Institut des Sciences de la Mécanique et Applications Industrielles

MF208: AEROACOUSTICS AND ACOUSTIC PROPAGATION IN MOVING MEDIA

Course notes 2022-2023

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Contents

Introduction

These course notes are intended for the students of the course MF208 "Aeroacoustics and acoustic propagation in moving media" of ENSTA Paris.

Chapter 1 presents the equations for the propagation of acoustic waves in homogeneous moving media. The linearized Euler equations are derived from the equations of fluid mechanics. Several wave equations are then obtained in the time domain, and the corresponding Helmholtz equations are presented in the frequency domain.

Chapter 2 is devoted to the acoustic radiation of stationary and moving elementary sources in a homogeneous medium at rest. The notion of Green's function, that will be useful in Chapter 5, is introduced, and the acoustic radiation of a monopole in subsonic rectilinear motion is detailed.

Chapter 3 explains the effects of absorption and refraction in fluid media. The acoustic absorption mechanisms are presented first, considering both atmospheric and oceanic media. Then, some examples of acoustic refraction by sound speed gradients and and wind speed gradients are shown.

Chapter 4 describes the geometrical acoustics approximation, that can be used to model acoustic wave propagation at high frequencies. Ray-tracing equations are presented, and the calculation of wave amplitude along the rays is explained.

Chapter 5 describes several acoustic analogies that can be used to calculate aeroacoustic sources. Lighthill analogy can be applied in the absence of obstacles, while Curle's analogy accounts for the presence of static obstacles, and Ffowcs Williams and Hawkings analogy is dedicated to the acoustic radiation of moving bodies.

Chapter 1

Equations for acoustical waves in an inhomogeneous moving medium

This chapter is based mostly on the books of Pierce (1989) and Ostashev (1997).

1.1 Equations of fluid mechanics

We start here from the equations of fluid mechanics written for the pressure $p(\underline{x}, t)$, the velocity $\underline{v}(\underline{x}, t)$ the density $\rho(\underline{x}, t)$ and the entropy $s(\underline{x}, t)$ that are functions of space $\underline{x}$ and time t. We consider that there is only one component in the medium. For media with different components (such as water and salt in the ocean, or water vapor and air in the atmosphere), more general formulations exist (Ostashev, 1997).

First, the conservation of mass or equation of continuity is written:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{v}) = \frac{D\rho}{Dt} + \rho \nabla.\underline{v} = 0, \tag{1.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}.\nabla$$

is called the **material or total derivative**.

Second, the Navier-Stokes equation or law of momentum conservation is given by:

$$\rho \left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}.\nabla)\underline{v} \right] = -\nabla p + \nabla.\tau + \rho g \underline{e}_z, \tag{1.2}$$

where τ is the **viscous stress tensor** and $\rho g \underline{e}_z$ corresponds to the **gravitational force**. The viscous stress tensor is important to model aeroacoustic source generation, but will be neglected here in the context of acoustic propagation. The Navier-Stokes equation is called **Euler equation** for an inviscid fluid.

The last equations come from the conservation of the total energy and from thermodynamic laws. The total energy per unit mass is the sum of the internal energy e, associated to molecular motion, and of the kinetic energy $v^2/2$. The conservation of energy can be formulated in terms of the specific entropy s (entropy per unit mass). In loose terms, entropy measures the degree of disorder of a system. This yields, in the absence of external source of heat (Rienstra and Hirschberg, 2021, Equation (1.9)):

$$\rho \frac{Ds}{Dt} = \rho \left(\frac{\partial s}{\partial t} + \underline{v}.\nabla s \right) = -\frac{1}{T}\nabla.\underline{q} + \frac{1}{T}\tau.\nabla \underline{v}, \tag{1.3}$$

where T is the temperature and $\underline{q}$ is the heat flux. The entropy of a fluid flow can never decrease (second law of thermodynamics). It can increase due to **irreversible processes** such as viscous dissipation or heat transfer from outside. When we neglect heat conduction (adiabatic flow) and viscous dissipation (inviscid fluid), the flow is called isentropic. As a result, energy changes are only due to reversible processes and entropy is conserved along streamlines (Pierce, 1989; Rienstra and Hirschberg, 2021):

$$\frac{Ds}{Dt} = 0 \Leftrightarrow \frac{\partial s}{\partial t} + \underline{v}.\nabla s = 0. \tag{1.4}$$

Finally, an equation of state is necessary to have a a number of equations equal to the number of unknowns. In the more general form, the entropy s is a function of two independent thermodynamic variables:

$$s = s(\rho, p) \quad \text{or} \quad p = p(\rho, s). \tag{1.5}$$

For an ideal gas, the equation of state can be written:

$$dp = \left.\frac{\partial p}{\partial \rho}\right|_s d\rho + \left.\frac{\partial p}{\partial s}\right|_\rho ds = c^2 d\rho + \frac{p}{c_v} ds, \tag{1.6}$$

with c_v the specific heat at constant volume and

$$c^2 = \gamma r T = \frac{\gamma p}{\rho}, \tag{1.7}$$

where c is the sound speed for an ideal gas, T is the temperature, $\gamma = c_p/c_v$ is the ratio of specific heat and r is a gas constant. Using Equations (1.4) and (1.6), we finally obtain:

$$\frac{Dp}{Dt} = \left.\frac{\partial p}{\partial \rho}\right|_s \frac{D\rho}{Dt} + \left.\frac{\partial p}{\partial s}\right|_\rho \cancel{\frac{Ds}{Dt}} = c^2 \frac{D\rho}{Dt} = -c^2 \rho \nabla.\underline{v}, \tag{1.8}$$

where the equation of continuity (1.1) has been used. Note that some fluids such as water cannot be considered as an ideal gas. Water sound speed depends on temperature, pressure, and salinity in a complex way (Pierce, 1989, Section 1-9). In the general case, the sound speed x is defined as:

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s . \tag{1.9}$$

1.2 Linearized Euler equations

We now linearize the equations of fluid mechanics about a base flow (or mean flow) that is independent of time, described by the variables $\underline{v_0}(\underline{x})$, $p_0(\underline{x})$, $\rho_0(\underline{x})$ and $s_0(\underline{x})$. Let us introduce the variables associated to acoustic waves in a fluid medium:

- acoustic pressure [Pa]: $p'(\underline{x}, t) = p(\underline{x}, t) - p_0(\underline{x})$;
- particle velocity [m/s]: $\underline{v}'(\underline{x}, t) = \underline{v}(\underline{x}, t) - \underline{v_0}(\underline{x})$;
- density associated to acoustic fluctuations [kg/m^3]: $\rho'(\underline{x}, t) = \rho(\underline{x}, t) - \rho_0(\underline{x})$;
- entropy associated to acoustic fluctuations [J/K]: $s'(\underline{x}, t) = s(\underline{x}, t) - s_0(\underline{x})$.

Generally, the acoustic fluctuations are small perturbations with respect to the mean quantities. The approximation of linear acoustics is considered valid if (Pierce, 1989)

$$|p'| \ll \rho_0 c^2, \quad |v'| \ll c, \quad |\rho'| \ll \rho_0. \tag{1.10}$$

These conditions will be explained in more details in Section 1.3. We also suppose the base flow **incompressible**: $\nabla.\underline{v_0} = 0$. This is generally a good approximation because the Mach number $M = v_0/c \ll 1$ in most situations in the atmosphere and the ocean. Finally, we introduce the mean sound speed $c_0(\underline{x})$ associated with the base flow:

$$c_0^2 = \gamma r T_0 = \frac{\gamma p_0}{\rho_0} = c^2 - c_f^2, \tag{1.11}$$

with $c_f^2 \ll c_0^2$ the fluctuations of the squared sound speed. For air, $\gamma = 1.4$ and $r = 287\,\mathrm{J/kg/K}$ so $c_0 \approx 340\,\mathrm{m/s}$ at 15^oC.

Let us now introduce the acoustic variables in the continuity equation:

$$\frac{\partial(\rho_0 + \rho')}{\partial t} + (\underline{v_0} + \underline{v}').\nabla(\rho_0 + \rho') + (\rho_0 + \rho')\nabla.(\underline{v_0} + \underline{v}') = 0. \tag{1.12}$$

It is possible to group terms of order 0 (base flow quantities only), of order 1 (one small quantity only), etc. This yields:

$$\begin{aligned}
&\text{order 0} \quad \cancel{\frac{\partial \rho_0}{\partial t}} + (\underline{v_0}.\nabla)\rho_0 + \rho_0 \cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 1} \quad \frac{\partial \rho'}{\partial t} + (\underline{v_0}.\nabla)\rho' + (\underline{v}'.\nabla)\rho_0 + \rho_0\nabla.\underline{v}' + \rho'\cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 2} \quad (\underline{v}'.\nabla)\rho' + \rho'\nabla.\underline{v}' = 0.
\end{aligned}$$

A similar procedure can be applied to the momentum equation (1.2):

$$\begin{aligned}
&\text{order 0} \quad \rho_0 \cancel{\frac{\partial \underline{v_0}}{\partial t}} + \rho_0(\underline{v_0}.\nabla)\underline{v_0} = -\nabla p_0 + \rho_0 g \underline{e}_z,\\
&\text{order 1} \quad \rho_0\left(\frac{\partial \underline{v}'}{\partial t} + (\underline{v_0}.\nabla)\underline{v}' + (\underline{v}'.\nabla)\underline{v_0}\right) + \rho'(\underline{v_0}.\nabla)\underline{v_0} = -\nabla p' + \rho' g \underline{e}_z,\\
&\text{order 2} \quad \rho'\frac{\partial \underline{v}'}{\partial t} + \rho_0(\underline{v}'.\nabla)\underline{v}' + \rho'\left[(\underline{v_0}.\nabla)\underline{v}' + (\underline{v}'.\nabla)\underline{v_0}\right] = 0,\\
&\text{order 3} \quad \rho'(\underline{v}'.\nabla)\underline{v}' = 0,
\end{aligned}$$

and to the equation of state for an ideal gas (1.8):

$$\begin{aligned}
&\text{order 0} \quad \cancel{\frac{\partial p_0}{\partial t}} + (\underline{v_0}.\nabla)\underline{p_0} + c_0^2\rho_0\cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 1} \quad \frac{\partial p'}{\partial t} + (\underline{v_0}.\nabla)p' + (\underline{v}'.\nabla)p_0 + \rho_0 c_0^2\nabla.\underline{v}' + \left(c_0^2\rho' + \rho_0 c_f^2\right)\cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 2} \quad (\underline{v}'.\nabla)p' + \left(c_0^2\rho' + \rho_0 c_f^2\right)\nabla.\underline{v}' + \rho' c_f^2\cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 3} \quad \rho' c_f^2\nabla.\underline{v}' = 0.
\end{aligned}$$

The following set of equations for the base flow (order 0) is obtained:

$$(\underline{v_0}.\nabla)\rho_0 = 0, \tag{1.13}$$

$$(\underline{v_0}.\nabla)\underline{v_0} = -\frac{\nabla p_0}{\rho_0} + g\underline{e}_z, \tag{1.14}$$

$$(\underline{v_0}.\nabla)p_0 = 0. \tag{1.15}$$

The set of equations at order 1 are called the linearized Euler equations for an ideal gas:

$$\frac{\partial \rho'}{\partial t} + (\underline{v_0}.\nabla)\rho' + (\underline{v}'.\nabla)\rho_0 + \rho_0\nabla.\underline{v}' = 0, \tag{1.16}$$

$$\rho_0\left(\frac{\partial \underline{v}'}{\partial t} + (\underline{v_0}.\nabla)\underline{v}' + (\underline{v}'.\nabla)\underline{v_0}\right) + \nabla p' - \frac{\rho'}{\rho_0}\nabla p_0 = 0, \tag{1.17}$$

$$\frac{\partial p'}{\partial t} + (\underline{v_0}.\nabla)p' + (\underline{v}'.\nabla)p_0 + \rho_0 c_0^2\nabla.\underline{v}' = 0. \tag{1.18}$$

Note that in order to obtain Equation (1.17), we have used that:

$$\rho'(\underline{v_0}.\nabla)\underline{v_0} - \rho' g\underline{e}_z = -\frac{\rho'\nabla p_0}{\rho_0},$$

which makes use of the momentum equation (1.14) at order 0.

In many situations, it is possible to neglect the terms proportional to ∇p_0. For instance, for a harmonic plane wave of the form $p'(x,t) = Ae^{-i\omega(t-x/c_0)}$ in a medium at rest ($v_0 = 0$), we will see in Section 2.1.1 that $v'(x,t) = p'(x,t)/(\rho_0 c_0)$ and $\rho'(x,t) = p'(x,t)/c_0^2$. Thus the orders of magnitude of the different terms in Equation (1.17) are:

$$\underbrace{\rho_0\frac{\partial \underline{v}'}{\partial t}}_{\sim\omega\frac{A}{c_0}} + \underbrace{\nabla p'}_{\sim\frac{\omega}{c_0}A} - \underbrace{\frac{\rho'}{\rho_0}\nabla p_0}_{\sim\frac{A}{c_0^2}g} = 0, \tag{1.19}$$

where we have used that $\nabla p_0 = \rho_0 g\underline{e}_z$ from Equation (1.14) in a medium at rest. Thus it appears that the term proportional to ∇p_0 can be neglected at sufficiently high frequencies such that $\omega \gg g/c_0$. This corresponds to $f \gg 10^{-3}$ Hz in air and $f \gg 5\times10^{-3}$ Hz in water. Ostashev *et al.* (2005) note that the terms proportional to ∇p_0 are important for internal gravity waves, but can be neglected for acoustic waves. It will thus be neglected in the rest of the course.

The linearized Euler equations become the following set of equations:

$$\frac{\partial \rho'}{\partial t} + (\underline{v_0}.\nabla)\rho' + (\underline{v}'.\nabla)\rho_0 + \rho_0 \nabla.\underline{v}' = 0, \tag{1.20}$$

$$\frac{\partial \underline{v}'}{\partial t} + (\underline{v_0}.\nabla)\underline{v}' + (\underline{v}'.\nabla)\underline{v_0} + \frac{\nabla p'}{\rho_0} = 0, \tag{1.21}$$

$$\frac{\partial p'}{\partial t} + (\underline{v_0}.\nabla)p' + \rho_0 c_0^2 \nabla.\underline{v}' = 0. \tag{1.22}$$

Note that Equations (1.21) and (1.22) do not depend on ρ'. These equations are the basis of many numerical solvers of the linearized Euler equations. It is also common to rewrite these equations under the following conservative form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} + \mathbf{H} = \mathbf{S}, \tag{1.23}$$

where $\mathbf{U} = (p', \rho_0 v_x', \rho_0 v_y', \rho_0 v_z')^T$ and $\mathbf{S} = (\rho_0 c^2 Q, F_x, F_y, F_z)^T$ corresponds to the source terms, with Q the volume velocity and $\underline{F}$ the exterior forces. The Eulerian fluxes $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are written:

$$\begin{aligned}
\mathbf{E} &= \begin{pmatrix} v_{0x}p' + \rho_0 c_0^2 v_x' \\ v_{0x}\rho_0 v_x' + p' \\ v_{0x}\rho_0 v_y' \\ v_{0x}\rho_0 v_z' \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} v_{0y}p' + \rho_0 c_0^2 v_y' \\ v_{0y}\rho_0 v_x' \\ v_{0y}\rho_0 v_y' + p' \\ v_{0y}\rho_0 v_z' \end{pmatrix}, \\
\mathbf{G} &= \begin{pmatrix} v_{0z}p' + \rho_0 c_0^2 v_z' \\ v_{0z}\rho_0 v_x' \\ v_{0z}\rho_0 v_y' \\ v_{0z}\rho_0 v_z' + p' \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 0 \\ \rho_0(\mathbf{v}'.\nabla)v_{0x} \\ \rho_0(\mathbf{v}'.\nabla)v_{0y} \\ \rho_0(\mathbf{v}'.\nabla)v_{0z} \end{pmatrix}.
\end{aligned} \tag{1.24}$$

From Equations (1.23) and (1.24), the following coupled equations can be retrieved:

$$\frac{\partial p'}{\partial t} + (\underline{v_0}.\nabla)p' + p'(\nabla.\underline{v_0}) + \rho_0 c_0^2 \nabla.\underline{v}' + (\underline{v_0}.\nabla)(\rho_0 c_0^2) = 0, \tag{1.25}$$

$$\frac{\partial \underline{v}'}{\partial t} + (\underline{v_0}.\nabla)\underline{v}' + (\nabla.\underline{v_0})\underline{v}' + \frac{\underline{v}'}{\rho_0}(\underline{v_0}.\nabla)\rho_0 + \frac{\nabla p'}{\rho_0} + (\underline{v}'.\nabla)\underline{v_0} = 0. \tag{1.26}$$

Equation (1.25) is equivalent to Equation (1.22) because the base flow is incompressible ($\nabla.\underline{v_0} = 0$) and the last term can be written:

$$(\underline{v_0}.\nabla)(\rho_0 c_0^2) = \gamma(\underline{v_0}.\nabla)p_0 = 0, \tag{1.27}$$

using Equations (1.11) and (1.15). Similarly, Equation (1.26) is equivalent to Equation (1.21) because the base flow is incompressible and $(\underline{v_0}.\nabla)\rho_0 = 0$ from Equation (1.13).

1.3 Validity of the linear acoustics approximation

From the equations of continuity and momentum conservation, it is clear that the terms of order 1 will be small compared to the terms of order 0 if $|\rho'| \ll \rho_0$ and $|p'| \ll p_0$. From the equation of state for a perfect gas, $p_0 = \rho_0 c_0^2/\gamma$ thus we obtain the condition $|p'| \ll \rho_0 c_0^2$. The condition for the particle velocity is less straightforward to obtain. Let us consider a plane wave propagating in the fluid. We will see in Section 2.1.1 that in this case:

$$v' = \frac{p'}{\rho_0 c_0} \Rightarrow |v'| = \frac{|p'|}{\rho_0 c_0} \ll c_0. \tag{1.28}$$

Note that it is not necessary than $|v'| \ll v_0$, such that linear acoustics is also valid in a medium at rest ($v_0 = 0$).

The linear acoustics approximation is valid in many applications. For instance, the amplitude of acoustic pressure corresponding to the threshold of pain is around 90 Pa (about 130 dB re. 20 muPa), which is still two orders of magnitude smaller than the atmospheric pressure that is close to 10^5 Pa. This corresponds to an amplitude of particle velocity of 0.2 m/s, which is much smaller than the sound speed in air.

1.4 Wave equations in the time and frequency domains

1.4.1 Propagation in a homogeneous medium at rest

The simplest case that can be considered corresponds to a homogeneous medium at rest: $\underline{v_0} = 0$, where ρ_0, p_0 and c_0 are constant. The linearized

Euler equations become simply:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla . \underline{v}' = 0, \tag{1.29}$$

$$\rho_0 \frac{\partial \underline{v}'}{\partial t} + \nabla p' = 0, \tag{1.30}$$

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \nabla . \underline{v}' = 0. \tag{1.31}$$

It is possible to obtain a wave equation for the acoustic pressure p', by substracting the time derivative of Equation (1.31) and the divergence of Equation (1.30) multiplied by c_0^2:

$$\boxed{\frac{\partial^2 p'}{\partial t^2} - c_0^2 \nabla^2 p' = 0} \, . \tag{1.32}$$

The operator $\nabla^2 = \Delta$ is called Laplacian and is written in Cartesian coordinates:

$$\nabla^2 p' = \Delta p' = \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2}. \tag{1.33}$$

Note also that a simple expression can be obtained between acoustic density and pressure from Equations (1.29) and (1.31):

$$\frac{\partial p'}{\partial t} = c_0^2 \frac{\partial \rho'}{\partial t} \Rightarrow \boxed{p' = c_0^2 \rho'} \, . \tag{1.34}$$

For a harmonic wave at angular frequency $\omega = 2\pi f$, the pressure can be written $p'(\underline{x}, t) = A(\underline{x}) \cos(\omega t + \phi(\underline{x}))$, where A is the amplitude and ϕ is the phase that are both functions of space. It is useful to introduce the following complex notation:

$$p'(\underline{x}, t) = \text{Re}\left[p_c(\underline{x}) e^{-i\omega t}\right], \tag{1.35}$$

where Re denotes the real part and the $p_c(\underline{x}) = A(\underline{x}) e^{-i\phi(\underline{x})}$ is the complex pressure amplitude. Introducing $p_c(\underline{x}) e^{-i\omega t}$ into the wave equation:

$$\boxed{\Delta p_c + k_0^2 p_c = 0} \, , \tag{1.36}$$

where $k_0 = \omega / c_0 = 2\pi / \lambda_0$ is the acoustic wave number, and λ_0 is the wavelength. Equation (1.36) is called the **Helmholtz equation**. Many computational methods assume a harmonic sound field as any sound signal can be decomposed into harmonic components using the Fourier transform (spectral decomposition), and it is easier to solve in the frequency domain as there is no time derivative to evaluate.

Remark: it is also possible to use the $e^{j\omega t}$ convention instead of the $e^{-i\omega t}$ convention. In this case, we would have:

$$p'(\underline{x},t) = \mathrm{Re}\left[p_c(\underline{x})e^{j\omega t}\right], \tag{1.37}$$

with $p_c(\underline{x}) = A(\underline{x})e^{j\phi(\underline{x})}$. The Helmholtz equation remains the same with both notations!

Finally, it is possible to introduce an acoustic velocity potential Φ' associated with the particle velocity $\underline{v}'$. Taking the curl of Equation (1.30):

$$\frac{\partial}{\partial t}\left(\nabla \times \underline{v}'\right) = 0, \tag{1.38}$$

since $\nabla \times \nabla p = 0$. This means that the rotational of particle velocity is independent of time. If the acoustic field is irrotational ($\nabla \times \underline{v}' = 0$), then the particle velocity derives from a potential Φ': $\underline{v}' = \nabla \Phi'$. The relationship between p' and Φ' is obtained from Equation (1.30):

$$p'(r,t) = -\rho_0 \frac{\partial \Phi'}{\partial t}. \tag{1.39}$$

Replacing this expression into Equation (1.32), we see that Φ' satisfies the same wave equation as p':

$$\boxed{\frac{\partial^2 \Phi'}{\partial t^2} - c_0^2 \nabla^2 \Phi' = 0}\,. \tag{1.40}$$

It is convenient to solve for the acoustic potential because acoustic pressure and particle velocity can be deduced by taking the temporal or spatial derivative of Φ'.

1.4.2 Propagation in an inhomogeneous medium at rest

We now consider that all the mean quantities depend on space in a medium at rest: $\underline{v_0} = 0$, with $\rho_0(\underline{x})$, $p_0(\underline{x})$ and $c_0(\underline{x})$. The linearized Euler equations reduce to:

$$\frac{\partial \rho'}{\partial t} + (\underline{v}'.\nabla)\rho_0 + \rho_0 \nabla.\underline{v}' = 0, \quad (1.41)$$

$$\frac{\partial \underline{v}'}{\partial t} + \frac{\nabla p'}{\rho_0} = 0, \quad (1.42)$$

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \nabla.\underline{v}' = 0. \quad (1.43)$$

As done previously, let us calculate $\frac{\partial}{\partial t}(1.43) - \rho_0 c_0^2 \nabla.(1.42)$:

$$\frac{\partial^2 p'}{\partial t^2} + \rho_0 c_0^2 \frac{\partial(\nabla.\underline{v}')}{\partial t} - \rho_0 c_0^2 \nabla. \left(\frac{\partial \underline{v}'}{\partial t}\right) - \rho_0 c_0^2 \nabla. \left(\frac{\nabla p'}{\rho_0}\right) = 0. \quad (1.44)$$

Since the operators ∇ and $\frac{\partial}{\partial t}$ commute, the terms involving the particle velocity $\underline{v}'$ cancel out and we obtain the following wave equation:

$$\boxed{\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \rho_0 \nabla. \left(\frac{1}{\rho_0}\nabla p'\right) = 0}\,. \quad (1.45)$$

Introducing $p_c(\underline{x})e^{-i\omega t}$ into the wave equation, we obtain:

$$-\frac{\omega^2}{c_0^2} p_c - \rho_0 \nabla. \left(\frac{1}{\rho_0}\nabla p_c\right) = 0 \Leftrightarrow k_0^2 p_c + \rho_0 \nabla. \left(\frac{1}{\rho_0}\nabla p_c\right) = 0. \quad (1.46)$$

Since $c_0^2 = \gamma p_0/\rho_0$ for an ideal gas from Equation (1.11), we have:

$$\rho_0 \nabla. \left(\frac{1}{\rho_0}\nabla p_c\right) = \frac{\gamma p_0}{c_0^2}\nabla. \left(\frac{c_0^2}{\gamma p_0}\nabla p_c\right) = \frac{1}{c_0^2}\nabla. \left(c_0^2 \nabla p_c\right) - \frac{1}{c_0^2 p_0}\nabla p_0. \left(c_0^2 \nabla p_c\right). \quad (1.47)$$

Since we neglect the pressure gradient term, we obtain:

$$\rho_0 \nabla. \left(\frac{1}{\rho_0}\nabla p_c\right) = \frac{k_0^2}{\omega^2}\nabla. \left(\frac{\omega^2}{k_0^2}\nabla p_c\right) = k_0^2 \nabla. \left(\frac{1}{k_0^2}\nabla p_c\right), \quad (1.48)$$

which yields the following Helmholtz equation in a inhomogeneous medium at rest:

$$\boxed{k_0^2 p_c + k_0^2 \nabla. \left(\frac{1}{k_0^2}\nabla p_c\right) = 0}\,. \quad (1.49)$$

This equation is the starting point of several frequency-domain numerical models such as the parabolic equation.

1.4.3 Propagation in a uniform moving medium

Except in a few simple cases, it is very difficult or even impossible to derive a wave equation in a moving medium. One of these simple cases correspond to a uniform moving medium where $\underline{v_0} = v_{0x}\underline{e}_x$ and v_{0x}, ρ_0 and c_0 are constant. In this case, the linearized Euler equations become:

$$\left(\frac{\partial}{\partial t} + v_{0x}\frac{\partial}{\partial x}\right)\rho' + \rho_0 \nabla.\underline{v}' = 0 \quad \text{or} \quad \frac{D\rho'}{Dt} + \rho_0 \nabla.\underline{v}' = 0, \tag{1.50}$$

$$\rho_0 \left(\frac{\partial}{\partial t} + v_{0x}\frac{\partial}{\partial x}\right)\underline{v}' + \nabla p' = 0 \quad \text{or} \quad \rho_0 \frac{D\underline{v}'}{Dt} + \nabla p' = 0, \tag{1.51}$$

$$\left(\frac{\partial}{\partial t} + v_{0x}\frac{\partial}{\partial x}\right)p' + \rho_0 c_0^2 \nabla.\underline{v}' = 0 \quad \text{or} \quad \frac{Dp'}{Dt} + \rho_0 c_0^2 \nabla.\underline{v}' = 0, \tag{1.52}$$

where the total derivative can be written:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_{0x}\frac{\partial}{\partial x}.$$

The **convected wave equation** is obtained by calculating $\frac{D}{Dt}(1.52) - c_0^2 \nabla.(1.51)$:

$$\boxed{\frac{D^2 p'}{Dt^2} - c_0^2 \nabla^2 p' = 0}\,. \tag{1.53}$$

Equation (1.53) is exact in a homogeneous moving medium. Ostashev (1997, Section 2.3) shows that it is also a good approximation in an inhomogeneous moving medium if the acoustic wavelength λ is small compared to the length scale l of variation in the ambient quantities $\underline{v_0}$, ρ_0 and c_0, i.e. at sufficiently high frequencies. He also derives more accurate wave equations for acoustic propagation in an inhomogeneous moving medium; see Ostashev (1997, Section 2.3) and Ostashev *et al.* (1997). Note that these equations are the basis of various vector parabolic equations that have been used to calculation the acoustic propagation in an inhomogeneous moving medium (Dallois *et al.*, 2001; Blanc-Benon *et al.*, 2001).

1.5 Acoustic energy, intensity and source power

1.5.1 Energy conservation law

Let us derive an acoustic energy conservation law in the simple case of a homogeneous medium at rest. The linearized equations (1.30) and (1.31) are reminded here:

$$\rho_0 \frac{\partial \underline{v}'}{\partial t} + \nabla p' = 0, \tag{1.54}$$

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \nabla . \underline{v}' = 0. \tag{1.55}$$

To obtain an energy conservation law, we sum the first equation multiplied by $\underline{v}'$ and the second equation mulitplied by $p'/(\rho_0 c_0^2)$:

$$\rho_0 \underline{v}' . \frac{\partial \underline{v}'}{\partial t} + \frac{1}{\rho_0 c_0^2} p' \frac{\partial p'}{\partial t} + \underline{v}' . \nabla p' + p' \nabla . \underline{v}' = 0. \tag{1.56}$$

This equation can be rewritten:

$$\frac{\partial w}{\partial t} + \nabla . \underline{I} = 0, \tag{1.57}$$

where

$$w = \frac{1}{2} \rho_0 v'^2 + \frac{1}{2} \frac{p'^2}{\rho_0 c_0^2} \tag{1.58}$$

is the energy density per unit volume, and $\underline{I} = p' \underline{v}'$ is the acoustic intensity vector. By integrating over an arbitrary volume V and applying Gauss theorem, we obtain the following law of energy conservation:

$$\frac{d}{dt} \int_V w dV + \int_S \underline{I} . \underline{n} dS = 0, \tag{1.59}$$

where $\underline{n}$ is the unit normal vector pointing out of the surface S enclosing V. This means that in the absence of sources, the variation of energy in the volume V is compensated by the flux of acoustic energy across its surface.

1.5.2 Time-averaged acoustic intensity and power

Let us introduce the time-averaged acoustic intensity as:

$$< \underline{I} > = \frac{1}{T} \int_{t_0}^{t_0+T} \underline{I}(t) dt = \frac{1}{T} \int_{t_0}^{t_0+T} p(t) \underline{v}(t) dt. \tag{1.60}$$

Starting from Equation (1.59), Pierce (1989, page 40) shows that the time-averaged of the first term is zero, thus:

$$\int_S < \underline{I} > . \underline{n} dS = 0 \quad \text{in the absence of sources.} \tag{1.61}$$

If a source is placed inside the volume V, this integral is not zero anymore, and we define the time-averaged acoustic power as:

$$< W_a >= \int_S < \underline{I} > .\underline{n} dS = \int_S < p\underline{v} > .\underline{n} dS, \tag{1.62}$$

where $\underline{n}$ is the normal to the surface S.

For harmonic waves, let $p(\underline{x}, t) = Re\{p_c(\underline{x})e^{-i\omega t}\}$ and $\underline{v}(\underline{x}, t) = Re\{\underline{v}_c(\underline{x})e^{-i\omega t}\}$. The time-averaged acoustic intensity for sinusoidal waves becomes:

$$< \underline{I} >= \frac{1}{2} Re\{p_c \underline{v}_c^*\}. \tag{1.63}$$

Demonstration:

Let $p_c = |p_c|e^{-i\phi_p}$ and $v_c = |v_c|e^{-i\phi_v}$. We thus obtain $p(t) = |p_c|\cos(\omega t + \phi_p)$ and $v_c(t) = |v_c|\cos(\omega t + \phi_v)$. As a result

$$< I >= \frac{1}{T} \int_{t_0}^{t_0+T} p(t)v(t)dt = \frac{|p_c||v_c|}{T} \int_{t_0}^{t_0+T} \cos(\omega t + \phi_p)\cos(\omega t + \phi_v)dt, \tag{1.64}$$

with $T = 1/f = 2\pi/\omega$. Using:

$$\cos(\omega t + \phi_p)\cos(\omega t + \phi_v) = \frac{1}{2}\left[\cos(2\omega t + \phi_p + \phi_v) + \cos(\phi_p - \phi_v)\right], \tag{1.65}$$

we obtain:

$$< I >= \frac{|p_c||v_c|}{2} \cos(\phi_p - \phi_v), \tag{1.66}$$

since the integral of the first term is zero. Since

$$\frac{1}{2} Re\{p_c \underline{v}_c^*\} = \frac{|p_c||v_c|}{2} Re\{e^{-i(\phi_p - \phi_v)}\} = \frac{|p_c||v_c|}{2} \cos(\phi_p - \phi_v),$$

Equation (1.63) is retrieved.

1.5.3 Sound pressure level and sound power level

The sound pressure level (SPL) is defined as:

$$L_p = 10 \log_{10}\left(\frac{p_{rms}^2}{p_{ref}^2}\right) = 20 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right), \tag{1.67}$$

where p_{ref} is a reference pressure and p_{rms} is the time-averaged or rms pressure:

$$p_{rms}^2 = < p^2 > = \frac{1}{T} \int_{t_0}^{t_0+T} p^2(t) dt. \tag{1.68}$$

For a harmonic wave, $p_{rms} = \max |p| / \sqrt{2}$. Similarly, the sound power level (SWL) is defined as:

$$L_W = 10 \log_{10} \left(\frac{< W_a >}{W_{ref}} \right), \tag{1.69}$$

with W_{ref} a reference power. The reference pressure p_{ref} is typically 2×10^{-5} Pa in air (threshold of hearing at 1 kHz) and 10^{-6} Pa in water.

Chapter 2

Acoustic radiation from stationary and moving sources

In this chapter, we consider the acoustic propagation in a homogeneous medium at rest ($\underline{v}_0 = 0$), with constant density ρ_0 and sound speed c_0. We drop the primes in the acoustic variables, so that the acoustic pressure is simply $p(\underline{x}, t)$, the particle velocity $v(\underline{x}, t)$, and the velocity potential $\Psi(\underline{x}, t)$.

2.1 Elementary solutions to the wave equation

2.1.1 Plane waves

Plane waves correspond to specific solutions to the wave equation where the wavefronts are planar, as seen in Figure 2.1. Considering the velocity potential Φ, the general solution to Equation (1.40) is given by:

$$\Phi(x, t) = F_+\left(t - \frac{x}{c_0}\right) + F_-\left(t + \frac{x}{c_0}\right), \tag{2.1}$$

where the function F_+ describes the wave propagation in the positive x direction, and F_- describes the wave propagation in the negative x direction.

The associated pressure field is:

$$\begin{aligned} p(x,t) = -\rho_0 \frac{\partial \Phi}{\partial t} &= -\rho_0 F'_+ \left(t - \frac{x}{c_0}\right) - \rho_0 F'_- \left(t + \frac{x}{c_0}\right) \\ &= G_+ \left(t - \frac{x}{c_0}\right) + G_- \left(t + \frac{x}{c_0}\right). \end{aligned} \tag{2.2}$$

The associated particle velocity field is $\underline{v} = \nabla\Phi = v_x \underline{e_x}$, with:

$$\begin{aligned} v_x(x,t) = \frac{\partial \Phi}{\partial x} &= -\frac{1}{c_0} F'_+ \left(t - \frac{x}{c_0}\right) + \frac{1}{c_0} F'_- \left(t + \frac{x}{c_0}\right) \\ &= \frac{1}{\rho_0 c_0} \left[G_+ \left(t - \frac{x}{c_0}\right) - G_- \left(t + \frac{x}{c_0}\right)\right]. \end{aligned} \tag{2.3}$$

Let us consider a special case of interest, that is a harmonic plane wave traveling along the positive x axis, with $p(\underline{x},t) = Re\{p_c(x)e^{-i\omega t}\}$ and $\underline{v}(x,t) = Re\{\underline{v}_c(x)e^{-i\omega t}\}$:

$$p_c(x) = P_0 e^{ik_0 x}, \tag{2.4}$$

$$\underline{v}_c(x) = \frac{p_c(x)}{\rho_0 c_0} \underline{e_x}, \tag{2.5}$$

$$< \underline{I} > = \frac{|p_c|^2}{2\rho_0 c_0} \underline{e_x} = \frac{\rho_0 c_0 |v_c|^2}{2} \underline{e_x}. \tag{2.6}$$

With this type of waves the amplitude remains constant with distance. As a result, the ratio of pressure to velocity is constant for a plane wave and equal to $Z_{c,fluid} = \rho_0 c_0$. The quantity $Z_{c,fluid}$ is called the **characteristic acoustic impedance** of the fluid.

2.1.2 Spherical waves

We now consider waves with spherical symmetry, which means that the variables do not depend on the spherical coordinates θ and ϕ: $p = p(r,t)$ and $\underline{v} = v(r,t)\underline{e_r}$. The wavefronts are spheres, and the acoustic intensity vector is along along the r direction: $\underline{I} = I_r \underline{e_r}$. This solution corresponds to the case of a point source with spherical symmetry.

Rewriting the homogeneous wave equation (1.40) for the velocity potential in spherical coordinates:

$$\frac{1}{c_0^2}\frac{\partial^2 \Phi}{\partial t^2} - \frac{1}{r}\frac{\partial^2 (r\Phi)}{\partial r^2} = 0 \Leftrightarrow \frac{1}{c_0^2}\frac{\partial^2 (r\Phi)}{\partial t^2} - \frac{\partial^2 (r\Phi)}{\partial r^2} = 0.$$

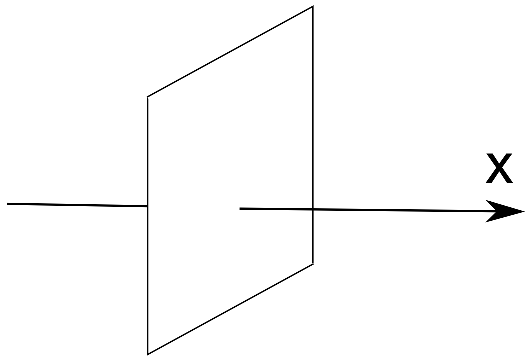


Figure 2.1: Plane wave traveling along the x-direction.

This means that $r\Phi$ can be written as a sum of a function of $t - r/c$ and a function of $t + r/c$, as done in Section 2.1.1 for plane waves. If we keep only the outward-going wave:

$$\Phi(r,t) = \frac{1}{r}F\left(t - \frac{r}{c_0}\right), \tag{2.7}$$

and thus:

$$p(r,t) = -\rho_0\frac{\partial\Phi}{\partial t} = -\frac{\rho_0}{r}F'\left(t - \frac{r}{c_0}\right), \tag{2.8}$$

$$v(r,t) = \frac{\partial\Phi}{\partial r} = \frac{p(r,t)}{\rho_0 c_0} - \frac{1}{r^2}F\left(t - \frac{r}{c_0}\right). \tag{2.9}$$

It appears that the pressure amplitude decreases as $1/r$. Also, the particle velocity is composed of two terms. Since the second term decreases as $1/r^2$, it becomes negligible if r is sufficiently large (far-field) and $v(r,t) \approx \frac{p(r,t)}{\rho_0 c_0}$, which corresponds to the relationship for plane waves.

It is possible to calculate the acoustic power of this wave by integrating over a sphere of radius r. From Equation (1.62), considering that the acoustic intensity is constant on the sphere and that $\underline{n} = \underline{e}_r$:

$$< W_a >= \int_S < \underline{I} > .\underline{e}_r dS = 4\pi r^2 < I_r > . \tag{2.10}$$

If we consider a harmonic spherical wave of the form $p(\underline{x},t) = Re\{p_c(r)e^{-i\omega t}\}$, with

$$p_c(r) = \frac{A}{r}e^{ik_0 r}, \tag{2.11}$$

the following time-averaged acoustic intensity is obtained from Equation (1.63):

$$< I(r) >= \frac{|p_c|^2}{2\rho_0 c_0} = \frac{< p^2 >}{\rho_0 c_0}. \tag{2.12}$$

From Equations (2.10) and (2.12), the acoustic power is thus:

$$< W_a >= 4\pi r^2 \frac{< p^2 >}{\rho_0 c_0} = \frac{2\pi |A|^2}{\rho_0 c_0}. \tag{2.13}$$

It appears clearly that the acoustic power is independent of the distance r since A is a constant; the acoustic power $< W_a >$ is a characteristics of the source(s) inside the sphere S.

From the previous expression, it is possible to derive a simple relationship between the sound pressure level and the sound power level:

$$\boxed{L_p = L_W - 10\log_{10}(4\pi r^2)}\,, \tag{2.14}$$

where $W_{ref} = p_{ref}^2/(\rho_0 c_0)$. In air, we consider typically $p_{ref} = 20 \times 10^{-6}\,\text{Pa}$ and $\rho_0 c_0 \approx 415\,\text{kg/m}^2/\text{s}$, thus $W_{ref} \approx 10^{-12}\,\text{W}$. The term $10\log_{10}(4\pi r^2)$ is called **geometrical spreading**. This means that there is an attenuation of $10\log_{10}(4) \approx 6\,\text{dB}$ of the sound pressure level L_p when the distance r is doubled (6 dB attenuation per doubling distance).

2.1.3 Green's function

The Green's function is the solution of the wave equation with a unit point impulsive source term:

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right) G(\underline{x}, t|\underline{y}, \tau) = \delta(\underline{x} - \underline{y})\delta(t - \tau), \tag{2.15}$$

with τ the source time, t the receiver time, $\underline{y}$ the source position, and $\underline{x}$ the receiver position. The Green's function should be zero for $t < \tau$ due to causality considerations. Let $\tilde{G}(\omega, \underline{x}|\underline{y})$ be the Fourier transform of the Green's function:

$$\tilde{G}(\omega, \underline{x}|\underline{y}) = \int_{-\infty}^{+\infty} G(\underline{x}, t|\underline{y}, \tau) e^{i\omega t} dt, \tag{2.16}$$

$$G(\underline{x}, t|\underline{y}, \tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{G}(\omega, \underline{x}|\underline{y}) e^{-i\omega t} d\omega. \tag{2.17}$$

To derive the inhomogeneous Helmholtz equation for $\tilde{G}(\omega, \underline{x}|\underline{y})$, we need to use the following property of the Dirac delta function:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega t} d\omega = \delta(t). \tag{2.18}$$

Using Equations (2.17) and (2.18), Equation (2.15) becomes:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\left(\tilde{G}(\omega, \underline{x}|\underline{y})e^{-i\omega t}\right) d\omega = \delta(\underline{x}-\underline{y})\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega(t-\tau)} d\omega$$
$$\Rightarrow \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(-\frac{\omega}{c_0^2}\tilde{G}(\omega, \underline{x}|\underline{y}) - \Delta\tilde{G}(\omega, \underline{x}|\underline{y})\right) e^{-i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(\underline{x}-\underline{y})e^{i\omega\tau}e^{-i\omega t} d\omega.$$

For each frequency ω, the integrand can be equated so that:

$$\Delta\tilde{G}(\omega, \underline{x}|\underline{y}) + k_0^2\tilde{G}(\omega, \underline{x}|\underline{y}) = -\delta(\underline{x}-\underline{y})e^{i\omega\tau} = -\delta(\underline{x}-\underline{y}), \tag{2.19}$$

with $k_0 = \omega/c_0$, if the origin in time of the impulse is chosen at $\tau = 0$.

In free field, spherical waves are propagating from the point source and the Green's function noted $\tilde{G}_0(\omega, \underline{x}|\underline{y})$ should follow the form given by Equation (2.11):

$$\tilde{G}_0(\omega, \underline{x}|\underline{y}) = \frac{A}{r}e^{ik_0 r}, \tag{2.20}$$

where $r = |\underline{x}-\underline{y}|$ is the source-receiver distance and using the $e^{-i\omega t}$ convention. By integration over a small sphere of radius ϵ (Rienstra and Hirschberg, 2021, Section 6.3), the constant A can be determined and the free field Green's function in the frequency domain is written:

$$\tilde{G}_0(\omega, \underline{x}|\underline{y}) = \frac{e^{ik_0 r}}{4\pi r}e^{i\omega\tau} = \frac{e^{ik_0|\underline{x}-\underline{y}|}}{4\pi|\underline{x}-\underline{y}|}e^{i\omega\tau}, \tag{2.21}$$

with $e^{i\omega\tau}$ a phase term equal to 1 if the origin of the impulse is chosen at $\tau = 0$. The expression in the time domain is obtained by taking the inverse Fourier transform using Equation (2.17):

$$\begin{aligned} G_0(\underline{x}, t|\underline{y}, \tau) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ik_0 r}}{4\pi r}e^{i\omega\tau}e^{-i\omega t} d\omega \\ &= \frac{1}{4\pi r}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[-i\omega\left(t-\tau-\frac{r}{c_0}\right)\right] d\omega\right) \\ &= \frac{1}{4\pi r}\delta\left(t-\tau-\frac{r}{c_0}\right), \end{aligned} \tag{2.22}$$

using Equation (2.18).

The Green's function in free field meets the reciprocity property, which means that the Green's function remains the same if source and receiver positions are interchanged in a medium at rest:

$$\text{time domain:} \quad G_0(\underline{x}, t|\underline{y}, \tau) = \frac{1}{4\pi r}\delta\left(t - \tau - \frac{r}{c_0}\right) = G_0(\underline{y}, -\tau|\underline{x}, -t)$$
$$\text{frequency domain:} \quad \tilde{G}_0(\omega, \underline{x}|\underline{y}) = \frac{e^{ik_0 r}}{4\pi r} = \tilde{G}_0(\omega, \underline{y}|\underline{x}).$$

The Green's function can be used to find integral solutions to inhomogeneous wave equation of the following type:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = S(\underline{x}, t), \tag{2.23}$$

with $S(\underline{x}, t)$ a volume source distribution. Using the free-field Green's function, the solution can be written (Rienstra and Hirschberg, 2021; Glegg and Devenport, 2017):

$$p(\underline{x}, t) = \int_{-\infty}^{+\infty}\int_V S(\underline{y}, \tau)G_0(\underline{x}, t|\underline{y}, \tau)dV(\underline{y})d\tau = \int_V \frac{S(\underline{y}, t - r/c_0)}{4\pi r}dV(\underline{y}). \tag{2.24}$$

This result can be seen as a particular case of Curle's integral solution derived in Section 5.2.

Note finally that in some references the $e^{j\omega t}$ convention is used, so that the frequency-domain Green's function in free field is given by (with $\tau = 0$):

$$\tilde{G}_0(\omega, \underline{y}|\underline{x}) = \frac{e^{-jk_0|\underline{x}-\underline{y}|}}{4\pi|\underline{x} - \underline{y}|} = \frac{e^{-jk_0 r}}{4\pi r}. \tag{2.25}$$

2.2 Acoustic radiation of stationary elementary sources

2.2.1 Inhomogenous wave equation with source terms

One way to introduce acoustic sources is to include source terms on the right hand-side of the linearized equations of Chapter 1. This is done here in a homogeneous medium at rest. In presence of a source of mass, the linearized continuity equation becomes:

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \rho_0 \nabla.\underline{v} = \dot{m}_V, \tag{2.26}$$

with $\dot{m}_V$ the injected mass per unit volume and per unit time (in kg/m^3/s). An example of such a source is a air bubble oscillating in a liquid or a loudspeaker inserted in a baffle. Similarly, in presence of external forces in the fluid, the linearized Euler equation is written:

$$\rho_0 \frac{\partial \underline{v}}{\partial t} + \nabla p = \underline{f}_V, \tag{2.27}$$

with $\underline{f}_V$ the exterior forces imposed to the fluid per unit volume. This corresponds to oscillating sources, or to a loudspeaker without baffle.

To obtain the wave equation, we substract the time derivative of Equation (2.26) and the divergence of Equation (2.27) as done in Section 1.4, and we obtain:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = \ddot{m}_V - \nabla.\underline{f}_V. \tag{2.28}$$

We obtain two additional source terms on the right-hand side, that will be analyzed in the following sections.

2.2.2 Acoustic field radiated by a monopole

The monopole is an elementary source obtained by considering a point mass source, which means that the injected mass per unit volume is written:

$$\dot{m}_V(\underline{x}, t) = \rho_0 q(t)\delta(\underline{x} - \underline{y}) = \rho_0 q(t)\delta(x_1 - y_1)\delta(x_2 - y_2)\delta(x_3 - y_3), \tag{2.29}$$

where $q(t)$ is the volume velocity (in m^3/s) of the source, and $\underline{y} = (y_1, y_2, y_3)$ is the position of the point source. As a result, in the absence of exterior forces ($\underline{f}_V = 0$), the inhomogeneous wave equation (2.28) becomes:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = \ddot{m}_V = \rho_0 \dot{q}(t)\delta(\underline{x} - \underline{y}). \tag{2.30}$$

The solution to this solution is given by Equation (2.24) with a volume source distribution $S(\underline{x}, t) = \rho_0 \dot{q}(t)\delta(\underline{x} - \underline{y})$. Using a variable of integration $\underline{z}$ that is different from the source position $\underline{y}$ we obtain:

$$
\begin{aligned}
p(\underline{x},t) &= \int_V S\left(\underline{z}, t - \frac{|\underline{x}-\underline{z}|}{c_0}\right)\frac{dV(\underline{z})}{4\pi|\underline{x}-\underline{z}|} \\
&= \int_V \rho_0 \dot{q}\left(t - \frac{|\underline{x}-\underline{z}|}{c_0}\right)\delta(\underline{z}-\underline{y})\frac{dV(\underline{z})}{4\pi|\underline{x}-\underline{z}|}.
\end{aligned}
$$

Using the sifting property of the Dirac delta function:

$$
\boxed{p(\underline{x},t) = \frac{\rho_0}{4\pi|\underline{x}-\underline{y}|}\dot{q}\left(t - \frac{|\underline{x}-\underline{y}|}{c_0}\right) = \frac{\rho_0}{4\pi r}\dot{q}\left(t - \frac{r}{c_0}\right)} \tag{2.31}
$$

since $r = |\underline{x}-\underline{y}|$.

For a harmonic motion at angular frequency $\omega = 2\pi f$, $q(t) = q_0 e^{-i\omega t}$ and the complex acoustic pressure $p_c(r)$ is given by:

$$
p_c(r) = \frac{-i\omega\rho_0 q_0}{4\pi r}e^{ik_0 r} = -i\omega\rho_0 q_0 \tilde{G}_0(\omega, \underline{x}|\underline{y}). \tag{2.32}
$$

This corresponds to the solution obtained in the MF207 course (Cotté and Doaré, 2022-2023) for a pulsating sphere of surface velocity V_a whose radius a is much smaller than the acoustic wavelength ($k_0 a \ll 1$). See Figure 2.2(a). The time-averaged acoustic intensity and power associated with the monopole are:

$$
< I(r) > = \frac{\rho_0 c}{32\pi^2}\frac{k_0^2|q_0|^2}{r^2}, \tag{2.33}
$$

$$
< W_m > = \frac{\rho_0 c}{8\pi}k_0^2|q_0|^2. \tag{2.34}
$$

From Equation (2.31), it is apparent that the acoustic pressure radiated by a monopole is proportional to the time derivative of the volume velocity $q(t) = 4\pi a^2 V_a(t)$, with $\dot{q}(t)$ in m^3/s^2. Thus a surface acceleration must be present to obtain a nonzero acoustic pressure. As $k_0 q_0 = 4\pi k_0 a^2 V_a$ with $k_0 a \ll 1$, the acoustic power of the monopole is a small quantity.

2.2.3 Acoustic field radiated by a dipole

The dipole is another elementary source obtained by considering a point force, which means that the exterior forces imposed to the fluid per unit volume is written:

$$
\underline{f}_V(\underline{x},t) = \underline{f}(\underline{x},t)\delta(\underline{x}-\underline{y}) = \underline{F}(\underline{x},t)\delta(x_1-y_1)\delta(x_2-y_2)\delta(x_3-y_3). \tag{2.35}
$$

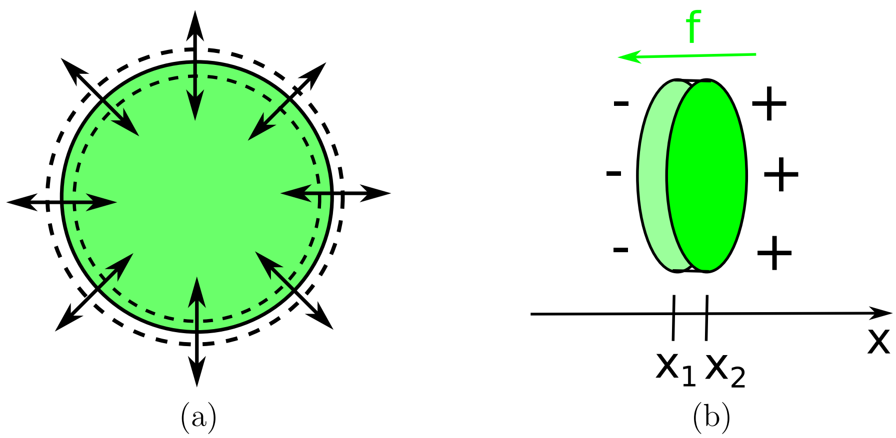


Figure 2.2: Schematics for (a) a pulsating sphere of radius a, (b) an oscillating thin disk of radius a

where $\underline{f}$ is the vector force and $\underline{y} = (y_1, y_2, y_3)$ is the position of the point force. As a result, in the absence of injected mass ($\dot{m}_V = 0$), the inhomogeneous wave equation (2.28) becomes:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = -\nabla.\underline{f}_V = -\underline{f}.\nabla\delta(\underline{x} - \underline{y}). \tag{2.36}$$

Let us suppose, without loss of generality, that the force is acting in the x direction, as shown in Figure 2.2(b). Then:

$$\underline{f}.\nabla\delta(\underline{x} - \underline{y}) = f_x \frac{\partial \delta(\underline{x} - \underline{y})}{\partial x}. \tag{2.37}$$

Since

$$\delta'(x_1) = \lim_{d \to 0} \frac{\delta(x_1 + d) - \delta(x_1)}{d}, \tag{2.38}$$

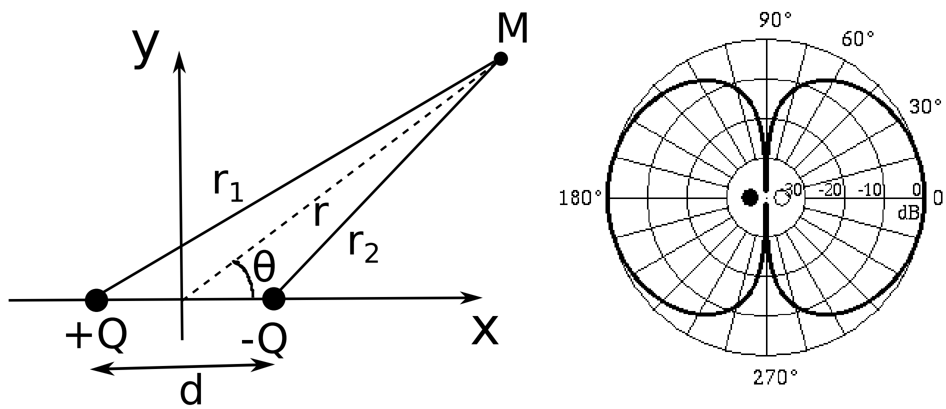
the inhomogeneous wave equation can be rewritten:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = f_x(t) \frac{\delta(\underline{x} - \underline{x_1}) - \delta(\underline{x} - \underline{x_2})}{d}, \tag{2.39}$$

with a small distance $d = x_2 - x_1$. The point force can thus be seen as the superposition of two monopoles of opposite signs.

For a harmonic oscillation $f_x(t) = F_x e^{-i\omega t}$, we obtain the following inhomogeneous Helmholtz equation:

$$\Delta p_c + k_0^2 p_c = -\frac{F_x}{d}\left[\delta(\underline{x} - \underline{x_1}) - \delta(\underline{x} - \underline{x_2})\right]. \tag{2.40}$$



(a) (b)

Figure 2.3: (a) Acoustic dipole seen as the combination of two monopoles of volume velocities $\pm Q$, and (b) associated directivity.

For each of the two monopoles, the solution is given by Equation (2.32) with $F_x/d = -i\omega\rho_0 q_0$, thus:

$$p_c(\underline{x}) = -i\omega\rho_0 q_0 \left[\tilde{G}_0(\omega, \underline{x}|\underline{x_1}) - \tilde{G}_0(\omega, \underline{x}|\underline{x_2})\right] = \frac{-i\omega\rho_0 q_0}{4\pi}\left[\frac{e^{ik_0|\underline{x}-\underline{x_1}|}}{|\underline{x}-\underline{x_1}|} - \frac{e^{ik_0|\underline{x}-\underline{x_2}|}}{|\underline{x}-\underline{x_2}|}\right].$$

Using the notations of Figure 2.3, with $\underline{x} = r\underline{e}_r$, we perform a Taylor series expansion for $d \ll r$ and $k_0 d \ll 1$:

$$\tilde{G}_0\left(r\underline{e_r}| \pm \frac{d}{2}\underline{e_x}\right) = \tilde{G}_0(r\underline{e_r}|\underline{0}) + \nabla\tilde{G}_0.\left(\pm\frac{d}{2}\underline{e_x}\right) + ... \qquad (2.41)$$

The pressure field radiated by a dipole oriented along x is thus given by:

$$p_c(r,\theta) = -i\omega\rho_0 q_0 d\cos\theta\frac{\partial\tilde{G}_0}{\partial r} = i\omega\rho_0 q_0 d\left(-ik_0 + \frac{1}{r}\right)\cos\theta\frac{e^{ik_0 r}}{4\pi r}, \qquad (2.42)$$

with $\cos\theta = \underline{e}_x.\underline{e}_r$.

This expression can be extended to a dipole of direction $\underline{n}$:

$$p_c(\underline{x}) = -\underline{F}.\nabla[\tilde{G}_0(\omega, \underline{x}|\underline{y})] = -\underline{F}.\underline{e}_r\left(-ik_0 + \frac{1}{r}\right)\frac{e^{ik_0 r}}{4\pi r}, \qquad (2.43)$$

where $\underline{F} = -i\omega\rho_0 q_0 d\underline{n}$. The angle θ is defined from the axis of the dipole along the unit vector $\underline{n}$: $\cos\theta = \underline{n}.\underline{e}_r = \underline{n}.(\underline{x} - \underline{y})/r$. Noting that $k_0 = \omega/c_0$, it is possible to obtain the expression in the time domain using an inverse Fourier transform:

$$p(\underline{x}, t) = \frac{1}{4\pi r}\underline{e}_r.\left(\frac{1}{c_0}\frac{\partial}{\partial t} + \frac{1}{r}\right)\underline{f}\left(t - \frac{r}{c_0}\right) = \frac{(x_i - y_i)}{4\pi r^2}\left(\frac{1}{c_0}\frac{\partial}{\partial t} + \frac{1}{r}\right)f_i\left(t - \frac{r}{c_0}\right). \tag{2.44}$$

In the far-field ($|\underline{y}| \ll |\underline{x}|$), $x_i - y_i \approx x_i$ and the second term in the parenthesis can be neglected thus:

$$p(\underline{x}, t) \approx \frac{1}{4\pi c_0 r}\underline{e}_r.\underline{f}'\left(t - \frac{r}{c_0}\right) = \frac{1)}{4\pi r c_0}\frac{x_i}{r}f_i'\left(t - \frac{r}{c_0}\right). \tag{2.45}$$

One important feature of the dipole that is apparent in Equations (2.43) and (2.44) is that the acoustic pressure radiation depends on the angle θ. It is maximal in the axis of the dipole ($\theta_n = n\pi$) and minimal in the perpendicular direction ($\theta_n = (2n+1)\pi/2$). The associated directivity factor is given by:

$$D(\theta) = \frac{|p_c(r,\theta)|}{\max_\theta |p_c(r,\theta)|} = |\cos\theta|, \tag{2.46}$$

and is plotted in Figure 2.3(b). The time-averaged acoustic intensity and power associated with the dipole are (Cotté and Doaré, 2022-2023):

$$< I(r) > = \frac{|p_c(r,\theta)|^2}{2\rho_0 c_0} = \frac{\rho_0 c_0 k_0^4 d^2 |q_0|^2}{32\pi^2 r^2}\cos^2\theta, \tag{2.47}$$

$$< W_d > = \frac{\rho_0 c_0 k_0^4 d^2 |q_0|^2}{24\pi}. \tag{2.48}$$

The ratio $< W_d > / < W_m >= (k_0 d)^2/3 \ll 1$, which shows that the dipole is a much less efficient acoustic source compared to the monopole.

2.3 Acoustic radiation of moving elementary sources

This section is mostly based on Morse and Ingard (1968, Section 11.2) and Rienstra and Hirschberg (2021, Section 9.2).

2.3.1 Monopole in uniform translation

For a monopole of volume velocity $q(t)$ moving along the axis x_1 at velocity V, as shown in Figure 2.4, the sound field is governed by the following wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = \rho_0 \frac{\partial}{\partial t}\left[q(t)\delta(x_1 - Vt)\delta(x_2 - y_2)\delta(x_3 - y_3)\right]. \tag{2.49}$$

Introducing the velocity potential Φ, related to pressure by Equation (1.39):

$$p(\underline{x}, t) = -\rho_0 \frac{\partial \Phi(\underline{x}, t)}{\partial t}, \tag{2.50}$$

the wave equation can be rewritten:

$$\frac{1}{c_0^2}\frac{\partial^2 \Phi}{\partial t^2} - \Delta \Phi = -q(t)\delta(x_1 - Vt)\delta(x_2 - y_2)\delta(x_3 - y_3). \tag{2.51}$$

In the following, we set $y_2 = y_3 = 0$, as can be seen in Figure 2.4.

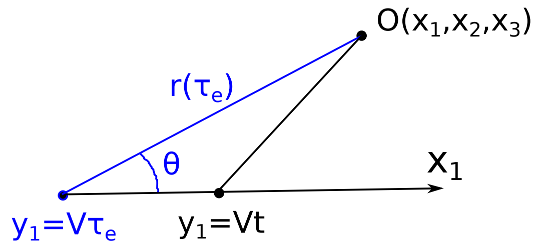


Figure 2.4: Source in uniform rectilinear motion.

This equation can be solved using different methods. Morse and Ingard (1968) use a coordinate transformation called Lorentz transformation to retrieve the problem of the noise radiation by a stationary source. We choose here to use the method of the Green's function, as it will be used later in Chapter 5 to find integral solutions of various acoustic analogies. Using Equation (2.24) along with the free-field Green's function given by:

$$G_0(\underline{x}, t|\underline{y}, \tau) = \frac{1}{4\pi r(\tau)}\delta\left(t - \tau - \frac{r(\tau)}{c_0}\right), \tag{2.52}$$

with $r(\tau) = |\underline{x} - \underline{y}(\tau)| = \sqrt{(x_1 - Vt)^2 + x_2^2 + x_3^2}$, the integral solution for the velocity potential Φ is given by:

$$\Phi(\underline{x}, t) = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{q(\tau)}{r(\tau)} \delta\left(t - \tau - \frac{r(\tau)}{c_0}\right) d\tau. \tag{2.53}$$

We can now find the solution using the following property of the delta function (Rienstra and Hirschberg, 2021, Equation (C.28)):

$$\int_{-\infty}^{+\infty} \delta[g(\tau)] h(\tau) d\tau = \sum_i \frac{h(\tau_i)}{|\frac{\partial g}{\partial \tau}(\tau_i)|}, \tag{2.54}$$

where τ_i are the solutions of $g(\tau_i) = 0$. As $g(\tau) = t - \tau - r(\tau)/c_0$, the emission times τ_i are given by:

$$\tau_i = t - \frac{r(\tau_i)}{c_0} = t - \frac{1}{c_0}\sqrt{(x_1 - V\tau_i)^2 + x_2^2 + x_3^2}. \tag{2.55}$$

Squaring this equation, we find the following quadratic equation for τ_i:

$$(1 - M^2)\tau_i^2 + 2\left(\frac{x_1 M}{c_0} - t\right)\tau_i + \left(t^2 - \frac{x_1^2 + x_2^2 + x_3^2}{c_0^2}\right), \tag{2.56}$$

with $M = V/c_0$ the Mach number, and the emission times are:

$$\tau_i = \frac{(c_0 t - M x_1) \pm \sqrt{(x_1 - Vt)^2 + (1 - M^2)(x_2^2 + x_3^2)}}{c_0(1 - M^2)}. \tag{2.57}$$

There are two possible solutions for this equation, but for subsonic motion ($M < 1$), only the negative sign is possible in order to meet the causality condition $\tau < t$. The emission time τ_e is thus given for subsonic motion ($M < 1$) by:

$$\tau_e = \frac{(c_0 t - M x_1) - \sqrt{(x_1 - Vt)^2 + (1 - M^2)(x_2^2 + x_3^2)}}{c_0(1 - M^2)}. \tag{2.58}$$

Furthermore:

$$\frac{\partial g}{\partial \tau} = -1 - \frac{1}{c_0}\frac{\partial r}{\partial \tau} = -1 - \frac{1}{c_0}\frac{\partial y_1}{\partial \tau}\frac{\partial r}{\partial y_1} = -1 + \frac{V}{c_0}\frac{r_1}{r} = -1 + M_r, \tag{2.59}$$

with $r_1 = (x_1 - y_1)$, and noting that $y_1 = V\tau$. The quantity M_r is the Mach number of the source in the direction of the observer. Note also that:

$$\frac{\underline{\underline{M}}.\underline{r}}{r} = M\frac{r_1}{r} = M_r = M\cos\theta. \tag{2.60}$$

As a result, the application of Equation (2.54) for a monopole in subsonic motion yields:

$$\Phi(\underline{x},t) = -\frac{1}{4\pi}\frac{q(\tau_e)}{r(\tau_e)[1-M_r(\tau_e)]} = -\frac{1}{4\pi}\frac{q(\tau_e)}{r(\tau_e)[1-M\cos\theta(\tau_e)]}, \tag{2.61}$$

with τ_e the emission time given by Equation 2.58.

We thus deduce for the acoustic pressure:

$$\begin{aligned} p(\underline{x},t) = -\rho_0\frac{\partial\Phi(\underline{x},t)}{\partial t} &= \frac{\rho_0}{4\pi r(\tau_e)[1-M\cos\theta(\tau_e)]}\frac{\partial q}{\partial t}(\tau_e) \\ &\quad + \frac{\rho_0 q(\tau_e)}{4\pi}\frac{\partial}{\partial t}\left(\frac{1}{r(\tau_e)[1-M\cos\theta(\tau_e)]}\right). \end{aligned} \tag{2.62}$$

Since

$$\frac{\partial r}{\partial \tau} = \frac{\partial y_1}{\partial \tau}\frac{\partial r}{\partial y_1} = -V\frac{r_1}{r} = -c_0 M_r, \tag{2.63}$$

$$\frac{\partial \tau}{\partial t} = \left(\frac{\partial t}{\partial \tau}\right)^{-1} = \left(1 + \frac{1}{c_0}\frac{\partial r}{\partial \tau}\right)^{-1} = (1-M_r)^{-1}, \tag{2.64}$$

with $M_r = M\cos\theta$, the derivative in the first term is calculated as:

$$\frac{\partial q}{\partial t}(\tau_e) = \frac{\partial \tau}{\partial t}\frac{\partial q}{\partial \tau}(\tau_e) = \frac{q'(\tau_e)}{1-M_r}. \tag{2.65}$$

The derivative in the second term is evaluated using:

$$\frac{\partial}{\partial t}\left(\frac{1}{r(\tau_e)[1-M_r(\tau_e)]}\right) = -\frac{1}{\left(r(\tau_e)[1-M_r(\tau_e)]\right)^2}\frac{\partial}{\partial t}\left(r(\tau_e)[1-M_r(\tau_e)]\right).$$

This derivative can be calculated as follows:

$$\frac{\partial r}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial r}{\partial \tau} = -\frac{c_0 M_r}{1-M_r} \tag{2.66}$$

$$\frac{\partial}{\partial t}(rM_r) = \frac{\partial \tau}{\partial t}\frac{\partial\left(rM\frac{r_1}{r}\right)}{\partial \tau} = \frac{M}{1-M_r}\frac{\partial(x_1 - V\tau)}{\partial \tau} = -\frac{VM}{1-M_r}, \tag{2.67}$$

where we have used Equation (2.60). Finally, the acoustic pressure due a monopole moving at a uniform velocity $V = c_0 M$ is given by:

$$p(\underline{x}, t) = \frac{\rho_0 q'(\tau_e)}{4\pi r(\tau_e)[1 - M\cos\theta(\tau_e)]^2} + \frac{\rho_0 q(\tau_e) c_0 M\,(\cos\theta(\tau_e) - M)}{4\pi r(\tau_e)^2[1 - M\cos\theta(\tau_e)]^3}. \quad (2.68)$$

The first term is proportional to $1/r$ and dominates the far-field, while the second term is only important in the near-field. The amplitude of these two terms are increased by a power of the Doppler factor $(1 - M_r)^{-1} = (1 - M\cos\theta(\tau_e))^{-1}$, due to the relative motion between source and receivers. This effect is called **convective amplification.**

Another consequence of the relative motion between source and receivers is the frequency shift that appears for harmonic sources. Let us consider $q(t) = q_0 \exp(-i\omega_0 t)$ in the far-field approximation of Equation (2.68):

$$p(\underline{x}, t) \approx -\frac{i\rho_0 q_0 \omega_0 \exp\left[-i\omega_0(t - r(\tau_e)/c_0)\right]}{4\pi r(\tau_e)[1 - M\cos\theta(\tau_e)]^2}. \quad (2.69)$$

The phase

$$\phi = \omega_0 \left(t - \frac{r(\tau_e)}{c_0}\right) \quad (2.70)$$

is not proportional to observe time t as the source-observe distance r is time-dependent. If we generalize the concept of frequency and define it as the time derivative of the phase, then:

$$\omega = \frac{\partial \phi}{\partial t} = \omega_0 \left(1 - \frac{1}{c_0}\frac{\partial r(\tau_e)}{\partial t}\right) = \omega_0 \left(1 + \frac{M\cos\theta(\tau_e)}{1 - M\cos\theta(\tau_e)}\right) = \frac{\omega_0}{1 - M\cos\theta(\tau_e)}, \quad (2.71)$$

using Equation (2.66). This is the Doppler formula, that shows that the frequency decreases from $\omega_0/(1-M)$ to $\omega_0/(1+M)$ when the sources moves past the observer (θ varies from 0 to π). Figure 2.5 shows the frequency shift due to the Doppler effect and the pressure amplitude ratio that is quantified by the factor:

$$\left|\frac{p(\underline{x}, t)}{p_0(\underline{x}, t)}\right| = \frac{r_0}{r(\tau_e)[1 - M\cos\theta(\tau_e)]^2}, \quad (2.72)$$

with p_0 the acoustic pressure for a stationary source at a distance $r_0 = \sqrt{x_2^2 + x_3^2}$ from the observer. The Doppler shift is better understood when looking at a map of the real part of the acoustic pressure at a given observer time t as shown in Figure 2.6 for Mach numbers of 0.25 and 0.5. It appears that as the source velocity is increased, the wavefronts become closer and closer in front of the source (frequency increase at the receiver), and farther away behind the source (frequency decrease at the receiver).

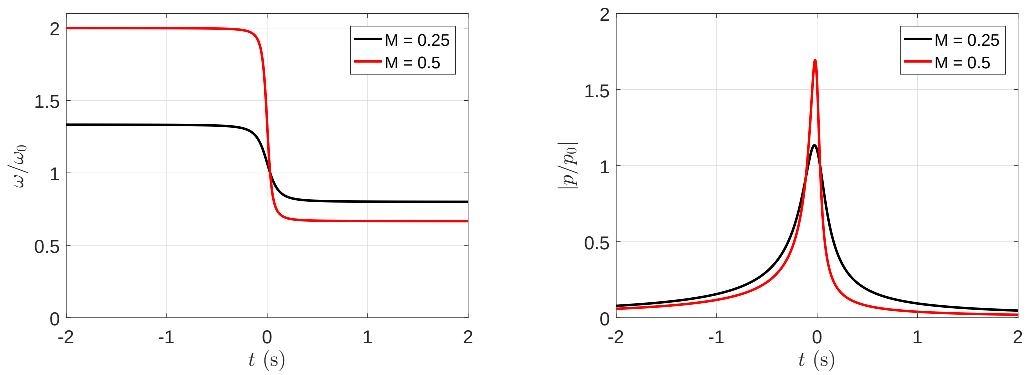


Figure 2.5: (a) Doppler factor ω/ω_0 and (b) pressure amplitude ratio $|p/p_0|$ for an observer at $r_0 = 10\,\mathrm{m}$ moving at Mach numbers 0.25 or 0.5.

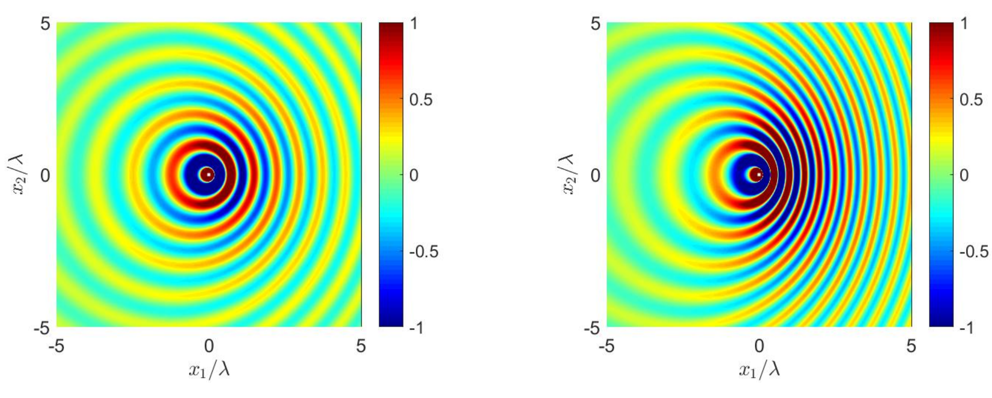


Figure 2.6: Map of the real part of the acoustic pressure at a given observer time t at (a) $M = 0.25$ or (b) $M = 0.5$.

2.3.2 Acoustic radiation of moving dipoles

The previous derivation for a moving monopole can be generalized to any moving point source:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = S(\underline{x}, t) = g(\underline{x}, t)\delta(\underline{x} - \underline{y}(t)), \tag{2.73}$$

with $\underline{y}(t)$ the motion of the source. For instance, for a moving dipole the source term is given by:

$$g(\underline{x}, t) = \frac{\partial f_i}{\partial x_i}, \tag{2.74}$$

and the far-field acoustic pressure is (Gloerfelt, 2016):

$$p(\underline{x}, t) \approx -\frac{1}{4\pi r(\tau_e) c_0}\frac{x_i}{x}\frac{f_i'}{[1 - M_r(\tau_e)]^2}. \tag{2.75}$$

Chapter 3

Absorption and refraction effects in inhomogeneous moving media

We use slightly different notations in this chapter. The mean sound speed associated with the base flow is written $c(\underline{x})$, and $k(\underline{x}) = \omega/c(\underline{x})$ is the associated acoustic wave number.

3.1 Acoustic absorption

3.1.1 Attenuation due to acoustic absorption

Acoustic absorption can be modeled using a complex wave number $k^* = k + i\alpha$, with α the absorption coefficient in Np/m. As a result, a harmonic spherical wave is now written:

$$p_c(r) = S\frac{e^{ikr}}{r}e^{-\alpha r}. \tag{3.1}$$

As a result, the relationship (2.14) between the sound pressure level and the sound power level in free field becomes (Salomons, 2001):

$$\boxed{L_p = L_W - 10\log_{10}(4\pi r^2) - ar}\ , \tag{3.2}$$

with $a = 20/\ln(10)\alpha \approx 8.686\alpha$ the absorption coefficient in dB/m.

3.1.2 Mechanisms of acoustic absorption in the atmosphere and in the ocean

A sound wave loses energy due to various irreversible processes that remove energy from an acoustic wave and convert it to heat:

- viscous losses and heat conduction losses (so-called classical absorption);
- relaxation losses of constituents.

The relaxation losses exist for polyatomic gases, and are associated with the change of rotational or translational energy of the molecules into internal energy (Evans *et al.*, 1972).

Pierce (1989, Section 10-8) obtains the following dispersion equation for a plane traveling wave including classical absorption and various relaxation processes ν:

$$k^* = \frac{\omega}{c_0} + i\ \alpha_{cl} + \frac{1}{\pi}\frac{\omega}{c}\sum_{\nu}(\alpha_\nu\lambda)_{\max}\frac{i\omega\tau_\nu}{1-i\omega\tau_\nu}, \qquad (3.3)$$

where α_{cl} is the classical absorption coefficient that is proportional to ω^2, $(\alpha_\nu\lambda)_{\max}$ corresponds to the maximum absorption per wavelength associated with the ν-type relaxation process, τ_ν is the relaxation time for the vibrational energy of type ν, and:

$$c_0 = \frac{c}{1+\frac{1}{\pi}\sum_\nu(\alpha_\nu\lambda)_{\max}}. \qquad (3.4)$$

Since $\lim_{x\to 0}\frac{k}{\omega} = \frac{1}{c_0}$, c_0 corresponds to the phase velocity in the limit of zero frequency, while c corresponds to the phase velocity in the high-frequency limit where $\omega\tau\nu \gg 1$ for all relaxation processes ν.

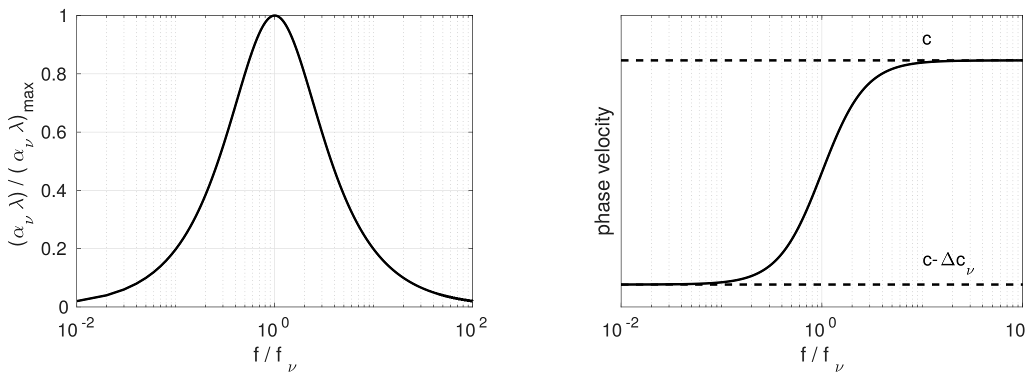
The absorption coefficient α is the imaginary part of k^*, and is thus written from Equation (3.3):

$$\alpha(f) = \alpha_{cl}(f) + \sum_\nu \alpha_\nu(f) = A_{vt}f^2 + \sum_\nu \frac{2}{c}(\alpha_\nu\lambda)_{\max}\frac{f_\nu f^2}{f_\nu^2+f^2}, \qquad (3.5)$$

with $f_\nu = 1/(2\pi\tau_\nu)$ the relaxation frequency of constituent ν. Pierce (1989, Section 10-8) shows that the absorption per wavelength of the relaxation process ν can be written:

$$\frac{\alpha_\nu \lambda}{(\alpha_\nu \lambda)_{\max}} = \frac{2}{f_\nu / f + f / f_\nu}. \tag{3.6}$$

As shown in Figure 3.1(a), the absorption is maximum at the relaxation frequency f_ν, and goes to zero for $f \ll f_\nu$ and $f \gg f_\nu$.



(a) (b)

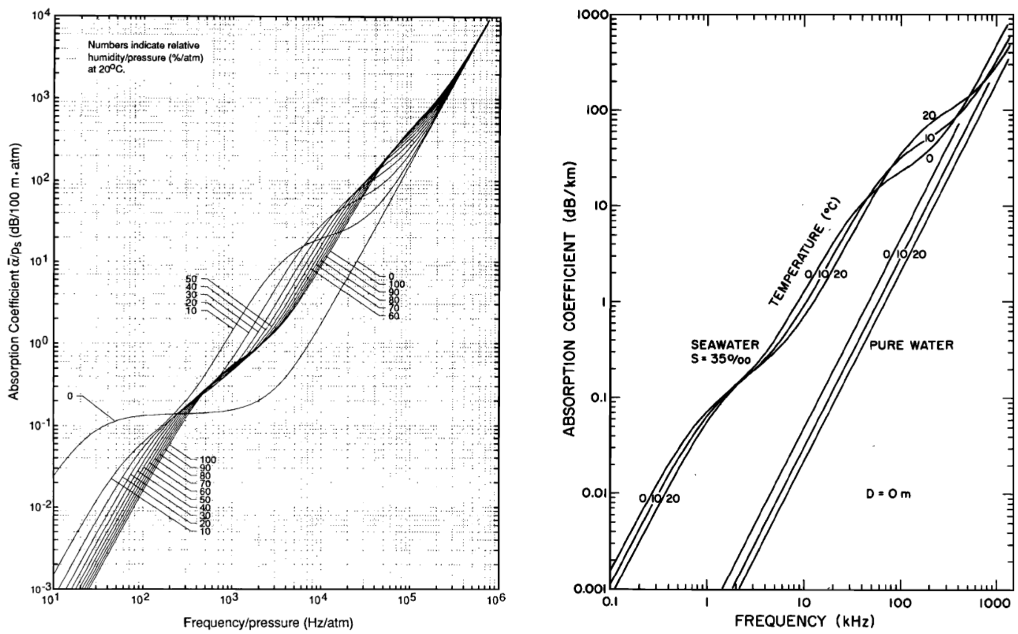
Figure 3.1: (a) Absorption per wavelength, and (b) phase velocity with respect to the normalized frequency f/f_ν for a single relxation process ν. Adapted from Pierce (1989, Fig. 10-12)

Equation (3.3) also shows that the phase velocity $v_{ph} = \omega / k_R$, with k_R the real part of k^*, depends on frequency because of relaxation processes, which means that the medium is dispersive. The phase velocity can be written:

$$v_{ph} = \frac{\omega}{k_R} = c - \sum_\nu \frac{\Delta c_\nu}{1 + (f/f_\nu)^2}, \tag{3.7}$$

with $\Delta c_\nu = (\alpha_\nu \lambda)_{\max} c / \pi$. This phase velocity is plotted in Figure 3.1(b) for a single relaxation process ν. In practice, Δc_ν is small and the approximation $k_R = \omega / c$ is generally used.

In the atmosphere, the acoustic absorption of air depends on pressure, temperature and humidity. The relaxation processes to take into account are due to nitrogen (N_2) and oxygen molecules (O_2), where $f_{N_2} \ll f_{O_2}$. The expressions for the absorption coefficient can be found for instance in Pierce (1989, Chapter 10), Bass *et al.* (1995, 1996) and Salomons (2001, Appendix B). See Figure 3.2(a).



(a) (b)

Figure 3.2: Absorption coefficient (a) for air in dB/100m/atm at 20^{o}C and for various relative humidities in % (taken from Bass *et al.* (1995)), and (b) for seawater in dB/km for a salinity of 35‰and a pH of 8 (from Francois and Garrison (1982b)).

In the ocean, the acoustic absorption of seawater depends on on pressure (or depth), temperature, salinity and acidity (pH). The relaxation processes to take into account are due to boric acid ($B(OH)_3$) and magnesium sulphate ($MgSO_4$), where $f_{B(OH)_3} \ll f_{MgSO_4}$. The expressions for the absorption coefficient can be found for instance in Francois and Garrison (1982a,b) and in Ainslie and McColm (1998). See Figure 3.2(b).

3.2 Refraction effects

3.2.1 Refraction due to vertical sound speed gradients

Refraction happens when sound waves propagate through fluid layers of varying sound speeds. Let us consider the simple case of Figure 3.3, where a sound ray propagates through two fluid layers of sound speed c_1 and c_2. For the moment, a sound ray is defined as a narrow beam of high frequency sound. A more precise definition will be given in Chapter 4 devoted to geometrical acoustics.

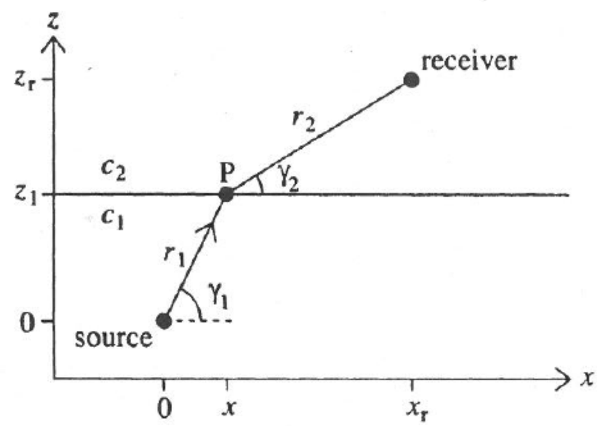


Figure 3.3: Refraction of sound between 2 layers with different sound speeds c_1 and $c_2 > c_1$. Taken from Salomons (2001).

According to Fermat's principle, the wave takes the path where the travel time is minimum. The travel time between source and receiver is given here by:

$$\tau = \frac{r_1}{c_1} + \frac{r_2}{c_2} = \frac{\sqrt{x^2 + z_1^2}}{c_1} + \frac{\sqrt{(x_r - x)^2 + (z_r - z_1)^2}}{c_2}. \tag{3.8}$$

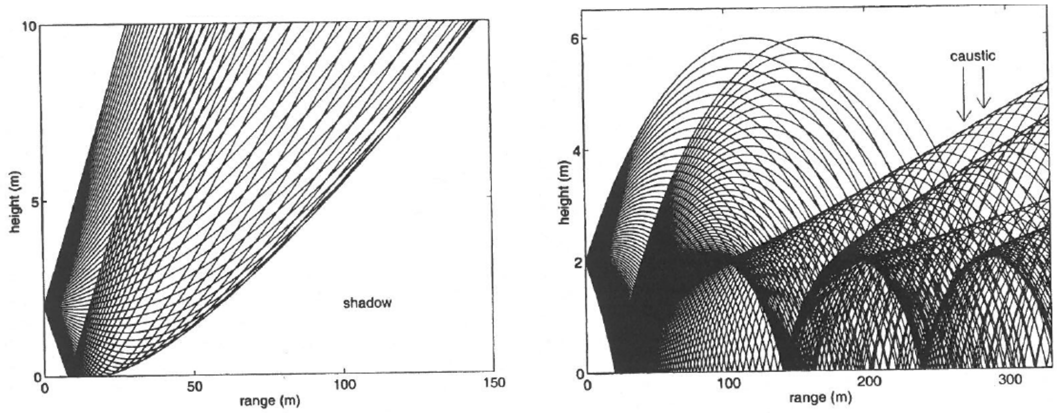
Let us find the coordinate x of point P that minimizes τ:

$$\frac{\partial \tau}{\partial x} = \frac{x/r_1}{c_1} - \frac{(x_r - x)/r_2}{c_2} = 0 \Rightarrow \frac{\cos\gamma_1}{c_1} = \frac{\cos\gamma_2}{c_2} \tag{3.9}$$

This expression is known as the **Snell-Descartes law**. It can be generalized to multiple layers of fluid, or to a stratified medium with sound speed $c(z)$:

$$\boxed{\frac{\cos\gamma(z)}{c(z)} = \text{constant along a sound ray}}\ . \tag{3.10}$$

This generalized Snell-Descartes law states that **the sound ray bends towards the region of lower sound speed**.



(a) (b)

Figure 3.4: Sound rays from a source at 2 meter height using a logarithmic sound speed profile $c(z) = c_0 + b \ln(1 + z/z_0)$ (a) in an upward-refracting atmosphere ($b = -1\,\mathrm{m/s}$), and (b) in a downward-refracting atmosphere ($b = 1\,\mathrm{m/s}$). Taken from Salomons (2001).

This variation of sound speed with altitude (in the atmosphere) or depth (in the ocean) z is commonly found. This comes from the fact that temperature typically varies with z, thus for an ideal fluid:

$$c(z) = \sqrt{\gamma r T(z)} = c_0 \sqrt{\frac{T(z)}{T_0}}. \tag{3.11}$$

Two ray-tracing examples Salomons (2001) are plotted in Figure 3.4 to illustrate refraction effects in a layered atmosphere. A logarithmic sound speed profile is considered:

$$c(z) = c_0 + b \ln\left(1 + \frac{z}{z_0}\right), \tag{3.12}$$

with $c_0 = 340\,\mathrm{m/s}$, $z_0 = 0.1\,\mathrm{m}$, and $b = \pm 1\,\mathrm{m/s}$.

When $b = -1\,\mathrm{m/s}$, the sound speed decreases with height so the sound rays bend upwards according to Snell-Descartes law. This is called an **upward-refracting atmosphere**. This is a typical daytime situation, also referred to as normal lapse. The sun heats the ground, so the air close to the ground is warmer than the air at higher altitudes. As a result, a shadow zone forms

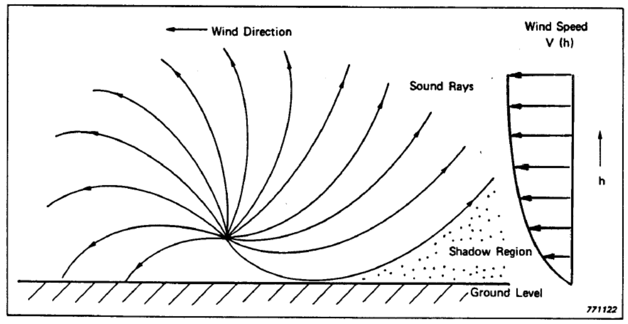


Figure 3.5: Sound rays illustrating the effect of refraction by wind speed vertical gradients. Taken from Lamancusa (2009, Section 10).

close to the ground where no sound can penetrate in the geometric approximation, as explained in Section 4.3. In reality, sound goes into the shadow zone due to diffraction effects.

When $b = +1\,\mathrm{m/s}$, the sound speed increases with height so the sound rays bend downwards according to Snell-Descartes law. This is called an **downward-refracting atmosphere**. This is a typical nighttime situation, also referred to as normal inversion. As a result, there can be multiple rays between source and receiver with multiple reflections on the ground, which is a favorable situation for acoustic propagation over longer distance.

3.2.2 Refraction due to wind speed gradients

Because of friction, the wind speed in the atmospheric boundary layer decreases to zero at the ground. Strong wind speed gradients are thus encountered close to the ground, and typically decrease with height.

An equivalence can be made between the effect of vertical gradients of wind speed and temperature on sound waves. Indeed, as shown in Figure 3.5, downward refraction occurs in the **downwind direction** or for **temperature inversion**, while upward refraction occurs in the **upwind direction** or for **temperature lapse**.

It is possible to take into account wind speed gradients in an approximate way using the **effective sound speed approximation**. The effective sound spped is defines as:

$$c_{eff}(z) = c(z) + U(z)\cos\phi, \tag{3.13}$$

where ϕ is the angle between wind direction and propagation direction. This approximation is generally valid for source and receivers close to the ground, as will be seen in Chapter 4.

Chapter 4

Geometrical acoustics

We use the same notations as in Chapter 3, with $c(\underline{x})$ the mean sound speed associated with the base flow and $k(\underline{x}) = \omega/c(\underline{x})$ the associated acoustic wave number. This chapter is mostly based on the work of Pierce (1989, Chapter 8), on the course notes of Bailly (2020) and on the PhD thesis of Gainville (2008).

4.1 Equations of geometrical acoustics

In the geometrical acoustics approximation, we are looking for a high-frequency solution, in the limit $\lambda \ll \ell$, where ℓ the characteristic scale of variation of the ambient quantities in space. The solution is sought as a local plane wave:

$$p'(\underline{x}, t) = \hat{P}(\underline{x}) e^{i\Theta(\underline{x},t)}. \tag{4.1}$$

For a harmonic plane wave, the amplitude $\hat{P}(\underline{x})$ is a constant and the phase $\Theta(\underline{x}, t) = \underline{k}.\underline{x} - \omega t$, where $\underline{k}$ and ω are constants. We generalize this concept, and define the local wavenumber vector $\underline{k}$ and the local angular frequency ω as:

$$\underline{k}(\underline{x}, t) = \nabla\Theta, \tag{4.2}$$

$$\omega(\underline{x}, t) = -\frac{\partial\Theta}{\partial t}. \tag{4.3}$$

The amplitude $\hat{P}$ and the wavenumber k are supposed to vary slowly with position x over a scale $\lambda = 2\pi/k$, which corresponds to $\lambda \ll \ell$. We will thus introduce a small parameter $\epsilon \sim \lambda/\ell$, and develop the pressure amplitude in an asymptotic series in power of ϵ:

$$\hat{P}(\underline{x}) = \hat{P}_0(\underline{x}) + \epsilon \hat{P}_1(\underline{x}) + \epsilon^2 \hat{P}_2(\underline{x}) + ... \tag{4.4}$$

We apply this approximation first to the wave equation in a homogeneous medium at rest, and then to the linearized Euler equations in an inhomogeneous moving medium.

4.1.1 In a homogeneous medium at rest

Let us start from the wave equation in a homogeneous medium at rest of constant sound speed c_0:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0. \tag{4.5}$$

Introducing $p'(\underline{x}, t) = \hat{P}(\underline{x}) e^{i\Theta(\underline{x},t)}$, the time derivative of the acoustic pressure becomes:

$$\frac{\partial p'}{\partial t} = i\frac{\partial \Theta}{\partial t}\hat{P}e^{i\Theta} = -i\omega \hat{P} e^{i\Theta} \Rightarrow \frac{\partial^2 p'}{\partial t^2} = -\omega^2 \hat{P} e^{i\Theta}.$$

Noting that $\nabla \left(e^{i\Theta(\underline{x},t)}\right) = i(\nabla\Theta)e^{i\Theta} = i\underline{k}e^{i\Theta}$, the Laplacian term is written:

$$\nabla^2 p' = \left(\nabla^2 \hat{P} + 2i\underline{k}.\nabla\hat{P} + i(\nabla.\underline{k})\hat{P} - k^2\hat{P}\right) e^{i\Theta}. \tag{4.6}$$

As a result, the Helmholtz equation is rewritten:

$$\underbrace{\nabla^2 \hat{P}}_{\propto 1/\ell^2} + \underbrace{\left(2i\underline{k}.\nabla\hat{P} + i(\nabla.\underline{k})\hat{P}\right)}_{\propto 1/\lambda\ell} - \hat{P}\underbrace{\left(k^2 - \frac{\omega^2}{c_0^2}\right)}_{\propto 1/\lambda^2} = 0. \tag{4.7}$$

As the amplitude $\hat{P}$ and the wavenumber k are supposed to vary slowly with position x, the terms $\nabla\hat{P}$ and $\nabla.\underline{k}$ are proportional to $1/\ell$. We can now introduce the asymptotic expression for the pressure amplitude in terms of the small parameter $\epsilon \sim \lambda/\ell$. Grouping terms by power of ϵ:

$$\epsilon^0: \quad \hat{P}_0\left(k^2 - \frac{\omega^2}{c_0^2}\right) = 0 \tag{4.8}$$

$$\epsilon^1: \quad i\left(2\underline{k}.\nabla\hat{P}_0 + (\nabla.\underline{k})\hat{P}_0\right) - \hat{P}_1\cancel{\left(k^2 - \frac{\omega^2}{c_0^2}\right)} = 0 \tag{4.9}$$

$$\epsilon^2: \quad ... \tag{4.10}$$

The first equation corresponds to the Eikonal equation or dispersion relation:

$$k^2 = \frac{\omega^2}{c_0^2}. \tag{4.11}$$

The solution to this equation corresponds to the **rays** connecting the wave-front surfaces (straight lines for a homogeneous medium), as will be seen in Section 4.2. The second equation corresponds to the transport equation where we suppose $\hat{P} \approx \hat{P}_0$ at first order:

$$2\underline{k}.\nabla\hat{P} + (\nabla.\underline{k})\hat{P} \quad \text{or} \quad \nabla.(\hat{P}^2\underline{k}) = 0. \tag{4.12}$$

The transport equation gives the **variation of amplitude** along ray paths as will be seen in Section 4.3.

To find the domain of validity of the geometrical acoustics approximation, we notice that we basically neglected the first term in Equation (4.7) in order to obtain the Eikonal and transport equations. As a result, the geometrical acoustics is valid when:

$$\left|\frac{\nabla^2\hat{P}}{\hat{P}}\right| \ll \left(\frac{\omega}{c_0}\right)^2. \tag{4.13}$$

Thus the higher the frequency the better the approximation.

4.1.2 In a moving medium

Let us start for the set of linearized Euler equations (1.20)-(1.22) given in Section 1.2. We will focus on Equations (1.21) and (1.22) that do not depend on ρ'. The expression for the particle velocity is similar to the expression (4.1) for the acoustic pressure:

$$\underline{v}'(\underline{x}, t) = \underline{\hat{V}}(\underline{x})e^{i\Theta(\underline{x},t)}. \tag{4.14}$$

Introducing these expressions in the momentum equation (1.21), we obtain:

$$\rho_0\left(-i\omega + i\underline{k}.\underline{v_0}\right)\underline{\hat{V}} + i\underline{k}\hat{P} = -\rho_0(\underline{v_0}.\nabla)\underline{\hat{V}} - \rho_0(\underline{\hat{V}}.\nabla)\underline{v_0} - \nabla\hat{P} \tag{4.15}$$

Terms on the left hand-side of the equation are proportional to $\omega \sim 1/\lambda$, so they are dominant with respect to the terms on the right hand-side that are proportional to $1/\ell$, as $\hat{V}$, $\hat{P}$ and v_0 are supposed to vary slowly with $\underline{x}$. Similarly, we obtain from Equation (1.22):

$$\left(-i\omega + i\underline{k}.\underline{v_0}\right)\underline{\hat{P}} + \rho_0 c^2 i\underline{k}.\underline{\hat{V}} = -(\underline{v_0}.\nabla)\hat{P} - \rho_0 c^2\nabla.\underline{\hat{V}}. \tag{4.16}$$

The system of equations can be recast in the following form:

$$\rho_0 id\underline{\hat{V}} + i\underline{k}\hat{P} = \underline{A} \tag{4.17}$$

$$id\underline{\hat{P}} + \rho_0 c^2 i\underline{k}.\underline{\hat{V}} = B, \tag{4.18}$$

with $d = \underline{k}.\underline{v_0} - \omega$, where the terms $\underline{A}$ and B are small compared to the terms on the left hand-side.

This system of equations can be solved using the general eikonal method. In a 2D space, with $\underline{\hat{V}} = (\hat{V}_x, \hat{V}_y)$, $\underline{A} = (A_x, A_y)$ and $\underline{k} = (k_x, k_y)$, the system of equations can be rewritten in matrix form:

$$\mathbf{H}\begin{pmatrix}\hat{V}_x\\ \hat{V}_y\\ \hat{P}\end{pmatrix} = \mathbf{\Lambda} \Leftrightarrow \begin{pmatrix}\rho_0 id & 0 & ik_x\\ 0 & \rho_0 id & ik_y\\ \rho_0 c^2 ik_x & \rho_0 c^2 ik_y & id\end{pmatrix}\begin{pmatrix}\hat{V}_x\\ \hat{V}_y\\ \hat{P}\end{pmatrix} = \begin{pmatrix}A_x\\ A_y\\ B\end{pmatrix}, \tag{4.19}$$

with $d = k_x v_{0x} + k_y v_{0y} - \omega$. Introducing the asymptotic series (4.4) of $\hat{P}$ in power of ϵ, and noting that $\mathbf{\Lambda}/\mathbf{H} = \mathcal{O}(\epsilon)$, we obtain the condition $\mathbf{H}(\hat{V}_{0x}, \hat{V}_{0y}, \hat{P}_0)^T = 0$ at order 0. The condition $\det(\mathbf{H}) = 0$ for having non-trivial solutions yields the dispersion relation $D(\underline{k}, \omega, \underline{x}) = 0$:

$$\det(\mathbf{H}) = \rho_0 id\left(-\rho_0 d^2 + \rho_0 c^2 k_y^2\right) + ik_x\left(\rho_0^2 c^2 k_x^2 d\right) = \rho_0^2 id\left(c^2 k_x^2 + c^2 k_y^2 - d^2\right), \tag{4.20}$$

thus the dispersion relation is given by:

$$D(\underline{k}, \omega, \underline{x}) = \left(\underline{k}.\underline{v_0} - \omega\right)\left[c^2 k^2 - \left(\underline{k}.\underline{v_0} - \omega\right)^2\right] = 0. \tag{4.21}$$

From the dispersion relation it appears that two types of waves can exist. The first solution corresponds to $\underline{k}.\underline{v_0} - \omega = 0$. The associated group velocity is:

$$\underline{v}_g = \left.\frac{\partial \omega}{\partial \underline{k}}\right|_{\underline{x}} = \underline{v_0}. \tag{4.22}$$

For this type of wave $d = 0$ thus we obtain at order 0 $\hat{P}_0 = 0$ and $\underline{k}.\underline{\hat{V}}_0 = 0$. Thus the particle velocity fluctuations are perpendicular to the propagation direction $\underline{n} = \underline{k}/k$, which corresponds to **transverse waves**. This is called the **vortical mode**.

The second solution corresponds to $c^2k^2 - \left(\underline{k}.\underline{v_0} - \omega\right)^2 = 0$ hence $\omega = \underline{k}.\underline{v_0} \pm ck$. The associated group velocity is:

$$\underline{v}_g = \left.\frac{\partial \omega}{\partial \underline{k}}\right|_{\underline{x}} = \underline{v_0} \pm c\frac{\underline{k}}{k} = \underline{v_0} \pm c\underline{n}. \tag{4.23}$$

For this type of wave $d = \mp ck$ thus we obtain at order 0 using the two first lines of the matrix system 4.19:

$$\mp i\rho_0 ck\hat{V}_{0x} + ik_x\hat{P}_0 = 0 \quad \Rightarrow \quad \hat{V}_{0x} = \pm\frac{\hat{P}_0}{\rho_0 c}\frac{k_x}{k} \tag{4.24}$$

$$\mp i\rho_0 ck\hat{V}_{0y} + ik_y\hat{P}_0 = 0 \quad \Rightarrow \quad \hat{V}_{0y} = \pm\frac{\hat{P}_0}{\rho_0 c}\frac{k_y}{k}. \tag{4.25}$$

As a result $\underline{\hat{V}}_0 = \pm\frac{\hat{P}_0}{\rho_0 c}\underline{n}$. This means that the particle velocity fluctuations are aligned with the propagation direction $\underline{n} = \underline{k}/k$, which corresponds to **longitudinal waves**. This is called the **acoustic mode**.

If the flow hadn't been considered as isentropic, a third type of solution called **entropy mode** would have been found, that corresponds to the convection of temperature fluctuations (Ostashev *et al.*, 2005; Boutillon, 2017). These three modes are generally decoupled only in the high frequency approximation (geometrical acoustics) or when the mean flow is homogeneous. In this case, any unsteady perturbation can be decomposed into these three types of modes or waves.

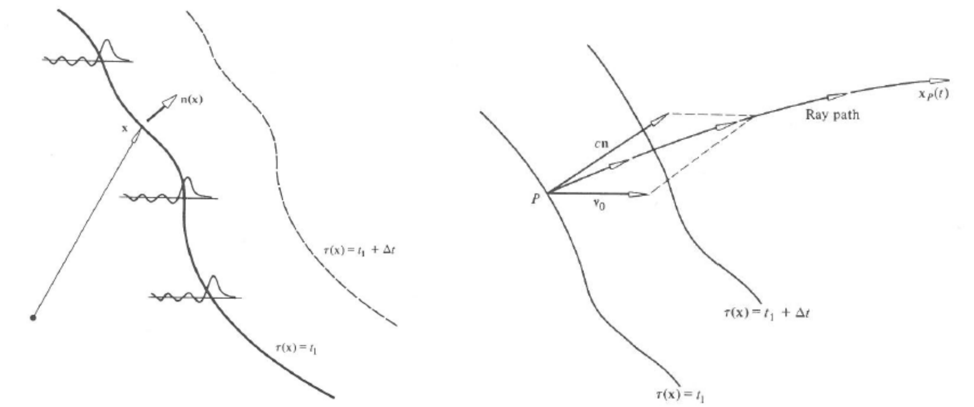
In order to calculate the amplitude of the acoustic mode, it is necessary to solve the equations at order 1:

$$\mathbf{H}\begin{pmatrix}\hat{V}_{1x}\\ \hat{V}_{1y}\\ \hat{P}_1\end{pmatrix} = \mathbf{\Lambda}_0, \tag{4.26}$$

under the condition $\det(\mathbf{H}) = 0$ associated with the dispersion relation $\omega = \underline{k}.\underline{v_0} \pm ck$. This yields a generalization of the transport equation (4.12) obtained in a homogeneous medium at rest (Bailly, 2020).

4.2 Wavefronts and ray equations

A wavefront is a moving surface along which a waveform feature is received (constant phase). We saw in Section 2.1 that it is a plane for plane waves, or a sphere for spherical waves as long as the propagation medium is homogeneous. However, in inhomogeneous moving media, the wavefronts are distorted and generally have a complex shape, as illustrated in Figure 4.1(a).



(a) (b)

Figure 4.1: (a) Wavefront defined by the function $\tau(\underline{x})$ and the normal $\underline{x}$, and (b) wavefront in a moving medium as a function of $c(\underline{x})\underline{n}$ and $\underline{v}_0$, with $\underline{n} = \nabla\tau/|\nabla\tau| = \underline{k}/k$. Taken from Pierce (1989, Chapter 8).

Pierce (1989) defines the function $\tau(\underline{x})$ (in second) constant along a wavefront, and defines tge normal to the wavefront as $\underline{n} = \nabla\tau/|\nabla\tau| = \underline{k}/k$. It can be shown that $\underline{k} = \omega\nabla\tau$.

The dispersion relation $\omega = \underline{k}.\underline{v_0} + ck = \underline{k}.\underline{v_g}$ obtained in Section 4.1 can be solved by the method of characteristics, commonly called rays. The rays are the curves $\underline{x}(t)$ tangent to the group velocity at each point, thus they are defined as:

$$\frac{d\underline{x}}{dt} = \underline{v}_g = \underline{v_0} + c\frac{\underline{k}}{k} = \underline{v_0} + c\underline{n}, \tag{4.27}$$

following Equation (4.23). In a medium at rest ($\underline{v_0} = 0$), the rays are perpendicular to wavefronts, but this is not true anymore in a moving medium, as shown in Figure 4.1(b).

The orientation of the vector $\underline{n} = \frac{\underline{k}}{k}$ is determined through the evolution of $\underline{k}(\underline{x}, t)$ along the ray:

$$\frac{\partial\underline{k}}{\partial t} = \frac{\partial(\nabla\Theta)}{\partial t} = -\nabla\omega = -\nabla(\underline{k}.\underline{v_g}) = -(\nabla\underline{v_g}).\underline{k} - (\nabla\underline{k}).\underline{v_g}. \tag{4.28}$$

This can also be written using Einstein notations:

$$\frac{\partial\underline{k}}{\partial t} = -\frac{\partial\omega}{\partial x_i} = -\frac{\partial}{\partial x_i}(v_{gj}k_j) = -v_{gj}\frac{\partial k_j}{\partial x_i} - k_j\frac{\partial v_{gj}}{\partial x_i}. \tag{4.29}$$

As $\underline{k} = \nabla\Theta$:

$$\nabla \times \underline{k} = 0 \Leftrightarrow \frac{\partial k_j}{\partial x_i} - \frac{\partial k_i}{\partial x_j} = 0. \tag{4.30}$$

As a result:

$$\frac{\partial \underline{k}}{\partial t} = -v_{gj}\frac{\partial k_i}{\partial x_j} - k_j \frac{\partial v_{gj}}{\partial x_i} = -(\underline{v_g}.\nabla)\underline{k} - (\nabla \underline{v_g}).\underline{k}, \tag{4.31}$$

and the equation for the evolution of $\underline{k}$ becomes:

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + (\underline{v_g}.\nabla)\underline{k} = -(\nabla \underline{v_g}).\underline{k} = -k\nabla c - (\nabla \underline{v_0}).\underline{k}. \tag{4.32}$$

Finally, the ray-tracing equations can be written using Einstein notations:

$$\frac{dx_i}{dt} = cn_i + v_{0i}, \tag{4.33}$$

$$\frac{dk_i}{dt} = -k\frac{\partial c}{\partial x_i} - k_j \frac{\partial v_{0j}}{\partial x_i}, \tag{4.34}$$

with $n_i = k_i/k$.

4.3 Wave amplitude along rays and caustics

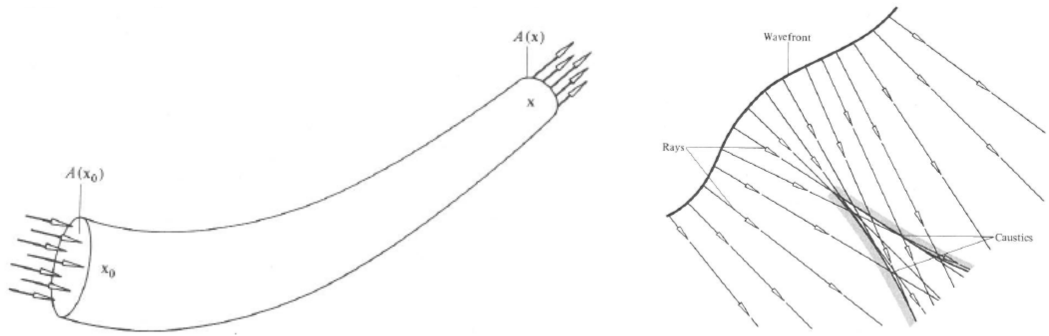
Let us consider a ray tube, that corresponds to the envelope of all the rays passing through a tiny area $A(\underline{x}_0)$, as shown in Figure 4.2(a). Applying Gauss's theorem to the transport equation (4.12) in a homogeneous medium at rest:

$$\begin{aligned} \int_{V_{\text{ray tube}}} \nabla.(\hat{P}^2\underline{k})dV &= \int_{A_{\text{ray tube}}} \hat{P}^2(\underline{k}.\underline{n})dA \\ &= \hat{P}^2(\underline{x})A(\underline{x})(\underline{k}.\underline{n})_{\underline{x}} - \hat{P}^2(\underline{x}_0)A(\underline{x}_0)(\underline{k}.\underline{n})_{\underline{x}_0} = 0. \end{aligned} \tag{4.35}$$

In a homogeneous medium at rest the rays are straight lines hence $(\underline{k}.\underline{n})_{\underline{x}} = (\underline{k}.\underline{n})_{\underline{x}_0}$, which yields the following amplitude variation along rays :

$$\hat{P}(\underline{x}) = \hat{P}(\underline{x}_0)\sqrt{\frac{A(\underline{x}_0)}{A(\underline{x})}} \quad \text{or} \quad \hat{P}^2(\underline{x})A(\underline{x}) = \text{constant}. \tag{4.36}$$

The amplitude is inversely proportional to the square root of ray tube area.



(a) (b)

Figure 4.2: (a) Schematics of a ray tube, and (b) example of a caustic. Taken from Pierce (1989, Chapter 8).

In an inhomogeneous medium, the rays are now curved because of sound speed and/or velocity gradients. The previous result has been generalized by Pierce (1989, Section 8-6) that shows the **conservation of energy along rays**. The amplitude variation along rays in an inhomogeneous medium at rest is written:

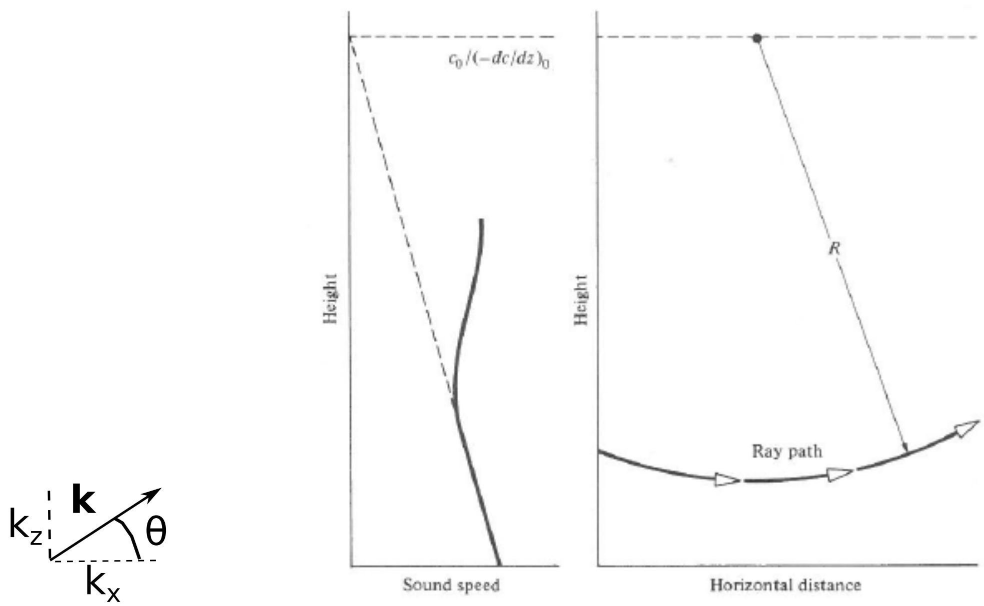
$$\frac{\hat{P}^2 A}{\rho_0 c} = \text{constant} \tag{4.37}$$

This result can be extended to an inhomogeneous moving medium (Pierce, 1989, Section 8.6).

When two rays intersect, the ray tube areas go to zero ($A(\underline{x}) = 0$), thus the pressure amplitude goes to infinity from the previous expressions. The envelope formed by a family of intersecting rays is called a **caustics**. An example is shown in Figure 4.2(b). When this happens, there is a need to think in terms of waveforms instead of rays. This is the goal of the geometrical theory of diffraction which is an extension of geometrical acoustics. Salomons (2001, Appendix L) presents a ray model that employs the theory of Ludwig and Kravtsov for the effects of caustics.

4.4 Solutions to the ray-tracing equations

It is possible to calculate analytically the solutions to the ray-tracing equations in some simple cases. For instance in a medium at rest, $\underline{v}_0 = 0$ and the



(a) (b)

Figure 4.3: (a) Ray direction, and (b) ray path considering a linear sound speed profile. Taken from Pierce (1989, Chapter 8).

ray-tracing equations are simply:

$$\frac{d\underline{x}}{dt} = c\underline{n} \quad \text{or} \quad \frac{dx_i}{dt} = cn_i, \tag{4.38}$$

$$\frac{d\underline{k}}{dt} = -k\nabla c \quad \text{or} \quad \frac{dk_i}{dt} = -k\frac{\partial c}{\partial x_i}, \tag{4.39}$$

with $n_i = k_i/k$. These equations can be integrated in time for a given initial direction at $t = 0$. We can assume without loss of generality: $\underline{k}_0 = k_{0x}\underline{e}_x + k_{0z}\underline{e}_z$, as shown in Figure 4.3(a).

In a layered atmosphere at rest, such that $c = c(z)$, the equations can be further simplified. We see that $\frac{dk_x}{dt} = \frac{dk_y}{dt} = 0$ which means that $k_x = k_{x0}$ and $k_y = k_{y0} = 0$. As a result $n_y = 0$ and the rays remain in the original plane of propagation. From the dispersion relation (Eikonal equation):

$$k^2 = k_x^2 + k_z^2 = \frac{\omega^2}{c^2} \Rightarrow k_x = \frac{\omega\cos\theta}{c} \quad \text{and} \quad k_z = \frac{\omega\sin\theta}{c}. \tag{4.40}$$

As a result the Snell-Descartes law is retrieved:

$$k_x = \frac{\omega \cos\theta}{c} = \text{constant} \Rightarrow \frac{\cos\theta(z)}{c(z)} = \text{constant}. \tag{4.41}$$

If a simple linear sound speed profile of the form $c(z) = c_0 + az$ is chosen, then $\nabla c = a\underline{e}_z$. In this specific case, it is possible to show that the ray trajectories are **arcs of circle** of radius $R = \omega/(ak_{x0}) = c(z_0)/(a\cos\theta_0)$, with $\theta_0 = \theta(t = 0)$, centered at height c_0/a (Pierce, 1989, Section 8-3). Figure 4.3(b) illustrates the shape of the ray paths for a linear sound speed profile. The greater the sound speed gradient a is, the smaller the radius of curvature R will be.

Pierce (1989, Section 8-3) extends the analysis to a moving medium, and shows that the rays bend with a radius of curvature:

$$R = c/\left(\frac{dc}{dz}\cos\theta + \frac{dv_{0x}}{dz}\right). \tag{4.42}$$

For rays propagating in nearly horizontal directions, $\cos\theta \approx 1$ and $R \approx c/\left(\frac{dc_{eff}}{dz}\right)$, where c_{eff} is the effective sound speed defined as:

$$c_{eff}(z) = c(z) + v_{0x}(z). \tag{4.43}$$

If θ is greater than approximately 30^o, the influence of a wind speed gradient is greater than the influence of a sound speed gradient of same amplitude, and the effective sound speed approximation becomes less accurate.

Chapter 5

Introduction to aeroacoustics

This chapter of introduction to aeroacoustics is based mostly on the books of Glegg and Devenport (2017), of Rienstra and Hirschberg (2021), and on the course notes of Gloerfelt (2016). The general notations of Chapter 1 are used.

5.1 Lighthill's analogy

The idea of Lighthill is to rewrite the general equations of fluid mechanics given in Chapter 1 as an inhomogeneous wave equation. All the non-linear terms will be included as source terms on the right-hand side of the equation. While this equation is formally exact, it is only useful when considering an observer outside the source region, in a homogeneous medium at rest. We thus introduce the constant sound speed c_∞ and density ρ_∞ in the observer region.

To derive Lighthill's equation, let us start from the equation of continuity (1.1) written:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0, \tag{5.1}$$

using Einstein notation, and the from momentum equation (1.2) rewritten in a slightly different form:

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j + p_{ij})}{\partial x_j} = 0, \tag{5.2}$$

where the gravitational force has been neglected. The tensor $p_{ij} = (p - p_\infty)\delta_{ij} - \tau_{ij}$ includes both the pressure forces and the viscous stresses. The term $\rho v_i v_j$ is commonly called Reynolds stress tensor while p_{ij} is called the compressive stress tensor.

By taking the time derivative of the continuity equation and substracting the divergence of the momentum equation, we obtain:

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2(\rho v_i v_j + p_{ij})}{\partial x_i \partial x_j}. \tag{5.3}$$

To obtain a wave equation on the density perturbation $\rho' = \rho - \rho_\infty$, we subtract $\frac{\partial^2(\rho' c_\infty^2)}{\partial x_i^2}$ on each side:

$$\boxed{\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}} \quad \text{with} \quad \boxed{T_{ij} = \rho v_i v_j + p_{ij} - (\rho - \rho_\infty) c_\infty^2 \delta_{ij}}, \tag{5.4}$$

because $\frac{\partial \rho}{\partial t} = \frac{\partial \rho'}{\partial t}$. This is known as Lighthill's wave equation (Lighthill, 1952). The Lighthill's tensor T_{ij} can also be rewritten:

$$T_{ij} = \rho v_i v_j + [(p - p_\infty) - (\rho - \rho_\infty) c_\infty^2] \delta_{ij} - \tau_{ij} = \rho v_i v_j + [p' - \rho' c_\infty^2] \delta_{ij} - \tau_{ij}, \tag{5.5}$$

with $p' = (p - p_\infty)$.

In the acoustic region far from the source, the approximation of linear acoustics holds thus $p' = \rho' c_\infty^2$, and the Lighthill's tensor $T_{ij} = 0$ because the medium is stationary ($v_i = 0$) and uniform ($p \approx p_\infty$ and $\rho \approx \rho_\infty$), and because viscous effects can be ignored (linear acoustics approximation).

Some important remarks and limitations can be made:

- Lighhill's equation is exact but it cannot be solved since there is only one equation for 5 unknowns (v_i, p and ρ). The interpretation that can be made is that the left hand side is the classical wave equation and the right hand side is a source term that is supposed to be known.

- Nearly incompressible flow: at relatively low Mach numbers and for homentropic flows ($p' \approx \rho' c_\infty^2$), Lighthill's tensor is often approximated as $T_{ij} \approx \rho_\infty v_i v_j$, neglecting viscous effects. This approximation is not valid in cases where acoustic waves have an influence on the mean flow (acoustic retroaction).

- The choice of the speed of sound c_∞ is not arbitrary, it must be chosen as the speed of sound in the stationary medium.

- As the medium is homogeneous and at rest around the observer, convection, refraction and diffraction/scattering effects are not accounted for by Lighthill's analogy.

5.2 Curle's theorem

5.2.1 Derivation of the general formulation

To find a solution to Lighthill's wave equation, it is possible to use the method of Green's functions. The Green's function is the solution of the wave equation with a unit point impulsive source term, given in Equation (2.15), and can be rewriten:

$$\left(\frac{1}{c_\infty^2}\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial y_i^2}\right) G(\underline{x}, t|\underline{y}, \tau) = \delta(\underline{x} - \underline{y})\delta(t - \tau), \tag{5.6}$$

with τ the source time, t the receiver time, $\underline{y}$ the source position, and $\underline{x}$ the receiver position. Let us rewrite Lighthill Equation (5.4) in terms of $\underline{y}$ and τ:

$$\frac{1}{c_\infty^2}\frac{\partial^2(\rho' c_\infty^2)}{\partial\tau^2} - \frac{\partial^2(\rho' c_\infty^2)}{\partial y_i^2} = \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}. \tag{5.7}$$

To obtain the solution, let us multiply Equation (5.7) by G, and subtract it from Equation (5.6) times $\rho' c_\infty^2$:

$$\begin{aligned} &\frac{1}{c_\infty^2}\left(\rho' c_\infty^2 \frac{\partial^2 G}{\partial\tau^2} - G\frac{\partial^2(\rho' c_\infty^2)}{\partial\tau^2}\right) - \left(\rho' c_\infty^2 \frac{\partial^2 G}{\partial y_i^2} - G\frac{\partial^2(\rho' c_\infty^2)}{\partial y_i^2}\right) \\ &\quad = \delta(\underline{x} - \underline{y})\delta(t - \tau)\rho' c_\infty^2 - G\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}. \end{aligned} \tag{5.8}$$

Now let us integrate this equation over source time τ and the volume $V(\underline{y})$ which includes the receiver position $\underline{x}$:

$$\begin{aligned} &\int_V \int_{-T}^{T} \left[\frac{1}{c_\infty^2}\left(\rho' c_\infty^2 \frac{\partial^2 G}{\partial\tau^2} - G\frac{\partial^2(\rho' c_\infty^2)}{\partial\tau^2}\right) - \left(\rho' c_\infty^2 \frac{\partial^2 G}{\partial y_i^2} - G\frac{\partial^2(\rho' c_\infty^2)}{\partial y_i^2}\right)\right] d\tau dV(\underline{y}) \\ &\quad = \rho'(\underline{x}, t)c_\infty^2 - \int_V \int_{-T}^{T} G\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} d\tau dV(\underline{y}), \end{aligned} \tag{5.9}$$

The above equation holds if $\underline{x}$ is inside V and if $-T < t < T$ (property of the dirac functions).

The first term in the integrand can be rewritten:

$$\rho' c_\infty^2 \frac{\partial^2 G}{\partial \tau^2} - G \frac{\partial^2 (\rho' c_\infty^2)}{\partial \tau^2} = \frac{\partial}{\partial \tau} \left((\rho' c_\infty^2) \frac{\partial G}{\partial \tau} - G \frac{\partial (\rho' c_\infty^2)}{\partial \tau} \right). \tag{5.10}$$

A a result, the integral of this first term can be calculated as follows:

$$\int_V \int_{-T}^{T} \left(\rho' c_\infty^2 \frac{\partial^2 G}{\partial \tau^2} - G \frac{\partial^2 (\rho' c_\infty^2)}{\partial \tau^2} \right) d\tau dV(\underline{y}) = \int_V \left[\rho' c_\infty^2 \frac{\partial G}{\partial \tau} - G \frac{\partial (\rho' c_\infty^2)}{\partial \tau} \right]_{\tau=-T}^{\tau=T} dV(\underline{y}). \tag{5.11}$$

The initial conditions can be chosen so that ρ' and $\frac{\partial \rho'}{\partial \tau}$ are equal to zero at $\tau = -T$. The causality condition also imposes that the Green's function and its time derivative are zero at $\tau = T > t$. As a result, this first term is equal to zero.

Similarly, the second term in the integrand can be rewritten:

$$\left(\rho' c_\infty^2 \frac{\partial^2 G}{\partial y_i^2} - G \frac{\partial^2 (\rho' c_\infty^2)}{\partial y_i^2} \right) = \frac{\partial}{\partial y_i} \left(\rho' c_\infty^2 \frac{\partial G}{\partial y_i} - G \frac{\partial (\rho' c_\infty^2)}{\partial y_i} \right). \tag{5.12}$$

This corresponds to the divergence of a vector. As a result, we can use the divergence theorem to turn the volume integral into a surface integral:

$$\int_V \left(\rho' c_\infty^2 \frac{\partial^2 G}{\partial y_i^2} - G \frac{\partial^2 (\rho' c_\infty^2)}{\partial y_i^2} \right) dV(\underline{y}) = \int_{S_{\text{tot}}} \left(\rho' c_\infty^2 \frac{\partial G}{\partial y_i} - G \frac{\partial (\rho' c_\infty^2)}{\partial y_i} \right) n_i dS(\underline{y}), \tag{5.13}$$

where the unit vector n_i has been chosen normal to the surface and pointing into the volume, as shown in Figure 5.1, and $S_{\text{tot}} = S \cup S_\infty$ is the surface surrounding V. In free field applications the integral on the surface S_∞ is equal to zero due to the Sommerfeld radiation condition, that states that there are no incoming waves coming from infinity Pierce (1989); Rienstra and Hirschberg (2021).

The integral equation (5.9) can thus be written:

$$\begin{aligned} \rho'(\underline{x}, t) c_\infty^2 = & \int_{-T}^{T} \int_S \left(\rho' c_\infty^2 \frac{\partial G}{\partial y_i} - G \frac{\partial (\rho' c_\infty^2)}{\partial y_i} \right) n_i dS(\underline{y}) d\tau \\ & + \int_V \int_{-T}^{T} G \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} d\tau dV(\underline{y}). \end{aligned} \tag{5.14}$$

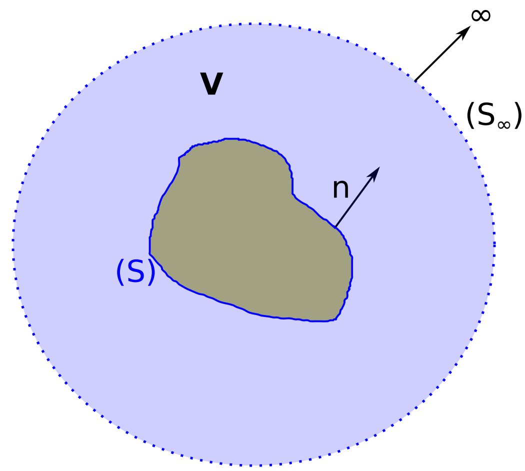


Figure 5.1: Schematics for Curle's method.

As explained by Glegg and Devenport (2017), it is difficult to calculate the second derivative of the Lighthill's tensor in practice, so it is useful to apply partial integral twice to pass the derivatives on the Green's function. After some manipulations, they obtain Glegg and Devenport (2017, Equation (4.3.6)):

$$\begin{aligned}\rho'(\underline{x},t)c_\infty^2 = &\int_{-T}^{T}\int_S \left((p_{ij} + \rho v_i v_j)\frac{\partial G}{\partial y_i} + G\frac{\partial(\rho v_j)}{\partial \tau} \right) n_j dS(\underline{y}) d\tau \\ &+ \int_V \int_{-T}^{T} \frac{\partial^2 G}{\partial y_i \partial y_j} T_{ij} d\tau dV(\underline{y}).\end{aligned} \tag{5.15}$$

This result due to Curle (1955) is very general, as any Green's function can be used. The only restriction is that the surface S bounding the fluid (or S_{tot} for confined flows) is stationary, and that the medium is homogeneous and at rest outside the source region, as in the Lighthill's equation.

In the absence of aerodynamic sources, $T_{ij} = 0$ and $\rho v_i v_j = 0$. Furthermore, $p'(\underline{x},t) = \rho'(\underline{x},t)c_\infty^2$ and $p_{ij} = p'\delta_{ij}$ so that we retrieve the Kirchhoff-Helmholtz integral equation (Pierce, 1989; Glegg and Devenport, 2017):

$$p'(\underline{x},t) = \int_{-T}^{T}\int_S \left(p'\frac{\partial G}{\partial y_i} + \rho_0 G\frac{\partial v_i}{\partial \tau} \right) n_i dS(\underline{y}) d\tau. \tag{5.16}$$

5.2.2 Interpretation of Curle's theorem using the free-field Green's function

To understand the physical meaning of each term of Equation (5.15), it is interesting to consider the free field Green's function given by Equation (2.22):

$$G_0(\underline{x}, t|\underline{y}, \tau) = \frac{1}{4\pi|\underline{x} - \underline{y}|}\delta\left(t - \tau - \frac{r}{c_\infty}\right), \tag{5.17}$$

with $r = |\underline{x} - \underline{y}|$ the source-receiver distance. The free field Green's function and it has the following properties:

$$\frac{\partial G_0}{\partial y_i} = -\frac{\partial G_0}{\partial x_i} \quad \text{and} \quad \frac{\partial^2 G_0}{\partial y_i y_j} = \frac{\partial^2 G_0}{\partial x_i x_j}.$$

As a result, Equation (5.15) can be rewritten with the spatial derivatives outside the integrals:

$$\begin{aligned}\rho'(\underline{x}, t)c_\infty^2 = &\int_{-T}^{T}\int_S G_0\frac{\partial(\rho v_j)}{\partial \tau}n_j dS(\underline{y})d\tau - \frac{\partial}{\partial x_i}\int_{-T}^{T}\int_S ((p_{ij} + \rho v_i v_j)G_0)\, n_j dS(\underline{y})d\tau \\ &+ \frac{\partial^2}{\partial x_i \partial x_j}\int_V\int_{-T}^{T} G_0 T_{ij} d\tau dV(\underline{y}),\end{aligned} \tag{5.18}$$

or equivalently:

$$\begin{aligned}\rho'(\underline{x}, t)c_\infty^2 = &\int_S \left[\frac{\partial(\rho v_j)}{\partial \tau}\right]_{\tau=\tau^*}\frac{n_j dS(\underline{y})}{4\pi|\underline{x} - \underline{y}|} - \frac{\partial}{\partial x_i}\int_S [(p_{ij} + \rho v_i v_j)]_{\tau=\tau^*}\frac{n_j dS(\underline{y})}{4\pi|\underline{x} - \underline{y}|} \\ &+ \frac{\partial^2}{\partial x_i \partial x_j}\int_V [T_{ij}]_{\tau=\tau^*}\frac{dV(\underline{y})}{4\pi|\underline{x} - \underline{y}|},\end{aligned} \tag{5.19}$$

where the source terms are evaluated at the retarded times $\tau = \tau^* = t - |\underline{x} - \underline{y}|/c_\infty$.

The first term in Equation (5.19) depends on the normal velocity $u_n = v_i n_i$ on the surface S:

$$(\rho'(\underline{x}, t)c_\infty^2)_{\text{monopole}} = \int_S \left[\frac{\partial(\rho u_n)}{\partial \tau}\right]_{\tau=\tau^*}\frac{dS(\underline{y})}{4\pi|\underline{x} - \underline{y}|}. \tag{5.20}$$

As for the pulsating sphere, it is related to the flux of mass across the surface. This term is zero if the surface is rigid or impenetrable. If the surface is acoustically compact, i.e. if it is sufficiently small that retarded time effects can be ignored, this term becomes in the acoustic far-field ($|\underline{x}| \gg |\underline{y}|$):

$$(\rho'(\underline{x},t)c_\infty^2)_{\text{monopole}} \approx \frac{1}{4\pi|\underline{x}|}\left[\int_S \frac{\partial(\rho u_n)}{\partial \tau} dS(\underline{y})\right]_{\tau=\tau^*} = \frac{1}{4\pi|\underline{x}|}\dot{Q}(t-|\underline{x}|/c). \tag{5.21}$$

This term is thus ominidirectional in the far-field, and is similar to the expression, for a monopole expression.

The second term in Equation (5.19) is related to the surface loading:

$$(\rho'(\underline{x},t)c_\infty^2)_{\text{dipole}} = -\frac{\partial}{\partial x_i}\int_S [(p_{ij}+\rho v_i v_j)]_{\tau=\tau^*}\frac{n_j dS(\underline{y})}{4\pi|\underline{x}-\underline{y}|}. \tag{5.22}$$

The space derivative can be evaluated as follows:

$$\frac{\partial}{\partial x_i}\left(\frac{f(\tau^*)}{|\underline{x}-\underline{y}|}\right) = \frac{1}{|\underline{x}-\underline{y}|}\frac{\partial f(\tau^*)}{\partial x_i} - \frac{(x_i-y_i)}{|\underline{x}-\underline{y}|^3}f(\tau^*), \tag{5.23}$$

where $\frac{\partial f(\tau^*)}{\partial x_i}$ is obtained using the chain rule:

$$\frac{\partial f(\tau^*)}{\partial x_i} = \frac{\partial \tau^*}{\partial x_i}\left[\frac{\partial f(\tau)}{\partial \tau}\right]_{\tau=\tau^*} = -\frac{(x_i-y_i)}{|\underline{x}-\underline{y}|c_\infty}\left[\frac{\partial f(\tau)}{\partial \tau}\right]_{\tau=\tau^*}, \tag{5.24}$$

as $\tau^* = t - |\underline{x}-\underline{y}|/c_\infty$. Thus:

$$(\rho'(\underline{x},t)c_\infty^2)_{\text{dipole}} = \int_S \left[\frac{\partial(p_{ij}+\rho v_i v_j)}{\partial \tau} + \frac{(p_{ij}+\rho v_i v_j)c_\infty}{|\mathbf{x}-\mathbf{y}|}\right]_{\tau=\tau^*}\frac{(x_i-y_i)\,n_j dS(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^2 c_\infty}. \tag{5.25}$$

If the surface is rigid or impenetrable, $\rho v_i v_j n_j = 0$ and the only significant term is the compressive stress tensor p_{ij}. In the far-field, the second term in the brackets becomes negligible, thus for a rigid stationary surface:

$$(\rho'(\underline{x},t)c_\infty^2)_{\text{dipole}} = \int_S \left[\frac{\partial p_{ij}}{\partial \tau}\right]_{\tau=\tau^*}\frac{(x_i-y_i)\,n_j dS(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^2 c_\infty}, \tag{5.26}$$

If the surface is acoustically compact so that the effects of retarded time can be neglected:

$$(\rho'(\underline{x},t)c_\infty^2)_{\text{dipole}} \approx \frac{x_i}{4\pi|\mathbf{x}|^2 c_\infty}\left[\frac{dF_i}{d\tau}\right]_{\tau=\tau^*} \quad \text{with} \quad F_i(\tau) = \int_S p_{ij} n_j dS(\mathbf{y}), \tag{5.27}$$

where F_i is the force applied by the surface to the fluid. This resembles a dipole oriented in the direction of the force.

Finally, the last term in Equation (5.19) is the volume integral which is referred to as a quadrupole term. It is possible to change the space derivatives to derivatives over source times. In the acoustic far-field only the dominant term is kept yielding:

$$(\rho'(\underline{x},t)c_\infty^2)_{\text{quadrupole}} \approx \frac{x_i x_j}{4\pi c_\infty^2 r^3} \int_V \left[\frac{\partial^2 T_{ij}}{\partial \tau^2}\right]_{\tau=\tau^*} dV(\underline{y}). \tag{5.28}$$

If the turbulence volume is acoustically compact (smaller than the acoustic wavelength), the effects of retarded time can be ignored thus:

$$(\rho'(\underline{x},t)c_\infty^2)_{\text{quadrupole}} \approx \frac{x_i x_j}{4\pi c_\infty^2 |\mathbf{x}|^3} \left[\int_V \frac{\partial^2 T_{ij}}{\partial \tau^2} dV(\underline{y})\right]_{\tau=\tau^*}. \tag{5.29}$$

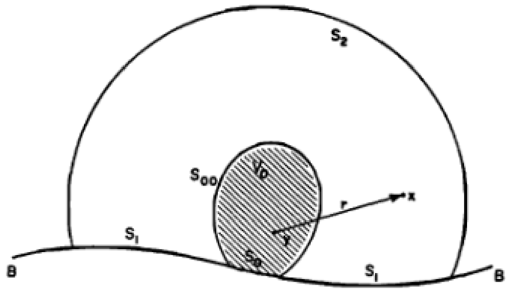
The acoustic directivity is the same as for elementary quadrupole sources.

Calling the different terms of Equation (5.19) monopole, dipole and quadrupole terms can be misleading, as this classification is only valid for compact sources, which means that the characteristic scale L of the sources is small compared to the acoustic wavelength: $L \ll \lambda \Leftrightarrow kL \ll 1$, with $k = 2\pi/\lambda$ (Rienstra and Hirschberg, 2021). One example of this is the reflection theorem of Powell (1960). Powell (1960) studies the effect of a plane boundary in the vicinity of a noise generating region, as shown in Figure 5.2(a). The original system contains a volume V_0 with the noise-generating region ($T_{ij} = 0$ outside V_0), and is delimited by the surfaces S_0, S_1 and S_2. The virtual image system is introduced to account for reflections at the boundaries S_0 and S_1, as shown in Figure 5.2(b), where primes are used for the quantities referring to the image. Let us consider an observer $\underline{x}$ located in the real volume enclosed by the surfaces S_0, S_1 and S_2. Assuming the plane surface BB is rigid and the fluid is inviscid ($p_{ij} = p'\delta_{ij}$), Equation (5.19) yields:

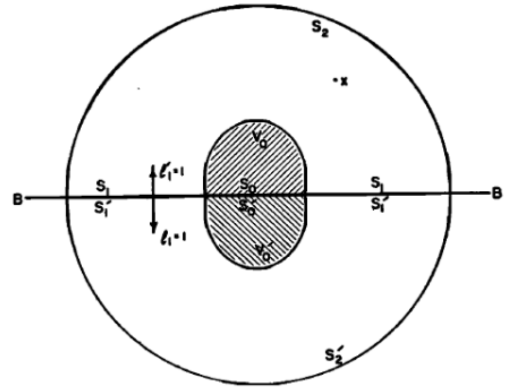
$$4\pi p'(\underline{x},t) = -\frac{\partial}{\partial x_i}\int_{S_0\cup S_1} [p']_{\tau=\tau^*} \frac{n_i dS(\underline{y})}{|\underline{x}-\underline{y}|} + \frac{\partial^2}{\partial x_i \partial x_j}\int_{V_0} [T_{ij}]_{\tau=\tau^*} \frac{dV(\underline{y})}{|\underline{x}-\underline{y}|}. \tag{5.30}$$

Let us now consider the image volume enclosed by the surfaces S_0', S_1' and S_2'. Since the observer $\underline{x}$ is not included in this image volume, we find:

$$0 = -\frac{\partial}{\partial x_i}\int_{S_0'\cup S_1'} [p']_{\tau=\tau^*} \frac{n_i' dS(\underline{y})}{|\underline{x}-\underline{y}|} + \frac{\partial^2}{\partial x_i \partial x_j}\int_{V_0'} [T_{ij}]_{\tau=\tau^*} \frac{dV(\underline{y})}{|\underline{x}-\underline{y}|}, \tag{5.31}$$



(a)



(b)

Figure 5.2: Schematics taken from Powell (1960) for (a) the general configuration of a noise-generating region enclosed in the volume V_0 in the vicinity of a boundary BB, and (b) the special case of a plane boundary, with an image system on the other side of the boundary BB (Powell reflection theorem).

with $n'_i = -n_i$. The surface pressure contributions cancel out when the sum of Equations (5.30) and (5.31) is taken, and we obtain simply:

$$4\pi p'(\underline{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{V_0 \cup V'_0} [T_{ij}]_{\tau=\tau^*} \frac{dV(\underline{y})}{|\underline{x} - \underline{y}|}, \tag{5.32}$$

From Equations (5.30) and (5.32) we deduce:

$$-\frac{\partial}{\partial x_i} \int_{S_0 \cup S_1} [p']_{\tau=\tau^*} \frac{n_i dS(\underline{y})}{|\underline{x} - \underline{y}|} = \frac{\partial^2}{\partial x_i \partial x_j} \int_{V'_0} [T_{ij}]_{\tau=\tau^*} \frac{dV(\underline{y})}{|\underline{x} - \underline{y}|}, \tag{5.33}$$

which is the statement of Powell's reflection theorem (Powell, 1960):
The pressure dipole distribution on a plane, infinite and rigid surface accounts for the reflection in that surface of the volume distribution of acoustic quadrupole generators of a contiguous inviscid fluid flow, and for nothing more, when these distributions are determined in accordance with Lighthill's concept of aerodynamic noise generation and its natural extension.

5.2.3 Scaling laws

Let us deduce from Equations (5.29) and (5.27) the scaling law for the noise radiated by a compact turbulence volume in free field (quadrupole term) and by a compact surface loading (dipole term).

For a turbulent flow in the absence of obstacles such as a jet (high Reynolds number, weakly inhomogeneous), the Lighthill tensor can be approximated as $T_{ij} = \rho_0 v_i v_j$, with ρ_0 the mean jet density that can be different from ρ_∞ (for a hot jet for instance). Let us note ℓ a characteristic turbulence length scale in the flow and U its characteristic velocity, then Equation (5.29):

$$T_{ij} \sim \rho_0 U^2 \quad \Rightarrow \quad \frac{\partial^2 T_{ij}}{\partial \tau^2} \sim \rho_0 U^4/\ell^2 \quad \Rightarrow \quad (p)'_{\text{quadrupole}} \sim \frac{\rho_0 U^4 V}{c_\infty^2 |\underline{x}| \ell^2},$$

with V the turbulence volume, and where the direction of maximum radiation has been chosen ($x_i x_j/|\underline{x}|^2 \approx 1$). As the far-field acoustic intensity is $< I > = |p'^2|/(2\rho_\infty c_\infty)$, it scales as:

$$< I >_{\text{quadrupole}} \sim \frac{\rho_0^2 U^8 V^2}{\rho_\infty c_\infty^5 |\underline{x}|^2 \ell^4} = \frac{\rho_0^2 U^3 M^5 V^2}{\rho_\infty |\underline{x}|^2 \ell^4}, \tag{5.34}$$

with $M = U/c_\infty$. The noise from turbulence scales with the eighth power of flow speed, which is called the u^8 law (Lighthill, 1952). For a circular free jet of diameter D, one can choose $\ell = D$ and $V = D^3$ such that (Rienstra and Hirschberg, 2021):

$$< I >_{\text{quadrupole}} \sim \frac{\rho_0^2 U^8 D^2}{\rho_\infty c_\infty^5 |\underline{x}|^2} = \frac{\rho_0^2 U^3 M^5 D^2}{\rho_\infty |\underline{x}|^2}. \tag{5.35}$$

For a cold jet $\rho_0 \approx \rho_\infty$ and this expression is simplified.

For the surface loading term, the net force F_i scales as $\rho_\infty U^2 S$, with S the surface of the body. As a result, Equation (5.27) for a compact surface yields:

$$(p)'_{\text{dipole}} \sim \frac{\rho_\infty U^3 S}{c_\infty |\underline{x}| \ell} \Rightarrow < I >_{\text{dipole}} \sim \frac{\rho_\infty U^6 S^2}{c_\infty^3 |\underline{x}|^2 \ell^2} = \frac{\rho_\infty U^3 M^3 S^2}{|\underline{x}|^2 \ell^2}.$$

The far-field noise scales with the sixth power of flow speed.

The ratio of the scaling laws for the quadrupole and dipole terms yields:

$$\frac{< I >_{\text{quadrupole}}}{< I >_{\text{dipole}}} \sim \frac{U^2 V^2}{c_\infty^2 \ell^2 S^2} = \frac{M^2 V^2}{\ell^2 S^2},$$

considering $\rho_0 = \rho_\infty$. This shows that if the ratio $V/(\ell S)$ is of order of magnitude one, the quadrupole source strength scales as M^2 times dipole source strength. The volume source terms can thus be neglected for $M \ll 1$.

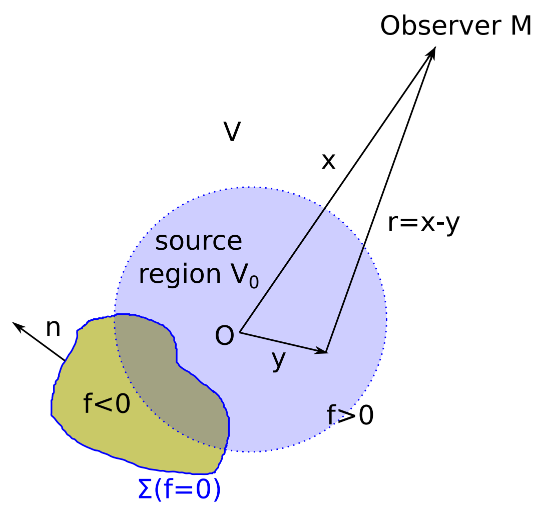


Figure 5.3: Schematics for Ffowcs Williams and Hawkings method.

5.3 Ffowcs Williams et Hawkings analogy

5.3.1 Ffowcs Williams et Hawkings equation

Lighhill analogy has been extended to a moving surface by Williams and Hawkings (1969). Let us define a surface Σ defined by $f(\underline{x}, t) = 0$, that can be partially included in the source region V_0, as can be seen in Figure 5.3. The region in the fluid outside Σ corresponds to $f > 0$, and the region inside Σ corresponds to $f < 0$. The normal to Σ is $\underline{n} = \nabla f / |\nabla f|$ and points out of the region towards $f > 0$. The surface velocity is noted $\underline{v}^{\Sigma}$, such that:

$$\frac{\partial f}{\partial t} + \underline{v}^{\Sigma}.\nabla f = 0 \Leftrightarrow \frac{\partial f(\underline{x}, t)}{\partial t} + v_i^{\Sigma} \frac{\partial f(\underline{x}, t)}{\partial x_i} = \frac{\partial f(\underline{y}, \tau)}{\partial \tau} + v_i^{\Sigma} \frac{\partial f(\underline{y}, \tau)}{\partial y_i} = 0. \tag{5.36}$$

In order to obtain the Ffowcs Williams et Hawkings equation, we will replace a problem on the unknowns $\underline{v}$, p and ρ with a discontinuity on the boundary Σ by a problem on the unknowns $\underline{v}H(f)$, $pH(f)$ and $\rho H(f)$ valid everywhere, where $H(f)$ is the Heaviside step function defined by:

$$H(f) = \begin{cases} 1 & \text{when} \quad f > 0, \\ 0 & \text{when} \quad f < 0. \end{cases} \tag{5.37}$$

It can be seen that the derivative of this function is very large when $x = 0$, such that:

$$\int_a^b \frac{\partial H(x)}{\partial x} dx = 1 \quad \text{if} \quad a < 0 < b. \tag{5.38}$$

The derivative of the Heaviside function has the same property as the Dirac delta function thus:

$$\frac{\partial H(x)}{\partial x} = \delta(x) \quad \text{and} \quad \begin{cases} \nabla H(f) & = \frac{\partial H}{\partial f} \nabla f = \delta(f) |\nabla f| \underline{n} \\ \frac{\partial H(f)}{\partial t} & = \delta(f) \frac{\partial f}{\partial t} = -\delta(f) \underline{v}^{\Sigma} . \nabla f. \end{cases} \tag{5.39}$$

To derive the continuity equation on the new variables $\rho H(f)$ and $\underline{v} H(f)$, let us develop the following expression:

$$\frac{\partial[\rho' H(f)]}{\partial t} + \frac{\partial[\rho v_i H(f)])}{\partial x_i} = H(f) \left[\frac{\partial \rho'}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} \right] + \left(\rho' \frac{\partial H(f)}{\partial t} + \rho v_i \nabla H \right). \tag{5.40}$$

The first term on the right hand side is zero because it corresponds to the continuity equation (1.1) where the flow is defined ($f > 0$), and because $H(f) = 0$ when $f < 0$. As a result, using Equation (5.39):

$$\frac{\partial[\rho' H(f)]}{\partial t} + \frac{\partial[\rho v_i H(f)]])}{\partial x_i} = \left(\rho v_i - \rho' v_i^{\Sigma} \right) \delta(f) \frac{\partial f}{\partial x_i}. \tag{5.41}$$

Similarly, the momentum equation can be rewritten:

$$\frac{\partial[\rho v_i H(f)]}{\partial t} + \frac{\partial[(\rho v_i v_j + p_{ij}) H(f)]}{\partial x_j} = \left(\rho v_i (v_j - v_j^{\Sigma}) + p_{ij} \right) \delta(f) \frac{\partial f}{\partial x_j}. \tag{5.42}$$

Subtracting the time derivative of Equation (5.41) to the divergence of Equation (5.42), and subtracting $\frac{\partial^2 [\rho' c_\infty^2 H(f)]}{\partial x_i^2}$ on each side, as in Section 5.1 yields:

$$\frac{\partial^2[\rho' H(f)]}{\partial t^2} - c_\infty^2 \frac{\partial^2[\rho' H(f)]}{\partial x_i^2} = \frac{\partial^2[T_{ij} H(f)]}{\partial x_i \partial x_j} + \frac{\partial[F_i \delta(f)]}{\partial x_i} + \frac{\partial[Q \delta(f)]}{\partial t}, \tag{5.43}$$

where

$$T_{ij} = \rho v_i v_j + [(p - p_\infty) - (\rho - \rho_\infty) c_\infty^2] \delta_{ij} - \tau_{ij}, \tag{5.44}$$

$$F_i = -[\rho v_i (v_j - v_j^{\Sigma}) + p_{ij}] \frac{\partial f}{\partial x_j}, \tag{5.45}$$

$$Q = \left(\rho v_i - \rho' v_i^{\Sigma} \right) \frac{\partial f}{\partial x_i} = [\rho (v_i - v_i^{\Sigma}) + \rho_\infty v_i^{\Sigma}] \frac{\partial f}{\partial x_i}. \tag{5.46}$$

This is the Ffowcs Williams et Hawkings equation. In the absence of Σ, we recover Lighthill's equation.

5.3.2 Ffowcs Williams and Hawkings integral solution

It is possible to consider the method of Green's functions, as in Section 5.2, to obtain an integral solution under the form:

$$\begin{aligned}\rho'(\underline{x},t)c_\infty^2 H(f) = &\int_{-T}^{T}\int_V G\frac{\partial^2[T_{ij}H(f)]}{\partial y_i\partial y_j}d\tau dV(\underline{y})\\ &+\int_{-T}^{T}\int_V G\frac{\partial[F_i\delta(f)]}{\partial y_i}dV(\underline{y})d\tau + \int_{-T}^{T}\int_V G\frac{\partial[Q\delta(f)]}{\partial \tau}dV(\underline{y})d\tau.\end{aligned} \tag{5.47}$$

We can now use the following property of the delta function, which can be seen as a generalization of Equation (2.54) (Rienstra and Hirschberg, 2021, Eq. (C.39)):

$$\int_V \delta(h(\underline{x}))g(\underline{x})dV(\underline{x}) = \int_\Sigma \frac{g(\underline{x})}{|\nabla h|}dS(\underline{x}), \tag{5.48}$$

where the summation is performed on the surface Σ corresponding to $h(\underline{x}) = 0$. This yields:

$$\begin{aligned}\rho'(\underline{x},t)c_\infty^2 H(f) = &\int_{-T}^{T}\int_{V_0(f>0)} G\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}d\tau dV(\underline{y})\\ &+\int_{-T}^{T}\int_{\Sigma(f=0)} \frac{G}{|\nabla f|}\frac{\partial F_i}{\partial y_i}dS(\underline{y})d\tau + \int_{-T}^{T}\int_{\Sigma(f=0)} \frac{G}{|\nabla f|}\frac{\partial Q}{\partial \tau}dS(\underline{y})d\tau.\end{aligned} \tag{5.49}$$

As explained by Glegg and Devenport (2017, Section 5.1), it is convenient to pass the derivatives to the Green's functions:

$$\begin{aligned}\rho'(\underline{x},t)c_\infty^2 H(f) = &\int_{-T}^{T}\int_{V_0(f>0)} \frac{\partial^2 G}{\partial y_i\partial y_j}T_{ij}d\tau dV(\underline{y})\\ &-\int_{-T}^{T}\int_{\Sigma(f=0)} \frac{F_i}{|\nabla f|}\frac{\partial G}{\partial y_i}dS(\underline{y})d\tau - \int_{-T}^{T}\int_{\Sigma(f=0)} \frac{Q}{|\nabla f|}\frac{\partial G}{\partial \tau}dS(\underline{y})d\tau.\end{aligned} \tag{5.50}$$

Let us now consider the free field Green's function $G_0(\underline{x},t|\underline{y},\tau) = \frac{\delta(g)}{4\pi r}$, with $g = t-\tau-r/c_\infty$ and $r = |\underline{x}-\underline{y}|$. Since

$$\frac{\partial^2 G_0}{\partial y_i y_j} = \frac{\partial^2 G_0}{\partial x_i x_j} \quad , \quad \frac{\partial G_0}{\partial y_i} = -\frac{\partial G_0}{\partial x_i} \quad \text{and} \quad \frac{\partial G_0}{\partial \tau} = -\frac{\partial G_0}{\partial t},$$

the derivatives can be placed outside the integrals:

$$\rho'(\underline{x},t)c_\infty^2 H(f) = \frac{\partial^2}{\partial x_i \partial x_j}\int_{-T}^{T}\int_{V_0(f>0)} \frac{\delta(g)}{4\pi r} T_{ij} d\tau dV(\underline{y})$$
$$+ \frac{\partial}{\partial x_i}\int_{-T}^{T}\int_{\Sigma(f=0)} \frac{F_i}{|\nabla f|}\frac{\delta(g)}{4\pi r} dS(\underline{y})d\tau + \frac{\partial}{\partial t}\int_{-T}^{T}\int_{\Sigma(f=0)} \frac{Q}{|\nabla f|}\frac{\delta(g)}{4\pi r} dS(\underline{y})d\tau. \tag{5.51}$$

We can now perform the time integration using the property of the dirac function given in Equation (2.54), and rewritten here as:

$$\int_{-\infty}^{+\infty} \delta[g(\tau)]h(\tau)d\tau = \left[\frac{h(\tau)}{|\frac{\partial g}{\partial \tau}(\tau_i)|}\right]_{g(\tau)=0} = \left[\frac{h(\tau)}{|1-M_r|}\right]_{\tau=\tau^*}, \tag{5.52}$$

with $\tau^* = t - r(\tau^*)/c_\infty$ as in Equation (5.19), and using Equation (2.59) to calculate $\partial g/\partial \tau$. Note that the motion of the source is now arbitrary, so that the Mach number M_r of the source in the direction of the observer is now defined in a more general way as:

$$M_r = -\frac{1}{c_\infty}\frac{\partial r}{\partial \tau} = -\frac{v_i^\Sigma}{c_\infty}\frac{r_i}{r} = \frac{\underline{M}.\underline{r}}{r}, \tag{5.53}$$

with $r_i = (x_i - y_i)$. As a result, Equation (5.51) takes the following form:

$$\begin{aligned}\rho'(\underline{x},t)c_\infty^2 H(f) &= \frac{\partial^2}{\partial x_i \partial x_j}\int_{V_0(f>0)} \left[\frac{T_{ij}}{4\pi r|1-M_r|}\right]_{\tau=\tau^*} dV(\underline{y}) \\ &- \frac{\partial}{\partial x_i}\int_{\Sigma(f=0)} \left[\frac{(\rho v_i(v_j - v_j^\Sigma) + p_{ij})n_j}{4\pi r|1-M_r|}\right]_{\tau=\tau^*} dS(\underline{y}) \\ &+ \frac{\partial}{\partial t}\int_{\Sigma(f=0)} \left[\frac{(\rho(v_j - v_j^\Sigma) + \rho_\infty v_j^\Sigma)n_j}{4\pi r|1-M_r|}\right]_{\tau=\tau^*} dS(\underline{y}),\end{aligned} \tag{5.54}$$

using the expressions for F_i and Q given in Equations (5.45) and (5.46), and using $\frac{\partial f}{\partial x_i} = \nabla f = |\nabla f| n_i$.

If the surface is stationary ($v_j^\Sigma = 0$ and $M_r = 0$), the Curle integral solution (5.19) is retrieved. If the surface is impermeable and non vibrating

($v_j = v_j^\Sigma$), the Ffowcs Williams and Hawkings integral solution reduces to:

$$
\begin{aligned}
\rho'(\underline{x},t)c_\infty^2 H(f) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{V_0(f>0)} \left[\frac{T_{ij}}{4\pi r|1-M_r|}\right]_{\tau=\tau^*} dV(\underline{y}) \\
- \frac{\partial}{\partial x_i} \int_{\Sigma(f=0)} \left[\frac{p_{ij} n_j}{4\pi r|1-M_r|}\right]_{\tau=\tau^*} d\Sigma(\underline{y}) \\
+ \frac{\partial}{\partial t} \int_{\Sigma(f=0)} \left[\frac{\rho_\infty v_j^\Sigma}{4\pi r|1-M_r|}\right]_{\tau=\tau^*} d\Sigma(\underline{y}),
\end{aligned}
\tag{5.55}
$$

Many rotor noise analysis are performed based on Equation (5.55); see for instance Glegg and Devenport (2017, Chapter 16). The second term in Equation (5.55) is commonly referred to as the loading noise. At low speeds this is usually the dominant source of sound.

5.4 Loading noise from rotating blades

At low Mach number, the loading noise that corresponds to the second term in Equation (5.55) is generally the dominant source of sound (Glegg and Devenport, 2017, Chapter 16). If the blades are sufficiently thin so that the acoustic wavelength at the frequencies of interest are much larger than the blade thickness, the integral of the complete surface of the rotor blade Σ can be replaced by an integration over the blade planform S_{blade}, which is the projection of the rotor geometry into the rotor plane. Thus:

$$
(p'(\underline{x},t))_{\text{loading}} = -\frac{\partial}{\partial x_i} \int_{S_{\text{blade}}} \left[\frac{f_i(\underline{y},\tau)}{4\pi r|1-M_r|}\right]_{\tau=\tau^*} dS(\underline{y}), \tag{5.56}
$$

where $f_i dS = \left([p_{ij}n_j]_{\text{upper}} - [p_{ij}n_j]_{\text{lower}}\right) d\Sigma$ is the force per unit area applied to the fluid due to the pressure difference between the upper and lower surfaces.

To obtain a precise calculation of the loading noise, it is interesting to shift the spatial derivatives to sources times, as suggested by Farassat (1981). To this aim, let us start from the form of the loading noise found in Equation (5.50):

$$
(p'(\underline{x},t))_{\text{loading}} = -\int_{-T}^{T} \int_{\Sigma(f=0)} \frac{F_i}{|\nabla f|} \frac{\partial G_0}{\partial y_i} dS(\underline{y}) d\tau, \tag{5.57}
$$

where the free field Green's function G_0 has been used. The derivative of the Green's function is given by:

$$\frac{\partial G_0}{\partial y_i} = -\frac{\partial G_0}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(\frac{\delta(g)}{4\pi r}\right) = -\frac{\partial r}{\partial x_i}\left(\frac{1}{4\pi r}\frac{\partial \delta(g)}{\partial r} - \frac{\delta(g)}{4\pi r^2}\right), \tag{5.58}$$

where $\frac{\partial r}{\partial x_i} = -(x_i - y_i)/r$, and

$$\frac{\partial \delta(g)}{\partial r} = \frac{\partial g}{\partial r}\frac{\partial \delta(g)}{\partial g} = -\frac{1}{c_\infty}\frac{\partial \delta(g)}{\partial g} = -\frac{1}{c_\infty}\frac{\partial \delta(g)}{\partial t}, \tag{5.59}$$

since

$$\frac{\partial \delta(g)}{\partial t} = \frac{\partial g}{\partial t}\frac{\partial \delta(g)}{\partial g} = \frac{\partial \delta(g)}{\partial g}. \tag{5.60}$$

Finally:

$$\frac{\partial G_0}{\partial y_i} = \frac{(x_i - y_i)}{r}\left(\frac{1}{4\pi r c_\infty}\frac{\partial \delta(g)}{\partial t} + \frac{\delta(g)}{4\pi r^2}\right) \approx \frac{x_i}{4\pi |x|^2 c_\infty}\frac{\partial \delta(g)}{\partial t} \tag{5.61}$$

in the far-field. Equation (5.57) thus becomes in the far-field:

$$\begin{aligned}
(p'(\underline{x},t))_{\text{loading}} &\approx -\frac{x_i}{4\pi|x|^2 c_\infty}\frac{\partial}{\partial t}\int_{-T}^{T}\int_{\Sigma(f=0)}\frac{F_i}{|\nabla f|}\delta(g)d\tau \\
&\approx -\frac{x_i}{4\pi|x|^2 c_\infty}\frac{\partial}{\partial t}\int_{\Sigma(f=0)}\left[\frac{F_i}{|\nabla f||1-M_r|}\right]_{\tau=\tau^*} dS(\underline{y}) \\
&\approx \frac{x_i}{4\pi|x|^2 c_\infty}\frac{\partial}{\partial t}\int_{S_{\text{blade}}}\left[\frac{f_i(\underline{y},\tau)}{|1-M_r|}\right]_{\tau=\tau^*} dS(\underline{y}).
\end{aligned} \tag{5.62}$$

We can shift the time derivatives inside the integral and use the property

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \left(\frac{\partial t}{\partial \tau}\right)^{-1}\frac{\partial}{\partial \tau} = \frac{1}{1 + c_\infty \frac{\partial r}{\partial \tau}}\frac{\partial}{\partial \tau} = \frac{1}{1-M_r}\frac{\partial}{\partial \tau}. \tag{5.63}$$

As a result Equation (5.62) becomes:

$$\begin{aligned}
(p'(\underline{x},t))_{\text{loading}} &\approx \frac{x_i}{4\pi|x|^2 c_\infty}\int_{S_{\text{blade}}}\left[\frac{1}{1-M_r}\frac{\partial}{\partial \tau}\left(\left[\frac{f_i(\underline{y},\tau)}{|1-M_r|}\right]_{\tau=\tau^*}\right) dS(\underline{y})\right. \\
&\approx \frac{x_i}{4\pi|x|^2 c_\infty}\int_{S_{\text{blade}}}\left[\frac{1}{(1-M_r)^2}\left(\frac{\partial f_i}{\partial \tau} + \frac{f_i}{(1-M_r)}\frac{\partial M_r}{\partial \tau}\right)\right]_{\tau=\tau^*} dS(\underline{y})
\end{aligned} \tag{5.64}$$

If the source is acoustically compact, we can ignore the differences in retarded time from different points on the surface and we obtain:

$$(p'(\underline{x},t))_{\text{loading}} \approx \frac{x_i}{4\pi|x|^2 c_\infty}\left[\frac{1}{(1-M_r)^2}\left(\frac{\partial F_i}{\partial \tau} + \frac{F_i}{(1-M_r)}\frac{\partial M_r}{\partial \tau}\right)\right]_{\tau=\tau^*} \tag{5.65}$$

with $F_i = \int_{S_{\text{blade}}} f_i dS(\mathbf{y})$. This corresponds to the solution obtained by Lowson (1965). The first term is similar to the dipole term given in Equation 5.27, and is equal to the zero if the force is a constant (steady loading). The second term is an acceleration term, that can be nonzero even if the force is constant.

Dimensional analysis:

For a compact dipole of characteristic length ℓ, the the force F_i scales as $\rho_\infty U^2 \ell^2$, the Mach number scales as U/c_∞, and the time derivatives scales as U/ℓ, thus:

$$(p)'_{\text{loading}} \sim \frac{\rho_\infty}{c_\infty}\frac{\ell}{x}\left(\frac{U^3}{(1-M_r)^2} + \frac{U^4}{(1-M_r)^3 c_\infty}\right).$$

The first term scales as U^3 and can be interpreted as a convected dipole, with the same Doppler factor as in Equation (2.75). The second term scales as U^4 and can be interpreted as a convected quadrupole (acceleration term), with the same Doppler factor as in Equation (**??**).

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