the imaginary part of the matrix M for δ small enough.

We consider that the cell contour Γ can be parametrized by polar coordinates. We have, up to $O(\delta^3)$,

$$M + \beta\rho^{-1}|\Gamma| = -\beta^2 \int_{\rho^{-1}\Gamma} n \mathcal{L}_{\rho^{-1}\Gamma}[n] \, ds, \tag{5.7}$$

where again we have assumed that $\sigma_0 = 1$ and $\epsilon_0 = 0$.

Recall that

$$\beta = \frac{\delta \sigma_m}{\sigma_m^2 + \omega^2 \varepsilon_m^2} - i \frac{\delta \omega \varepsilon_m}{\sigma_m^2 + \omega^2 \varepsilon_m^2}.$$

Hence, the positivity of $\mathcal{L}_{\rho^{-1}\Gamma}$ yields

$$\Im M \geq \frac{\delta\omega\varepsilon_m}{2\rho(\sigma_m^2 + \omega^2\varepsilon_m^2)}|\Gamma|I$$

for δ small enough, where I is the identity matrix.

Finally, by using (5.7) one can see that the eigenvalues of $\Im M$ have one maximum each with respect to the frequency. Let $l_i, i = 1, 2,\ l_1 \geq l_2$, be the eigenvalues of $\int_{\rho^{-1}\Gamma} n\mathcal{L}_{\rho^{-1}\Gamma}[n]ds$. We have

$$\lambda_i = \frac{\delta\omega\varepsilon_m}{\rho(\sigma_m^2 + \omega^2\varepsilon_m^2)}|\Gamma| - \frac{2\delta^2\omega\varepsilon_m\sigma_m}{(\sigma_m^2 + \omega^2\varepsilon_m^2)^2}l_i, \quad i = 1, 2. \tag{5.8}$$

Therefore, τ_i is the inverse of the positive root of the following polynomial in ω:

$$-\varepsilon_m^4|\Gamma|\omega^4 + 6\delta\varepsilon_m^2\sigma_m l_i\rho\omega^2 + \sigma_m^4|\Gamma|.$$

5.5 Anisotropy measure

Anisotropic electrical properties can be found in biological tissues such as muscles and nerves. In this subsection, based on formula (2.9), we introduce a natural measure of the conductivity anisotropy and derive its dependence on the frequency of applied current. Assessment of electrical anisotropy of muscle may have useful clinical application. Because neuromuscular diseases produce substantial pathological changes, the anisotropic pattern of the muscle is likely to be highly disturbed [21, 32]. Neuromuscular diseases could lead to a reduction in anisotropy for a range of frequencies as the muscle fibers are replaced by isotropic tissue.

Let $\lambda_1 \leq \lambda_2$ be the eigenvalues of the imaginary part of the membrane polarization tensor $M(\omega)$. The function

$$\omega \mapsto \frac{\lambda_1(\omega)}{\lambda_2(\omega)}$$

can be used as a measure of the anisotropy of the conductivity of a dilute suspension. Assume $\epsilon_0 = 0$. As frequency ω increases, the factor βk_0 decreases. Therefore, for large ω, using the expansions in (5.8) we obtain that

$$\frac{\lambda_1(\omega)}{\lambda_2(\omega)} = 1 + (l_1 - l_2)\frac{2\delta\sigma_m\rho}{(\sigma_m^2 + \omega^2\varepsilon_m^2)|\Gamma|} + O(\delta^2), \tag{5.9}$$

where $l_1 \leq l_2$ are the eigenvalues of $\int_{\rho^{-1}\Gamma} n\mathcal{L}_{\rho^{-1}\Gamma}[n]ds$.

Formula (5.9) shows that as the frequency increases, the conductivity anisotropy decreases. The anisotropic information can not be captured for

$$\omega \gg \frac{1}{\varepsilon_m}((l_1 - l_2)\frac{2\delta\sigma_m\rho}{|\Gamma|} - \sigma_m^2)^{1/2}.$$

6 Spectroscopic imaging of a dilute suspension

6.1 Spectroscopic conductivity imaging

We now make use of the asymptotic expansion of the effective admittivity in terms of the volume fraction $f = \rho^2$ to image a permittivity inclusion. Consider D to be a bounded domain in Ω with admittivity $1 + fM(\omega)$, where $M(\omega)$ is a membrane polarization tensor and f is the volume fraction of the suspension in D. The inclusion D models a suspension of cells in the background Ω. For simplicity, we neglect the permittivity ϵ_0 of Ω and assume that its conductivity $\sigma_0 = 1$. We also assume that $M(\omega)$ is isotropic. At the macroscopic scale, if we inject a current g on $\partial\Omega$, then the electric potential satisfies:

$$\begin{cases} \nabla \cdot (1 + fM(\omega)\chi_D)\nabla u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega} = g, \quad \displaystyle\int_{\partial\Omega} g(x)ds(x) = 0, \quad \int_\Omega u(x)dx = 0. \end{cases} \tag{6.1}$$

The imaging problem is to detect and characterize D from measurements of u on $\partial\Omega$.

Integrating by parts and using the trace theorem for the double-layer potential [23, 45], we obtain, $\forall\, x \in \partial\Omega$,

$$\begin{aligned} &\frac{1}{2}u(x) + \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)\cdot n(x)}{|x-y|^2} u(y)ds(y) + \frac{1}{2\pi}\int_{\partial\Omega} g(y)\ln|x-y|ds(y) \\ &\quad = \frac{f}{2\pi}M(\omega)\int_D \nabla u(y)\cdot \frac{(x-y)}{|x-y|^2}dy. \end{aligned} \tag{6.2}$$

Since f is small,

$$\int_D \nabla u(y)\cdot\frac{(x-y)}{|x-y|^2}dy \simeq \int_D \nabla U(y)\cdot\frac{(x-y)}{|x-y|^2}dy$$

holds uniformly for $x \in \partial\Omega$, where U is the background solution, that is,

$$\begin{cases} \Delta U = 0 & \text{in } \Omega, \\ \dfrac{\partial U}{\partial n}\Big|_{\partial\Omega} = g, \quad \displaystyle\int_\Omega U(x)dx = 0, \end{cases}$$

Therefore, taking the imaginary part of (6.2) yields

$$\frac{1}{2}\Im u(x) + \frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y)\cdot n(x)}{|x-y|^2}\Im u(y)ds(y) \simeq \frac{f}{2\pi}\Im M(\omega)\int_D \nabla U(y)\cdot\frac{(x-y)}{|x-y|^2}dy, \tag{6.3}$$

uniformly for $x \in \partial\Omega$, provided that g is real. Finally, taking the argument of the maximum of the right-hand side in (6.3) with respect to the frequency ω gives the Debye relaxation time of the suspension in D.

6.2 Selective spectroscopic imaging

A challenging applied problem is to design a selective spectroscopic imaging approach for suspensions of cells. Using a pulsed imaging approach [33, 37], we propose a simple way to selectively image dilute suspensions. Again, we assume for the sake of simplicity that $\epsilon_0 = 0$ and $\sigma_0 = 1$.

In the time-dependant regime, the electrical model for the cell (2.1) is replaced with

$$u(x,t) = \int \hat{h}(\omega)\hat{u}(x,\omega)e^{i\omega t}d\omega,$$

where $\hat{u}(x,\omega)$ is the solution to

$$\begin{cases} \Delta\hat{u}(\cdot,\omega) = 0 & \text{in } D \setminus \overline{C}, \\ \Delta\hat{u}(\cdot,\omega) = 0 & \text{in } C, \\ \dfrac{\partial\hat{u}(\cdot,\omega)}{\partial n}\Big|_+ = \dfrac{\partial\hat{u}(\cdot,\omega)}{\partial n}\Big|_- & \text{on } \Gamma, \\ \hat{u}(\cdot,\omega)|_+ - \hat{u}(\cdot,\omega)|_- - \beta(\omega)\dfrac{\partial\hat{u}(\cdot,\omega)}{\partial n} = 0 & \text{on } \Gamma, \\ \dfrac{\partial\hat{u}(\cdot,\omega)}{\partial n}\Big|_{\partial D} = f, \quad \displaystyle\int_{\partial D}\hat{u}(\cdot,\omega)ds = 0, \end{cases} \tag{6.4}$$

and

$$h(t) = \int \hat{h}(\omega)e^{i\omega t}d\omega$$

is the pulse shape. The support of h is assumed to be compact.

At the macroscopic scale, if we inject a pulsed current, $g(x)h(t)$, on $\partial\Omega$, then the electric potential $u(x,t)$ in the presence of a suspension occupying D is given by

$$u(x,t) = \int \hat{h}(\omega)\hat{u}(x,\omega)e^{i\omega t}d\omega,$$

where

$$\begin{cases} \nabla\cdot(1 + fM(\omega)\chi_D)\nabla\hat{u}(\cdot,\omega) = 0 & \text{in } \Omega, \\ \dfrac{\partial\hat{u}(\cdot,\omega)}{\partial n}\Big|_{\partial\Omega} = g, \quad \displaystyle\int_{\partial\Omega}\hat{u}(\cdot,\omega)ds = 0. \end{cases}$$

Assume that we are in the presence of two suspensions occupying the domains D_1 and D_2 inside Ω. From (6.2) it follows that

$$\begin{aligned} &\frac{1}{2}\hat{u}(x,\omega) + \frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y)\cdot n(x)}{|x-y|^2}\hat{u}(y,\omega)ds(y) + \frac{1}{2\pi}\int_{\partial\Omega}g(y)\ln|x-y|ds(y) \\ &\quad \simeq \frac{f_1}{2\pi}M_1(\omega)\int_{D_1}\nabla U(y)\cdot\frac{(x-y)}{|x-y|^2}dy + \frac{f_2}{2\pi}M_2(\omega)\int_{D_2}\nabla U(y)\cdot\frac{(x-y)}{|x-y|^2}dy, \end{aligned} \tag{6.5}$$

and therefore,

$$\begin{aligned} &\frac{1}{2}u(x,t) + \frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y)\cdot n(x)}{|x-y|^2}u(y,t)ds(y) + \frac{1}{2\pi}h(t)\int_{\partial\Omega}g(y)\ln|x-y|ds(y) \\ &\quad \simeq \frac{f_1}{2\pi}\mathcal{M}_1(t)\int_{D_1}\nabla U(y)\cdot\frac{(x-y)}{|x-y|^2}dy + \frac{f_2}{2\pi}\mathcal{M}_2(t)\int_{D_2}\nabla U(y)\cdot\frac{(x-y)}{|x-y|^2}dy, \end{aligned} \tag{6.6}$$

uniformly in $x \in \partial\Omega$ and $t \in \operatorname{supp} h$, where

$$\mathcal{M}_i(t) := \int \hat{h}(\omega) M_i(\omega) e^{i\omega t} d\omega, \quad i = 1, 2.$$

As it will be shown in section 8, by comparing the Debye relaxation times associated to M_1 and M_2, one can design the pulse shape h in order to image selectively D_1 or D_2. For example, one can selectively image D_1 by taking $\hat{h}(\omega)$ close to zero around the Debye relaxation time of M_2 and close to one around the Debye relaxation time of M_1.

6.3 Spectroscopic measurement of anisotropy

In this subsection we assume that M is anisotropic and consider the solution u to (6.1). We want to assess the anisotropy of the inclusion D of admittivity $1 + fM(\omega)$ from measurements of u on the boundary $\partial\Omega$.

From (6.3) it follows that

$$\begin{aligned} &\int_{\partial\Omega} g(x) \left[\frac{1}{2} \Im u(x) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x-y)\cdot n(x)}{|x-y|^2} \Im u(y) ds(y) \right] ds(x) \\ &\quad \simeq \frac{f}{2\pi} \int_D \Im M(\omega) \nabla U(y) \cdot \nabla U(y) dy, \end{aligned} \tag{6.7}$$

provided that g is real. Now, taking constant current sources corresponding to $g = a \cdot n$, where $a \in \mathbb{R}^2$ is a unit vector, yields

$$\mathcal{S}[a] := \int_{\partial\Omega} g(x) \left[\frac{1}{2} \Im u(x) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x-y)\cdot n(x)}{|x-y|^2} \Im u(y) ds(y) \right] ds(x) \simeq \frac{f}{2\pi} \Im M(\omega) |a|^2 |D|.$$

Since

$$\frac{\min_a \mathcal{S}[a]}{\max_a \mathcal{S}[a]} \simeq \frac{\lambda_1(\omega)}{\lambda_2(\omega)},$$

where λ_1 and λ_2 (with $\lambda_1 \leq \lambda_2$) are the eigenvalues of $\Im M$, it follows from subsection 5.5 that

$$\omega \mapsto \frac{\min_a \mathcal{S}[a]}{\max_a \mathcal{S}[a]}$$

is a natural measure of conductivity anisotropy. This measure may be used for the detection and classification of neuromuscular diseases via measurement of muscle anisotropy [21, 32].

7 Stochastic homogenization of randomly deformed conductivity resistant membranes

The first main result of this section is to show that a rigorous homogenization theory can be derived when the cells (and hence interfaces) are randomly deformed from a periodic structure, and the random deformation is ergodic and stationary in the sense of (2.13).

7.1 Auxiliary problem: proof of Theorem 2.3

In this subsection, we prove Theorem 2.3, that is the existence and uniqueness of the auxiliary problem. As in many stochastic homogenization problems, this is the key step. The main difficulty, as usual, lies in the loss of compactness.

Our strategy is as follows: First, an absorption term is added to regularize the problem which gains back some compactness: the sequence of regularized solutions, which correspond to a sequence of vanishing regularization, have a converging gradient. Secondly, the potential field corresponds to the limiting gradient is shown to be a solution to the auxiliary problem. Finally, using regularity results and sub-linear growth of potential field with stationary gradient, we prove that the solution to the auxiliary problem is unique.

Proof of Theorem 2.3. Step 1: The regularized auxiliary problem. Fix $p \in \mathbb{R}^2$. Consider the following regularized problem where an absorption $\alpha > 0$ is added.

$$\begin{cases} \nabla \cdot k_0(\nabla w^{\pm}_{p,\alpha}(y) + p) + \alpha w^{\pm}_{p,\alpha} = 0 & \text{in} \quad \Phi(\mathbb{R}^{\pm}_2, \gamma), \\ n \cdot k_0 \nabla w^{-}_{p,\alpha}(y) = n \cdot k_0 \nabla w^{-}_{p,\alpha}(y), & \text{and} \quad w^{+}_{p,\alpha} - w^{-}_{p,\alpha} = \beta k_0 n \cdot \nabla w^{-}_{p,\alpha} \text{ in } \Phi(\Gamma_2, \gamma), \\ w^{\pm}_{p,\alpha}(y,\gamma) = \widetilde{w}^{\pm}_{p,\alpha}(\Phi^{-1}(y,\gamma),\gamma), & \text{and} \quad \widetilde{w}^{\pm}_{p,\alpha} \text{ are stationary.} \end{cases} \tag{7.1}$$

Define the space $\mathcal{H} = \{w = \widetilde{w} \circ \Phi^{-1} \mid \widetilde{w} \in H^1_{\text{loc}}(\mathbb{R}^+_2) \times H^1_{\text{loc}}(\mathbb{R}^+_2), \widetilde{w} \text{ is stationary}\}$. More precisely, this means the space of functions $w = \widetilde{w} \circ \Phi^{-1}$ where $\widetilde{w}$ restricted in $\mathbb{R}^+_2$ (respectively $\mathbb{R}^-_2$) is in $\in H^1_{\text{loc}}(\mathbb{R}^+_2)$ (respectively $H^1_{\text{loc}}(\mathbb{R}^-_2)$), and in addition, $\widetilde{w}^{\pm}$ are stationary.

Equip $\mathcal{H}$ with the inner product

$$(u,v)_{\mathcal{H}} = \mathbb{E}\left(\int_{Y^+} \nabla\tilde{u} \cdot \nabla\overline{\tilde{v}} dx + \int_{Y^-} \nabla\tilde{u} \cdot \nabla\overline{\tilde{v}} dx + \int_Y \tilde{u}\overline{\tilde{v}} dx\right). \tag{7.2}$$

Then $\mathcal{H}$ is a Hilbert space. Define the bilinear form

$$A_\alpha(u,v) = \mathbb{E}\left(\int_{\Phi(Y^+)} k_0 \nabla u^+ \cdot \overline{\nabla v^+} dx + \int_{\Phi(Y^-)} k_0 \nabla u^- \cdot \overline{\nabla v^-} dx \right. \\ \left. + \alpha \int_{\Phi(Y)} u\tilde{v} dx + \frac{1}{\beta}\int_{\Phi(\Gamma_0)} (u^+ - u^-)\overline{(w^+ - w^-)} ds\right),$$

and the linear functional

$$b_p(v) = -k_0 \mathbb{E}\left(\int_{\Phi(Y^+)} p \cdot \overline{\nabla v^+} dx + \int_{\Phi(Y^-)} p \cdot \overline{\nabla v^-} dx + \int_{\Phi(\Gamma_0)} (n(x)\cdot p)\overline{(v^+ - v^-)}(x) ds(x)\right).$$

For a fixed $\alpha > 0$, we verify that A_α is bounded and coercive, and b_p is bounded. By the Lax–Milgram theorem, there exists a unique $w_{p,\alpha} \in \mathcal{H}$ such that

$$A_\alpha(w_{p,\alpha}, \varphi) = b_p(\varphi), \quad \forall \varphi \in \mathcal{H}. \tag{7.3}$$

In fact, the solution satisfies (7.1) in the distributional sense $\mathbb{P}$-a.s. in $\mathcal{O}$. Furthermore, the following estimates are immediate:

$$\mathbb{E}\int_{Y^\pm} |\nabla \tilde{w}^{\pm}_{p,\alpha}|^2 \le C, \quad \mathbb{E}\int_{\Gamma_0} |\tilde{w}^{+}_{p,\alpha} - \tilde{w}^{-}_{p,\alpha}|^2 \le C, \quad \mathbb{E}\int_{Y^\pm} |\tilde{w}^{\pm}_{p,\alpha}|^2 \le \frac{C}{\alpha}. \tag{7.4}$$

Apply the extension operators in Corollary A.1 and Corollary A.2. We get the sequences $\{\tilde{w}_{p,\alpha}^{\text{ext}} = P\tilde{w}_{p,\alpha}^{+}\} \subset H^1_{\text{loc}}(\mathbb{R}^2)$ and $\{w_{p,\alpha}^{\text{ext}} = P_\gamma w_{p,\alpha}^{+}\}$. Further, $\{\tilde{w}_{p,\alpha}^{\text{ext}}\}$ are stationary. They satisfy that $w_{p,\alpha}^{\text{ext}} = \tilde{w}_{p,\alpha}^{\text{ext}} \circ \Phi^{-1}$ and that

$$\mathbb{E}\int_Y |\nabla \tilde{w}_{p,\alpha}^{\text{ext}}|^2 \le C, \quad \mathbb{E}\int_{\Gamma_0} |\tilde{w}_{p,\alpha}^{\text{ext}} - \tilde{w}_{p,\alpha}^{-}|^2 \le C, \quad \mathbb{E}\int_Y |\tilde{w}_{p,\alpha}^{\text{ext}}|^2 \le \frac{C}{\alpha}. \tag{7.5}$$

Step 2: Converging subsequences as the regularization parameter vanishes. Thanks to the above estimates, there exists some subsequence, still denoted by $\nabla \tilde{w}_{p,\alpha}^{\text{ext}}$, which converges weakly as $\alpha \downarrow 0$ to a function $\tilde{\eta}_p^{\text{ext}} \in [L^2_{\text{loc}}(\mathbb{R}^2, L^2(\mathcal{O}))]^2$, where $\tilde{\eta}_p^{\text{ext}}$ is stationary. By a change of variable, we also have that $\nabla w_{p,\alpha}^{\text{ext}}$ converges in $[L^2_{\text{loc}}(\mathbb{R}^2, L^2(\mathcal{O}))]^2$ to η_p^{ext} and

$$\eta_p^{\text{ext}}(y,\gamma) = \nabla_y \Psi(y,\gamma)\tilde{\eta}_p^{\text{ext}}(\tilde{y},\gamma), \tag{7.6}$$

where $\Psi = \Phi^{-1}$ and $\tilde{y} = \Psi(y)$. Moreover, as gradients, $\nabla_{\tilde{y}} \tilde{w}_{p,\alpha}^{\text{ext}}$ and $\nabla_y w_{p,\alpha}^{\text{ext}}$ are curl free. This property is preserved by their limits:

$$\partial_{y_i}(\eta_p^{\text{ext}})_j = \partial_{y_j}(\eta_p^{\text{ext}})_i, \quad \partial_{\tilde{y}_i}(\tilde{\eta}_p^{\text{ext}})_j = \partial_{\tilde{y}_j}(\tilde{\eta}_p^{\text{ext}})_i, \quad i,j = 1,\cdots,d. \tag{7.7}$$

That is to say, η_p^{ext} and $\tilde{\eta}_p^{\text{ext}}$ are also gradient functions. Consequently, there exist w_p^{ext} and $\tilde{w}_p^{\text{ext}}$ such that $\eta_p^{\text{ext}} = \nabla_y w_p^{\text{ext}}$ and $\tilde{\eta}_p^{\text{ext}} = \nabla_{\tilde{y}} \tilde{w}_p^{\text{ext}}$. The relation (7.6) implies that $w_p^{\text{ext}}(y) = \tilde{w}_p^{\text{ext}}(\Psi(y,\gamma),\gamma) + C_p(\gamma)$ where $C_p(\gamma)$ is a random constant. We hence redefine $\tilde{w}_p^{\text{ext}}$ by adding to it the random variable C_p so that $w_p^{\text{ext}} = \tilde{w}_p^{\text{ext}} \circ \Psi$. By the same token, we have that $\nabla \tilde{w}_{p,\alpha}^{-}$ and $\nabla w_{p,\alpha}^{-}$ converge (along the above subsequence) to $\tilde{\eta}_p^{-} \in [L^2_{\text{loc}}(\mathbb{R}^-_2)]^2$ and $\eta_p^{-} \in [L^2_{\text{loc}}(\Phi(\mathbb{R}^-_2))]^2$ respectively. Further, η_p^{-} is stationary; in addition, for some $\tilde{w}_p^{-} \in H^1_{\text{loc}}(\mathbb{R}^-_2)$ and $w_p^{-} \in H^1_{\text{loc}}(\mathbb{R}^-_2)$ satisfying that $w_p^{-} = \tilde{w}_p^{-} \circ \Psi$, we have $\tilde{\eta}_p^{-} = \nabla \tilde{w}_p^{-}$ and $\eta_p^{-} = \nabla w_p^{-}$.

Repeating the above argument with the help of the second inequality in (7.5), one observes that $\{\tilde{w}_{p,\alpha}^{\text{ext}} - \tilde{w}_{p,\alpha}^{-}\}$ restricted to the interface $\mathbf{\Gamma}$ converges to some $\tilde{\zeta}_p \in L^2_{\text{loc}}(\mathbf{\Gamma})$ and $\tilde{\zeta}_p$ is stationary. Similarly, by a change of variable, $w_{p,\alpha}^{\text{ext}} - w_{p,\alpha}^{-}$ converges to $\zeta_p = \tilde{\zeta}_p \circ \Psi$ and $\zeta_p \in L^2_{\text{loc}}(\Phi(\mathbf{\Gamma}))$.

Since $\tilde{w}_{p,\alpha}^{\text{ext}}$ is an extension of $\tilde{w}_{p,\alpha}^{+}$, the inequality (A.6) holds. Also, since $\tilde{w}_{p,\alpha}^{\text{ext}}$ is stationary, one has $\mathbb{E}\int_Y \nabla \tilde{w}_{p,\alpha}^{\text{ext}} d\tilde{y} = 0$. Passing to the limit, we get

$$\mathbb{E}\int_Y \nabla_{\tilde{y}} \tilde{w}_p^{\text{ext}}(\tilde{y}) d\tilde{y} = 0, \quad \text{and} \quad \mathbb{E}\int_Y |\nabla_{\tilde{y}} \tilde{w}_p^{\text{ext}}|^s d\tilde{y} \le C\mathbb{E}\int_{Y^+} |\nabla_{\tilde{y}} \tilde{w}_p^{\text{ext}}|^s d\tilde{y}, \tag{7.8}$$

where C depends on the same parameters as in (A.6). Similarly, we also have that

$$\mathbb{E}\int_{\Phi(Y)} |\nabla_y w_p^{\text{ext}}|^s dy \le C\mathbb{E}\int_{\Phi(Y^+)} |\nabla_y w_p^{\text{ext}}|^s dy, \tag{7.9}$$

where C depends on the same parameters as in (A.8). Here and above, $s \ge 1$ is some parameter so that the right-hand sides are finite.

Step 3: The limit solves the auxiliary problem. Take the limiting functions w_p^{ext} and $\tilde{w}_p^{\text{ext}}$ from last step. Let w_p^{+} be the restriction of w_p^{ext} to $\Phi(\mathbb{R}^+_2)$, $\tilde{w}_p^{+}$ be the restriction of $\tilde{w}_p^{\text{ext}})$

to $\mathbb{R}_2^+$. Then by the above construction and (7.8), the last two equations of (2.24) are satisfied.

Let us verify that (w_p^+, w_p^-) satisfies the equation and the boundary conditions in (2.24). Recall the weak formulation (7.3) for the regularized equation (7.1). Pass to the limit $\alpha \to 0$ along the subsequence found above. In particular, we observe that Cauchy–Schwarz and the last inequality of (7.4) imply that

$$\left|\alpha\, \mathbb{E}\int_{\Phi(Y)} w_{p,\alpha}\overline{\varphi}\, dy\right| \le \sqrt{\alpha}\left(\mathbb{E}\int_{\Phi(Y)} \alpha|w_{p,\alpha}|^2 dy\right)^{\frac{1}{2}}\left(\mathbb{E}\int_{\Phi(Y)} |\varphi|^2 dy\right)^{\frac{1}{2}} \longrightarrow 0.$$

As a result, we obtain in the limit that for all $(\varphi^+, \varphi^-) \in \mathcal{H}$,

$$\begin{aligned}\mathbb{E}\left(\int_{\Phi(Y^+)} k_0(p+\nabla w_p^+)\cdot\overline{\nabla\varphi^+}dx + \int_{\Phi(Y^-)} k_0(p+\nabla w_p^-)\cdot\overline{\nabla\varphi^-}dx\right)&\\ +\frac{1}{\beta}\mathbb{E}\left(\int_{\Phi(\Gamma_0)} \zeta_p\overline{(\varphi^+-\varphi^-)}ds\right) = 0&\end{aligned} \tag{7.10}$$

By letting φ in $C_0^\infty(\Phi(Y^+))\cap\mathcal{H}$ and $C_0^\infty(\Phi(Y^-))\cap\mathcal{H}$ respectively, we see that the first line of (2.24) is satisfied in the distributional sense. It is worth mentioning that $C_0^\infty(\Phi(Y^\pm))\cap\mathcal{H}$ is the set of test functions in $\mathcal{H}$ (being stationary) and whose restrictions to the unit cell is in the first space, $C_0^\infty(\Phi(Y^\pm))$, but the function itself is not compactly supported there. To verify the second line of (2.24), we first take $\varphi \in C_0^\infty(\Phi(Y))\cap\mathcal{H}$ and apply integration by parts to get

$$\mathbb{E}\int_{\Phi(\Gamma_0)} k_0(n\cdot\nabla w_p^+ - n\cdot\nabla w_p^-)\overline{\varphi}ds = 0, \qquad \forall\varphi\in C_0^\infty(\Phi(Y))\cap\mathcal{H}.$$

This implies the first boundary condition. For the second condition, consider arbitrary $\varphi \in C^\infty(\overline{\Phi(Y^-)})\cap\mathcal{H}$ and let the test function in (7.10) be $\varphi\chi_{\Phi(\mathbb{R}_2^-)}$. By the first boundary condition and integration by parts formula, we get

$$\mathbb{E}\int_{\Phi(\Gamma_0)}\left[k_0 n\cdot(p+\nabla w_p^-) - \frac{1}{\beta}\zeta_p\right]\overline{\varphi}ds = 0, \qquad \forall\varphi\in C^\infty(\overline{\Phi(Y^-)})\cap\mathcal{H}.$$

This implies that $\zeta_p = \beta k_0 n(x)\cdot(p+\nabla w_p^+)$. It suffices to link ζ_p with $w_p^+ - w_p^-$ on $\Phi(\Gamma_0)$. Hence fix an arbitrary ϕ in $C^\infty(\Phi(\Gamma_0))\cap\mathcal{H}$ such that $\int_{\Phi(\Gamma_0)}\phi ds = 0$ and $\phi\circ\Phi$ is stationary. Then we can construct $\varphi = \varphi^+\chi_{\Phi(Y^+)} + \varphi^-\chi_{\Phi(Y^-)}$ in $\mathcal{H}$ by solving:

$$\begin{cases}\Delta\varphi^+ = 0 \text{ in } \Phi(Y^+), \quad \text{and} \qquad \Delta\varphi^- = 0 \text{ in } \Phi(Y^-),\\ \varphi^+ = 0 \text{ on } \partial\Phi(Y,) \quad \text{and} \quad n(x)\cdot\nabla\varphi^\pm = \phi \text{ on } \Phi(\Gamma_0).\end{cases}$$

Using φ as test function in (7.3) and by integration by parts, one obtains

$$\mathbb{E}\int_{\Phi(Y^+)}\nabla w_{p,\alpha}^+\cdot\overline{\nabla\varphi^+}dx + \mathbb{E}\int_{\Phi(Y^-)}\nabla w_{p,\alpha}^-\cdot\overline{\nabla\varphi^-}dx = \mathbb{E}\int_{\Phi(\Gamma_0)}(-w_{p,\alpha}^+ + w_{p,\alpha}^-)\overline{\phi}ds.$$

Pass to the limit and recall that $w_{p,\alpha}^+ - w_{p,\alpha}^-$ converges to ζ_p; we obtain

$$\mathbb{E}\int_{\Phi(Y^+)}\nabla w_p^+\cdot\overline{\nabla\varphi^+}dx + \mathbb{E}\int_{\Phi(Y^-)}\nabla w_p^-\cdot\overline{\nabla\varphi^-}dx = -\mathbb{E}\int_{\Phi(\Gamma_0)}\zeta_p\overline{\phi}ds.$$

Since φ is constructed so that $\Delta\varphi^\pm = 0$ in $\Phi(Y^\pm)$, we have also that

$$\mathbb{E}\int_{\Phi(Y^+)} \nabla w_p^+ \cdot \nabla\overline{\varphi^+}dx + \mathbb{E}\int_{\Phi(Y^-)} \nabla w_p^- \cdot \nabla\overline{\varphi^-}dx = -\mathbb{E}\int_{\Phi(\Gamma_0)} (w_p^+ - w_p^-)\overline{\phi}ds.$$

It follows that

$$\mathbb{E}\int_{\Phi(\Gamma_0)} (w_p^+ - w_p^- - \zeta_p)\overline{\phi}ds = 0, \quad \forall\phi \in C^\infty(\Phi(\Gamma_0)) \text{ s.t.} \int_{\Phi(\Gamma_0)} \phi ds = 0.$$

Note that both ζ_p and $w_p^+ - w_p^+$ are stationary. The above identity shows that $\zeta_p = w_p^+ - w_p^- + C_2(\gamma)$ where $C_2(\gamma)$ is a random constant. Re-define w_p^- by subtracting from it the constant C_2; then the second boundary condition in the second line of (2.24) is satisfied. Note that, by subtracting the same constant from $\tilde{w}_p^-$, the change of variable $w_p^- = \tilde{w}_p^- \circ \Psi$ remains valid. We summarize that (w_p^+, w_p^-) obtained above provides a solution to the auxiliary problem (2.24).

Step 4: Uniqueness of the auxiliary problem. Suppose otherwise, then there exist v_0^+ and v_0^- satisfying (2.24) with $p = 0$. In addition, there is an extension of $\tilde{v}_0^+$ denoted by $\tilde{v}_0^{\text{ext}}$, such that

$$\nabla\tilde{v}_0^{\text{ext}} \text{ is stationary}, \quad \text{and} \quad \mathbb{E}\int_Y \nabla\tilde{v}_0^{\text{ext}}dx = 0. \tag{7.11}$$

On the one hand, by the standard elliptic regularity theory, we know that v_0^+ and v_0^- are in $W^{1,\infty}_{\text{loc}}(\Phi(\mathbb{R}^+_2))$ and $W^{1,\infty}_{\text{loc}}(\mathbb{R}^-_2)$. Then this is true also for $\tilde{v}_0^+$ and $\tilde{v}_0^-$. Consequently, we have that: $\nabla\tilde{v}_0^{\text{ext}}$ is stationary; $\mathbb{E}\|\nabla\tilde{v}_0^{\text{ext}}\|^s_{L^s(Y)} \le C$ for some $s > 2$. These properties of $\tilde{v}_0^{\text{ext}}$ imply that it grows sub-linearly, thanks to [13, Lemma A.5].

Let us take the weak formulation of the equations satisfied by (v_0^+, v_0^-), and take this function itself as the test function. Integrate over $\Phi(NY)$ for a large integer N. We get

$$\int_{\Phi(NY\cap\mathbb{R}^+_2)} k_0|\nabla v_0^+|^2dx + \int_{\Phi(NY\cap\mathbb{R}^-_2)} k_0|\nabla v_0^-|^2dx \\ + \beta^{-1}\int_{\Phi(NY\cap\Gamma)} |v_0^+ - v_0^-|^2ds = \int_{\partial\Phi(NY)} k_0 n\cdot\nabla v_0^+\overline{v_0^+}ds.$$

Since v_0^+ grows sub-linearly at infinity, for sufficiently large N, one has $|v_0^+| = o(N)$. Consequently, the right-hand side is of order $o(N^2)$. Take the real part of the left-hand side and divided it by N^2, we have

$$\frac{1}{N^2}\sum_{n\in\mathcal{I}(N)}\left[\int_{\Phi(Y_n^+)} \sigma_0|\nabla v_0^+|^2dx + \int_{\Phi(Y_n^-)} \sigma_0|\nabla v_0^-|^2dx + \Re\beta^{-1}\int_{\Phi(\Gamma_n)} |v_0^+ - v_0^-|^2ds\right] \longrightarrow 0,$$

where $\mathcal{I}(N)$ are the indices of cubes $\{Y_n \subset NY\}$. By a change of variable with bounds (2.17) and (2.18), we also have that

$$\frac{1}{N^2}\sum_{n\in\mathcal{I}(N)}\left[\int_{Y_n^+} \sigma_0|\nabla\tilde{v}_0^+|^2d\tilde{x} + \int_{Y_n^-} \sigma_0|\nabla\tilde{v}_0^-|^2d\tilde{x} + \beta^{-1}\int_{\Gamma_n} |\tilde{v}_0^+ - \tilde{v}_0^-|^2(\tilde{x})ds(\tilde{x})\right] \longrightarrow 0.$$

Since the integrands above are stationary and the number of elements in $\mathcal{I}(N)$ is also N^2, we can apply ergodic theorem and conclude that

$$\mathbb{E}\int_{Y}|\nabla\tilde{v}_0^+|^2 d\tilde{x} = \mathbb{E}\int_{Y^-}|\nabla\tilde{v}_0^-|^2 d\tilde{x} = \mathbb{E}\int_{\Gamma_0}|\tilde{v}_0^+ - \tilde{v}_0^-|^2(\tilde{x})ds(\tilde{x}) = 0.$$

This implies that $\tilde{v}_0^+ = \tilde{v}_0^- = C$ for some constant. By the change of variables, $v_0^+ = v_0^- = C$ as well, proving the uniqueness. □

7.2 Proof of the homogenization theorem

In this section, we prove the homogenization theorem using the energy method, *i.e.*, the method of oscillating test functions [44].

7.2.1 Oscillating test functions

We first build the oscillating test functions using the solutions to the auxiliary problem. Fix a vector $p \in \mathbb{R}^2$. Let (w_p^+, w_p^-) be the unique solution to the auxiliary problem (2.24). In particular, w_p^+ has an extension w_p^{ext}. We define

$$\begin{cases} w_{1p}^{\varepsilon}(x,\gamma) = x\cdot p + \varepsilon w_p^{\text{ext}}(\frac{x}{\varepsilon},\gamma), & x\in\mathbb{R}^2,\\ w_{2p}^{\varepsilon}(x,\gamma) = x\cdot p + \varepsilon Q w_p^{-}(\frac{x}{\varepsilon},\gamma), & x\in\mathbb{R}^2. \end{cases} \tag{7.12}$$

Here and in the sequel, Q denotes the trivial extension operator which sets $Qf = 0$ outside the domain of f. By scaling the auxiliary problem, we verify that $(w_p^{\varepsilon+}, w_p^{\varepsilon-})$, where $w_p^{\varepsilon+}$ is the restriction of w_{1p}^{ε} in $\varepsilon\Phi(\mathbb{R}_2^+)$ and $w_p^{\varepsilon-}$ is the restriction of w_{2p}^{ε} in $\varepsilon\Phi(\mathbb{R}_2^-)$, satisfies

$$\begin{cases} \nabla\cdot k_0\nabla w_p^{\varepsilon+} = 0 \quad\text{and}\quad \nabla\cdot k_0\nabla w_p^{\varepsilon-} = 0 \text{ in } \varepsilon\Phi(\mathbb{R}_2^{\pm}),\\ k_0 n\cdot\nabla w_p^{\varepsilon+} = k_0 n\cdot\nabla w_p^{\varepsilon-} \quad\text{and}\quad w_p^{\varepsilon+} - w_p^{\varepsilon-} = \varepsilon\beta k_0 n\cdot\nabla w_p^{\varepsilon-} \text{ on } \varepsilon\Phi(\mathbf{\Gamma}). \end{cases}$$

In particular, for any bounded open set $\mathscr{O}\subset\mathbb{R}^2$ and any test function $\varphi = (\varphi^+,\varphi^-)$ such that $\varphi^{\pm}\in H^1_{\text{loc}}(\mathscr{O}\cap\Phi(\mathbb{R}_2^{\pm}))$ and φ supported in $\mathscr{O}$, we have that

$$\begin{aligned} \int_{\mathscr{O}\cap\varepsilon\Phi(\mathbb{R}_2^+)} k_0\nabla w_p^{\varepsilon+}\cdot\overline{\nabla\varphi^+}dx + \int_{\mathscr{O}\cap\varepsilon\Phi(\mathbb{R}_2^-)} k_0\nabla w_p^{\varepsilon-}\cdot\overline{\nabla\varphi^-}dx \\ + (\varepsilon\beta)^{-1}\int_{\mathscr{O}\cap\varepsilon\Phi(\mathbf{\Gamma})}(w_p^{\varepsilon-} - w_p^{\varepsilon-})\overline{(\varphi^+ - \varphi^-)}ds = 0. \end{aligned} \tag{7.13}$$

Define also the vector fields $\eta_p^{\varepsilon\pm} = k_0\nabla w_p^{\varepsilon\pm}$. We derive the following convergence results.

Lemma 7.1. *Let $w_p^{\varepsilon\pm}$ and the vector fields $\eta_p^{\varepsilon\pm}$ be defined as above and let $\mathscr{O}\subset\mathbb{R}^2$ be a bounded open set. Then as $\varepsilon\to 0$, we have the following:*

$$w_{1p}^{\varepsilon}\to x\cdot p, \quad \textit{uniformly in } \mathscr{O} \textit{ a.s. in } \mathcal{O}; \tag{7.14}$$

$$w_{2p}^{\varepsilon} \to x \cdot p, \quad \textit{in } L^2(\mathscr{O}) \textit{ a.s. in } \mathcal{O}. \tag{7.15}$$

$$Q\eta_p^{\varepsilon\pm} \rightharpoonup \det\left(\mathbb{E}\int_Y \nabla\Phi(x,\cdot)dx\right)^{-1} \mathbb{E}\int_{\Phi(Y^\pm)} k_0\left(\nabla w_p^{\pm}(x,\cdot)+p\right)dx \textit{ in } [L^2(\mathscr{O})]^2 \textit{ a.s. in } \mathcal{O}. \tag{7.16}$$

Proof. The first limit holds since $w_p^{\text{ext}}(x)$ grows sub-linearly as $|x|$ tends to infinity, a fact we have already proved in Step 4 in the proof of Theorem 2.3. Indeed, we have

$$w_{1p}^{\varepsilon} - x\cdot p = |x|\left(|\varepsilon^{-1}x|^{-1}w_p^{\text{ext}}(\varepsilon^{-1}x)\right) \to 0 \text{ uniformly in } \mathscr{O}.$$

To prove the second convergence result, we write

$$w_{2p}^{\varepsilon} - x\cdot p = \varepsilon\left(w_p^{-}(\frac{x}{\varepsilon}) - w_p^{\text{ext}}(\frac{x}{\varepsilon})\right)\chi_{\varepsilon\Phi(\mathbb{R}_2^-)} + \varepsilon w_p^{\text{ext}}(\frac{x}{\varepsilon})\chi_{\varepsilon\Phi(\mathbb{R}_2^-)}.$$

The second item on the right converges uniformly in $\mathscr{O}$ to zero. Therefore, it suffices to prove that $J_\varepsilon := \|\varepsilon w_p^-(\varepsilon^{-1}x) - \varepsilon w_p^{\text{ext}}(\varepsilon^{-1}x)\|_{L^2(\varepsilon\Phi(\mathbb{R}_2^-)\cap\mathscr{O})}$ converges to zero. Given $\mathscr{O}$ and ε, we can find $\mathcal{I}_\varepsilon(\mathscr{O}) \subset \mathbb{Z}^2$ such that $\mathscr{O} \subset \cup_{k\in\mathcal{I}_\varepsilon}\varepsilon\Phi(Y_n)$ and $|\mathcal{I}_\varepsilon| \lesssim C(\mathscr{O})\varepsilon^{-d}$. Then

$$\begin{aligned} J_\varepsilon &\le \sum_{n\in\mathcal{I}_\varepsilon}\int_{\varepsilon\Phi(Y_n^-)} \varepsilon^2\left|w_p^{\text{ext}}(\frac{x}{\varepsilon}) - w_p^-(\frac{x}{\varepsilon})\right|^2 dx = \varepsilon^{2+d}\sum_{n\in\mathcal{I}_\varepsilon}\int_{\Phi(Y_n^-)}\left|w_p^{\text{ext}}(x) - w_p^-(x)\right|^2 dx \\ &\le C\varepsilon^{2+d}\sum_{n\in\mathcal{I}_\varepsilon}\int_{Y_n^-}\left|\tilde{w}_p^{\text{ext}}(\tilde{x}) - \tilde{w}_p^-(\tilde{x})\right|^2 d\tilde{x}. \end{aligned}$$

In the last inequality, we used the change of variable $\tilde{x} = \Phi^{-1}(x)$ and the bounds (2.17) and (2.18). Using the estimate (C.4), we have

$$J_\varepsilon \le C\varepsilon^2\left[\frac{1}{|\mathcal{I}_\varepsilon|}\sum_{n\in\mathcal{I}_\varepsilon}\left(\int_{\Gamma_n}\left|\tilde{w}_p^+(\tilde{x}) - \tilde{w}_p^-(\tilde{x})\right|^2 ds(\tilde{x}) + \int_{Y_n^-}\left|\nabla\tilde{w}_p^{\text{ext}}(\tilde{x}) - \nabla\tilde{w}_p^-(\tilde{x})\right|^2 d\tilde{x}\right)\right].$$

Note that the integrands above are stationary and the item inside the bracket is ready for applying ergodic theorem. This item converges to

$$\mathbb{E}\int_{\Gamma_0}|\tilde{w}_p^+ - \tilde{w}_p^-|^2(\tilde{x})ds(\tilde{x}) + \mathbb{E}\int_Y|\nabla\tilde{w}_p^{\text{ext}} - \nabla\tilde{w}_p^-|^2 d\tilde{x},$$

which is bounded for example by (7.4) and (7.5). Consequently, $J_\varepsilon \to 0$, proving (7.15).

Now we prove the last convergence result. Let $\tilde{\eta}_p^{\pm} = (k_0[p + (\nabla\Phi)^{-1}\nabla\tilde{w}_p^{\pm}])\chi_{\Phi(\mathbb{R}_2^{\pm})}$; then they are stationary and the relation $\eta_p^{\varepsilon\pm} = \tilde{\eta}_p^{\pm}\left(\Phi^{-1}(\frac{x}{\varepsilon},\gamma)\right)$ holds. By Lemma 2.2. of [19], we obtain (7.16). □

7.2.2 Proof of the homogenization theorem

In this subsection we prove the homogenization theorem using Tartar's energy method. There are two main steps. In the first step, we use the energy estimates to extract converging subsequences. In the second step, we identify the limit as a solution to a homogenized equation which has unique solution.

Proof of Theorem 2.4. Step 1: Extraction of converging subsequences. Let $(u^+_\varepsilon, u^-_\varepsilon)$ be the solution to the heterogeneous problem (2.2). In particular, u^+_ε has an extension $u^{\rm ext} \in H^1(\Omega)$. Let the vector fields $\xi^\pm_\varepsilon$ be $k_0 \nabla u^\pm_\varepsilon$. Then the estimates (3.14) and (3.11) show that

$$\|u^{\rm ext}_\varepsilon\|_{H^1(\Omega)} + \|Q\xi^+_\varepsilon\|_{[L^2(\Omega)]^2} + \|Q\xi^+_\varepsilon\|_{[L^2(\Omega)]^2} \le C.$$

Consequently, there exists a subsequence and functions $u_0 \in H^1(\Omega)$ and $\xi_1, \xi_2 \in [L^2(\Omega)]^2$, such that

$$\begin{aligned} &u^{\rm ext}_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega), && u^{\rm ext}_\varepsilon \to u_0 \text{ strongly in } L^2(\Omega);\\ &Q\xi^+_\varepsilon \rightharpoonup \xi_1 \text{ weakly in } [L^2(\Omega)]^2, && Q\xi^-_\varepsilon \rightharpoonup \xi_2 \text{ weakly in } [L^2(\Omega)]^2. \end{aligned} \tag{7.17}$$

In the proof of Proposition 3.4, we also proved that

$$u^{\rm ext}_\varepsilon \chi^-_\varepsilon - Qu^-_\varepsilon \to 0 \text{ strongly in } L^2(\Omega). \tag{7.18}$$

Now fix an arbitrary test function $\varphi \in C^\infty_0(\Omega)$. Take $(\varphi\chi^+_\varepsilon, \varphi\chi^-_\varepsilon)$ as a test function in (3.5). Then the interface term disappears and we get

$$\int_\Omega k_0(Q\xi^+_\varepsilon) \cdot \overline{\nabla\varphi} dx + \int_\Omega k_0(Q\xi^-_\varepsilon) \cdot \overline{\nabla\varphi} dx = 0.$$

Passing to the limit $\varepsilon \to 0$ along the subsequence above, one finds

$$\int_\Omega (\xi_1 + \xi_2) \cdot \overline{\nabla\varphi} dx = 0, \quad \forall \varphi \in C^\infty_0(\Omega). \tag{7.19}$$

Therefore, the limiting vector field $\xi_1 + \xi_2$ satisfies that

$$\nabla \cdot (\xi_1 + \xi_2) = 0, \quad \text{in } \mathcal{D}'(\Omega), \tag{7.20}$$

where $\mathcal{D}'(\Omega)$ denotes the space of tempered distributions on Ω. Now for any $\phi \in C^\infty(\partial\Omega)$, we may lift it to a smooth function $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi = \phi$ on $\partial\Omega$. Take $(\varphi\chi^+_\varepsilon, \varphi\chi^-_\varepsilon)$ as the test function in (3.5) and pass to the limit; we get

$$\int_\Omega (\xi_1 + \xi_2) \cdot \overline{\nabla\varphi} dx = \int_{\partial\Omega} g\overline{\phi}\, ds.$$

Since $\xi_1 + \xi_2 \in L^2(\Omega)$ and $\nabla \cdot (\xi_1 + \xi_2) \in L^2(\Omega)$, the trace $n \cdot (\xi_1 + \xi_2)$ on the boundary $\partial\Omega$ is well defined. Applying the divergence theorem and (7.20) we get

$$\int_{\partial\Omega} n \cdot (\xi_1 + \xi_2)\overline{\phi}\, ds = \int_{\partial\Omega} g\overline{\phi}\, ds, \quad \forall \phi \in C^\infty(\overline{\Omega}).$$

This shows that, $n \cdot (\xi_1 + \xi_2) = g$ at $\partial\Omega$. Further, since the trace of $Q\xi^-_\varepsilon$ is zero for all ε, the same argument above shows that $n \cdot \xi_2 = 0$ at $\partial\Omega$. We hence get

$$n \cdot \xi_1 = g \quad \text{at } \partial\Omega. \tag{7.21}$$

Step 2: Weak convergence of Qu^-_ε. We can write Qu^-_ε as $u^{\rm ext}_\varepsilon \chi^-_\varepsilon + (Qu^-_\varepsilon - u^{\rm ext}_\varepsilon \chi^-_\varepsilon)$. Due to (7.18) and the fact that $u^{\rm ext}_\varepsilon$ converges strongly to u_0, we only need to verify that χ^-_ε converges weakly to θ. To this purpose, fix an arbitrary open set K compactly supported in Ω, and observe that for sufficiently small ε, K is compactly supported in E_ε defined in (2.21). Then we have

$$\int_K \chi_{\Omega_\varepsilon^-} dx = \int_{K\cap\varepsilon\Phi(\mathbb{R}_2^-)} dx = \int_{\varepsilon\Phi^{-1}\left(\frac{K}{\varepsilon}\right)} \chi_{\mathbb{R}_2^-}\left(\frac{z}{\varepsilon}\right) \det\nabla\Phi\left(\frac{z}{\varepsilon},\gamma\right)dz.$$

In [19, 18], it is shown that the characteristic function $\varepsilon\Phi^{-1}\left(\frac{K}{\varepsilon}\right)$ converges strongly in $L^1(\mathbb{R}^2)$ to that of the set $[\mathbb{E}\int_Y \nabla\Phi(y,\cdot)dy]^{-1}K$. On the other hand, since the function $\chi_{\mathbb{R}_2^-}\det\nabla\Phi$ is stationary, by ergodic theorem, we have

$$\chi_{\mathbb{R}_2^-}\left(\frac{z}{\varepsilon}\right)\det\nabla\Phi\left(\frac{z}{\varepsilon},\gamma\right) \overset{*}{\rightharpoonup} \mathbb{E}\int_Y \chi_{\mathbb{R}_2^-}\det\nabla\Phi(z,\gamma)dz = \theta\varrho^{-1}, \quad \text{in } L^\infty(\mathbb{R}^2).$$

Consequently, we observe that for any open set K compactly supported in Ω, we have

$$\int \chi_K\chi_{\Omega_\varepsilon^-}dx \to \theta\varrho^{-1}\int_{[\mathbb{E}\int_Y\nabla\Phi(y,\cdot)dy]^{-1}K} dx = \theta\varrho^{-1}\det\left(\mathbb{E}\int_Y \nabla\Phi(y,\cdot)dy\right)^{-1}|K| = \theta|K|.$$

Here, we used the fact that $\det\left(\mathbb{E}\int_Y\nabla\Phi(y,\cdot)dy\right) = \varrho^{-1}$, a fact also proved in [19, 18]. Since linear combinations of characteristic functions of compact sets in Ω are dense in $L^2(\Omega)$, we get the desired result. The fact that $\theta < 1$ is easily deduced from the assumption on Y^- and the assumption (2.19), we omit the proof. This completes the proof of item two of the theorem up to a subsequence.

Step 3: Identifying the limit. Fix an arbitrary test function $\varphi \in C_0^\infty(\Omega)$. By the constructions of $\Omega_\varepsilon^-, K_\varepsilon$ and E_ε defined in (2.20) and (2.21), for sufficiently small ε, the function φ is compactly supported in E_ε.

Fix a $p \in \mathbb{R}^2$. Let w_{1p}^ε and w_{2p}^ε be as in (7.12). In the weak formulation (7.13) of the equations satisfied by them, take $(\overline{\varphi u_\varepsilon^+}, \overline{\varphi u_\varepsilon^-})$ as a test function; we get

$$\int_\Omega (Q\eta_p^{\varepsilon+})\cdot\nabla(\overline{\varphi}u_\varepsilon^+)dx + \int_\Omega (Q\eta_p^{\varepsilon-})\cdot\nabla(\overline{\varphi}u_\varepsilon^-)dx + \frac{1}{\varepsilon\beta}\int_{\Gamma_\varepsilon}(w_{1p}^\varepsilon - w_{2p}^\varepsilon)\overline{\varphi}(u_\varepsilon^+ - u_\varepsilon^-)ds = 0.$$

Similarly, in the weak formulation (3.5), take $(\overline{\varphi w_{1p}^\varepsilon}, \overline{\varphi w_{2p}^\varepsilon})$ as the test function; we get

$$\int_\Omega (Q\xi_\varepsilon^+)\cdot\nabla(\overline{\varphi}w_{1p}^\varepsilon)dx + \int_\Omega (Q\xi_\varepsilon^-)\cdot\nabla(\overline{\varphi}w_{2p}^\varepsilon)dx + \frac{1}{\varepsilon\beta}\int_{\Gamma_\varepsilon}(u_\varepsilon^+ - u_\varepsilon^-)\overline{\varphi}(w_{1p}^\varepsilon - w_{2p}^\varepsilon)ds = 0.$$

Note that the integrating domains in the first formula can be taken as above because φ is compactly supported in E_ε, which implies that $\varepsilon\Phi(\mathbf{\Gamma}) \cap \operatorname{supp}\varphi = \Gamma_\varepsilon \cap \operatorname{supp}\varphi$. Subtracting the two formulas above and noticing in particular that the interface terms cancel out, we get

$$\begin{aligned}\int_\Omega (Q\eta_p^{\varepsilon+})\cdot\overline{\nabla\varphi}u_\varepsilon^{\text{ext}}dx + \int_\Omega (Q\eta_p^{\varepsilon-})\cdot\overline{\nabla\varphi}u_\varepsilon^{\text{ext}}dx + \int_\Omega (Q\eta_p^{\varepsilon-})\cdot\overline{\nabla\varphi}(Qu_\varepsilon^- - u_\varepsilon^{\text{ext}}\chi_\varepsilon^-)dx \\ - \int_\Omega (Q\xi_\varepsilon^+)\cdot\overline{\nabla\varphi}w_{1p}^\varepsilon dx - \int_\Omega (Q\xi_\varepsilon^-)\cdot\overline{\nabla\varphi}w_{2p}^\varepsilon dx = 0.\end{aligned}$$

By the convergence results (7.16), (7.14), (7.15), (7.17) and (7.18), we observe that each integrand above is a product of a strong converging term with a weak converging term. Therefore, we can pass the above to the limit $\varepsilon \to 0$ and get

$$\int_\Omega (\eta_{1p} + \eta_{2p})u_0 \cdot \overline{\nabla\varphi}dx = \int_\Omega (\xi_1 + \xi_2)(x\cdot p)\cdot\overline{\nabla\varphi}dx,$$

where η_{1p} (resp. η_{2p}) is defined as the right-hand side of (7.16) with the "+" (resp. "−") sign. The integral on the right can be written as

$$\int_\Omega (\xi_1 + \xi_2) \cdot [\overline{\nabla(\varphi x \cdot p)} - p\overline{\varphi}]dx = -\int_\Omega (\xi_1 + \xi_2) \cdot p\,\overline{\varphi}dx,$$

where we have used (7.19). For the integral involving $\eta_{1p} + \eta_{2p}$, we check that the derivation of (7.20) works for $\eta_{1p} + \eta_{2p}$, which shows that the integral can be written as

$$-\int_\Omega [\nabla \cdot (\eta_{1p} + \eta_{2p})u_0 + (\eta_{1p} + \eta_{2p})\nabla u_0]\overline{\varphi}dx = -\int_\Omega [(\eta_{1p} + \eta_{2p})\nabla u_0]\overline{\varphi}dx.$$

Due to the linearity in p of the auxiliary problem (2.24) in p, we verify that its solutions satisfy that $w_p^\pm = \sum_{j=1}^2 w_{e_j}^\pm p_j$, where $\{e_j\}_{j=1}^2$ is the Euclidean basis of $\mathbb{R}^2$. Consequently, we have that

$$\begin{aligned} e_i \cdot (\eta_{1p} + \eta_{2p}) &= k_0 \det\left(\mathbb{E}\int_Y \nabla\Phi(x,\cdot)dx\right)^{-1} \\ &\times \sum_{j=1}^2 \left[\mathbb{E}\int_{\Phi(Y^+)} \left(e_i \cdot \nabla w_{e_j}^+(x,\cdot) + \delta_{ij}\right)dx + \mathbb{E}\int_{\Phi(Y^-)} \left(e_i \cdot \nabla w_{e_j}^-(x,\cdot) + \delta_{ij}\right)dx\right] p_j. \end{aligned}$$

Recall the definition of the matrix (K_{ij}^*) in (2.26). We check that $\eta_{1p} + \eta_{2p} = (K^*)^t p$ where $(K^*)^t$ denotes the transpose of K^*. Combining the above formulas, we finally obtain that

$$\int_\Omega p^t(\xi_1 + \xi_2 - K^*\nabla u_0)\overline{\varphi}dx = 0, \quad \forall \varphi \in C_0^\infty(\Omega),\ p \in \mathbb{R}^2.$$

We hence conclude that $\xi_1 + \xi_2 = K^*\nabla u_0$, and by (7.20) we verify the first line of (2.25). Similarly, by (7.21), we verify the boundary condition in (2.25). Finally, the facts that

$$\int_\Omega Qu_\varepsilon^+ dx = 0, \quad \text{and} \quad Qu_\varepsilon^- \rightharpoonup \theta u_0 \text{ weakly in } L^2$$

indicate that

$$\int_\Omega u_0 dx = \lim_{\varepsilon\to 0}\int_\Omega (Qu_\varepsilon^+ + Qu_\varepsilon^-)dx = \lim_{\varepsilon\to 0}\int_\Omega Qu_\varepsilon^- dx = \theta\int_\Omega u_0 dx.$$

Since $\theta < 1$, we see that the integral condition in (2.24) is satisfied. Therefore, the subsequence obtained in step one satisfies the homogenized problem (2.25). Since this problem has a unique solution, all converging subsequences converge to the unique solution. Finally, the whole sequence $(u_\varepsilon^+, u_\varepsilon^-)$ converges, completing the proof. □

7.3 Effective admittivity of a dilute suspension

In this subsection, we consider the case when the cells are dilute. We aim to derive a formal first order asymptotic expansion of the effective admittivity in terms of the volume fraction of the dilute cells.

In the formula of the homogenized coefficient (2.26), the integral term has the form

$$J_{ij} = \mathbb{E} \int_{\Phi(Y^+)} e_j \cdot \nabla w^+_{e_i}(y, \cdot)\, dy + \mathbb{E} \int_{\Phi(Y^-)} e_j \cdot \nabla w^-_{e_i}(y, \cdot)\, dy.$$

Thanks to the ergodic theorem, J_{ij} also takes the form

$$J_{ij} = \lim_{N\to\infty} \frac{1}{N^2} \sum_{n\in\mathcal{I}(N)} \left(\int_{\Phi(Y_n^+)} e_j \cdot \nabla w^+_{e_i}(y, \cdot)\, dy + \int_{\Phi(Y_n^-)} e_j \cdot \nabla w^-_{e_i}(y, \cdot)\, dy \right).$$

Here, $\mathcal{I}(N)$ is the indices for the cubes $\{Y_n\}$ inside the big cube NY. Now using integration by parts, we simplify the above expression to

$$J_{ij} = \lim_{N\to\infty} \frac{1}{N^2} \left(\int_{\partial\Phi(NY)} n_j w^+_{e_i}(y, \cdot)\, ds(y) - \sum_{n\in\mathcal{I}(N)} \int_{\Gamma_n} (w^+_{e_i} - w^-_{e_i})(y, \cdot) n_j\, ds(y) \right).$$

Here, n denotes the outer normal vector along the boundary of $\Phi(NY)$ and $\Phi(Y_n^-)$, $n \in \mathcal{I}(N)$; $n_j = n \cdot e_j$ denotes its j-th component. Note that the boundary terms at $\{\partial\Phi(Y_n)\}_{n\in\mathcal{I}(N)} \cap \Phi(NY)$ are cancelled because two adjacent cubes share the same outer normal vector at their common boundary except for reversed signs.

Finally, we have seen that $w^+_{e_i}$ has sub-linear growth. Since the surface $\Phi(NY)$ has volume of order $O(N)$, the sub-linear growth indicates that the boundary integral at $\partial\Phi(NY)$ is of order $o(N^2)$. Consequently, when divided by N^2 this term goes to zero. By applying the ergodic theorem again, we obtain that

$$J_{ij} = \lim_{N\to\infty} \frac{1}{N^2} \sum_{n\in\mathcal{I}(N)} \int_{\Gamma_n} (w^-_{e_i} - w^+_{e_i})(y, \cdot) n_j\, ds(y) = \mathbb{E} \int_{\partial\Phi(Y^-)} (w^-_{e_i} - w^+_{e_i})(y, \cdot) n_j\, ds(y). \tag{7.22}$$

In the next subsection, we investigate this integral further by deriving a formal representation for the jump $w^+_{e_i} - w^-_{e_i}$ in the case when the inclusions are dilute, *i.e.*, small and far away from each other.

To model the dilute suspension, we assume that the reference cell Y^- is of the form of ρB, where B is a domain of unit length scale, and $\rho \ll 1$ denotes the small length scale of the dilute inclusions. Due to the assumptions (2.17) and (2.18), the length scale of the cell $\Phi(Y^-)$ is still of order ρ. Further, due to the assumption (2.19), the distance of the cell $\Phi(Y^-)$ from the "boundary" $\partial\Phi(Y)$ is of order one, which is much larger than the size of the inclusion.

Since the distances between the inclusions are much larger than their sizes, we may use the single inclusion approximation. That is, $w^\pm_{e_i}$ can be approximated by the solutions to the following interface problem:

$$\begin{cases} \nabla \cdot k_0 \nabla w^\pm_{e_i} = 0 \text{ in } \Phi(Y^-) \text{ and } \mathbb{R}^2 \setminus \Phi(Y^-), \\ \dfrac{\partial w^+_{e_i}}{\partial n} = \dfrac{\partial w^-_{e_i}}{\partial n}, \text{ and } w^+_{e_i} - w^-_{e_i} = \rho\beta k_0 (\dfrac{\partial w^-_{e_i}}{\partial n} + n \cdot e_i) \text{ on } \Phi(\Gamma), \\ w^+_{e_i} \to 0 \text{ at } \infty. \end{cases}$$

Here, $\Phi(\Gamma)$ denotes the boundary of the inclusion. Note that the extra ρ in the jump condition is due to the fact that the length scale of the inclusion $\Phi(Y^-)$ is of order

ρ. Using double layer potentials, we represent $w_{e_i}^+$ and $w_{e_i}^-$ as $\mathcal{D}_{\Phi(\Gamma)}[\phi_i]$ restricted to $\Phi(Y^-)$ and $\mathbb{R}^2 \setminus \Phi(Y^-)$ respectively. Due to the trace formula of $\mathcal{D}_{\Phi(\Gamma)}$ and the jump conditions above, the function ϕ_i is determined by

$$-\phi_i = \rho\beta k_0 \left(\frac{\partial \mathcal{D}_{\Phi(\Gamma)}[\phi_i]}{\partial n} + n_i\right). \tag{7.23}$$

Let us define the operator $\mathcal{L}_{\Phi(\Gamma)}$ by $\frac{\partial \mathcal{D}_{\Phi(\Gamma)}}{\partial n}$, then we have that

$$w_{e_i}^+ - w_{e_i}^- = -\phi_i = \rho\beta k_0 (I + \rho\beta k_0 \mathcal{L}_{\Phi(\Gamma)})^{-1}[n_i], \quad \text{on } \Phi(\Gamma).$$

As a consequence, we have also that

$$J_{ij} \simeq -\rho\beta k_0 \mathbb{E} \int_{\Phi(\Gamma)} (I + \rho\beta k_0 \mathcal{L}_{\Phi(\Gamma)})^{-1}[n_i] n_j ds.$$

Let us define ψ_i to be $-(I + \rho\beta k_0 \mathcal{L}_{\Phi(\Gamma)})^{-1}[n_i]$, that is $\psi_i + \rho\beta k_0 n \cdot \nabla \mathcal{D}_{\Phi(\Gamma)}[\psi_i](x) = -n_i$. Define the scaled function $\tilde{\psi}_i(\tilde{x}) = \psi_i(\rho\tilde{x})$ on the scaled curve $\rho^{-1}\Phi(\Gamma)$. Using the homogeneity of the gradient of the Newtonian potential, we verify that

$$\mathcal{D}_{\Phi(\Gamma)}[\psi_i](x) = \mathcal{D}_{\rho^{-1}\Phi(\Gamma)}[\tilde{\psi}_i](\tilde{x}), \quad \text{and} \quad \rho n \cdot \nabla \mathcal{D}_{\Phi(\Gamma)}[\psi_i](x) = n \cdot \nabla \mathcal{D}_{\rho^{-1}\Phi(\Gamma)}[\tilde{\psi}_i](\tilde{x}),$$

where $\tilde{x} = \rho^{-1}x$. This shows that $\tilde{\psi}_i = -(I + \beta k_0 \mathcal{L}_{\rho^{-1}\Phi(\Gamma)})^{-1}[n_i]$. Using the change of variable $y \to \rho\tilde{y}$ in the previous integral representation of J_{ij}, we rewrite it as

$$J_{ij} \simeq \rho\beta k_0 \mathbb{E} \int_{\rho^{-1}\Phi(\Gamma)} \psi_i(\rho\tilde{y}) n_j ds(\rho\tilde{y}) = \rho^2 \beta k_0 \mathbb{E} \int_{\rho^{-1}\Phi(\Gamma)} \tilde{\psi}_i n_j ds(\tilde{y}).$$

Finally, the approximation (2.27) of the effective permittivity for the dilute suspension holds, where $f = \varrho\rho^2$ is the volume fraction where ϱ accounts for the averaged change of volume due to the random diffeomorphism; the polarization matrix M is defined by (2.28) and is associated to the deformed inclusion scaled to the unit length scale. Note that the imaging approach developed in subsection 6.2 can be applied here as well.

8 Numerical simulations

We present in this section some numerical simulations to illustrate the fact that the Debye relaxation times are characteristics of the microstructure of the tissue.

We use for the different parameters the following realistic values:

- the typical size of eukaryotes cells: $\rho \simeq 10 - 100\ \mu$m;
- the ratio between the membrane thickness and the size of the cell: $\delta/\rho = 0.7 \cdot 10^{-3}$;
- the conductivity of the medium and the cell: $\sigma_0 = 0.5$ S.m^{-1};
- the membrane conductivity: $\sigma_m = 10^{-8}$ S.m^{-1};

- the permittivity of the medium and the cell: $\epsilon_0 = 90 \times 8.85 \cdot 10^{-12}$ F.m^{-1};
- the membrane permittivity: $\varepsilon_m = 3.5 \times 8.85 \cdot 10^{-12}$ F.m^{-1};
- the frequency: $\omega \in [10^4, 10^9]$ Hz.

Note that the assumptions of our model $\delta \ll \rho$ and $\sigma_m \ll \sigma_0$ are verified.

We first want to retrieve the invariant properties of the Debye relaxation times. We consider (Figure 8.1) an elliptic cell (in green) that we translate (to obtain the red one), rotate (to obtain the purple one) and scale (to obtain the dark blue one). We compute the membrane polarization tensor, its imaginary part, and the associated eigenvalues which are plotted as a function of the frequency (Figure 8.2). The frequency is here represented on a logarithmic scale. One can see that for the two eigenvalues the maximum of the curves occurs at the same frequency, and hence that the Debye relaxation times are identical for the four elliptic cells. Note that the red and green curves are even superposed; this comes from the fact that M is invariant by translation.

Next, we are interested in the effect of the shape of the cell on the Debye relaxation times. We consider for this purpose, (Figure 8.3) a circular cell (in green), an elliptic cell (in red) and a very elongated elliptic cell (in blue). We compute similarly as in the preceding case, the polarization tensors associated to the three cells, take their imaginary part and plot the two eigenvalues of these imaginary parts with respect to the frequency. As shown in Figure 8.4, the maxima occur at different frequencies for the first and second eigenvalues. Hence, we can distinguish with the Debye relaxation times between these three shapes.

Finally, we study groups of one (in green), two (in blue) and three cells (in red) in the unit period (Figure 8.5) and the corresponding polarization tensors for the homogenized media. The associated relaxation times are different in the three configurations (Figure 8.6) and hence can be used to differentiate tissues with different cell density or organization.

These simulations prove that the Debye relaxation times are characteristics of the shape and organization of the cells. For a given tissue, the idea is to obtain by spectroscopy the frequency dependence spectrum of its effective admittivity. One then has access to the membrane polarization tensor and the spectra of the eigenvalues of its imaginary part. One compares the associated Debye relaxation times to the known ones of healthy and cancerous tissues at different levels. Then one would be able to know using statical tools with which probability the imaged tissue is cancerous and at which level.

9 Concluding remarks

In this paper we derived new formulas for the effective admittivity of suspensions of cells and characterized their dependance with respect to the frequency in terms of membrane polarization tensors. We applied the formulas in the dilute case to image suspensions of cells from electrical boundary measurements. We presented numerical results to illustrate the use of the Debye relaxation time in classifying microstructures. We also developed a selective spectroscopic imaging approach. We showed that specifying the pulse shape in terms of the relaxation times of the dilute suspensions gives rise to selective imaging.

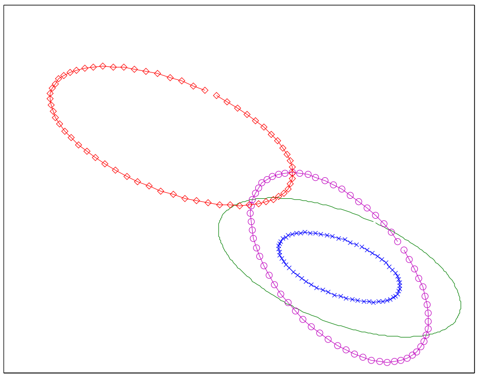

Figure 8.1: *An ellipse translated, rotated and scaled.*

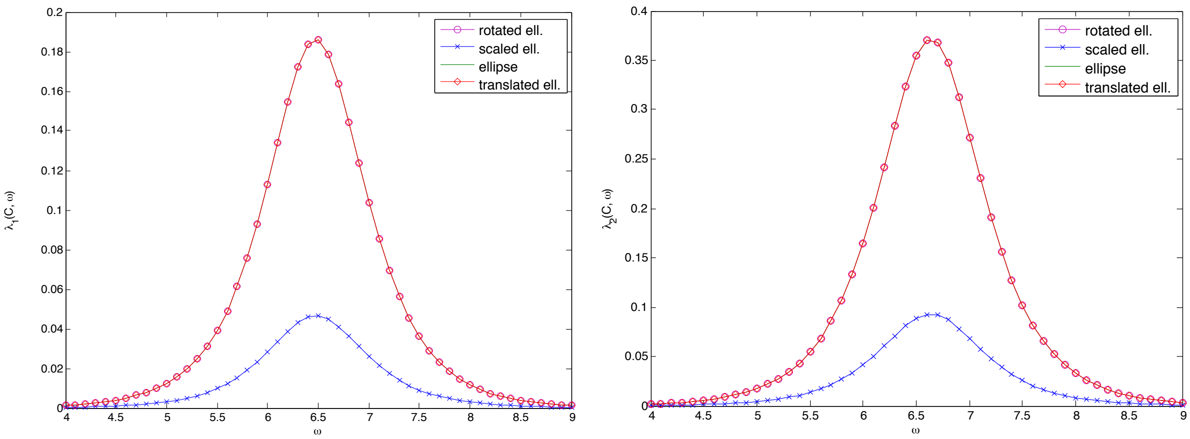

Figure 8.2: *Frequency dependence of the eigenvalues of* $\Im M$ *for the* 4 *ellipses in Figure 8.1.*

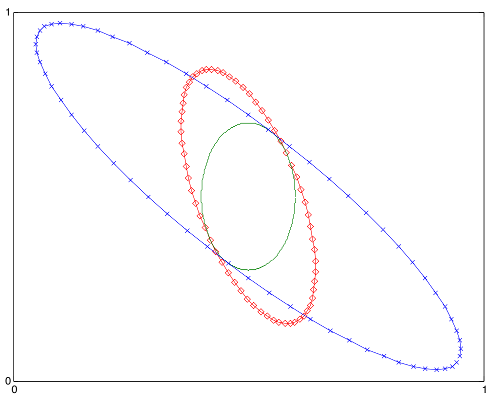

Figure 8.3: *A circle, an ellipse and a very elongated ellipse.*

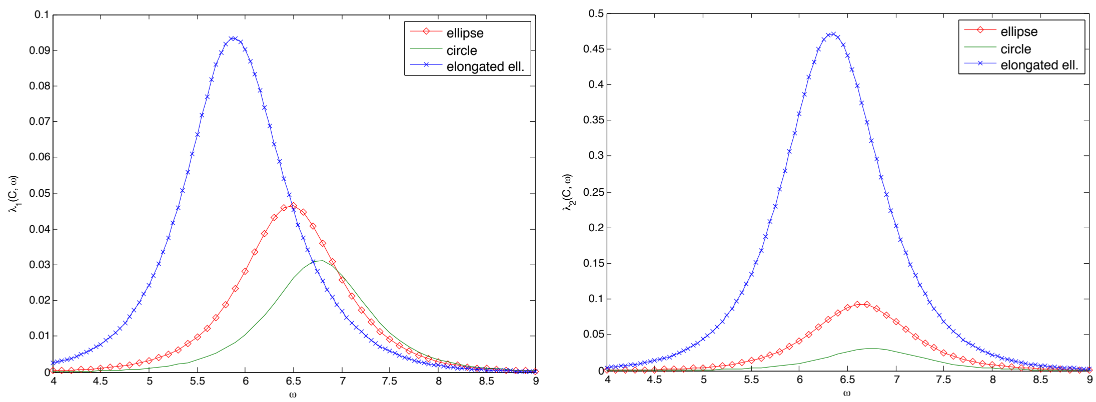

Figure 8.4: *Frequency dependence of the eigenvalues of $\Im M$ for the 3 different cell shapes in Figure 8.3.*

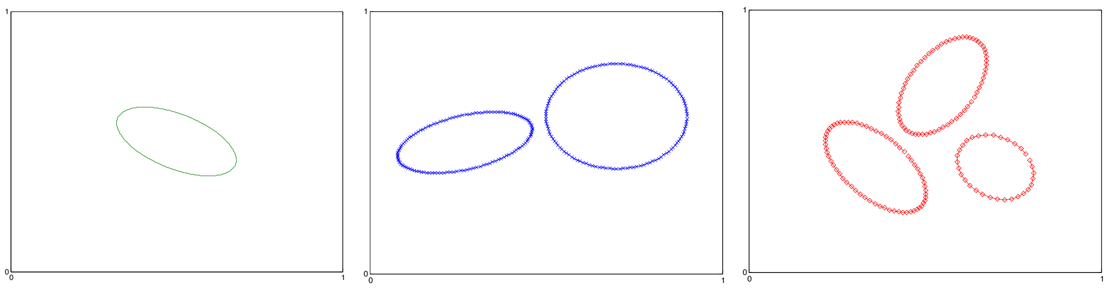

Figure 8.5: *Groups of one, two and three cells.*

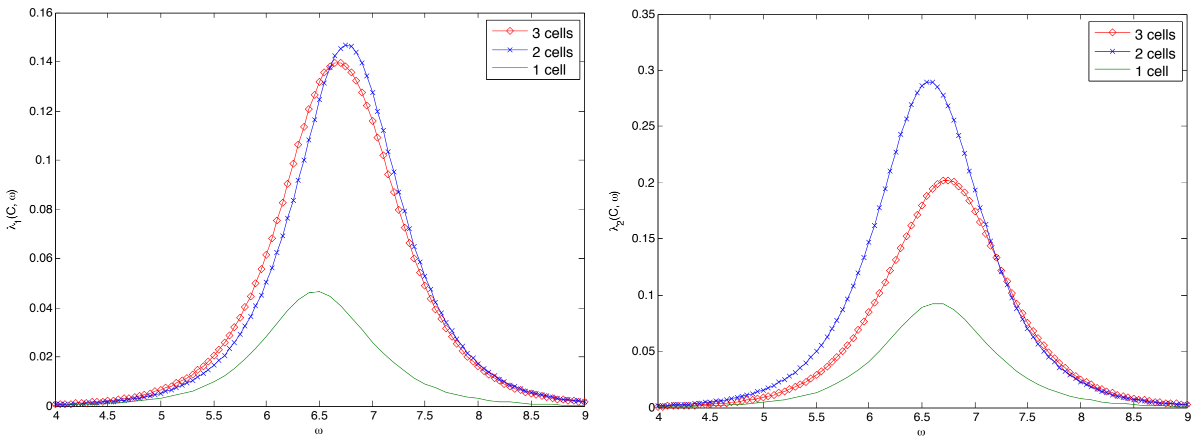

Figure 8.6: *Frequency dependence of the eigenvalues of $\Im M$ in the 3 different cases.*

A challenging problem is to extend our results to elasticity models of the cell. In [10, 8], formulas for the effective shear modulus and effective viscosity of dilute suspensions of elastic inclusions were derived. On the other hand, it was observed experimentally that the dependance of the viscosity of a biological tissue with respect to the frequency characterizes the microstructure [16, 24]. A mathematical justification and modeling for this important finding are under investigation and would be the subject of a forthcoming paper.

A Extension lemmas

Due to the problem settings of this paper, we need to study convergence properties of functions that are defined on the multiple connected sets $\mathbb{R}^+_2$, $\Phi(\mathbb{R}^+_2)$ and $\varepsilon\Phi(\mathbb{R} + d^-)$. Extension operators becomes useful to treat such functions.

Consider two open sets $U, V \subset \mathbb{R}^2$ with the relation $U \subset V$, and two Sobolev spaces $W^{1,p}(U)$ and $W^{1,p}(V)$, $p \in [1, \infty]$. What we call an *extension operator* is a bounded linear map $P : W^{1,p}(U) \to W^{1,p}(V)$, such that $Pu = u$ a.e. on U for all $u \in W^{1,p}(U)$. In this section, we introduce several extension operators of this kind that are needed in the paper. They extend functions that are defined on Y^-, $\mathbb{R}^+_2$, $\Phi(\mathbb{R}^+_2)$ and $\varepsilon\Phi(\mathbb{R}^+_2)$ (hence Ω^+_ε) respectively.

Throughout this section, the short hand notion $\mathcal{M}_A(f)$ for a measurable set $A \subset \mathbb{R}^2$ with positive volume and a function $f \in L^1(A)$ denotes the mean value of f in A, that is

$$\mathcal{M}_A(f) = \frac{1}{|A|} \int_A f(x)dx. \tag{A.1}$$

We start with an extension operator inside the unit cube Y. Since Y^- has smooth boundary, there exists an extension operator $S : W^{1,p}(Y^+) \to W^{1,p}(Y)$ such that for all $f \in W^{1,p}(Y^+)$ and $p \in [1, \infty)$,

$$\|Sf\|_{L^p(Y)} \le C\|f\|_{L^p(Y^+)}, \quad \|Sf\|_{W^{1,p}(Y)} \le C\|f\|_{W^{1,p}(Y^+)}, \tag{A.2}$$

where C only depends on p and Y^-. Such an S is given in [27, section 5.4], where the second estimate above is given; the first estimate easily follows from their construction as well. Cioranescu and Saint Paulin [22] constructed another extension operator which refines the second estimate above. For the reader's convenience, we state and prove their result in the following. Similar results can be found in [34] as well.

Theorem A.1. *Let Y, Y^+ and Y^- be as defined in section 2; in particular, ∂Y^- is smooth. Then there exists an extension operator $P : H^1(Y^+) \to H^1(Y)$ satisfying that for any $f \in H^1(Y^+)$ and $p \in [1, \infty)$,*

$$\|\nabla Pf\|_{L^p(Y)} \le C\|\nabla f\|_{L^p(Y^+)}, \quad \|Pf\|_{L^p(Y)} \le C\|f\|_{L^p(Y^+)}, \tag{A.3}$$

where C only depends on the dimension and the set Y^-.

Proof. Recall the mean operator $\mathcal{M}$ in (A.1) and the extension operator S in (A.2). Given f, we define Pf by

$$Pf = \mathcal{M}_{Y^+}(f) + S(f - \mathcal{M}_{Y^+}(f)). \tag{A.4}$$

Then by setting $\psi = f - \mathcal{M}_{Y^+}(f)$, we have that

$$\|\nabla Pf\|_{L^p(Y)} = \|\nabla S\psi\|_{L^p(Y)} \le C\|\psi\|_{W^{1,p}(Y^+)} \le C\|\nabla\psi\|_{L^p(Y^+)} = C\|\nabla f\|_{L^p(Y^+)}.$$

In the second inequality above, we used the Poincaré–Wirtinger inequality for ψ and the fact that ψ is mean-zero on Y^+. The L^2 bound of Pf follows from the observation

$$\|\mathcal{M}_{Y^+}(f)\|_{L^p(Y)} \le \left(\frac{|Y|}{|Y^+|}\right)^{\frac{1}{p}} \|f\|_{L^p(Y^+)}$$

and the L^p estimate of Sf in (A.2). This completes the proof. □

Apply the extension operator on each translated cubes in $\mathbb{R}^+_2$, we get the following.

Corollary A.1. *Recall the definition of Y_n, Y_n^+ and Y_n^- in section 2. Abuse notations and define*

$$(Pu)|_{Y_n} = P(u|_{Y_n^+}), \quad n \in \mathbb{Z}^2, u \in W^{1,p}_{\text{loc}}(\mathbb{R}^+_2). \tag{A.5}$$

Then P is an extension operator from $W^{1,p}_{\text{loc}}(\mathbb{R}^+_2)$ to $W^{1,p}_{\text{loc}}(\mathbb{R}^2)$. Further, with the same positive constant C in (A.3) *and for any $n \in \mathbb{Z}^2$, we have*

$$\|\nabla Pu\|_{L^p(Y_n)} \le C\|\nabla u\|_{L^p(Y_n^-)}, \quad \|Pu\|_{L^p(Y_n)} \le C\|u\|_{L^p(Y_n^-)}. \tag{A.6}$$

Given a diffeomorphism, the extension operator P can be transformed as follows. In the same manner, under the map of scaling, the extension operator is naturally defined.

Corollary A.2. *Let $\Phi(\cdot,\gamma)$ be a random diffeomorphism satisfying* (2.17) *and* (2.18)*. Denote the inverse function Φ^{-1} by Ψ. Define P_γ as*

$$P_\gamma u = [P(u \circ \Phi)] \circ \Psi, \quad u \in W^{1,p}_{\text{loc}}(\Phi(\mathbb{R}^+_2)). \tag{A.7}$$

Then P_γ is an extension operator from $W^{1,p}_{\text{loc}}(\Phi(\mathbb{R}^+_2))$ to $W^{1,p}_{\text{loc}}(\Phi(\mathbb{R}^2))$ which satisfies that

$$\|\nabla P_\gamma u\|_{L^p(\Phi(Y_n))} \le C\|\nabla u\|_{L^p(\Phi(Y_n^-))}, \quad \|P_\gamma u\|_{L^p(\Phi(Y_n))} \le C\|u\|_{L^p(\Phi(Y_n^-))}, \tag{A.8}$$

where the constant C depends further on the constants in (2.17) *and* (2.18)*.*

Corollary A.3. *Let $\Phi(\cdot,\gamma)$ and Ψ be as above. For each $\varepsilon > 0$, define P^ε_γ as follows: for any $u \in W^{1,p}_{\text{loc}}(\varepsilon\Phi(\mathbb{R}^+_2))$, $P^\varepsilon_\gamma u$ is defined on each deformed and scaled cube $\varepsilon\Phi(Y_n)$ by*

$$P^\varepsilon_\gamma u(x) = \varepsilon P\tilde{u}(\Psi(\frac{x}{\varepsilon})), \tag{A.9}$$

where $\tilde{u} = \varepsilon^{-1} u \circ \varepsilon\Phi$ and P is as in (A.6)*. Then P^ε_γ is an extension operator from $W^{1,p}_{\text{loc}}(\varepsilon\Phi(\mathbb{R}^+_2))$ to $W^{1,p}_{\text{loc}}(\varepsilon\Phi(\mathbb{R}^2))$ which satisfies that for any $n \in \mathbb{Z}^2$,*

$$\|\nabla P^\varepsilon_\gamma u\|_{L^p(\varepsilon\Phi(Y_n))} \le C\|\nabla u\|_{L^p(\varepsilon\Phi(Y_n^-))}, \quad \|P^\varepsilon_\gamma u\|_{L^p(\varepsilon\Phi(Y_n))} \le C\|u\|_{L^p(\varepsilon\Phi(Y_n^-))}, \tag{A.10}$$

where the constant C depends on the same parameters as stated below (A.8)*.*

Proof. We focus on proving (A.10). Under the change of variable $x = \varepsilon\Phi(y)$, we have

$$\nabla_x P^\varepsilon_\gamma u(x) = \nabla\Psi(\frac{x}{\varepsilon})\nabla_y P\tilde{u}(\Phi^{-1}(\frac{x}{\varepsilon})) = \nabla\Psi(\Phi(y))\nabla_y P\tilde{u}(y),$$

On each deformed and scaled cube $\varepsilon\Phi(Y_n)$, we calculate

$$\begin{aligned}\|\nabla P^\varepsilon_\gamma u\|^p_{L^p(\varepsilon\Phi(Y_n))} &= \int_{\varepsilon\Phi(Y_n)} |\nabla_x P^\varepsilon_\gamma u(x)|^p dx = \int_{Y_n} |\nabla\Psi(\Phi(y))\nabla_y P\tilde{u}(y)|^p \varepsilon^2 \det(\nabla\Phi(y))dy \\ &\le \varepsilon^2 \int_{Y_n} |\nabla\Psi(\Phi(y))|^p |\nabla_y P\tilde{u}(y)|^p \det(\nabla\Phi(y))dy \le C\varepsilon^2 \int_{Y_n} |\nabla_y P\tilde{u}(y)|^p dy.\end{aligned}$$

Here, we have used the Cauchy–Schwarz inequality and the bounds (2.17)-(2.18) on the Jacobian matrix and its determinant. Upon applying (A.3), we get

$$\|\nabla P^\varepsilon_\gamma u\|^p_{L^p(\varepsilon\Phi(Y_n))} \le C\varepsilon^2 \|\nabla_y \tilde{u}\|^p_{L^2(Y_n^+)}.$$

Since $\tilde{u}(y) = \frac{1}{\varepsilon}u(\varepsilon\Phi(y))$, we have $\nabla_y\tilde{u}(y) = \nabla_y\Phi(y)\nabla_x u(\varepsilon\Phi(y))$. Change variables in the last integral and repeat the analysis above to get

$$\|\nabla_y \tilde{u}\|^p_{L^p(Y_n^+)} \le C\varepsilon^{-d}\|\nabla_x u\|^p_{L^p(\varepsilon\Phi(Y_n^+))}.$$

Combining the above estimates, one finds some C independent of ε or γ such that (A.10) holds. Moreover, the constant C is uniform for all $\varepsilon\Phi(Y_n)$. The L^2 estimate for $P^\varepsilon_\gamma u$ is simpler and ignored. This completes the proof. □

Finally, we define the extension operator from $W^{1,p}(\Omega^+_\varepsilon)$ to $W^{1,p}(\Omega)$. This is essentially the same operator in Corollary A.3. Indeed, recall that Ω is decomposed to the cushion K_ε and the cell containers E_ε; see (2.21). We only need to apply P^ε_γ in E_ε.

Theorem A.2. *Let the domains $\Omega^\pm_\varepsilon$, K_ε and E_ε be as defined in section 2. Let $\Phi(\cdot,\gamma)$ be a random diffeomorphism satisfying* (2.17)-(2.19). *Define the linear operator P^ε_γ as follows: for $u \in W^{1,p}(\Omega^+_\varepsilon)$, let $P^\varepsilon_\gamma u$ be given by* (A.9) *in E_ε, and let $P^\varepsilon_\gamma u = u$ in K_ε. Then P^ε_γ is an extension operator from $W^{1,p}(\Omega^+_\varepsilon)$ to $W^{1,p}(\Omega)$ and it satisfies*

$$\|\nabla P^\varepsilon_\gamma u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega^+_\varepsilon)}, \quad \|P^\varepsilon_\gamma u\|_{L^p(\Omega)} \le C\|u\|_{L^p(\Omega^+_\varepsilon)}, \tag{A.11}$$

where the constants C's do not depend on ε or γ.

Proof. Since P^ε_γ leaves u unchanged in K_ε and it satisfies the estimates (A.10) uniformly in the cubes $E_\varepsilon = \cup_{n\in\mathcal{I}_\varepsilon}\varepsilon\Phi(Y_n)$, we have the following:

$$\begin{aligned}\|\nabla P^\varepsilon_\gamma u\|^p_{L^p(\Omega)} &= \|\nabla f\|^p_{L^p(K_\varepsilon)} + \sum_{n\in\mathcal{I}_\varepsilon}\|\nabla P^\varepsilon_\gamma u\|^p_{L^p(\varepsilon\Phi(Y_n))} \\ &\le \|\nabla u\|^p_{L^p(K_\varepsilon)} + C\sum_{n\in\mathcal{I}_\varepsilon}\|\nabla u\|^p_{L^p(\varepsilon\Phi(Y_n^+))} \le C\|\nabla u\|^p_{L^p(\Omega^+_\varepsilon)}.\end{aligned}$$

This completes the proof of the first estimate in (A.11). The second estimate follows in the same manner, completing the proof. □

B Poincaré–Wirtinger inequality

Our next goal is to derive a Poincaré–Wirtinger inequality for functions in $H^1(\Omega_\varepsilon^+)$ with a constant independent of ε and γ. The following fact of the fluctuation of a function is useful.

Lemma B.1. *Let $X \subset \mathbb{R}^2$ be an open bounded domain with positive volume and $f \in L^1(X)$. Assume that $X_1 \subset X$ is a subset with positive volume, then we have*

$$\|f - \mathcal{M}_{X_1}(f)\|_{L^2(X_1)} \le \|f - \mathcal{M}_X(f)\|_{L^2(X)}. \tag{B.1}$$

Proof. To simplify notations, let f_1 be the restriction of f on X_1, $m_1 = \mathcal{M}_{X_1}(f_1)$ and $\theta_1 = |X_1|/|X|$. Similarly, let f_2 be the restriction of f on $X_2 = X \setminus X_1$, $m_2 = \mathcal{M}_{X_2}(f_2)$. Let $m = \mathcal{M}_X(f)$. Then we have that

$$f - m = \begin{cases} f_1 - m_1 + (1-\theta)(m_1 - m_2), & x \in X_1, \\ f_2 - m_2 + \theta(m_2 - m_1), & x \in X_2. \end{cases}$$

Then basic computation plus the observation that $f_i - m_i$ integrates to zero on X_i for $i = 1, 2$ yield the following:

$$\|f - m\|^2_{L^2(X)} = \|f_1 - m_1\|^2_{L^2(X_1)} + \|f_2 - m_2\|^2_{L^2(X_2)} + (1-\theta)\theta|X|(m_2 - m_1)^2.$$

Since the items on the right-hand side are all non-negative, we obtain (B.1). □

Corollary B.1. *Assume the same conditions as in Theorem A.2. Then for any $u \in H^1_{\mathbb{C}}(\Omega_\varepsilon^+)$, we have that*

$$\|u\|_{L^2(\Omega_\varepsilon^+)} \le C\|\nabla u\|_{L^2(\Omega_\varepsilon^+)}, \tag{B.2}$$

where the constant C does not depend on ε or γ.

Proof. Thanks to Theorem A.2, we extend u to $P^\varepsilon_\gamma u$ which is in $H^1(\Omega)$. Use (B.1) and the fact that $\mathcal{M}_{\Omega_\varepsilon^+}(u) = 0$ to get

$$\|u\|_{L^2(\Omega_\varepsilon^+)} \le \|P^\varepsilon_\gamma u - \mathcal{M}_\Omega(P^\varepsilon_\gamma u)\|_{L^2(\Omega)}.$$

Now apply the standard Poincaré–Wirtinger inequality for functions in $H^1(\Omega)$, and then use (A.11). We get

$$\|P^\varepsilon_\gamma u - \mathcal{M}_\Omega(P^\varepsilon_\gamma u)\|_{L^2(\Omega)} \le C\|\nabla P^\varepsilon_\gamma u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega_\varepsilon^+)}.$$

The constant C depends on Ω and the parameters stated in Theorem A.2 but not on ε or γ. The proof is now complete. □

Another corollary of the extension lemma is that we have the following uniform estimate when taking the trace of $u \in W_\varepsilon$ on the fixed boundary $\partial\Omega$.

Corollary B.2. *Assume the same conditions as in Theorem A.2. Then there exists a constant C depending on Ω and the parameters as stated in Theorem A.2 but independent of ε and γ such that*

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\|\nabla u\|_{L^2(\Omega_\varepsilon^+)}, \tag{B.3}$$

for any $u \in H^1(\Omega_\varepsilon^+)$.

Proof. Thanks to Theorem A.2 we extend u to $P_\gamma^\varepsilon u$ which is in $H^1(\Omega)$. The trace inequality on Ω shows

$$\|P_\gamma^\varepsilon u\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C(\Omega)\|P_\gamma^\varepsilon u\|_{H^1(\Omega)}. \tag{B.4}$$

The desired estimate then follows from (A.11) and (B.2). □

C Equivalence of the two norms on W_ε

In this section, we prove Proposition 3.2 which establishes the equivalence between the two norms on W_ε. We essentially follow [43] where the periodic case was considered. The random deformation setting requires certain modification. The details of such modifications are provided here for the reader's convenience.

The first inequality of the proposition is proved by the following lemma together with the Poincaré–Wirtinger inequality (B.2):

Lemma C.1. *There exists a constant C independent of ε or γ, such that*

$$\|v^\pm\|^2_{L^2(\Gamma_\varepsilon)} \le C(\varepsilon^{-1}\|v^\pm\|^2_{L^2(\Omega_\varepsilon^\pm)} + \varepsilon\|\nabla v^\pm\|^2_{L^2(\Omega_\varepsilon^\pm)}) \tag{C.1}$$

for any $v^+ \in H^1(\Omega_\varepsilon^+)$ and $v^- \in H^1(\Omega_\varepsilon^-)$.

Proof. According to the set-up, the interface Γ_ε consists of $\varepsilon\Phi(\Gamma_i)$ where $i = 1, \cdots, N(\varepsilon)$ are the labels for the deformed cubes $\{\varepsilon\Phi(Y_i)\}$ inside Ω and Γ_i are the corresponding unit scale interfaces.

Let us consider the case of $v^+ \in H^1(\Omega_\varepsilon^+)$; the other case is proved in the same manner. Denote by v_i the restriction of v^+ on the deformed cube $\varepsilon\Phi(Y_i)$. We lift this function to $\tilde{v}_i(y) = v_i(\varepsilon\Phi(y))$ which is now defined on Y_i^+. For this function, we have the trace inequality

$$\|\tilde{v}_i\|^2_{L^2(\Gamma_i)} \le C(\|\tilde{v}_i\|^2_{L^2(Y_i^+)} + \|\nabla\tilde{v}_i\|^2_{L^2(Y_i^+)}). \tag{C.2}$$

Note that this constant depends on the reference shape Y^- but is uniform in i.

On the other hand, because for any $\gamma \in \mathcal{O}$, the diffeomorphism Φ satisfies (2.17) and (2.18), the Lebesgue measures $ds(x)$ on the curve $\varepsilon\Phi(\Gamma_i)$ and $ds(y)$ on Γ_i, which are related by the change of variable $x = \varepsilon\Phi(y)$, satisfy

$$C_1 ds(x) \le \varepsilon ds(y) \le C_2 ds(x)$$

for some constant $C_{1,2}$ which depend only on the constants in the assumptions and Y^- but uniform in ε and γ.

Consequently, we have

$$\|v^+\|^2_{L^2(\Gamma_\varepsilon)} = \sum_{i=1}^{N(\varepsilon)} \int_{\varepsilon\Phi(\Gamma_i)} |v_i(x)|^2 ds(x) \le C\varepsilon \sum_{i=1}^{N(\varepsilon)} \int_{\Gamma_i} |\tilde{v}_i(y)|^2 ds(y).$$

Apply (C.2) and change the variable back; use again $dx \sim \varepsilon^2 dy$ and $\nabla_y \tilde{v}_i = \varepsilon \nabla_x v_i$ to get

$$\begin{aligned}\|v^+\|^2_{L^2(\Gamma_\varepsilon)} &\le C\varepsilon \sum_{i=1}^{N(\varepsilon)} \int_{Y_i^+} |\tilde{v}_i(y)|^2 + |\nabla_y \tilde{v}(y)|^2 dy \\ &\le C\varepsilon^{-1} \sum_{i=1}^{N(\varepsilon)} \int_{\varepsilon\Phi(Y_i^+)} |v_i(x)|^2 + \varepsilon^2 |\nabla v(x)|^2 dx\end{aligned}$$

This completes the proof of (C.1). □

The other inequality in (3.10) is implied by the following lemma:

Lemma C.2. *There exists a constant $C > 0$ independent of ε or γ such that*

$$\|v\|_{L^2(\Omega_\varepsilon^-)} \le C\left(\sqrt{\varepsilon}\|v\|_{L^2(\Gamma_\varepsilon)} + \varepsilon\|\nabla v\|_{L^2(\Omega_\varepsilon^-)}\right) \tag{C.3}$$

for all $v \in H^1(\Omega_\varepsilon^-)$.

Proof. We first observe that on the reference cube Y with reference cell Y^-, we have that

$$\|v\|^2_{L^2(Y^-)} \le C\left(\|v\|^2_{L^2(\Gamma_0)} + \|\nabla v\|^2_{L^2(Y^-)}\right), \tag{C.4}$$

for any $v \in H^1(Y^-)$ where C only depends on Y^- and the dimension. Indeed, suppose otherwise, we could find a sequence $\{v_n\} \subset H^1(Y^-)$ such that $\|v_n\|_{L^2(Y^-)} \equiv 1$ but

$$\|v_n\|_{L^2(\Gamma_0)} + \|\nabla v_n\|_{L^2(Y^-)} \longrightarrow 0, \quad \text{as } n \to \infty.$$

Then since $\|v_n\|_{H^1}$ is uniformly bounded, there exists a subsequence, still denoted as $\{v_n\}$, and a function $v \in H^1(Y^-)$ such that

$$v_n \rightharpoonup v \text{ weakly in } H^1(Y^-), \quad \nabla v_n \rightharpoonup \nabla v \text{ weakly in } L^2(Y^-).$$

Consequently, $\|\nabla v\|_{L^2} \le \liminf \|\nabla v_n\|_{L^2} = 0$, which implies that $v = C$ for some constant. Moreover, since the embedding $H^1(Y^-) \hookrightarrow L^2(\Gamma_0)$ is compact, the convergence $v_n \to v$ holds strongly in $L^2(\Gamma_0)$ and $\|v\|_{L^2(\Gamma)} \le \lim \|v_n\|_{L^2(\Gamma_0)} = 0$. Consequently $v \equiv 0$. On the other hand, $v_n \to v$ holds strongly in $L^2(Y^-)$ and hence $\|v\|_{L^2(Y^-)} = \lim \|v_n\|_{L^2(Y^-)} = 1$. This contradicts with the fact that $v \equiv 0$.

To prove (C.3), we lift functions in $\varepsilon\Phi(Y_i^-)$ to functions in Y_i^- as in the proof of the previous lemma, and use the scaling relations of the measures: $dx \sim \varepsilon^2 dy$ and $ds(x) \sim \varepsilon ds(y)$. We calculate

$$\|v\|^2_{L^2(\Omega_\varepsilon^-)} = \sum_{i=1}^{N(\varepsilon)} \int_{\varepsilon\Phi(Y_i^-)} |v|^2 dx \le C\varepsilon^2 \int_{Y^-} |\tilde{v}|^2 dy \le C\varepsilon^2 \sum_{i=1}^{N(\varepsilon)} \int_{\Gamma_i} |\tilde{v}|^2 ds + \int_{Y_i^-} |\nabla \tilde{v}|^2 dy$$

where in the last inequality we used (C.4). Change the variables back to get

$$\|v\|^2_{L^2(\Omega_\varepsilon^-)} \le C\varepsilon^2 \sum_{i=1}^{N(\varepsilon)} \int_{\varepsilon\Phi(\Gamma_i)} \varepsilon^{-d+1}|v|^2 ds + \int_{\varepsilon\Phi(Y_i^-)} \varepsilon^{-d+2}|\nabla v|^2 dy.$$

Note that we used again $\nabla_y \tilde{v} = \varepsilon \nabla_x v$. The above inequality is precisely (C.3). □

Proof of Proposition 3.2. To prove the first inequality, we apply Lemma C.1 to get

$$\begin{aligned}\varepsilon\|u^+ - u^-\|^2_{L^2(\Gamma_\varepsilon)} &\leq 2(\varepsilon\|u^+\|^2_{L^2(\Gamma_\varepsilon)} + \|u^-\|^2_{L^2(\Gamma_\varepsilon)}) \\ &\leq C(\|u^+\|^2_{L^2(\Omega_\varepsilon^+)} + \|u^-\|^2_{L^2(\Omega_\varepsilon^-)} + \varepsilon^2\|\nabla u^+\|^2_{L^2(\Omega_\varepsilon^+)} + \varepsilon^2\|\nabla u^+\|^2_{L^2(\Omega_\varepsilon^+)}).\end{aligned}$$

Only the first term in (B.2) does not show in $\|\cdot\|_{H^1_{\mathbb{C}}\times H^1}$, but it is controlled by $\|\nabla u^+\|_{L^2(\Omega_\varepsilon^+)}$ uniformly in ε and γ thanks to (B.2).

For the second inequality, we only need to control $\|u^-\|_{L^2(\Omega_\varepsilon^-)}$. We apply Lemma C.2 and the triangle inequality:

$$\|u^-\|^2_{L^2(\Omega_\varepsilon^-)} \leq C\left(\varepsilon\|u^+\|^2_{L^2(\Gamma_\varepsilon)} + \varepsilon\|u^+ - u^-\|^2_{L^2(\Gamma_\varepsilon)} + \varepsilon^2\|\nabla u^-\|^2_{L^2(\Omega_\varepsilon^-)}\right).$$

Only the first term does not appear in $\|\cdot\|_{W_\varepsilon}$, but using Lemma C.1 and (B.2) we can bound it by

$$\varepsilon\|u^+\|^2_{L^2(\Gamma_\varepsilon)} \leq C(\|u^+\|^2_{L^2(\Omega_\varepsilon^+)} + \varepsilon^2\|\nabla u^+\|^2_{L^2(\Omega_\varepsilon^+)}) \leq C\|\nabla u^+\|^2_{L^2(\Omega_\varepsilon^+)}.$$

This completes the proof. □

D Technical lemma

Lemma D.1. *Let φ_1 be a function in $\mathcal{D}(\Omega, C^\infty_\sharp(Y^+)) \times \mathcal{D}(\Omega, C^\infty_\sharp(Y^-))$. There exists at least one function θ in $(\mathcal{D}(\Omega, H^1_\sharp(Y^+)) \times \mathcal{D}(\Omega, H^1_\sharp(Y^-)))^2$ solution of the following problem:*

$$\begin{cases} -\nabla_y \cdot \theta^+(x,y) = 0 & \text{in } Y^+, \\ -\nabla_y \cdot \theta^-(x,y) = 0 & \text{in } Y^-, \\ \theta^+(x,y)\cdot n = \theta^-(x,y)\cdot n & \text{on } \Gamma, \\ \theta^+(x,y)\cdot n = \varphi_1^+(x,y) - \varphi_1^-(x,y) & \text{on } \Gamma, \\ y \longmapsto \theta(x,y) Y-\text{periodic}. \end{cases} \tag{D.1}$$

Proof. We look for a solution under the form $\theta = \nabla_y \eta$. We hence introduce the following variational problem:

$$\begin{cases} \text{Find } \eta \in (H^1_\sharp(Y^+)/\mathbb{C}) \times (H^1_\sharp(Y^-)/\mathbb{C}) \text{ such that} \\ \displaystyle\int_{Y^+} \nabla\eta^+(y)\cdot\overline{\psi}^+(y)dy + \int_{Y^-}\nabla\eta^-(y)\cdot\overline{\psi}^-(y)dy \\ \qquad\qquad\qquad\displaystyle = \frac{1}{\beta k_0}\int_\Gamma(\varphi_1^+ - \varphi_1^-)(\overline{\psi}^+ - \overline{\psi}^-)(y)ds(y), \\ \text{for all } \psi \in (H^1_\sharp(Y^+)/\mathbb{C}) \times (H^1_\sharp(Y^-)/\mathbb{C}), \end{cases}$$

for a fixed $x \in \Omega$. Lax-Milgram theorem gives us existence and uniqueness of such an η. Since $\varphi_1 \in \mathcal{D}(\Omega, C^\infty_\sharp(Y^+)) \times \mathcal{D}(\Omega, C^\infty_\sharp(Y^-))$, there exists at least one function $\theta \in (\mathcal{D}(\Omega, H^1_\sharp(Y^+)) \times \mathcal{D}(\Omega, H^1_\sharp(Y^-))^2$ solution of (D.1). Note that we do not have uniqueness of such a solution. □

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