A discrete domain decomposition method for acoustics with uniform exponential rate of convergence using non-local impedance operators

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1 Introduction

We consider the Helmholtz equation in harmonic regime in a domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, and a first order absorbing condition on its boundary Γ with unit outward normal vector $\mathbf{n}$. Let $k \in \mathbb{R}$ be a constant wave number and $f \in L^2(\Omega)$, we seek $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u - k^2 u = f, & \text{in } \Omega, \\ (\partial_{\mathbf{n}} + ik)\, u = 0, & \text{on } \Gamma. \end{cases} \tag{1}$$

In previous works [2, 3, 5], a domain decomposition method (DDM) using non-local transmission operator with suitable properties was described. The relaxed Jacobi algorithm written at the continuous level was proven to converge exponentially. However, it was only a conjecture, hinted at by numerical experiments in [5, Section 8], that the discretized algorithm using finite elements has a rate of convergence *uniformly* bounded with respect to the discretization parameter, hence does not deteriorate when the mesh is refined. In this work we prove this conjecture for the case of Lagrange finite elements. Numerical experiments in [5, Section 8.3] highlighted that this important property is not shared by DDM based on local operators [4] or rational fractions of local operators [1].

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2 DDM algorithm: the continuous case

Impedance based transmission problem. We suppose that the domain is partitioned into two non-overlapping subdomains $\overline{\Omega} = \overline{\Omega}_- \cup \overline{\Omega}_+$. The transmission interface between the subdomains is noted Σ, with a unit normal vector $\mathbf{n}$ oriented from Ω_+ to Ω_-, and we suppose[1] that Σ does not intersect Γ. We then consider the following transmission problem

$$\begin{cases} -\Delta u_\pm - k^2 u_\pm = f|_{\Omega_\pm}, & \text{in } \Omega_\pm, \\ (\pm\partial_{\mathbf{n}} + ik\mathrm{T})\, u_\pm = (\pm\partial_{\mathbf{n}} + ik\mathrm{T})\, u_\mp, & \text{on } \Sigma, \end{cases} \tag{2}$$

with $(\partial_{\mathbf{n}} + ik)\, u_\pm = 0$ on $\Gamma \cap \partial\Omega_\pm$. T is a suitable impedance operator supposed to be injective, positive and self-adjoint so that the coupled problems (2) are well posed and equivalent to the model problem (1), see [2, Th. 3] and [5, Lem. 1].

Reformulation at the interface. Let $V_\pm = H^1(\Omega_\pm)$, $V = V_+ \times V_-$ and $V_\Sigma = H^{-1/2}(\Sigma)$. We define the lifting operator R

$$\mathrm{R} \ : \ V_\Sigma^2 \ni (x_+, x_-) \mapsto (\mathrm{R}_+ x_+, \mathrm{R}_- x_-) \in V. \tag{3}$$

where $u_\pm = \mathrm{R}_\pm x_\pm$ are solutions of the following decoupled boundary value problems

$$-\Delta u_\pm - k^2 u_\pm = 0, \ \text{in } \Omega_\pm, \qquad (\pm\partial_{\mathbf{n}} + ik\mathrm{T})\, u_\pm = x_\pm, \ \text{on } \Sigma, \tag{4}$$

and $(\partial_{\mathbf{n}} + ik)\, u_\pm = 0$ on $\Gamma \cap \partial\Omega_\pm$. We define the scattering operator S

$$\mathrm{S} \ : \ V_\Sigma^2 \ni (x_+, x_-) \mapsto (\mathrm{S}_+ x_+, \mathrm{S}_- x_-) \in V_\Sigma^2, \tag{5}$$

with $\mathrm{S}_\pm x = -x + 2ik\mathrm{T}(\mathrm{R}_\pm x)|_\Sigma$ for $x \in V_\Sigma$. We finally define the operator $\mathrm{A} = \Pi\mathrm{S}$ on V_Σ^2, where Π is an exchange operator: $\Pi\,(x_+, x_-) = (x_-, x_+)$ for a couple of traces $(x_+, x_-) \in V_\Sigma^2$. The following result provides equivalence between the decomposed problem (2) and a problem at the interface (6), see [2, Th. 5] and [5, Prop. 3].

Theorem 1 *If $u = (u_+, u_-) \in V$ is solution of* (2) *then the trace $x = (x_+, x_-) \in V_\Sigma^2$ defined as $x_\pm := (\pm\partial_{\mathbf{n}} + ik\mathrm{T})\, u_\mp|_\Sigma$ is solution of the interface problem*

$$x = Ax + b, \qquad \text{on } \Sigma, \tag{6}$$

where $b = 2ik\,(\mathrm{T}F_-|_\Sigma, \mathrm{T}F_+|_\Sigma) \in V_\Sigma^2$ and $F = (F_+, F_-) \in V$ is such that $-\Delta F_\pm - k^2 F_\pm = f|_{\Omega_\pm}$ in $\Omega_\pm$, $(\pm\partial_{\mathbf{n}} + ik\mathrm{T})\, F_\pm = 0$ on Σ and $(\partial_{\mathbf{n}} + ik)\, F_\pm = 0$ on $\Gamma \cap \partial\Omega_\pm$.

Reciprocally, if $x \in V_\Sigma^2$ is solution of (6)*, then $u = (u_+, u_-) \in V$ defined as $u = Rx + F$ is solution of* (2)*.*

Continuous DDM algorithm. The solution of (2) is computed iteratively using a relaxed Jacobi algorithm on the interface problem (6). From an initial trace $x^0 \in V_\Sigma^2$ and a relaxation parameter $r \in (0, 1)$, iteration n writes,

[1] In the presence of such intersections, the proof fails and as a matter of fact the exponential convergence is not observed numerically.

$$x^n = (1-r)x^{n-1} + r\mathrm{A}x^{n-1} + b. \tag{7}$$

Note that the application of A involves solving the decoupled local problems (4) which can be done in parallel. The previous theorem guarantees that the solution of (7) satisfies (2) at convergence. In the following we assume in addition that

$$\mathrm{T} \;:\; H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma) \quad \text{is a self-adjoint isomorphism.} \tag{8}$$

Only non-local operators, constructed in practice using integral operators with appropriate singular kernels, can fit in this framework. Under those additional assumptions the algorithm (7) converges exponentially, see [2, Th. 7] and [5, Th. 1].

3 DDM algorithm: the discrete setting

We consider two series $(V_{\pm,h})_h$ of finite dimensional subspaces $V_{\pm,h} \subset V_\pm$ conformal at the interface i.e. $V_{\Sigma,h} = \{u_{\pm,h}|_\Sigma \mid u_{\pm,h} \in V_{\pm,h}\} \subset V_\Sigma$. Let $V_h = V_{+,h} \times V_{-,h} \subset V$. We define the sesquilinear form $a_{\widetilde{\Omega}}$ for a domain $\widetilde{\Omega} \in \{\Omega, \Omega_+, \Omega_-\}$: for all $u, u' \in H^1(\widetilde{\Omega})$,

$$a_{\widetilde{\Omega}}(u,u') = (\nabla u, \nabla u')_{L^2(\widetilde{\Omega})} - k^2(u,u')_{L^2(\widetilde{\Omega})} + ik(u,u')_{L^2(\Gamma\cap\partial\widetilde{\Omega})}. \tag{9}$$

By Assumption (8), the transmission operator T induces a continuous and coercive sesquilinear form t on $H^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$ such that

$$t(z,z') = \langle \mathrm{T}z, z'\rangle_\Sigma, \qquad \forall z, z' \in H^{1/2}(\Sigma). \tag{10}$$

Reformulation at the interface. We follow the approach of the continuous setting and define the discrete version R_h of the lifting operator R given in (3) by

$$\mathrm{R}_h \;:\; V_{\Sigma,h}^2 \ni \left(x_{+,h}, x_{-,h}\right) \mapsto \left(\mathrm{R}_{+,h}x_{+,h}, \mathrm{R}_{-,h}x_{-,h}\right) \in V_h. \tag{11}$$

with $\mathrm{R}_{\pm,h}$, the discrete versions of $\mathrm{R}_\pm$ given in (4), such that $u_{\pm,h} = \mathrm{R}_{\pm,h}x_{\pm,h}$ satisfies

$$a_{\Omega_\pm}(u_{\pm,h}, u'_{\pm,h}) + ik\, t(u_{\pm,h}, u'_{\pm,h}) = \langle x_{\pm,h}, u'_{\pm,h}\rangle_\Sigma, \qquad \forall u'_{\pm,h} \in V_{\pm,h}. \tag{12}$$

Similarly, the discrete version S_h of the scattering operator S defined in (5) is

$$\mathrm{S}_h \;:\; V_{\Sigma,h}^2 \ni \left(x_{+,h}, x_{-,h}\right) \mapsto \left(\mathrm{S}_{+,h}x_{+,h}, \mathrm{S}_{-,h}x_{-,h}\right) \in V_{\Sigma,h}^2, \tag{13}$$

with the discrete versions $\mathrm{S}_{\pm,h}$ of $\mathrm{S}_\pm$ are such that: for all $w'_{\pm,h} \in V_{\Sigma,h}$,

$$\langle \mathrm{S}_{\pm,h}x_{\pm,h}, w'_{\pm,h}\rangle_\Sigma = -\langle x_{\pm,h}, w'_{\pm,h}\rangle_\Sigma + 2ik\, t\left(\mathrm{R}_{\pm,h}x_{\pm,h}, w'_{\pm,h}\right). \tag{14}$$

We finally define the discrete operator $\mathrm{A}_h = \Pi\mathrm{S}_h$ on $V_{\Sigma,h}^2$. It can then be proven, in a similar fashion as for the continuous case, that the discretization of the problem (2) is equivalent to a discrete counterpart of the interface problem (6): find $x_h \in V^2_{\Sigma,h}$ such that $x_h = \mathrm{A}_h x_h + b_h$, where b_h is the discrete counterpart of b.

Discrete DDM algorithm. In the following we analyse the convergence of the discretization of the DDM algorithm (7): from an initial trace $x_h^0 \in V^2_{\Sigma,h}$ and for a relaxation parameter $r \in (0,1)$, iteration n writes

$$x_h^n = (1-r)x_h^{n-1} + r\mathrm{A}_h x_h^{n-1} + b_h. \tag{15}$$

4 An abstract uniform exponential convergence result

We now state an abstract result specifying the conditions under which uniform exponential convergence is achieved.

Theorem 2 *If* A_h *is contractant in* $V^2_{\Sigma,h}$ *and* $\mathrm{I} - \mathrm{A}_h$ *is an isomorphism in* $V^2_{\Sigma,h}$ *with* uniformly bounded inverse*, then the relaxed Jacobi algorithm* (15) *with* $r \in (0,1)$ *converges exponentially* uniformly *(C and* τ *are independent of h below):*

$$\exists\, \tau \in (0,1),\ C > 0,\ h_0 > 0, \quad \forall h < h_0,\ n \in \mathbb{N}, \quad \|u_h^n - u_h\|_V \leq C\tau^n. \tag{16}$$

Proof At each iteration n, the surface error $\varepsilon_h^n = x_h^n - x_h$ satisfies

$$\varepsilon_h^{n+1} = (1-r)\varepsilon_h^n + r\mathrm{A}_h\varepsilon_h^n. \tag{17}$$

By hypothesis we have (for some $\delta \in (0,2]$ independent of h)

$$\|\mathrm{A}_h\varepsilon_h^n\|_{V^2_\Sigma} \leq \|\varepsilon_h^n\|_{V^2_\Sigma}, \qquad \text{and} \qquad \|(\mathrm{I}-\mathrm{A}_h)\varepsilon_h^n\|_{V^2_\Sigma} \geq \delta\|\varepsilon_h^n\|_{V^2_\Sigma}. \tag{18}$$

We have the identity for $r \in (0,1)$ and $a, b \in V^2_\Sigma$

$$\|(1-r)a + rb\|^2_{V^2_\Sigma} = (1-r)\|a\|^2_{V^2_\Sigma} + r\|b\|^2_{V^2_\Sigma} - r(1-r)\|a-b\|^2_{V^2_\Sigma}. \tag{19}$$

Using this identity (take $a = \varepsilon_h^n$ and $b = \mathrm{A}_h\varepsilon_h^n$) together with (17) and (18) we get

$$\|\varepsilon_h^{n+1}\|_{V^2_\Sigma} \leq \tau\, \|\varepsilon_h^n\|_{V^2_\Sigma}, \qquad \text{with } \tau = \sqrt{1 - r(1-r)\delta^2}, \tag{20}$$

and where τ is well defined in $\mathbb{R}$ since $\delta \in (0,2]$. Since we have $u_h^n - u_h = \mathrm{R}_h\varepsilon_h^n$, the well-posedness of the local problems yields the existence of a constant $c > 0$ independent of h such that, for h sufficiently small, $\|u_h^n - u_h\|_V \leq c\|\varepsilon_h^n\|_{V^2_\Sigma}$. □

Since T is assumed to be a self-adjoint isomorphism from $H^{1/2}(\Sigma)$ to $H^{-1/2}(\Sigma)$, the contractive nature of A_h and the fact that $\mathrm{I}-\mathrm{A}_h$ is an isomorphism can be proven, see [2, Th. 3 and Lem. 6] and [5, Lem. 2 and 3]. However, the *uniform* boundedness of the inverse of $\mathrm{I} - \mathrm{A}_h$ was recognized as an open question in [5, Rem. 3]. The previous proof highlights that this property is essential to prevent the convergence rate from potentially degenerating (tending to 1 as h goes to 0).

5 An abstract sufficient condition for exponential convergence

The next theorem states that a sufficient condition for the operator $\mathrm{I} - \mathrm{A}_h$ to be an isomorphism with uniformly continuous inverse relies on the existence of two liftings with suitable properties.

Theorem 3 *Assume that there exists two liftings $L_{\pm,h}$ from $V_{\Sigma,h}$ to $V_{\pm,h}$* uniformly *continuous and preserving Dirichlet boundary conditions: namely there exists $c > 0$, independent of h, such that for all $x_{\pm,h} \in V_{\Sigma,h}$,*

$$\left(L_{\pm,h} x_{\pm,h}\right)|_\Sigma = x_{\pm,h}, \quad \textit{and} \quad \|L_{\pm,h} x_{\pm,h}\|_{V_\pm} \leq c\|x_{\pm,h}\|_{V_\Sigma}. \tag{21}$$

Then $\mathrm{I} - \mathrm{A}_h$ is an isomorphism in $V^2_{\Sigma,h}$ with uniformly *bounded inverse (C is independent of h below):*

$$\exists\, C > 0,\ h_0 > 0, \quad \forall h < h_0,\ x_h \in V^2_{\Sigma,h}, \quad \|x_h\|_{V^2_\Sigma} \leq C\, \|(\mathrm{I} - \mathrm{A}_h)x_h\|_{V^2_\Sigma}, \tag{22}$$

To prove this result, we closely follow the lines of the proof in the continuous case, which we recall below. Let $y = (y_+, y_-) \in V^2_\Sigma$ we aim at finding $x = (x_+, x_-) \in V^2_\Sigma$ such that $(\mathrm{I}-\mathrm{A})x = y$. From the definitions of Section 2, this is equivalent to finding $u_\pm \in V_\pm$ and $x_\pm \in V_\Sigma$ such that (omitting the boundary condition on $\Gamma \cap \partial\Omega_\pm$ here and in the following for brevity)

$$\begin{cases} -\Delta u_\pm - k^2 u_\pm = 0, \quad \text{in } \Omega_\pm, \qquad (\pm\partial_{\mathbf{n}} + ik\,T)\, u_\pm = x_\pm, \quad \text{on } \Sigma, \\ x_\pm - (-x_\mp + 2ik\,T u_\mp) = y_\pm, \quad \text{on } \Sigma. \end{cases} \tag{23}$$

Step 1: Definition of two jumps. A key point is to recognize that the property (8) of T allows to define the Dirichlet and Neumann jumps u_D and u_N such that

$$u_D = (ik\,T)^{-1}\ \frac{y_+ - y_-}{2}\ \in H^{1/2}(\Sigma), \qquad u_N = \frac{y_- + y_+}{2}\ \in H^{-1/2}(\Sigma). \tag{24}$$

Step 2: Transmission problem. It is then straightforward to check that the system of equations (23) is equivalent to compute directly $x_\pm = (\pm\partial_{\mathbf{n}} + ik\,T)\, u_\pm$ where $(u_+, u_-) \in V$ is solution of the transmission problem

$$-\Delta u_\pm - k^2 u_\pm = 0,\ \text{in } \Omega_\pm, \qquad u_+ - u_- = u_D, \quad \partial_{\mathbf{n}} u_+ - \partial_{\mathbf{n}} u_- = u_N,\ \text{on } \Sigma. \tag{25}$$

Step 3: Construction of the solution. The solution $u_\pm$ of (25) is sought in the form $u_\pm = u^c|_{\Omega_\pm} + u^d_\pm$ with $(u^d_+, u^d_-) \in V$ (discontinuous across Σ) and $u^c \in H^1(\Omega)$ (continuous across Σ), constructed as follows. We first construct $u^d_\pm$ as the result of two liftings $L_\pm$ from V_Σ to $V_\pm$ such that $u^d_\pm = \pm\frac{1}{2}\mathrm{L}_\pm u_D$. The liftings $L_\pm$ can be obtained for instance by solving a modified (coercive) Helmholtz equation in the local domains. Having found such a $u^d_\pm$ which satisfies by construction $u^d_+ - u^d_- = u_D$, it is clear that $u_\pm = u^c|_{\Omega_\pm} + u^d_\pm$ solves (25) if $u^c \in H^1(\Omega)$ satisfies (writing $u^c_\pm = u^c|_{\Omega_\pm}$)

$$\begin{cases} -\Delta u^c_\pm - k^2 u^c_\pm = \Delta u^d_\pm + k^2 u^d_\pm, & \text{in } \Omega_\pm, \\ \partial_{\mathbf{n}} u^c_+ - \partial_{\mathbf{n}} u^c_- = u_N - \partial_{\mathbf{n}} u^d_+ + \partial_{\mathbf{n}} u^d_-, & \text{on } \Sigma. \end{cases} \tag{26}$$

The problem (26) is well-posed in $H^1(\Omega)$ by application of Fredholm's alternative. The solution $x = (x_+, x_-) \in V^2_\Sigma$ of (23) can finally be computed directly as $x_\pm = (\pm\partial_{\mathbf{n}} + ik\,T)\,u_\pm$.

The proof at the discrete level mimics this procedure but we need to systematically verify at each step that uniform bounds hold. In the following, C denotes a constant possibly taking different values from one inequality to another.

Proof (of Theorem 3) Let $y_h \in V^2_{\Sigma,h}$, using the definitions of Section 3 the problem of finding $x_h \in V^2_{\Sigma,h}$ such that $(\mathrm{I} - \mathrm{A}_h)x_h = y_h$ writes: find $u_{\pm,h} \in V_{\pm,h}$ and $x_{\pm,h} \in V_{\Sigma,h}$ such that, for all $u'_{\pm,h} \in V_{\pm,h}$ and $z'_{\pm,h} \in V_{\Sigma,h}$,

$$\begin{cases} a_{\Omega_\pm}(u_{\pm,h}, u'_{\pm,h}) + ik\,t(u_{\pm,h}, u'_{\pm,h}) = \langle x_{\pm,h}, u'_{\pm,h}\rangle_\Sigma, \\ \langle x_{\pm,h}, z'_{\pm,h}\rangle_\Sigma - \Big(-\langle x_{\mp,h}, z'_{\pm,h}\rangle_\Sigma + 2ik\,t(u_{\mp,h}, z'_{\pm,h})\Big) = \langle y_{\pm,h}, z'_{\pm,h}\rangle_\Sigma. \end{cases} \tag{27}$$

Step 1: Definition of two jumps. Let $v_{D,h}$ and $u_{N,h}$ be such that

$$v_{D,h} := (ik)^{-1}\,\frac{y_{+,h} - y_{-,h}}{2}, \qquad u_{N,h} := \frac{y_{-,h} + y_{+,h}}{2}. \tag{28}$$

Both quantities belong to $V_{\Sigma,h}$ and we have, with C independent of h,

$$\|v_{D,h}\|_{V_\Sigma} \le C\,\|y_h\|_{V^2_\Sigma}, \qquad \|u_{N,h}\|_{V_\Sigma} \le C\,\|y_h\|_{V^2_\Sigma}. \tag{29}$$

Note that $v_{D,h}$ is not the discrete counterpart of u_D. A good candidate would be $u_{D,h} = T_h^{-1} v_{D,h}$ where T_h is a discrete version of T. This leads us to the definition

$$u_{D,h} = T_h^{-1} v_{D,h} \qquad \Leftrightarrow \qquad t(u_{D,h}, z'_h) = \langle v_{D,h}, z'_h\rangle_\Sigma, \quad \forall z'_h \in V_{\Sigma,h}. \tag{30}$$

Since t is supposed to be strictly coercive, such a $u_{D,h}$ exists and it holds, with C independent of h,

$$\|u_{D,h}\|_{H^{1/2}(\Sigma)} \le C\,\|v_{D,h}\|_{V_\Sigma}. \tag{31}$$

Step 2: Transmission problem. The solutions $x_{\pm,h} \in V_{\Sigma,h}$ of (27) must satisfy

$$\langle x_{\pm,h}, u'_{\pm,h}\rangle_\Sigma = a_{\Omega_\pm}(u_{\pm,h}, u'_{\pm,h}) + ik\,t(u_{\pm,h}, u'_{\pm,h}), \quad \forall u'_{\pm,h} \in V_{\pm,h}, \tag{32}$$

where $u_{\pm,h} \in V_{\pm,h}$ must satisfy a discrete version of the transmission problem (25)

$$\begin{cases} t(u_{+,h} - u_{-,h}, z'_h) = \langle v_{D,h}, z'_h\rangle_\Sigma, & \forall z'_h \in V_{\Sigma,h}, \\ a_{\Omega_+}(u_{+,h}, u'_{+,h}) + a_{\Omega_-}(u_{-,h}, u'_{-,h}) = \langle u_{N,h}, u'_h\rangle_\Sigma, & \forall (u'_{+,h}, u'_{-,h}) \in V_h \cap H^1(\Omega). \end{cases} \tag{33}$$

where the equation on the first line is obtained by taking the difference of the two equations in the second line of (27) and the equation on the second line is obtained by summing all equations in (27) with a test function in $V_h \cap H^1(\Omega)$.

Reciprocally, let $u_{\pm,h} \in V_{\pm,h}$ and $x_{\pm,h} \in V_{\Sigma,h}$ be solutions of (33) and (32). For any $z'_h \in V_{\Sigma,h}$, there exists by assumption $u'_h = (u'_{+,h}, u'_{-,h}) \in V_h \cap H^1(\Omega)$ such that $u'_{\pm,h}|_\Sigma = z'_h$. By taking linear combinations of equations of (33) with these test functions z'_h and u'_h one obtains the two equations on the second line of (27).

Step 3: Construction of the solution. The solution $u_{\pm,h}$ of (33) is sought in the form $u_{\pm,h} = u^c_h|_{\Omega_\pm} + u^d_{\pm,h}$ with $(u^d_{+,h}, u^d_{-,h}) \in V_h$ and $u^c_h \in V_h \cap H^1(\Omega)$ constructed as follows. We first construct $u^d_{\pm,h} = \pm\frac{1}{2}\mathrm{L}_{\pm,h} u_{D,h}$, by hypothesis on the liftings we have

$$\|u^d_{\pm,h}\|_{V_\pm} \le C\, \|u_{D,h}\|_{H^{1/2}(\Sigma)}, \tag{34}$$

with C independent of h. By construction $t(u^d_{+,h} - u^d_{-,h}, z'_h) = \langle v_{D,h}, z'_h\rangle_\Sigma$ for all $z'_h \in V_{\Sigma,h}$. Hence, using the last equation in (33), $u_{\pm,h} = u^c_h|_{\Omega_\pm} + u^d_{\pm,h}$ will be solution of (33) if $u^c_h \in V_h \cap H^1(\Omega)$ is such that, for all $(u'_{+,h}, u'_{-,h}) \in V_h \cap H^1(\Omega)$,

$$a_\Omega(u^c_h, u'_h) = \langle u_{N,h}, u'_h\rangle_\Sigma - a_{\Omega_-}(u^d_{-,h}, u'_{-,h}) - a_{\Omega_+}(u^d_{+,h}, u'_{+,h}). \tag{35}$$

Since a_Ω is $H^1(\Omega)$-coercive, it is well known from the theory of Galerkin approximation of Fredholm type problem that for h sufficiently small, such a u^c_h exists and it holds, with C independent of h,

$$\|u^c_h\|_V \le C\left(\|u_{N,h}\|_{V_\Sigma} + \|u^d_{-,h}\|_{V_-} + \|u^d_{+,h}\|_{V_+}\right). \tag{36}$$

From $u_{\pm,h} = u^d_{\pm,h} + u^c_h|_{\Omega_\pm}$ in $V_{\pm,h}$ we have, with C independent of h,

$$\|u_{\pm,h}\|_{V_\pm} \le C\left(\|u^d_{\pm,h}\|_{V_\pm} + \|u^c_h|_{\Omega_\pm}\|_{V_\pm}\right). \tag{37}$$

The solution $x_{\pm,h} \in V_{\Sigma,h}$ of (32) hence (27) are computed using (33) hence satisfy, with C independent of h,

$$\|x_{\pm,h}\|_{V_\Sigma} \le C\, \|u_{\pm,h}\|_{V_\pm}. \tag{38}$$

Since all the quantities computed at each step are bounded uniformly by the data used for their construction, see (29), (31) , (34), (36) and (37), the uniform bound of Theorem 3 with respect to h is established. □

6 Application to finite element approximations

In this section we assume that $\Omega_\pm$ are bounded open polyhedral Lipchitz domains discretized using conforming simplicial mesh elements and consider classical Lagrange finite element spaces. The previous proof relies on the existence of two uniformly stable liftings $L_{\pm,h}$ from $V_{\Sigma,h}$ to $V_{\pm,h}$ which must preserve the Dirichlet trace on Σ. A theoretical construction of such liftings $L_{\pm,h}$ can be obtained as $L_{\pm,h} = P_h \circ L_\pm$ where $P_h : V_\pm \longrightarrow V_{\pm,h}$ is an interpolator and $L_\pm : V_\Sigma \longrightarrow V_\pm$ are two continuous liftings. The construction of $L_{\pm,h}$ is hence reduced to the construction of P_h. The classical Lagrange interpolator fails to provide a practical answer because it lacks the continuity property for non-smooth functions (point-wise function evaluations). The Clément interpolator is continuous but fails to preserve the prescribed trace on the boundary. An interpolator featuring the suitable properties have been proposed by Scott and Zhang [6] for general conforming Lagrange finite elements of any order in $\mathbb{R}^d$, $d = 2, 3$. For the sake of illustration, we briefly recall below the construction of this operator for $\mathbb{P}_1$ Lagrange finite elements on triangles.

For each vertex M_i of the mesh, choose arbitrarily σ_i an edge connected to M_i. The application $v \in \mathbb{P}_1(\sigma_i) \mapsto v(M_i) \in \mathbb{R}$ is a continuous linear form on $\mathbb{P}_1(\sigma_i) \subset L^2(\sigma_i)$. From Riesz theorem, there exists a unique $\psi_i \in \mathbb{P}_1(\sigma_i)$ such that, for all $v \in \mathbb{P}_1(\sigma_i)$, we have $v(M_i) = (\psi_i, v)_{L^2(\sigma_i)}$. Let w_i be the $\mathbb{P}_1$ Lagrange basis function associated to the vertex M_i. There is a natural definition of an interpolation operator P_h on $H^1(\Omega)$ such that: for all $v \in H^1(\Omega)$,

$$P_h v := \sum_i (\psi_i, v)_{L^2(\sigma_i)} w_i. \tag{39}$$

From the trace theorem, P_h is a continuous linear mapping from $H^1(\Omega)$ to V_h and is invariant on V_h. To preserve the trace on the boundary, we require in addition that for all vertices M_i on the boundary of Ω, the edge σ_i is chosen to belong to the boundary. This operator P_h is the Scott-Zhang operator and satisfies Hypothesis (21), see [6, Th. 2.1 and Cor. 4.1].

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References

1. Y. Boubendir, X. Antoine, and C. Geuzaine. A Quasi-Optimal Non-Overlapping Domain Decomposition Algorithm for the Helmholtz Equation. *J. Comp. Phys.*, 213(2):262–280, 2012.
2. F. Collino, S. Ghanemi, and P. Joly. Domain decomposition method for harmonic wave propagation: a general presentation. *CMAME*, 184(24):171–211, 2000.
3. F. Collino, P. Joly, and M. Lecouvez. Optimized quasi-local transmission conditions for non overlapping domain decomposition method. Submitted.
4. B. Després. *Méthodes de décomposition de domaine pour la propagation d'ondes en régime harmonique. Le théorème de Borg pour l'équation de Hill vectorielle*. PhD thesis, Université Paris IX Dauphine, 1991.
5. M. Lecouvez. *Méthodes itératives de décomposition de domaine sans recouvrement avec convergence géométrique pour l'équation de Helmholtz*. PhD thesis, Ecole polytechnique, 2015.
6. L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.