# AE-01: ACOUSTIC PROPAGATION IN INHOMOGENEOUS MOVING MEDIA

## Course notes 2019-2020

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# Contents

# Introduction

These course notes are intended for the students of the course "Acoustic propagation in inhomogeneous moving media" of the M2 program *Acoustical Engineering* of the University Paris-Saclay. These notes cover approximately half of the course, and do not provide a description of the numerical models of acoustic propagation that are studied in the second part of the course. They are divided into 4 chapters

Chapter 1 presents the equations for the propagation of acoustic waves in homogeneous moving media. The linearized Euler equations are derived from the equations of fluid mechanics. Several wave equations are then obtained in the time domain, and the corresponding Helmholtz equations are presented in the frequency domain.

Chapter 2 is devoted to acoustic propagation in a homogeneous medium at rest. After a brief presentation of some simple solutions to the wave equation in free field, a large part of this chapter focuses on the acoustic propagation above a flat ground surface, where both plane-wave and spherical-wave reflection by a finite impedance ground are considered.

Chapter 3 explains the effects of absorption and refraction in fluid media. The acoustic absorption mechanisms are presented first, considering both atmospheric and oceanic media. Then, some examples of acoustic refraction by sound speed gradients and and wind speed gradients are shown.

Chapter 4 describes the geometrical acoustics approximation, that can be used to model acoustic wave propagation at high frequencies. Ray-tracing equations are presented, and the calculation of wave amplitude along the rays is mentioned.

# Chapter 1

# Equations for acoustical waves in an inhomogeneous moving medium

This chapter is based mostly on the books of Pierce (1989) and Ostashev (1997).

## 1.1 Equations of fluid mechanics

We start here from the equations of fluid mechanics written for the pressure $p_t(\underline{x}, t)$, the velocity $\underline{v_t}(\underline{x}, t)$ the density $\rho_t(\underline{x}, t)$ and the entropy $S_t(\underline{x}, t)$ that are functions of space $\underline{x}$ and time $t$. The subscript $t$ means that these are the total values in the fluid medium. We consider that there is only one component in the medium. For media with different components (such as water and salt in the ocean, or water vapor and air in the atmosphere), more general formulations exist (Ostashev, 1997).

First, the conservation of mass or equation of continuity is written:

$$\frac{\partial \rho_t}{\partial t} + \nabla.(\rho_t \underline{v_t}) = \frac{D\rho_t}{Dt} + \rho_t \nabla.\underline{v_t} = 0, \tag{1.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v_t}.\nabla$$

is called the **material or total derivative**.

Second, the Navier-Stokes equation or law of momentum conservation is given by:

$$\rho_t \left[ \frac{\partial \underline{v_t}}{\partial t} + (\underline{v_t}.\nabla)\underline{v_t} \right] = -\nabla p_t + \nabla.\tau + \rho_t g \underline{e}_z, \tag{1.2}$$

where $\tau$ is the **viscous stress tensor** and $\rho g \underline{e}_z$ corresponds to the **gravitational force**. The viscous stress tensor is important to model aeroacoustic source generation, but will be neglected here in the context of acoustic propagation. The Navier-Stokes equation is called **Euler equation** for an inviscid fluid.

The last equations come from the conservation of the total energy and from thermodynamic laws. The total energy per unit mass is the sum of the internal energy $e_t$, associated to molecular motion, and of the kinetic energy $v_t^2/2$. The conservation of energy can be formulated in terms of the specific entropy $s_t$ (entropy per unit mass). In loose terms, entropy measures the degree of disorder of a system. This yields (see e.g. Panton *et al.* (2007, Section 1.3.5) or Boutillon (2017, Section 2.5)):

$$\rho_t \frac{Ds_t}{Dt} = \rho_t \left( \frac{\partial s_t}{\partial t} + \underline{v_t}.\nabla s_t \right) = -\frac{1}{T}\nabla.\underline{q} + \frac{q_e}{T} + \frac{\Phi}{T}, \tag{1.3}$$

where $\underline{q}$ is the heat flux, $q_e$ is the heat transfer from outside, and $\Phi$ corresponds to viscous dissipation. The entropy of a fluid flow can never decrease (second law of thermodynamics). It can increase due to **irreversible processes** such as viscous dissipation or heat transfer from outside. For an inviscid fluid, we neglect not only viscous dissipation but also heat transfer (adiabatic flow). As a result, energy changes are only due to reversible processes and entropy is conserved along streamlines (Pierce, 1989):

$$\frac{Ds_t}{Dt} = 0 \Leftrightarrow \frac{\partial s_t}{\partial t} + \underline{v_t}.\nabla s_t = 0. \tag{1.4}$$

Finally, an equation of state is necessary to have a a number of equations equal to the number of unknowns. In the more general form, the entropy $s_t$ is a function of two independent thermodynamic variables:

$$s_t = s_t(\rho_t, p_t) \quad \text{or} \quad p_t = p_t(\rho_t, s_t). \tag{1.5}$$

For an ideal gas, the equation of state can be written:

$$dp_t = \left.\frac{\partial p_t}{\partial \rho_t}\right|_{s_t} d\rho_t + \left.\frac{\partial p_t}{\partial s_t}\right|_{\rho_t} ds_t = c^2 d\rho_t + \frac{p_t}{c_v} ds_t, \tag{1.6}$$

with $c_v$ the specific heat at constant volume and

$$c^2 = \gamma r T = \frac{\gamma p_t}{\rho_t}, \tag{1.7}$$

where $c$ is the sound speed for an ideal gas, $T$ is the temperature, $\gamma = c_p/c_v$ is the ratio of specific heat and $r$ is a gas constant. Using Equations (1.4) and (1.6), we finally obtain:

$$\frac{Dp_t}{Dt} = \left.\frac{\partial p_t}{\partial \rho_t}\right|_{s_t} \frac{D\rho_t}{Dt} + \left.\frac{\partial p_t}{\partial s_t}\right|_{\rho_t} \cancel{\frac{Ds_t}{Dt}} = c^2 \frac{D\rho_t}{Dt} = -c^2 \rho_t \nabla . \underline{v_t}, \tag{1.8}$$

where the equation of continuity (1.1) has been used. Note that some fluids such as water cannot be considered as an ideal gas. Water sound speed depends on temperature, pressure, and salinity in a complex way (Pierce, 1989, Section 1-9).

As shown in various references (Ostashev *et al.*, 2005; Boutillon, 2017), any unsteady perturbation can be decomposed into three types of modes or waves:

- the vortical mode corresponds to transverse waves (particle velocity fluctuations perpendicular to the direction of propagation);
- the entropy mode corresponds to the convection of temperature fluctuations;
- the acoustic mode corresponds to locally longitudinal waves.

These 3 modes are generally decoupled only in the high frequency approximation (geometrical acoustics) or when the mean flow is homogeneous.

It is common to decompose the mean velocity into an irrotational part and a solenoidal part:

$$\underline{v_0} = \nabla \Phi_0 + \nabla \times \underline{\Psi_0}, \tag{1.9}$$

where $\Phi_0$ is a scalar velocity potential and $\underline{\Psi_0}$ is a vector velocity potential. The acoustic waves are related to the irrotational velocity fluctuations described by $\nabla \Phi_0$, called also potential flow by Rienstra and Hirschberg (2014).

## 1.2 Linearized Euler equations

We now linearize the equations of fluid mechanics about a base flow (or mean flow) that is independent of time, described by the variables $\underline{v_0}(\underline{x})$, $p_0(\underline{x})$, $\rho_0(\underline{x})$ and $s_0(\underline{x})$. Let us introduce the variables associated to acoustic waves in a fluid medium:

- acoustic pressure [Pa]: $p(\underline{x}, t) = p_t(\underline{x}, t) - p_0(\underline{x})$;
- particle velocity [m/s]: $\underline{v}(\underline{x}, t) = \underline{v_t}(\underline{x}, t) - \underline{v_0}(\underline{x})$;
- density associated to acoustic fluctuations [kg/m$^3$]: $\rho(\underline{x}, t) = \rho_t(\underline{x}, t) - \rho_0(\underline{x})$;
- entropy associated to acoustic fluctuations [J/K]: $s(\underline{x}, t) = s_t(\underline{x}, t) - s_0(\underline{x})$.

Generally, the acoustic fluctuations are small perturbations with respect to the mean quantities. The approximation of linear acoustics is considered valid if (Pierce, 1989)

$$|p| \ll \rho_0 c^2, \quad |v| \ll c, \quad |\rho| \ll \rho_0. \tag{1.10}$$

These conditions will be explained in more details in Section 1.3. We also suppose the base flow **incompressible**: $\nabla.\underline{v_0} = 0$. This is generally a good approximation because the Mach number $M = v_0/c \ll 1$ in most situations in the atmosphere and the ocean. Finally, we introduce the mean sound speed $c_0(\underline{x})$ associated with the base flow:

$$c_0^2 = \gamma r T_0 = \frac{\gamma p_0}{\rho_0} = c^2 - c_f^2, \tag{1.11}$$

with $c_f^2 \ll c_0^2$ the fluctuations of the squared sound speed. For air, $\gamma = 1.4$ and $r = 287\,\mathrm{J/kg/K}$ so $c_0 \approx 340\,\mathrm{m/s}$ at $15^o$C.

Let us now introduce the acoustic variables in the continuity equation:

$$\frac{\partial(\rho_0 + \rho)}{\partial t} + (\underline{v_0} + \underline{v}).\nabla(\rho_0 + \rho) + (\rho_0 + \rho)\nabla.(\underline{v_0} + \underline{v}) = 0. \tag{1.12}$$

It is possible to group terms of order 0 (base flow quantities only), of order 1 (one small quantity only), etc. This yields:

$$
\begin{aligned}
&\text{order 0} \quad \cancel{\frac{\partial \rho_0}{\partial t}} + (\underline{v_0}.\nabla)\rho_0 + \rho_0 \cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 1} \quad \frac{\partial \rho}{\partial t} + (\underline{v_0}.\nabla)\rho + (\underline{v}.\nabla)\rho_0 + \rho_0\nabla.\underline{v} + \rho\cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 2} \quad (\underline{v}.\nabla)\rho + \rho\nabla.\underline{v} = 0.
\end{aligned}
$$

A similar procedure can be applied to the momentum equation (1.2):

$$
\begin{aligned}
&\text{order 0} \quad \rho_0\cancel{\frac{\partial \underline{v_0}}{\partial t}} + \rho_0(\underline{v_0}.\nabla)\underline{v_0} = -\nabla p_0 + \rho_0 g\underline{e}_z,\\
&\text{order 1} \quad \rho_0\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v_0}.\nabla)\underline{v} + (\underline{v}.\nabla)\underline{v_0}\right) + \rho(\underline{v_0}.\nabla)\underline{v_0} = -\nabla p + \rho g\underline{e}_z,\\
&\text{order 2} \quad \rho\frac{\partial \underline{v}}{\partial t} + \rho_0(\underline{v}.\nabla)\underline{v} + \rho\left[(\underline{v_0}.\nabla)\underline{v} + (\underline{v}.\nabla)\underline{v_0}\right] = 0,\\
&\text{order 3} \quad \rho(\underline{v}.\nabla)\underline{v} = 0,
\end{aligned}
$$

and to the equation of state for an ideal gas (1.8):

$$
\begin{aligned}
&\text{order 0} \quad \cancel{\frac{\partial p_0}{\partial t}} + (\underline{v_0}.\nabla)p_0 + c_0^2\rho_0\cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 1} \quad \frac{\partial p}{\partial t} + (\underline{v_0}.\nabla)p + (\underline{v}.\nabla)p_0 + \rho_0 c_0^2\nabla.\underline{v} + \left(c_0^2\rho + \rho_0 c_f^2\right)\cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 2} \quad (\underline{v}.\nabla)p + \left(c_0^2\rho + \rho_0 c_f^2\right)\nabla.\underline{v} + \rho c_f^2\cancel{\nabla.\underline{v_0}} = 0,\\
&\text{order 3} \quad \rho c_f^2\nabla.\underline{v} = 0.
\end{aligned}
$$

The following set of equations for the base flow (order 0) is obtained:

$$
(\underline{v_0}.\nabla)\rho_0 = 0, \tag{1.13}
$$

$$
(\underline{v_0}.\nabla)\underline{v_0} = -\frac{\nabla p_0}{\rho_0} + g\underline{e}_z, \tag{1.14}
$$

$$
(\underline{v_0}.\nabla)p_0 = 0. \tag{1.15}
$$

The set of equations at order 1 are called the linearized Euler equations for an ideal gas:

$$\frac{\partial \rho}{\partial t} + (\underline{v_0}.\nabla)\rho + (\underline{v}.\nabla)\rho_0 + \rho_0 \nabla.\underline{v} = 0, \tag{1.16}$$

$$\rho_0 \left( \frac{\partial \underline{v}}{\partial t} + (\underline{v_0}.\nabla)\underline{v} + (\underline{v}.\nabla)\underline{v_0} \right) + \nabla p - \frac{\rho}{\rho_0}\nabla p_0 = 0, \tag{1.17}$$

$$\frac{\partial p}{\partial t} + (\underline{v_0}.\nabla)p + (\underline{v}.\nabla)p_0 + \rho_0 c_0^2 \nabla.\underline{v} = 0. \tag{1.18}$$

Note that in order to obtain Equation (1.17), we have used that:

$$\rho(\underline{v_0}.\nabla)\underline{v_0} - \rho g \underline{e}_z = -\frac{\rho \nabla p_0}{\rho_0},$$

which makes use of the momentum equation (1.14) at order 0.

In many situations, it is possible to neglect the terms proportional to $\nabla p_0$. For instance, for a harmonic plane wave of the form $p(x,t) = Ae^{-i\omega(t-x/c_0)}$ in a medium at rest ($v_0 = 0$), we will see in Section 2.2.1 that $v(x,t) = p(x,t)/(\rho_0 c_0)$ and $\rho(x,t) = p(x,t)/c_0^2$. Thus the orders of magnitude of the different terms in Equation (1.17) are:

$$\underbrace{\rho_0 \frac{\partial \underline{v}}{\partial t}}_{\sim \omega \frac{A}{c_0}} + \underbrace{\nabla p}_{\sim \frac{\omega}{c_0} A} - \underbrace{\frac{\rho}{\rho_0}\nabla p_0}_{\sim \frac{A}{c_0^2} g} = 0, \tag{1.19}$$

where we have used that $\nabla p_0 = \rho_0 g \underline{e}_z$ from Equation (1.14) in a medium at rest. Thus it appears that the term proportional to $\nabla p_0$ can be neglected at sufficiently high frequencies such that $\omega \gg g/c_0$. This corresponds to $f \gg 10^{-3}$ Hz in air and $f \gg 5 \times 10^{-3}$ Hz in water. Ostashev *et al.* (2005) note that the terms proportional to $\nabla p_0$ are important for internal gravity waves, but can be neglected for acoustic waves. It will thus be neglected in the rest of the course.

The linearized Euler equations become the following set of equations:

$$\frac{\partial \rho}{\partial t} + (\underline{v_0}.\nabla)\rho + (\underline{v}.\nabla)\rho_0 + \rho_0 \nabla.\underline{v} = 0, \tag{1.20}$$

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v_0}.\nabla)\underline{v} + (\underline{v}.\nabla)\underline{v_0} + \frac{\nabla p}{\rho_0} = 0, \tag{1.21}$$

$$\frac{\partial p}{\partial t} + (\underline{v_0}.\nabla)p + \rho_0 c_0^2 \nabla.\underline{v} = 0. \tag{1.22}$$

Note that Equations (1.21) and (1.22) do not depend on $\rho$. These equations are the basis of many numerical solvers of the linearized Euler equations. It is also common to rewrite these equations under the following conservative form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} + \mathbf{H} = \mathbf{S}, \tag{1.23}$$

where $\mathbf{U} = (p, \rho_0 v_x, \rho_0 v_y, \rho_0 v_z)^T$ and $\mathbf{S} = (\rho_0 c^2 Q, F_x, F_y, F_z)^T$ corresponds to the source terms, with $Q$ the volume velocity and $\underline{F}$ the exterior forces. The Eulerian fluxes $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are written:

$$\begin{aligned}
\mathbf{E} &= \begin{pmatrix} v_{0x}p + \rho_0 c_0^2 v_x \\ v_{0x}\rho_0 v_x + p \\ v_{0x}\rho_0 v_y \\ v_{0x}\rho_0 v_z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} v_{0y}p + \rho_0 c_0^2 v_y \\ v_{0y}\rho_0 v_x \\ v_{0y}\rho_0 v_y + p \\ v_{0y}\rho_0 v_z \end{pmatrix}, \\
\mathbf{G} &= \begin{pmatrix} v_{0z}p + \rho_0 c_0^2 v_z \\ v_{0z}\rho_0 v_x \\ v_{0z}\rho_0 v_y \\ v_{0z}\rho_0 v_z + p \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 0 \\ \rho_0(\mathbf{v}.\nabla)v_{0x} \\ \rho_0(\mathbf{v}.\nabla)v_{0y} \\ \rho_0(\mathbf{v}.\nabla)v_{0z} \end{pmatrix}.
\end{aligned} \tag{1.24}$$

From Equations (1.23) and (1.24), the following coupled equations can be retrieved:

$$\frac{\partial p}{\partial t} + (\underline{v_0}.\nabla)p + p(\nabla.\underline{v_0}) + \rho_0 c_0^2 \nabla.\underline{v} + (\underline{v_0}.\nabla)(\rho_0 c_0^2) = 0, \tag{1.25}$$

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v_0}.\nabla)\underline{v} + (\nabla.\underline{v_0})\underline{v} + \frac{\underline{v}}{\rho_0}(\underline{v_0}.\nabla)\rho_0 + \frac{\nabla p}{\rho_0} + (\underline{v}.\nabla)\underline{v_0} = 0. \tag{1.26}$$

Equation (1.25) is equivalent to Equation (1.22) because the base flow is incompressible ($\nabla.\underline{v_0} = 0$) and the last term can be written:

$$(\underline{v_0}.\nabla)(\rho_0 c_0^2) = \gamma(\underline{v_0}.\nabla)p_0 = 0, \tag{1.27}$$

using Equations (1.11) and (1.15). Similarly, Equation (1.26) is equivalent to Equation (1.21) because the base flow is incompressible and $(\underline{v_0}.\nabla)\rho_0 = 0$ from Equation (1.13).

## 1.3 Validity of the linear acoustics approximation

From the equations of continuity and momentum conservation, it is clear that the terms of order 1 will be small compared to the terms of order 0 if $|\rho| \ll \rho_0$

and $|p| \ll p_0$. From the equation of state for a perfect gas, $p_0 = \rho_0 c_0^2/\gamma$ thus we obtain the condition $|p| \ll \rho_0 c_0^2$. The condition for the particle velocity is less straightforward to obtain. Let us consider a plane wave propagating in the fluid. We will see in Section 2.2.1 that in this case:

$$v = \frac{p}{\rho_0 c_0} \Rightarrow |v| = \frac{|p|}{\rho_0 c_0} \ll c_0. \tag{1.28}$$

Note that it is not necessary than $|v| \ll v_0$, such that linear acoustics is also valid in a medium at rest ($v_0 = 0$).

The linear acoustics approximation is valid in many applications. For instance, the amplitude of acoustic pressure corresponding to the threshold of pain is around 90 Pa (about 130 dB re. 20 muPa), which is still two orders of magnitude smaller compared to the atmospheric pressure that is close to $10^5$ Pa. This corresponds to an amplitude of particle velocity of 0.2 m/s, which is much smaller than the sound speed in air.

## 1.4 Wave equations in the time and frequency domains

### 1.4.1 Propagation in a homogeneous medium at rest

The simplest case that can be considered corresponds to a homogeneous medium at rest: $\underline{v_0} = 0$, where $\rho_0$, $p_0$ and $c_0$ are constant. The linearized Euler equations become simply:

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla . \underline{v} = 0, \tag{1.29}$$

$$\rho_0 \frac{\partial \underline{v}}{\partial t} + \nabla p = 0, \tag{1.30}$$

$$\frac{\partial p}{\partial t} + \rho_0 c_0^2 \nabla . \underline{v} = 0. \tag{1.31}$$

It is possible to obtain a wave equation for the acoustic pressure $p$, by substracting the time derivative of Equation (1.31) and the divergence of Equation (1.30) multiplied by $c_0^2$:

$$\boxed{\frac{\partial^2 p}{\partial t^2} - c_0^2 \nabla^2 p = 0}\,. \tag{1.32}$$

The operator $\nabla^2 = \Delta$ is called Laplacian and is written in cartesian coordinates:

$$\nabla^2 p = \Delta p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}. \tag{1.33}$$

Note also that a simple expression can be obtained between acoustic density and pressure from Equations (1.29) and (1.31):

$$\frac{\partial p}{\partial t} = c_0^2 \frac{\partial \rho}{\partial t} \Rightarrow \boxed{p = c_0^2 \rho}\ . \tag{1.34}$$

For a harmonic wave at angular frequency $\omega = 2\pi f$, the pressure can be written $p(\underline{x}, t) = A(\underline{x}) \cos(\omega t + \phi(\underline{x}))$, where $A$ is the amplitude and $\phi$ is the phase that are both functions of space. It is useful to introduce the following complex notation:

$$p(\underline{x}, t) = \mathrm{Re}\left[p_c(\underline{x}) e^{-i\omega t}\right], \tag{1.35}$$

where Re denotes the real part and the $p_c(\underline{x}) = A(\underline{x}) e^{-i\phi(\underline{x})}$ is the complex pressure amplitude. Introducing $p_c(\underline{x}) e^{-i\omega t}$ into the wave equation:

$$\boxed{\Delta p_c + k_0^2 p_c = 0}\ , \tag{1.36}$$

where $k_0 = \omega / c_0 = 2\pi / \lambda_0$ is the acoustic wave number, and $\lambda_0$ is the wavelength. Equation (1.36) is called the **Helmholtz equation**. Many computational methods assume a harmonic sound field as any sound signal can be decomposed into harmonic components using the Fourier transform (spectral decomposition), and it is easier to solve in the frequency domain as there is no time derivative to evaluate.

Remark: it is also possible to use the $e^{j\omega t}$ convention instead of the $e^{-i\omega t}$ convention. In this case, we would have:

$$p(\underline{x}, t) = \mathrm{Re}\left[p_c(\underline{x}) e^{j\omega t}\right], \tag{1.37}$$

with $p_c(\underline{x}) = A(\underline{x}) e^{j\phi(\underline{x})}$. The Helmholtz equation remains the same with both notations!

Finally, it is possible to introduce an acoustic velocity potential $\Phi$ associated with the particle velocity $\underline{v}$. Taking the curl of Equation (1.30):

$$\frac{\partial}{\partial t} \nabla \times \underline{v} = 0, \tag{1.38}$$

since $\nabla \times \nabla p = 0$. This means that the rotational of particle velocity is independent of time. If the acoustic field is irrotational ($\nabla \times \underline{v} = 0$), then the particle velocity derives from a potential $\Phi$: $\underline{v} = \nabla\Phi$. The relationship between $p$ and $\Phi$ is obtained from Equation (1.30):

$$p(r,t) = -\rho_0 \frac{\partial \Phi}{\partial t}. \tag{1.39}$$

Replacing this expression into the wave equation (1.32), we see that $\Phi$ satisfies the same equation as $p$:

$$\boxed{\frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0}\,. \tag{1.40}$$

It is convenient to solve for the acoustic potential because acoustic pressure and particle velocity can be deduced by taking the temporal or spatial derivative of $\Phi$.

### 1.4.2 Propagation in an inhomogeneous medium at rest

We now consider that all the mean quantities depend on space in a medium at rest: $\underline{v_0} = 0$, with $\rho_0(\underline{x})$, $p_0(\underline{x})$ and $c_0(\underline{x})$. The linearized Euler equations reduce to:

$$\frac{\partial \rho}{\partial t} + (\underline{v}.\nabla)\rho_0 + \rho_0 \nabla.\underline{v} = 0, \tag{1.41}$$

$$\frac{\partial \underline{v}}{\partial t} + \frac{\nabla p}{\rho_0} = 0, \tag{1.42}$$

$$\frac{\partial p}{\partial t} + \rho_0 c_0^2 \nabla.\underline{v} = 0. \tag{1.43}$$

As done previously, let us calculate $\frac{\partial}{\partial t}(1.43) - \rho_0 c_0^2 \nabla.(1.42)$:

$$\frac{\partial^2 p}{\partial t^2} + \rho_0 c_0^2 \frac{\partial(\nabla.\underline{v})}{\partial t} - \rho_0 c_0^2 \nabla.\left(\frac{\partial \underline{v}}{\partial t}\right) - \rho_0 c_0^2 \nabla.\left(\frac{\nabla p}{\rho_0}\right) = 0. \tag{1.44}$$

Since the operators $\nabla$ and $\frac{\partial}{\partial t}$ commute, the terms involving the particle velocity $\underline{v}$ cancel and we obtain the following wave equation:

$$\boxed{\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \rho_0 \nabla.\left(\frac{1}{\rho_0}\nabla p\right) = 0}\,. \tag{1.45}$$

Introducing $p_c(\underline{x})e^{-i\omega t}$ into the wave equation, we obtain:

$$-\frac{\omega^2}{c_0^2}p_c - \rho_0\nabla.\left(\frac{1}{\rho_0}\nabla p_c\right) = 0 \Leftrightarrow k_0^2 p_c + \rho_0\nabla.\left(\frac{1}{\rho_0}\nabla p_c\right) = 0. \quad (1.46)$$

Since $c_0^2 = \gamma p_0/\rho_0$ for an ideal gas from Equation (1.11), we have:

$$\rho_0\nabla.\left(\frac{1}{\rho_0}\nabla p_c\right) = \frac{\gamma p_0}{c_0^2}\nabla.\left(\frac{c_0^2}{\gamma p_0}\nabla p_c\right) = \frac{1}{c_0^2}\nabla.\left(c_0^2\nabla p_c\right) - \frac{1}{c_0^2 p_0}\nabla p_0.\left(c_0^2\nabla p_c\right). \quad (1.47)$$

Since we neglect the pressure gradient term, we obtain:

$$\rho_0\nabla.\left(\frac{1}{\rho_0}\nabla p_c\right) = \frac{k_0^2}{\omega^2}\nabla.\left(\frac{\omega^2}{k_0^2}\nabla p_c\right) = k_0^2\nabla.\left(\frac{1}{k_0^2}\nabla p_c\right), \quad (1.48)$$

which yields the following Helmholtz equation in a inhomogeneous medium at rest:

$$\boxed{k_0^2 p_c + k_0^2\nabla.\left(\frac{1}{k_0^2}\nabla p_c\right) = 0}\,. \quad (1.49)$$

This equation is the starting point of several frequency-domain numerical models such as the parabolic equation.

### 1.4.3 Propagation in a uniform moving medium

Except in a few simple cases, it is very difficult or even impossible to derive a wave equation in a moving medium. One of these simple cases correspond to a uniform moving medium where $\underline{v_0} = v_{0x}\underline{e_x}$ and $v_{0x}$, $\rho_0$ and $c_0$ are constant. In this case, the linearized Euler equations become:

$$\left(\frac{\partial}{\partial t} + v_{0x}\frac{\partial}{\partial x}\right)\rho + \rho_0\nabla.\underline{v} = 0 \quad \text{or} \quad \frac{D\rho}{Dt} + \rho_0\nabla.\underline{v} = 0, \quad (1.50)$$

$$\rho_0\left(\frac{\partial}{\partial t} + v_{0x}\frac{\partial}{\partial x}\right)\underline{v} + \nabla p = 0 \quad \text{or} \quad \rho_0\frac{D\underline{v}}{Dt} + \nabla p = 0, \quad (1.51)$$

$$\left(\frac{\partial}{\partial t} + v_{0x}\frac{\partial}{\partial x}\right)p + \rho_0 c_0^2\nabla.\underline{v} = 0 \quad \text{or} \quad \frac{Dp}{Dt} + \rho_0 c_0^2\nabla.\underline{v} = 0, \quad (1.52)$$

where the total derivative can be written:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_{0x}\frac{\partial}{\partial x}.$$

The **convected wave equation** is obtained by calculating $\frac{D}{Dt}$(1.52)$-c_0^2\nabla$.(1.51):

$$\boxed{\frac{D^2p}{Dt^2} - c_0^2\nabla^2 p = 0}\ . \tag{1.53}$$

Equation (1.53) is exact in a homogeneous moving medium. Ostashev (1997, Section 2.3) shows that it is also a good approximation in an inhomogeneous moving medium if the acoustic wavelength $\lambda$ is small compared to the length scale $l$ of variation in the ambient quantities $\underline{v}_0$, $\rho_0$ and $c_0$, i.e. at sufficiently high frequencies. He also derives more accurate wave equations for acoustic propagation in an inhomogeneous moving medium; see Ostashev (1997, Section 2.3) and Ostashev (1997). Note that these equations are the basis of various vector parabolic equations that have been used to calculation the acoustic propagation in an inhomogeneous moving medium (Dallois *et al.*, 2001; Blanc-Benon *et al.*, 2001).

# Chapter 2

# Acoustic propagation in a homogeneous medium at rest

In this chapter, we consider the acoustic propagation in a homogeneous medium at rest ($\underline{v}_0 = 0$), with constant density $\rho_0$ and sound speed $c_0$.

## 2.1 Definitions

### 2.1.1 Acoustic intensity and power

The time-averaged acoustic power of a source is defined as:

$$< W_a >= \int_S < \underline{I} > .\underline{n} dS = \int_S < p\underline{v} > .\underline{n} dS, \tag{2.1}$$

where $\underline{n}$ is the normal to the surface $S$ and $< \underline{I} >$ is the time-averaged acoustic intensity given by:

$$< \underline{I} >= \frac{1}{T} \int_{t_0}^{t_0+T} \underline{I}(t) dt = \frac{1}{T} \int_{t_0}^{t_0+T} p(t) \underline{v}(t) dt. \tag{2.2}$$

For harmonic waves, let $p(\underline{x}, t) = Re\{p_c(\underline{x}) e^{-i\omega t}\}$ and $\underline{v}(\underline{x}, t) = Re\{\underline{v}_c(\underline{x}) e^{-i\omega t}\}$. The time-averaged acoustic intensity for sinusoidal waves becomes (Pierce, 1989, Section 1.8):

$$< \underline{I} >= \frac{1}{2} Re\{p_c \underline{v}_c^*\}. \tag{2.3}$$

### 2.1.2 Sound pressure level and sound power level

The sound pressure level (SPL) is defined as:

$$L_p = 10\log_{10}\left(\frac{p_{rms}^2}{p_{ref}^2}\right) = 20\log_{10}\left(\frac{p_{rms}}{p_{ref}}\right), \tag{2.4}$$

where $p_{ref}$ is a reference pressure and $p_{rms}$ is the time-averaged or rms pressure:

$$p_{rms}^2 =< p^2 >= \frac{1}{T}\int_{t_0}^{t_0+T} p^2(t)dt. \tag{2.5}$$

For a harmonic wave, $p_{rms} = \max|p|/\sqrt{2}$. Similarly, the sound power level (SWL) is defined as:

$$L_W = 10\log_{10}\left(\frac{< W_a >}{W_{ref}}\right), \tag{2.6}$$

with $W_{ref}$ a reference power. The reference pressure $p_{ref}$ is typically $2 \times 10^{-5}$ Pa in air (threshold of hearing at 1 kHz) and $10^{-6}$ Pa in water.

## 2.2 Simple solutions of the wave equation in free field

### 2.2.1 Plane waves

Plane waves correspond to specific solutions to the wave equation where the wavefronts are planar, as seen in Figure 2.1. Considering the velocity potential $\Phi$, the general solution to Equation (1.40) is given by:

$$\Phi(x,t) = F_+\left(t - \frac{x}{c_0}\right) + F_-\left(t + \frac{x}{c_0}\right), \tag{2.7}$$

where the function $F_+$ describes the wave propagation in the positive $x$ direction, and $F_-$ describes the wave propagation in the negative $x$ direction. The associated pressure field is:

$$\begin{aligned} p(x,t) = -\rho_0\frac{\partial\Phi}{\partial t} &= -\rho_0 F_+'\left(t - \frac{x}{c_0}\right) - \rho_0 F_-'\left(t + \frac{x}{c_0}\right) \\ &= G_+\left(t - \frac{x}{c_0}\right) + G_-\left(t + \frac{x}{c_0}\right). \end{aligned} \tag{2.8}$$

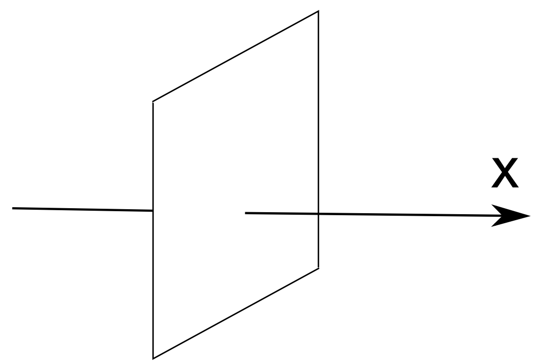



Figure 2.1: Plane wave traveling along the x-direction.

The associated particle velocity field is $\underline{v} = \nabla\Phi = v_x \underline{e_x}$, with:

$$\begin{aligned} v_x(x,t) = \frac{\partial \Phi}{\partial x} &= -\frac{1}{c_0} F'_+ \left( t - \frac{x}{c_0} \right) + \frac{1}{c_0} F'_- \left( t + \frac{x}{c_0} \right) \\ &= \frac{1}{\rho_0 c_0} \left[ G_+ \left( t - \frac{x}{c_0} \right) - G_- \left( t + \frac{x}{c_0} \right) \right]. \end{aligned} \tag{2.9}$$

Let us consider a special case of interest, that is a harmonic plane wave traveling along the positive $x$ axis, with $p(\underline{x},t) = Re\{p_c(x)e^{-i\omega t}\}$ and $\underline{v}(x,t) = Re\{\underline{v}_c(x)e^{-i\omega t}\}$:

$$p_c(x) = P_0 e^{ik_0 x}, \tag{2.10}$$

$$\underline{v}_c(x,t) = \frac{p_c(x,t)}{\rho_0 c_0} \underline{e_x}, \tag{2.11}$$

$$< \underline{I} > = \frac{|p_c|^2}{2\rho_0 c_0} \underline{e_x} = \frac{\rho_0 c_0 |v_c|^2}{2} \underline{e_x}. \tag{2.12}$$

With this type of waves the amplitude remains constant with distance. As a result, the ratio of pressure to velocity is constant for a plane wave and equal to $Z_{c,fluid} = \rho_0 c_0$. The quantity $Z_{c,fluid}$ is called the **characteristic acoustic impedance** of the fluid (Pierce, 1989, Section 3-3).

### 2.2.2 Spherical waves

We now consider waves with spherical symmetry, which means that the variables do not depend on the spherical coordinates $\theta$ and $\phi$: $p = p(r,t)$ and

$\underline{v} = v(r,t)\underline{e_r}$. The wavefronts are spheres, and the acoustic intensity vector is along along the $r$ direction: $\underline{I} = I_r\underline{e_r}$. This solution corresponds to the case of a point source with spherical symmetry.

Rewriting the homogeneous wave equation (1.40) for the velocity potential in spherical coordinates:

$$\frac{1}{c_0^2}\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{r}\frac{\partial^2(r\Phi)}{\partial r^2} = 0 \Leftrightarrow \frac{1}{c_0^2}\frac{\partial^2(r\Phi)}{\partial t^2} - \frac{\partial^2(r\Phi)}{\partial r^2} = 0.$$

This means that $r\Phi$ can be written as a sum of a function of $t - r/c$ and a function of $t + r/c$, as done in Section 2.2.1 for plane waves. If we keep only the outward-going wave:

$$\Phi(r,t) = \frac{1}{r}F\left(t - \frac{r}{c_0}\right), \tag{2.13}$$

and thus:

$$p(r,t) = -\rho_0\frac{\partial\Phi}{\partial t} = -\frac{\rho_0}{r}F'\left(t - \frac{r}{c_0}\right), \tag{2.14}$$

$$v(r,t) = \frac{\partial\Phi}{\partial r} = \frac{p(r,t)}{\rho_0 c_0} - \frac{1}{r^2}F\left(t - \frac{r}{c_0}\right). \tag{2.15}$$

It appears that the pressure amplitude decreases as $1/r$. Also, the particle velocity is composed of two terms. Since the second term decreases as $1/r^2$, it becomes negligible if $r$ is sufficiently large (far-field) and $v(r,t) \approx \frac{p(r,t)}{\rho_0 c_0}$, which corresponds to the relationship for plane waves.

It is possible to calculate the acoustic power of this wave by integrating over a sphere of radius $r$. From Equation (2.1), considering that the acoustic intensity is constant on the sphere and that $\underline{n} = \underline{e}_r$:

$$< W_a >= \int_S < \underline{I} > .\underline{e}_r dS = 4\pi r^2 < I_r > . \tag{2.16}$$

If we consider a harmonic spherical wave of the form $p(\underline{x},t) = Re\{p_c(r)e^{-i\omega t}\}$, with

$$p_c(r) = \frac{A}{r}e^{ik_0 r}, \tag{2.17}$$

the following time-averaged acoustic intensity is obtained from Equation (2.3):

$$< I(r) >= \frac{|p_c|^2}{2\rho_0 c_0} = \frac{< p^2 >}{\rho_0 c_0}. \tag{2.18}$$

From Equations (2.16) and (2.18), the acoustic power is thus:

$$< W_a >= 4\pi r^2 \frac{< p^2 >}{\rho_0 c_0} = \frac{2\pi |A|^2}{\rho_0 c_0}. \tag{2.19}$$

It appears clearly that the acoustic power is independent of the distance $r$ since $A$ is a constant; the acoustic power $< W_a >$ is a characteristics of the source(s) inside the sphere $S$.

From the previous expression, it is possible to derive a simple relationship between the sound pressure level and the sound power level:

$$\boxed{L_p = L_W - 10 \log_{10}(4\pi r^2)} \ , \tag{2.20}$$

where $W_{ref} = p_{ref}^2/(\rho_0 c_0)$. In air, we consider typically $p_{ref} = 20 \times 10^{-6}$ Pa and $\rho_0 c_0 \approx 415\,\mathrm{kg/m^2/s}$, thus $W_{ref} \approx 10^{-12}$ W. The term $10 \log_{10}(4\pi r^2)$ is called **geometrical spreading**. This means that there is an attenuation of $10 \log_{10}(4) \approx 6$ dB of the sound pressure level $L_p$ when the distance $r$ is doubled (6 dB attenuation per doubling distance).

### 2.2.3 Cylindrical waves

Cylindrical waves are obtained by solving the wave equation in cylindrical coordinates. This solution corresponds to the case of a line source. It can be shown that in the far-field, the pressure is proportional to $1/\sqrt{r}$, which corresponds to a 3 dB attenuation per doubling distance.

### 2.2.4 Green's function in free field

The Green's function is the solution of the following Helmholtz equation with a point source term:

$$\Delta G(\underline{x}, \underline{x_S}) + k_0^2 G(\underline{x}, \underline{x_S}) = -4\pi\delta(\underline{x} - \underline{x_S}), \tag{2.21}$$

with $\underline{x_S}$ the source position, $\underline{x}$ the receiver position, and $r = |\underline{x} - \underline{x_S}|$ the source-receiver distance. Here are some properties of the Green's functions:

- reciprocity relation: $G(\underline{x}, \underline{x_S}) = G(\underline{x_S}, \underline{x})$;
- superposition principle: for $N$ point sources of amplitudes $S_n$:

$$p_c(\underline{x}) = \sum_{n=1}^{N} S_n G(\underline{x}, \underline{x_{Sn}}). \tag{2.22}$$

The reciprocity relation means that the Green's function remains the same if source and receiver positions are interchanged in a medium at rest. This symmetry of acoustic fields does not apply to moving media. As mentioned by Ostashev (1997, Section 1.3.6), reciprocal acoustic transmission can be used to separate experimentally the effect of flow and sound speed variations. This is the basis of remote sensing techniques based on travel time measurements in various directions, also called acoustic tomography (Ostashev *et al.*, 2009; Brown *et al.*, 2016).

Using the $e^{-i\omega t}$ convention, the 3D Green's function in free field is given by (Pierce, 1989; Salomons, 2001):

$$G(\underline{x}, \underline{x_S}) = \frac{e^{ik_0|\underline{x}-\underline{x_S}|}}{|\underline{x}-\underline{x_S}|} = \frac{e^{ik_0 r}}{r}. \tag{2.23}$$

## 2.3 Acoustic propagation above a flat ground surface

### 2.3.1 Acoustic impedance of a ground surface

The specific acoustic impedance of a ground surface, also called surface impedance, at the angular frequency $\omega = 2\pi f$ is defined as the ratio of complex pressure and to normal particle velocity at the ground surface:

$$Z_s(\omega) = \left.\frac{p_c(\omega)}{v_{c,n}(\omega)}\right|_{\text{ground surface}}, \tag{2.24}$$

with $v_{c,n} = \underline{v}_c.\underline{n}$, where $\underline{n}$ is the unit vector normal to the ground surface (into the surface and out of the fluid) (Pierce, 1989, Section 3-3). This is a complex quantity defined in the frequency domain, with unit kg/(m$^2$s). It is also common to define a normalized (specific) acoustic impedance:

$$Z(\omega) = \frac{Z_s(\omega)}{Z_{c,fluid}} = \frac{Z_s(\omega)}{\rho_0 c_0}. \tag{2.25}$$

This quantity is dimensionless since $Z_s$ is normalized by the characteristic impedance of the fluid as introduced in Section 2.2.1. Sometimes, it is convenient to use the normalized (specific) acoustic admittance $\beta = 1/Z$.

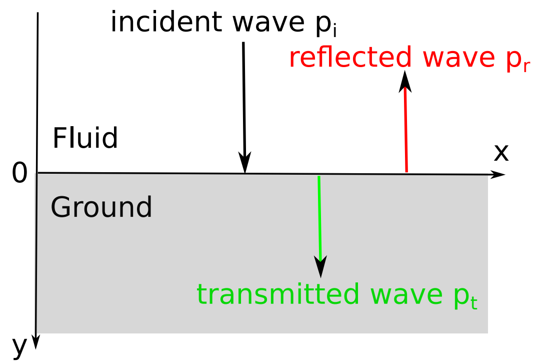



Figure 2.2: Schematics for the normal reflection of a plane wave by a flat ground surface.

### 2.3.2 Plane wave reflection by a ground surface

#### Plane wave reflection under normal incidence

First, let us consider the normal reflection of a plane wave by a flat ground surface located at $y = 0$, as illustrated in Figure 2.2. Since there is an impedance mismatch at $y = 0$, the incident wave (with subscript $i$) in the fluid breaks down into a reflected wave in the fluid (with subscript $r$) and a transmitted wave in the ground (with subscript $t$). For a harmonic plane wave at angular frequency $\omega$ ($e^{-i\omega t}$ convention), using the results of Section 2.2.1, the complex acoustic pressure and particle velocity fields in the fluid ($y < 0$) can be written:

$$p_c(y) = p_{c,i} + p_{c,r} = A\left[e^{ik_0 y} + R_p e^{-ik_0 y}\right], \tag{2.26}$$

$$v_c(y) = v_{c,i} + v_{c,r} = \frac{A}{\rho_0 c_0}\left[e^{ik_0 y} - R_p e^{-ik_0 y}\right], \tag{2.27}$$

with $A$ the complex pressure amplitude.

There is a continuity of pressure and normal velocity at $y = 0$, thus the ratio $p_c/v_c$ just above and below the ground surface is the same, and is equal to the specific acoustic impedance $Z_s$:

$$\frac{p_c(0^+)}{v_c(0^+)} = \frac{p_c(0^-)}{v_c(0^-)} = Z_s = \rho_0 c_0 Z, \tag{2.28}$$

using Equations (2.24) and (2.25). Note that we consider the continuity of normal velocity since the fluid is supposed inviscid, as explained in Chapter 1. Using Equations (2.26)-(2.28), it is now possible to express the plane wave reflection coefficient $R_p$ under normal incidence with respect to the normalized impedance $Z$ of the ground:

$$R_p = \frac{Z-1}{Z+1} = \frac{1-\beta}{1+\beta}. \tag{2.29}$$

In the case of a perfectly rigid ground, the velocity of the ground is zero so the particle velocity at $y = 0$ is also zero:

$$v_{c,n} = 0 \Leftrightarrow \frac{\partial p_c}{\partial n} = 0, \tag{2.30}$$

using the linearized Euler equation (1.30). This means that the normalized impedance $Z$ is infinite or the normalized admittance $\beta$ is zero, thus $R_p = 1$. The wave is completely reflected and remains in phase with the incident wave. As a first approximation, this case would correspond to a road surface or to a concrete wall. In the case of a pressure release surface, the acoustic pressure is zero at the surface, thus $Z = 0$ and $R_p = -1$. The wave is completely reflected and is out of phase compared to the incident wave. As a first approximation, this case would correspond to a water-air interface, since the characteristic acoustic impedance $\rho_0 c_0$ in air is much smaller than its value in water. In general $Z$ is a complex quantity which means that there is a phase difference between pressure and velocity.

### Plane wave reflection under oblique incidence

Let us now consider an incident plane wave in the direction $\underline{n} = (\sin\theta_i, \cos\theta_i, 0)$, as illustrated in Figure 2.3. The acoustic wave number $k_0 = \omega/c_0$ in the fluid is written:

$$k_0^2 = k_{0x}^2 + k_{0y}^2 = k_{0x}'^2 + k_{0y}'^2, \tag{2.31}$$

where $(k_{0x}, k_{0y})$ are the components of the incident wave and $(k'_{0x}, k'_{0y})$ the components of the reflected wave:

$$\begin{aligned} k_{0x} &= k_0 \sin\theta_i, \quad k_{0y} = k_0 \cos\theta_i, \\ k'_{0x} &= k_0 \sin\theta_r, \quad k'_{0y} = k_0 \cos\theta_r. \end{aligned}$$

The complex pressure and normal velocity fields in the fluid ($y < 0$) thus become:

$$p_c(x,y) = A\left[e^{i(k_{0x}x + k_{0y}y)} + R_p e^{i(k'_{0x}x - k'_{0y}y)}\right], \tag{2.32}$$

$$v_{c,y}(x,y) = \frac{A}{\rho_0 c_0}\left[\cos\theta_i e^{i(k_{0x}x + k_{0y}y)} - R_p \cos\theta_r e^{i(k'_{0x}x - k'_{0y}y)}\right]. \tag{2.33}$$

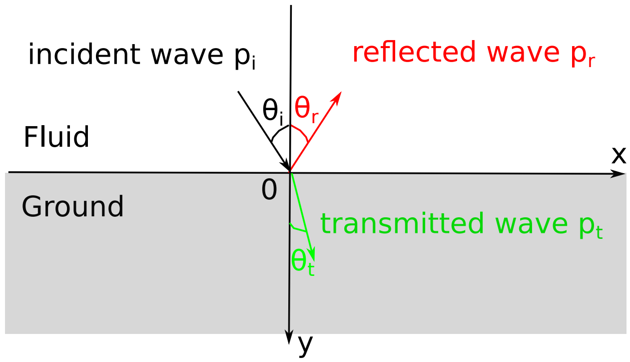



Figure 2.3: Schematics for the plane wave reflection by a flat ground surface under oblique incidence.

The continuity of pressure and normal velocity at $y = 0$ yields:

$$\frac{p_c(x,0^+)}{v_{c,y}(x,0^+)} = \frac{p_c(x,0^-)}{v_{c,y}(x,0^-)} = \rho_0 c_0 Z, \tag{2.34}$$

thus:

$$\frac{e^{ik_{0x}x} + R_p e^{ik'_{0x}x}}{\cos\theta_i e^{ik_{0x}x} - R_p \cos\theta_r e^{ik'_{0x}x}} = Z \Leftrightarrow e^{ik_{0x}x}\left(Z\cos\theta_i - 1\right) = e^{ik'_{0x}x} R_p \left(1 + Z\cos\theta_r\right). \tag{2.35}$$

This expression must be satisfied for all $x$ thus $k'_x = k_x$ and $\theta_r = \theta_i = \theta$. Finally, the plane wave reflection coefficient under oblique incidence is given by:

$$R_p = \frac{Z\cos\theta - 1}{Z\cos\theta + 1} = \frac{\cos\theta - \beta}{\cos\theta + \beta}. \tag{2.36}$$

This expression shows that:

- when $\theta$ approaches $\pi/2$ (grazing incidence), $R_p \approx -1$;
- for a rigid ground ($\beta = 0$), $R_p = 1$ for all values of $\theta$;
- for a pressure release surface ($Z = 0$), $R_p = -1$ for all values of $\theta$.

It is also interesting to look at the acoustic energy associated with plane-wave reflection. From Equations (2.3) and (2.24), the time-averaged acoustic intensity is given by:

$$< I(y=0) >= \frac{1}{2} Re\{p_c(y=0) v_{c,n}(y=0)^*\} = \frac{|v_{c,n}(y=0)|^2}{2} Re\{Z_s\}. \tag{2.37}$$

This quantity can be seen as the time-averaged acoustic power flowing into the surface per unit area. Thus for a passive surface, that is a surface that can only absorb energy, we have the condition $Re\{Z_s\} \geq 0$ (passivity condition). From Equations (2.32) and (2.33), we also obtain:

$$< I(y=0) >= \frac{|A|^2 \cos\theta}{2\rho_0 c_0}(1 - |R_p|^2), \tag{2.38}$$

because the real part of $(1+R_p)(1-R_p^*)$ is equal to $1-|R_p|^2$. We thus obtain the intensity reflection and transmission coefficients as:

$$R_I = \frac{< I_r >}{< I_i >} = |R_p|^2, \tag{2.39}$$

$$T_I = \frac{< I_t >}{< I_i >} = 1 - |R_p|^2, \tag{2.40}$$

where $< I_i >$, $< I_r >$ and $< I_t >$ are respectively the time-averaged intensity of the incident, reflected and transmitted wave. The fraction of energy transmitted into the ground is absorbed by the ground, so $T_I$ corresponds also to the absorption coefficient $\alpha$. This fraction of energy corresponds to the incident acoustic energy minus the acoustic energy that is reflected by the ground. Note that the coefficients $R_I$ and $T_I = \alpha$ are functions of the angle of incidence $\theta$ and of the angular frequency $\omega$.

### Local reaction approximation

Let us now consider the transmitted wave into the ground. The ground is seen as an equivalent fluid of density $\rho_g$ and sound speed $c_g$, where $\rho_g$ and $c_g$ can be complex numbers. We will see in Section 2.4 that this is generally a reasonable assumption for porous media.

The complex pressure and velocity fields in the ground ($y > 0$) are written:

$$p_c(x,y) = AT_p e^{i(k_{gx}x + k_{gy}y)}, \tag{2.41}$$

$$v_{c,y}(x,y) = \frac{AT_p}{\rho_g c_g} \cos\theta_t e^{i(k_{gx}x + k_{gy}y)}, \tag{2.42}$$

where $k_g = \sqrt{k_{gx}^2 + k_{gy}^2} = \omega/c_g$ is the (complex) wave number associated to the transmitted wave in the ground, with $k_{gx} = k_g \sin\theta_t$ and $k_{gy} = k_g \cos\theta_t$, and $T_p$ is the plane-wave transmission coefficient. Using Equations (2.24) and (2.25), the normalized surface impedance can be written:

$$Z = \frac{Z_s}{\rho_0 c_0} = \frac{1}{\rho_0 c_0} \frac{p_c(x,0)}{v_{c,y}(x,0)} = \frac{1}{\cos\theta_t} \frac{\rho_g c_g}{\rho_0 c_0}. \tag{2.43}$$

The continuity of pressure and normal velocity at $y = 0$ yields:

$$(1 + R_p) e^{ik_{0x}x} = T_p e^{ik_{gx}x}, \tag{2.44}$$

$$\frac{\cos\theta}{\rho_0 c_0}(1 - R_p) e^{ik_{0x}x} = \frac{\cos\theta_t}{\rho_g c_g} T_p e^{ik_{gx}x}. \tag{2.45}$$

These expressions should hold for arbitrary $x$ values, thus $k_{gx} = k_{0x}$. This corresponds to the Snell-Descartes law of refraction:

$$\frac{\sin\theta}{c_0} = \frac{\sin\theta_t}{c_g}. \tag{2.46}$$

From Equations (2.44) and (2.45), we obtain that $T_p = 1 + R_p$ with:

$$R_p = \frac{\rho_g c_g \cos\theta - \rho_0 c_0 \cos\theta_t}{\rho_g c_g \cos\theta + \rho_0 c_0 \cos\theta_t}. \tag{2.47}$$

Using Equations (2.43) and (2.47) one retrieves Equation (2.36) obtained previously. Also, it appears that $k_{gx}$ is real since $k_{gx} = k_{0x} = k_0 \sin\theta$. This means that $k_{gy} = \sqrt{k_g^2 - k_{gx}^2}$ is a complex number if $k_g = \omega/c_g$ is complex. As a result, the pressure field of Equation (2.41) in the ground can be rewritten:

$$p_c(x,y) = A T_p e^{i(k_{gx}x + Re[k_{gy}]y)} e^{-Im[k_{gy}]y}, \tag{2.48}$$

with $Im[k_{gy}] > 0$ to have a physical behavior when $y \to \infty$. The transmitted wave is thus evanescent.

For most grounds of interest, the sound speed in the ground is much smaller than the sound speed in the fluid, such that $|c_g| \ll c_0$ or $|k_g| \gg k_0$. From the Snell-Descartes law (2.46) this yields $\theta_t \approx 0$. This means from Equation (2.43) that the normalized impedance does not depend on the angle of incidence $\theta$:

$$Z = \frac{\rho_g c_g}{\rho_0 c_0} = \frac{Z_{c,ground}}{Z_{c,fluid}}, \tag{2.49}$$

where $Z_{c,ground} = \rho_g c_g$ is the characteristic impedance of the ground. Such grounds are called **locally reacting grounds**. In general, however, the dependence of $Z$ on $\theta$ does exist, and there is an extended reaction, which means that the reflected wave at a given point on the ground surface depends on the pressure distribution in a region around this point.

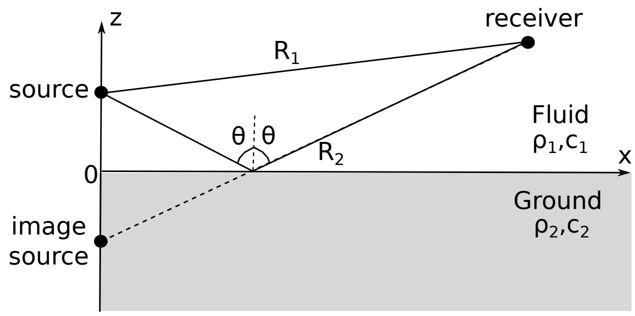



Figure 2.4: Schematics for the spherical wave reflection by a flat ground surface. $R_1$ is the distance from the source to the receiver, and $R_2$ is the distance from the image source to the receiver.

### 2.3.3 Spherical wave reflection by a ground surface

#### Description of the problem

We consider a point source of unit amplitude that radiates harmonic waves with spherical symmetry at angular frequency $\omega$ above a ground surface, as schematically shown in Figure 2.4. The derivation presented in this section is based on the work of Di and Gilbert (1993), also detailed by Salomons (2001, Appendix D). We will only consider the case of a locally reacting ground; the case of an extended reacting ground is also considered in these references.

The source coordinates are $(0, 0, z_S)$. The fluid has a characteristic impedance $\rho_1 c_1$, and the ground has a normalized impedance $Z = (\rho_2 c_2)/(\rho_1 c_1)$. The complex pressure field $p_c$ (using the $e^{-i\omega t}$ convention) can be written:

$$p_c(x,y,z) = \begin{cases} p_1(x,y,z) & \text{for} \quad z \geq 0, \\ p_2(x,y,z) & \text{for} \quad z \leq 0, \end{cases}$$

where $p_1$ and $p_2$ are solutions of the Helmholtz equations:

$$(\Delta + k_1^2)p_1 = -4\pi\delta(x)\delta(y)\delta(z - z_S) \quad \text{for} \quad z \geq 0, \tag{2.50}$$

$$(\Delta + k_2^2)p_2 = 0 \quad \text{for} \quad z \leq 0, \tag{2.51}$$

with $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$ the wave numbers in the fluid and in the ground respectively.

**Equations of the problem in the wave number domain**

A solution will be sought in the wave number domain, using a two-dimensional spatial Fourier transform. Let us define the Fourier transform pairs $(p_m, P_m)$, with $m = 1, 2$, as:

$$P_m(k_x, k_y, z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(ik_x x + ik_y y)} p_m(x, y, z) dx dy, \tag{2.52}$$

$$p_m(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(ik_x x + ik_y y)} P_m(k_x, k_y, z) dx dy. \tag{2.53}$$

We first apply the operator $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(ik_x x + ik_y y)} dx dy$ on both sides of Equation (2.50):

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(ik_x x + ik_y y)} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_1^2 \right) p_1 dx dy$$
$$= -4\pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(-ik_x x - ik_y y)} \delta(x)\delta(y)\delta(z - z_S) dx dy = -4\pi\delta(z - z_S),$$

where the following property of the Dirac $\delta$ function has been used:

$$\int_{-\infty}^{-\infty} f(x)\delta(x - x_0) dx = f(x_0). \tag{2.54}$$

It is possible to calculate easily the double derivatives of $p_1$, for instance the one with respect to $x$ is:

$$\frac{\partial^2 p_1}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left[ \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(ik_x x + ik_y y)} P_m(k_x, k_y, z) dx dy \right]$$
$$= \frac{-k_x^2}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(ik_x x + ik_y y)} P_m(k_x, k_y, z) dx dy = -k_x^2 p_1.$$

Finally we obtain the following system to solve for the complex pressure amplitudes $p_1$ and $p_2$:

$$\left[ \frac{\partial^2}{\partial z^2} + (k_1^2 - k_x^2 - k_y^2) \right] P_1 = -4\pi\delta(z - z_S) \quad \text{for} \quad z \geq 0, \tag{2.55}$$

$$\left[ \frac{\partial^2}{\partial z^2} + (k_2^2 - k_x^2 - k_y^2) \right] P_2 = 0 \quad \text{for} \quad z \leq 0. \tag{2.56}$$

These complex pressure amplitudes $p_1$ and $p_2$ must satisfy the boundary conditions:

$$P_1(k_x, k_y, 0) = P_2(k_x, k_y, 0), \tag{2.57}$$
$$\frac{1}{\rho_1}\left(\frac{\partial P_1}{\partial z}\right)_{z=0} = \frac{1}{\rho_2}\left(\frac{\partial P_2}{\partial z}\right)_{z=0}, \tag{2.58}$$
$$P_1(k_x, k_y, z_S + \epsilon) = P_1(k_x, k_y, z_S - \epsilon), \tag{2.59}$$
$$\left(\frac{\partial P_1}{\partial z}\right)_{z=z_S+\epsilon} - \left(\frac{\partial P_1}{\partial z}\right)_{z=z_S-\epsilon} = -4\pi, \tag{2.60}$$

with $0 < \epsilon \ll 1$. The first and third equations correspond to the continuity of pressure at the ground surface and at the source height $z_S$. The second equation corresponds to the continuity of normal velocity at the ground surface. From Euler's equation:

$$i\omega\rho_m V_{m,z} = -\frac{\partial P_m}{\partial z} \Rightarrow V_{m,z} = -\frac{1}{i\omega\rho_m}\frac{\partial P_m}{\partial z}.$$

The last equation is obtained by integrating Equation (2.55) between $z_S - \epsilon$ and $z_S + \epsilon$:

$$\int_{z_S-\epsilon}^{z_S+\epsilon} \frac{\partial^2 P_1}{\partial z^2} dz + (k_1^2 - k_x^2 - k_y^2) \int_{z_S-\epsilon}^{z_S+\epsilon} P_1 dz = -4\pi.$$

When $\epsilon \to 0$, the second term vanishes because of the continuity of pressure thus:

$$\left[\frac{\partial P_1}{\partial z}\right]_{z_S-\epsilon}^{z_S+\epsilon} = \left(\frac{\partial P_1}{\partial z}\right)_{z=z_S+\epsilon} - \left(\frac{\partial P_1}{\partial z}\right)_{z=z_S-\epsilon} = -4\pi.$$

This means the fluid velocity is discontinuous at the source height (mass injection).

### Solution of the problem in the wave number domain

We write the solution of Equations (2.55) and (2.55) under the form:

$$P_1 = C_1 e^{ik_{1z}z} \quad \text{for} \quad z \geq z_S, \tag{2.61}$$
$$P_1 = C_2 e^{ik_{1z}z} + C_3 e^{-ik_{1z}z} \quad \text{for} \quad 0 \leq z \leq z_S, \tag{2.62}$$
$$P_2 = C_4 e^{-ik_{2z}z} \quad \text{for} \quad z \leq 0, \tag{2.63}$$

where $C_1$, $C_2$, $C_3$ and $C_4$ are constants to determine and $k_{mz}^2 = k_m^2 - k_x^2 - k_y^2$, $m = 1, 2$. The boundary conditions give us four relationships between the constants:

$$C_2 + C_3 = C_4, \tag{2.64}$$

$$\frac{ik_{1z}}{\rho_1}(C_2 - C_3) = \frac{-ik_{2z}}{\rho_2} C_4, \tag{2.65}$$

$$C_1 e^{ik_{1z}z_S} = C_2 e^{ik_{1z}z_S} + C_3 e^{-ik_{1z}z_S}, \tag{2.66}$$

$$ik_{1z}\left(C_1 e^{ik_{1z}z_S} - C_2 e^{ik_{1z}z_S} + C_3 e^{-ik_{1z}z_S}\right) = -4\pi. \tag{2.67}$$

From Equations (2.64) and (2.65):

$$\frac{ik_{1z}}{\rho_1}(C_2 - C_3) = \frac{-ik_{2z}}{\rho_2}(C_2 + C_3) \Rightarrow C_2 = \frac{\rho_2 k_{1z} - \rho_1 k_{2z}}{\rho_2 k_{1z} + \rho_1 k_{2z}} C_3 = R(k_{1z}) C_3.$$

From Equations (2.66) and (2.67):

$$(C_1 - C_2) e^{ik_{1z}z_S} = C_3 e^{-ik_{1z}z_S} = -\frac{4\pi}{ik_{1z}} - C_3 e^{-ik_{1z}z_S} \Rightarrow C_3 = \frac{2\pi i}{k_{1z}} e^{ik_{1z}z_S} = A e^{ik_{1z}z_S},$$

with $A = 2\pi i / k_{1z}$. The other constants are easily obtained:

$$C_2 = R(k_{1z}) C_3 = A R(k_{1z}) e^{ik_{1z}z_S}, \tag{2.68}$$

$$C_1 = C_2 + C_3 e^{-2ik_{1z}z_S} = A R(k_{1z}) e^{ik_{1z}z_S} + A e^{-ik_{1z}z_S}, \tag{2.69}$$

which leads us to the following solution above the ground ($z \geq 0$):

$$P_1 = P_d + P_r = A e^{ik_{1z}|z - z_S|} + A R(k_{1z}) e^{ik_{1z}(z + z_S)}, \tag{2.70}$$

where $P_d$ and $P_r$ are respectively the direct and reflected waves, and $R(k_{1z})$ is the reflection coefficient. For a locally reacting ground, $\theta_t \approx 0$ so $k_{2z} = k_2 \cos\theta_t \approx k_2$ and:

$$R(k_{1z}) \approx \frac{\rho_2 k_{1z} - \rho_1 k_2}{\rho_2 k_{1z} + \rho_1 k_2} = \frac{k_{1z} - \frac{\rho_1 c_1 k_1}{\rho_2 c_2}}{k_{1z} + \frac{\rho_1 c_1 k_1}{\rho_2 c_2}} = \frac{k_{1z} - k_1/Z}{k_{1z} + k_1/Z}, \tag{2.71}$$

because $k_2 = \omega / c_2 = k_1 c_2 / c_1$ and $Z = (\rho_2 c_2)/(\rho_1 c_1)$.

#### Laplace transform solution

If we apply the inverse Fourier transform of the direct wave $P_d$ corresponds the Green's function in free field:

$$p_d = \frac{e^{ik_1\sqrt{r^2+(z-z_S)^2}}}{\sqrt{r^2+(z-z_S)^2}} = \frac{e^{ik_1 R_1}}{R_1} = G(\underline{x}, \underline{x}_S), \tag{2.72}$$

where $r^2 = x^2+y^2$, and $R_1 = \sqrt{r^2+(z-z_S)^2}$ is the source-receiver distance. It is however not possible to directly apply the Fourier transform to $P_r$. We will thus use a Laplace transform instead to find a more useful solution.

The Laplace transform $F(s)$ of a function $f(q)$, defined for real numbers $q \geq 0$, is defined as:

$$F(s) = \int_0^{+\infty} f(q) e^{-sq} dq, \tag{2.73}$$

where $s = \sigma + i\omega$ is a complex number frequency parameter. When $s = i\omega$ the Laplace transform reduces to the Fourier transform.

The reflection coefficient can be written as the following Laplace transform, replacing variable $s$ by $k_{1z}$:

$$R(k_{1z}) = \int_0^{\infty} s(q) e^{-qk_{1z}} dq,$$

where $s(q)$ is the following image source distribution:

$$s(q) = \delta(q) - 2\frac{k_1}{Z}\exp\left(-\frac{qk_1}{Z}\right).$$

Indeed we have:

$$\begin{aligned} R(k_{1z}) &= \int_0^{\infty} \left[\delta(q) - 2\frac{k_1}{Z}\exp\left(-\frac{qk_1}{Z}\right)\right] e^{-qk_{1z}} dq \\ &= 1 - 2\frac{k_1}{Z}\int_0^{\infty} \exp\left[-q\left(\frac{k_1}{Z} + k_{1z}\right)\right] dq \\ &= 1 - 2\frac{k_1}{Z}\left[-\frac{e^{-q(k_1/Z+k_{1z})}}{k_1/Z+k_{1z}}\right]_0^{\infty} = 1 - \frac{2k_1/Z}{k_1/Z+k_{1z}} = \frac{k_{1z}-k_1/Z}{k_{1z}+k_1/Z}, \end{aligned}$$

which is the result of Equation (2.71).

From Equations (2.70) and (2.3.3), the contribution of the reflected path in the Fourier domain is given by:

$$P_r = \int_0^\infty As(q)e^{ik_{1z}[z+(z_S+iq)]}dq. \tag{2.74}$$

Since the inverse Fourier transform of $P_d = Ae^{ik_{1z}|z-z_S|}$, corresponding to the direct path contribution $p_d$, is given by Equation (2.72), it is possible to calculate the inverse Fourier transform of $P_r$, noting that the argument of the exponential is similar with $-z_S$ being replaced by $z_S + iq$:

$$\begin{aligned} p_r &= \int_0^\infty s(q)\frac{e^{ik_1\sqrt{r^2+(z+z_S+iq)^2}}}{\sqrt{r^2+(z+z_S+iq)^2}}dq \\ &= \int_0^\infty \left[\delta(q) - 2\frac{k_1}{Z}\exp\left(-\frac{qk_1}{Z}\right)\right]\frac{e^{ik_1\sqrt{r^2+(z+z_S+iq)^2}}}{\sqrt{r^2+(z+z_S+iq)^2}}dq \\ &= \frac{e^{ik_1\sqrt{r^2+(z+z_S)^2}}}{\sqrt{r^2+(z+z_S)^2}} - 2\frac{k_1}{Z}\int_0^\infty \exp\left(-\frac{qk_1}{Z}\right)\frac{e^{ik_1\sqrt{r^2+(z+z_S+iq)^2}}}{\sqrt{r^2+(z+z_S+iq)^2}}dq. \end{aligned}$$

Thus the total pressure field in the fluid is:

$$p_1 = \frac{e^{ik_1R_1}}{R_1} + \frac{e^{ik_1R_2}}{R_2}\left[1 - 2\frac{k_1}{Z}\frac{R_2}{e^{ik_1R_2}}\int_0^\infty \exp\left(-\frac{qk_1}{Z}\right)\frac{e^{ik_1\sqrt{r^2+(z+z_S+iq)^2}}}{\sqrt{r^2+(z+z_S+iq)^2}}dq\right], \tag{2.75}$$

where $R_2 = \sqrt{x^2+y^2+(z+z_S)^2}$ is the image source-receiver distance. This expression can be rewritten under the following classical form:

$$\boxed{p_1 = \frac{\exp(ik_1R_1)}{R_1} + Q\frac{\exp(ik_1R_2)}{R_2}}, \tag{2.76}$$

where $Q$ is the spherical-wave reflection coefficient given by:

$$Q = 1 - 2\frac{k_1}{Z}\frac{R_2}{e^{ik_1R_2}}\int_0^\infty \exp\left(-\frac{qk_1}{Z}\right)\frac{e^{ik_1\sqrt{r^2+(z+z_S+iq)^2}}}{\sqrt{r^2+(z+z_S+iq)^2}}dq. \tag{2.77}$$

Note that this expression is for a point source of unit amplitude.

### Approximate solution in the limit of grazing incidence

In the limit of grazing incidence, i.e $R_2 \gg z + z_S$, the following approximate expression for the spherical wave reflection coefficient can be obtained:

$$Q = R_p + (1 - R_p)F(d) \quad \text{with} \quad F(d) = 1 + id\sqrt{\pi}e^{-d^2}\text{erfc}(-id), \tag{2.78}$$

where $d = \sqrt{ik_1R_2/2}(1/Z + \cos\theta)$ is the numerical distance, $R_p = (Z\cos\theta - 1)/(Z\cos\theta + 1)$ is the plane wave reflection coefficient, and erfc is the complimentary error function defined as:

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2)dt. \tag{2.79}$$

The expression (2.78), also known as the Weyl-Van der Pol formula in electromagnetic propagation theory, is the most widely used analytical solution for the propagation of a point source above a flat ground in a homogeneous medium (Attenborough *et al.*, 2007).

To obtain this expression, the square root expression in Equation (2.77) is written:

$$\begin{aligned}\sqrt{r^2 + (z + z_S + iq)^2} &= \sqrt{r^2 + (z + z_S)^2 + 2iq(z + z_S) - q^2} \\ &= R_2\sqrt{1 + \frac{2iq(z + z_S)}{R_2^2} - \frac{q^2}{R_2^2}}.\end{aligned}$$

For $R_2 \gg z + z_S$, we can use the following approximation in the numerator of the integral:

$$\sqrt{r^2 + (z + z_S + iq)^2} \approx R_2\left(1 + \frac{iq(z + z_S)}{R_2^2} - \frac{q^2}{2R_2^2}\right) = R_2 + iq\cos\theta - \frac{q^2}{2R_2},$$

using $\cos\theta = (z + z_S)/R_2$. This approximation is correct because the integral is dominated by small values of $q$, so we can assume $q \ll R_2$. In the denominator of the integral we use simply $\sqrt{r^2 + (z + z_S + iq)^2} \approx R_2$, and thus:

$$Q \approx 1 - 2\frac{k_1}{Z}\int_0^\infty \exp\left[-qk_1\left(\frac{1}{Z} + \cos\theta\right) - ik_1\frac{q^2}{2R_2}\right]dq. \tag{2.80}$$

Let us develop Equation (2.78) to show it corresponds to Equation (2.80):

$$Q = R_p + (1 - R_p)F(d) = \frac{Z\cos\theta - 1}{Z\cos\theta + 1} + \frac{2}{Z\cos\theta + 1}\left(1 + 2ide^{-d^2}\int_{-id}^\infty \exp(-t^2)dt\right).$$

Considering the following change of variable in the integral:

$$t = q\sqrt{\frac{ik_1}{2R_2}} - id,$$

then:

$$Q = 1 + \frac{4ide^{-d^2}\sqrt{\frac{ik_1}{2R_2}}}{Z\cos\theta + 1}\int_0^\infty \exp\left[-\left(q\sqrt{\frac{ik_1}{2R_2}} - id\right)^2\right] dq.$$

Since

$$\left(q\sqrt{\frac{ik_1}{2R_2}} - id\right)^2 = \frac{ik_1}{2R_2}q^2 - 2idq\sqrt{\frac{ik_1}{2R_2}} - d^2,$$

the expression becomes:

$$Q = 1 + \frac{4id\sqrt{\frac{ik_1}{2R_2}}}{Z\cos\theta + 1}\int_0^\infty \exp\left[\frac{-ik_1}{2R_2}q^2 + 2idq\sqrt{\frac{ik_1}{2R_2}}\right] dq.$$

Since

$$2id\sqrt{\frac{ik_1}{2R_2}} = -k_1\left(\frac{1}{Z} + \cos\theta\right),$$

we finally obtain the same expression as in Equation (2.80).

Remark: if we use $\sqrt{r^2 + (z + z_S + iq)^2} = R_2$ in both the numerator and the denominator, we would obtain:

$$\begin{aligned} p_1 &= \frac{e^{ik_1R_1}}{R_1} + \frac{e^{ik_1R_2}}{R_2}\left(1 - 2\frac{k_1}{Z}\int_0^\infty e^{-\frac{qk_1}{Z}} dq\right) \\ &= \frac{e^{ik_1R_1}}{R_1} + \frac{e^{ik_1R_2}}{R_2}\left(1 - 2\frac{k_1}{Z}\left[-\frac{Z}{k_1}e^{-\frac{qk_1}{Z}}\right]_0^\infty\right) = \frac{e^{ik_1R_1}}{R_1} - \frac{e^{ik_1R_2}}{R_2}. \end{aligned}$$

This crude approximation corresponds to $Q = -1$, which is also the value of the plane wave reflection coefficient obtained when $\theta \to \pi/2$.

### Sound pressure level relative to the free field

The sound pressure level relative to free field is defined as:

$$\Delta L = L_p - L_{p,FF} = 10\log_{10}\left(\frac{p_{rms}^2}{p_{rms,FF}^2}\right) = 10\log_{10}\left(\frac{|p_c|^2}{|p_{c,FF}|^2}\right), \qquad (2.81)$$

where the subscript $FF$ means "free-field". Since $L_{p,FF} = L_W - 10\log_{10}(4\pi R_1^2)$ from Equation (2.20), the relationship between sound pressure level and sound power level in the presence of ground becomes:

$$\boxed{L_p = L_W - 10\log_{10}(4\pi R_1^2) + \Delta L}\ . \tag{2.82}$$

For a point source of amplitude $S$, $p_{c,FF} = Se^{ik_1R_1}/R_1$, and from Equation (2.76):

$$p_c = S\frac{\exp(ik_1R_1)}{R_1} + QS\frac{\exp(ik_1R_2)}{R_2}. \tag{2.83}$$

As a result:

$$\boxed{\Delta L = 10\log_{10}\left|1 + Q\frac{R_1}{R_2}e^{ik_1(R_2-R_1)}\right|^2}\ . \tag{2.84}$$

Thus there will be constructive interferences between direct and reflected waves for $k_1(R_2 - R_1) = 0[2\pi]$, and destructive interferences for $k_1(R_2 - R_1) = \pi[2\pi]$. These interferences can be visualized as maxima and minima in the spectra of $\Delta L$ plotted in Figure 2.5 for both rigid and impedance grounds. In this example, source and receiver are close to the ground such that $R_2 \approx R_1$. For a rigid ground $Q = 1$ so $\Delta L$ varies between $-\infty$ and $10\log_{10}(4) \approx 6\,\text{dB}$. For an impedance ground, $|Q| < 1$ because there is some energy absorbed by the ground during the reflection. At high frequencies, $Q \to R_p$ because the wavefronts become planar. At low frequencies, however, it is clear from Figure 2.5 that the spherical-wave reflection coefficient $Q$ cannot be approximated by the plane-wave reflection coefficient $R_p$. The part of the spherical wave that is not accounted for by plane wave theory is called the **ground wave** in the literature (Embleton, 1996).

## 2.4 Ground impedance modeling and measurements

### 2.4.1 Acoustic propagation in porous media

Many natural grounds can be modeled as porous media, characterized by several parameters such as the open porosity $\Omega$, the flow resistivity $\sigma$ and the tortuosity $T$ (Salomons, 2001; Attenborough *et al.*, 2007). In the rigid frame approximation of Biot theory, the porous medium is seen as an equivalent

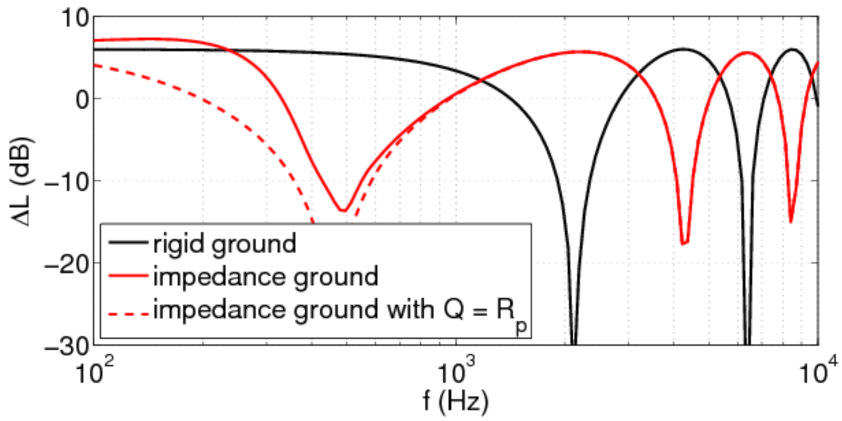



Figure 2.5: Spectrum of the sound pressure level relative to the free field for a rigid ground, for an impedance ground and for an impedance ground where the spherical-wave reflection coefficient is approximated by the plane-wave reflection coefficient ($Q = R_p$). The source and receiver are distant of 100 m, and their height is 2 m.

fluid medium of effective bulk modulus $K_{eff}$ and effective density $\rho_{eff}$; these two quantities are complex numbers. As a result, the pressure in the porous medium can be described by the following Helmholtz equation:

$$\frac{1}{\omega^2 \rho_{eff}} \Delta p_c + \frac{1}{K_{eff}} p_c = 0. \tag{2.85}$$

The effective sound speed of the porous medium is:

$$c_{eff} = \sqrt{\frac{K_{eff}}{\rho_{eff}}} = \frac{\omega}{k_c}, \tag{2.86}$$

where $k_c$ is the acoustic wave number in the porous medium, and the characteristic impedance is:

$$Z_c = \sqrt{\rho_{eff} K_{eff}}. \tag{2.87}$$

As a comparison, the bulk modulus in an ideal fluid medium is $\gamma p_0$ and the density is $\rho_0$, thus one retrieves the expression for the sound speed and the characteristic impedance in a fluid obtained previously:

$$c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}} = \frac{\omega}{k_0} \quad \text{and} \quad Z_0 = \sqrt{\rho_0 \gamma p_0} = \rho_0 c_0.$$

### 2.4.2 Ground impedance models

Among the simplest impedance models for the characteristic impedance $Z_c$ and the acoustic wavenumber $k_c$, Dragna *et al.* (2015) propose expressions for two families of models. The first family is called the square-root type impedance model, and has the form:

$$Z_c = \rho_0 c_0 \frac{\sqrt{T}}{\Omega} \alpha \sqrt{\frac{(\omega_1 - i\omega)(\omega_2 - i\omega)}{-i\omega(\omega_3 - i\omega)}}, \tag{2.88}$$

$$k_c = \frac{\omega}{c_0} \sqrt{T} \beta \sqrt{\frac{(\omega_1 - i\omega)(\omega_3 - i\omega)}{-i\omega(\omega_2 - i\omega)}}, \tag{2.89}$$

where $\alpha$, $\beta$, $\omega_1$, $\omega_2$ and $\omega_3$ are the positive coefficients of the model. Several phenomenological models (Zwikker and Kosten, Attenborough) belong to this family (Salomons, 2001).

The second family is called the polynomial model, and has the form:

$$Z_c = 1 + a \left(\frac{\sigma_e}{\rho_0 \omega}\right)^b + ic \left(\frac{\sigma_e}{\rho_0 \omega}\right)^d, \tag{2.90}$$

$$k_c = \sqrt{\omega} c_0 \left(1 + p \left(\frac{\sigma_e}{\rho_0 \omega}\right)^q + ir \left(\frac{\sigma_e}{\rho_0 \omega}\right)^s\right), \tag{2.91}$$

with $\sigma_e$ the effective flow resistivity of the ground, and $a$, $b$, $c$, $d$, $q$ and $s$ are constant coefficients. Several empirical models (Delany and Bazley, Miki) belong to this family.

If one assumes the ground is composed of a semi-infinite layer of porous medium, there is no reflected wave in the ground. As a result, the surface impedance is equal to the characteristic impedance and the normalized impedance is written:

$$Z_\infty = \frac{Z_s}{\rho_0 c_0} = \frac{Z_c}{\rho_0 c_0}. \tag{2.92}$$

For a layer of porous medium of thickness $d_e$ over a perfectly hard ground (rigidly backed layer), the normalized impedance is given by (Salomons, 2001; Attenborough *et al.*, 2007):

$$Z_{layer} = \frac{Z_s}{\rho_0 c_0} = \frac{Z_c}{\rho_0 c_0 \tanh(-ik_c d_e)} = \frac{-iZ_c}{\rho_0 c_0 \tan(k_c d_e)}. \tag{2.93}$$

A model closely-related to the square-root type models is the variable porosity model, which is an approximation for a porous medium in which the porosity decreases exponentially with depth at a rate $\alpha_e$:

$$Z_c = \sqrt{\frac{4\sigma}{-i\omega\gamma\rho_0}} + \frac{c_0\alpha_e}{-i\omega 4\gamma}, \tag{2.94}$$

where $\alpha_e$ is the effective rate of change of porosity. Dragna *et al.* (2015) show that this model is physically admissible and yield the best agreement with experimental data for sound propagation over grass and lawn grounds.

# Chapter 3

# Absorption and refraction effects in inhomogeneous moving media

We use slightly different notations in this chapter. The mean sound speed associated with the base flow is written $c(\underline{x})$, and $k(\underline{x}) = \omega/c(\underline{x})$ is the associated acoustic wave number.

## 3.1 Acoustic absorption

### 3.1.1 Attenuation due to acoustic absorption

Acoustic absorption can be modeled using a complex wave number $k^* = k + i\alpha$, with $\alpha$ the absorption coefficient in Np/m. As a result, a harmonic spherical wave is now written:

$$p_c(r) = S\frac{e^{ikr}}{r}e^{-\alpha r}. \tag{3.1}$$

As a result, the relationship (2.20) between the sound pressure level and the sound power level in free field becomes (Salomons, 2001):

$$\boxed{L_p = L_W - 10\log_{10}(4\pi r^2) - ar}\ , \tag{3.2}$$

with $a = 20/\ln(10)\alpha \approx 8.686\alpha$ the absorption coefficient in dB/m.

### 3.1.2 Mechanisms of acoustic absorption in the atmosphere and in the ocean

A sound wave loses energy due to various irreversible processes that remove energy from an acoustic wave and convert it to heat:

- viscous losses and heat conduction losses (so-called classical absorption);
- relaxation losses of constituents.

The relaxation losses exist for polyatomic gases, and are associated with the change of rotational or translational energy of the molecules into internal energy (Evans *et al.*, 1972).

Pierce (1989, Section 10-8) obtains the following dispersion equation for a plane traveling wave including classical absorption and various relaxation processes $\nu$:

$$k^* = \frac{\omega}{c_0} + i\ \alpha_{cl} + \frac{1}{\pi}\frac{\omega}{c}\sum_{\nu}(\alpha_\nu \lambda)_{\max}\frac{i\omega\tau_\nu}{1 - i\omega\tau\nu} \tag{3.3}$$

where $\alpha_{cl}$ is the classical absorption coefficient that is proportional to $\omega^2$, $(\alpha_\nu\lambda)_{\max}$ corresponds to the maximum absorption per wavelength associated with the $\nu$-type relaxation process, $\tau_\nu$ is the relaxation time for the vibrational energy of type $\nu$, and:

$$c_0 = \frac{c}{1 + \frac{1}{\pi}\sum_\nu(\alpha_\nu\lambda)_{\max}}. \tag{3.4}$$

Since $\lim_{x\to 0}\frac{k}{\omega} = \frac{1}{c_0}$, $c_0$ corresponds to the phase velocity in the limit of zero frequency, while $c$ corresponds to the phase velocity in the high-frequency limit where $\omega\tau\nu \gg 1$ for all relaxation processes $\nu$.

The absorption coefficient $\alpha$ is the imaginary part of $k^*$, and is thus written from Equation (3.3):

$$\alpha(f) = \alpha_{cl}(f) + \sum_\nu \alpha_\nu(f) = A_{vt}f^2 + \sum_\nu \frac{2}{c}(\alpha_\nu\lambda)_{\max}\frac{f_\nu f^2}{f_\nu^2 + f^2}, \tag{3.5}$$

with $f_\nu = 1/(2\pi\tau_\nu)$ the relaxation frequency of constituent $\nu$. Pierce (1989, Section 10-8) shows that the absorption per wavelength of the relaxation process $\nu$ can be written:

$$\frac{\alpha_\nu \lambda}{(\alpha_\nu \lambda)_{\max}} = \frac{2}{f_\nu/f + f/f_\nu}. \tag{3.6}$$

As shown in Figure 3.1(a), the absorption is maximum at the relaxation frequency $f_\nu$, and goes to zero for $f \ll f_\nu$ and $f \gg f_\nu$.

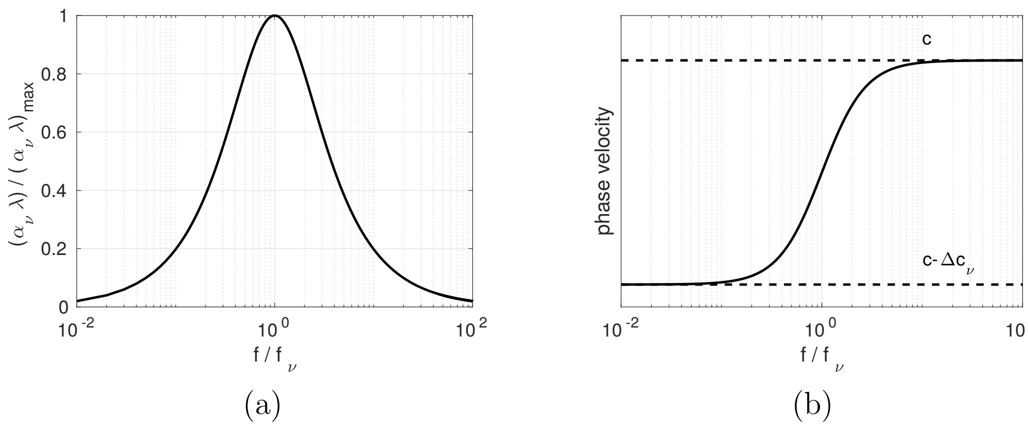


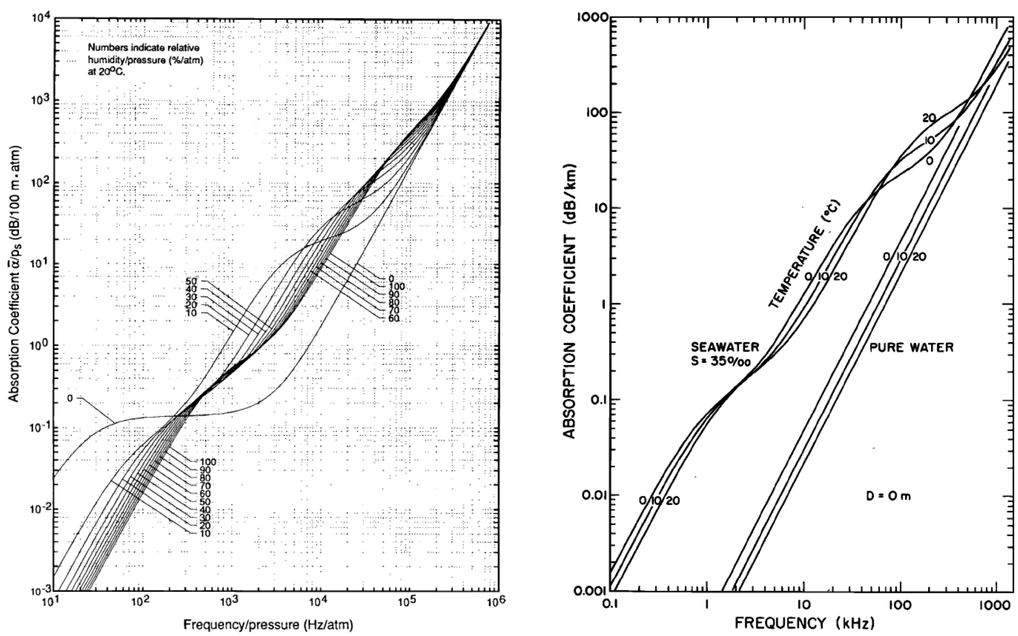

Figure 3.1: (a) Absorption per wavelength, and (b) phase velocity with respect to the normalized frequency $f/f_\nu$ for a single relxation process $\nu$. Taken adapted from Pierce (1989, Fig. 10-12)

Equation (3.3) also shows that the phase velocity $v_{ph} = \omega/k_R$, with $k_R$ the real part of $k^*$, depends on frequency because of relaxation processes, which means that the medium is dispersive. The phase velocity can be written:

$$v_{ph} = \frac{\omega}{k_R} = c - \sum_\nu \frac{\Delta c_\nu}{1 + (f/f_\nu)^2}, \tag{3.7}$$

with $\Delta c_\nu = (\alpha_\nu \lambda)_{\max} c/\pi$. This phase velocity is plotted in Figure 3.1(b) for a single relaxation process $\nu$. In practice, $\Delta c_\nu$ is small and the approximation $k_R = \omega/c$ is generally used.

In the atmosphere, the acoustic absorption of air depends on pressure, temperature and humidity. The relaxation processes to take into account are due to nitrogen ($N_2$) and oxygen molecules ($O_2$), where $f_{N_2} \ll f_{O_2}$. The expressions for the absorption coefficient can be found for instance in Pierce (1989, Chapter 10), Bass *et al.* (1995, 1996) and Salomons (2001, Appendix B). See Figure 3.2(a).




(a) (b)

Figure 3.2: Absorption coefficient (a) for air in dB/100m/atm at 20$^o$C and for various relative humidities in % (taken from Bass *et al.* (1995)), and (b) for seawater in dB/km for a salinity of 35‰and a pH of 8 (from Francois and Garrison (1982b)).

In the ocean, the acoustic absorption of seawater depends on on pressure (or depth), temperature, salinity and acidity (pH). The relaxation processes to take into account are due to boric acid ($B(OH)_3$) and magnesium sulphate ($MgSO_4$), where $f_{B(OH)_3} \ll f_{MgSO_4}$. The expressions for the absorption coefficient can be found for instance in Francois and Garrison (1982a,b) and in Ainslie and McColm (1998). See Figure 3.2(b).

## 3.2 Refraction effects

### 3.2.1 Refraction due to vertical sound speed gradients

Refraction happens when sound waves propagate through fluid layers of varying sound speeds. Let us consider the simple case of Figure 3.3, where a sound ray propagates through two fluid layers of sound speed $c_1$ and $c_2$. For the moment, a sound ray is defined as a narrow beam of high frequency sound. A more precise definition will be given in Chapter 4 devoted to geometrical acoustics.

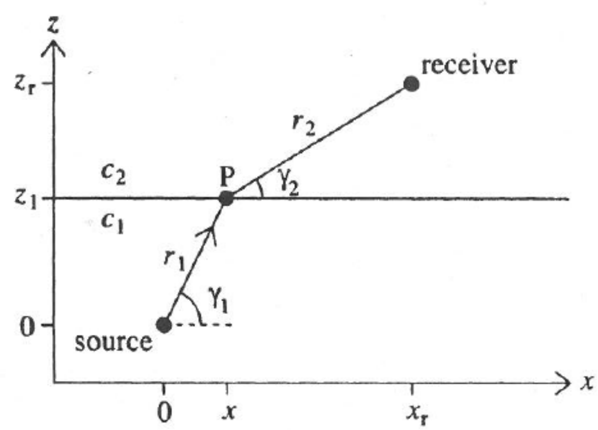



Figure 3.3: Refraction of sound between 2 layers with different sound speeds $c_1$ and $c_2 > c_1$. Taken from Salomons (2001).

According to Fermat's principle, the wave takes the path where the travel time is minimum. The travel time between source and receiver is given here by:

$$\tau = \frac{r_1}{c_1} + \frac{r_2}{c_2} = \frac{\sqrt{x^2 + z_1^2}}{c_1} + \frac{\sqrt{(x_r - x)^2 + (z_r - z_1)^2}}{c_2}. \tag{3.8}$$

Let us find the coordinate $x$ of point P that minimizes $\tau$:

$$\frac{\partial\tau}{\partial x} = \frac{x/r_1}{c_1} - \frac{(x_r - x)/r_2}{c_2} = 0 \Rightarrow \frac{\cos\gamma_1}{c_1} = \frac{\cos\gamma_2}{c_2} \tag{3.9}$$

This expression is known as the **Snell-Descartes law**. It can be generalized to multiple layers of fluid, or to a stratified medium with sound speed $c(z)$:

$$\boxed{\frac{\cos\gamma(z)}{c(z)} = \text{constant along a sound ray}}\,. \tag{3.10}$$

This generalized Snell-Descartes law states that **the sound ray bends towards the region of lower sound speed**.

This variation of sound speed with altitude (in the atmosphere) or depth (in the ocean) $z$ is commonly found. This comes from the fact that temperature typically varies with $z$, thus for an ideal fluid:

$$c(z) = \sqrt{\gamma r T(z)} = c_0\sqrt{\frac{T(z)}{T_0}}. \tag{3.11}$$

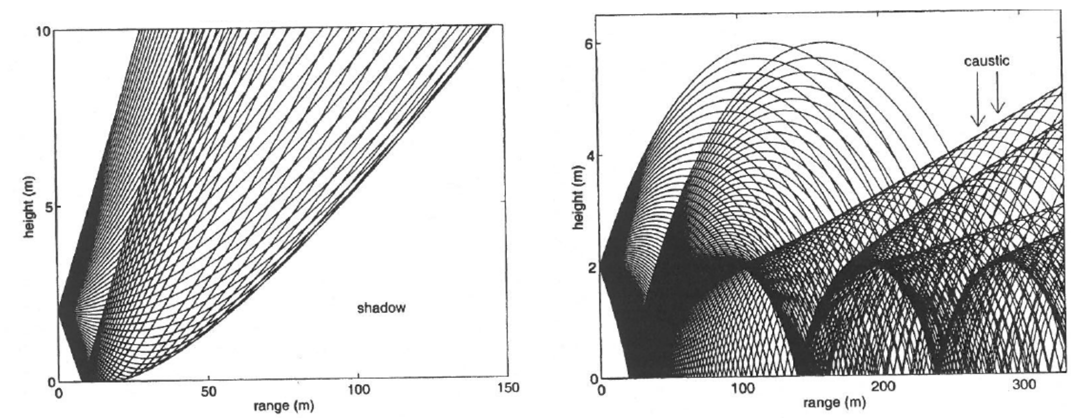
Two ray-tracing examples Salomons (2001) are plotted in Figure 3.4 to illustrate refraction effects in a layered atmosphere. A logarithmic sound speed profile is considered:

$$c(z) = c_0 + b\ln\left(1 + \frac{z}{z_0}\right), \tag{3.12}$$

with $c_0 = 340\,\text{m/s}$, $z_0 = 0.1\,\text{m}$, and $b = \pm 1\,\text{m/s}$.

When $b = -1\,\text{m/s}$, the sound speed decreases with height so the sound rays bend upwards according to Snell-Descartes law. This is called an **upward-refracting atmosphere**. This is a typical daytime situation, also referred to as normal lapse. The sun heats the ground, so the air close to the ground is warmer than the air at higher altitudes. As a result, a shadow zone forms close to the ground where no sound can penetrate in the geometric approximation, as explained in Section 4.3. In reality, sound goes into the shadow zone due to diffraction effects.

When $b = +1\,\text{m/s}$, the sound speed increases with height so the sound rays bend downwards according to Snell-Descartes law. This is called an **downward-refracting atmosphere**. This is a typical nighttime situation, also referred to as normal inversion. As a result, there can be multiple rays between source and receiver with multiple reflections on the ground, which is a favorable situation for acoustic propagation over longer distance.




(a) (b)

Figure 3.4: Sound rays from a source at 2 meter height using a logarithmic sound speed profile $c(z) = c_0 + b \ln(1 + z/z_0)$ (a) in an upward-refracting atmosphere ($b = -1$ m/s), and (b) in a downward-refracting atmosphere ($b = 1$ m/s). Taken from Salomons (2001).

### 3.2.2 Refraction due to wind speed gradients

Because of friction, the wind speed in the atmospheric boundary layer decreases to zero at the ground. Strong wind speed gradients are thus encountered close to the ground, and typically decrease with height.

An equivalence can be made between the effect of vertical gradients of wind speed and temperature on sound waves. Indeed, as shown in Figure 3.5, downward refraction occurs in the **downwind direction** or for **temperature inversion**, while upward refraction occurs in the **upwind direction** or for **temperature lapse**.

It is possible to take into account wind speed gradients in an approximate way using the **effective sound speed approximation**. The effective sound spped is defines as:

$$c_{eff}(z) = c(z) + U(z) \cos \phi, \tag{3.13}$$

where $\phi$ is the angle between wind direction and propagation direction. This approximation is generally valid for source and receivers close to the ground, as will be seen in Chapter 4.

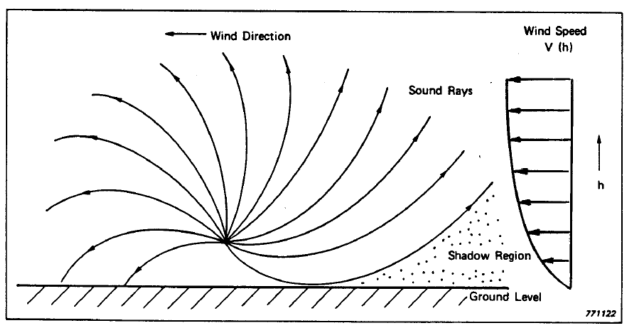



Figure 3.5: Sound rays illustrating the effect of refraction by wind speed vertical gradients. Taken from Lamancusa (2009, Section 10).

# Chapter 4

# Geometrical acoustics

We use the same notations as in Chapter 3, with $c(\underline{x})$ the mean sound speed associated with the base flow and $k(\underline{x}) = \omega/c(\underline{x})$ the associated acoustic wave number. This chapter is mostly based on the work of Pierce (1989, Chapter 8).

## 4.1 Wavefronts and ray equations

### 4.1.1 In a medium at rest

A wavefront is a moving surface along which a waveform feature is received (constant phase). We saw in Chapter 2 that it is a plane for plane waves, or a sphere for spherical waves as long as the propagation medium is homogeneous. However, in inhomogeneous media, the wavefronts are distorted and generally have a complex shape, as illustrated in Figure 4.1(a).

Let us define the function $\tau(\underline{x})$ (in second) constant along a wavefront, and let $\underline{n}$ be the normal to the wavefront. For a medium at rest ($\underline{v}_0 = 0$), the wavefront (ray) velocity is:

$$\underline{v}_{\text{ray}} = \frac{d\underline{x}}{dt} = c(\underline{x})\underline{n}(\underline{x}, t). \tag{4.1}$$

Following Pierce (1989, Chapter 8), we will now show that the relationship $\nabla\tau.\frac{d\underline{x}}{dt} = 1$ holds along a ray trajectory. To this end, let us use the Taylor series expansion for the function $\tau$ between $\underline{x}$ and $\underline{x} + \underline{\delta x}$:

$$\tau(\underline{x} + \underline{\delta x}) = \tau(\underline{x}) + \underline{\delta x}.\nabla\tau(\underline{x}) + ... \tag{4.2}$$

The position of the wavefront at $t + \delta t$ is approximately $\underline{x}(t) + \frac{dx}{dt}\delta t$ for a small time $\delta t$. As a result:

$$t + \delta t \approx \tau\left(\underline{x} + \frac{d\underline{x}}{dt}\delta t\right) \approx \tau(\underline{x}) + \delta t \frac{d\underline{x}}{dt}.\nabla\tau(\underline{x}), \tag{4.3}$$

using Equation (4.2). Since $t = \tau(\underline{x})$, the following relationship is obtained:

$$\nabla\tau.\frac{d\underline{x}}{dt} = 1 \Leftrightarrow c(\underline{x})\nabla\tau.\underline{n} = 1. \tag{4.4}$$

As a result, $\nabla\tau(\underline{x}) = \underline{n}/c(\underline{x})$ since $\nabla\tau$ is directed towards the normal to the wavefront given by the unit vector $\underline{n}$. This yields the following Eikonal equation:

$$\boxed{s^2(\underline{x}) = |\nabla\tau(\underline{x})|^2 = \frac{1}{c^2(\underline{x})}}. \tag{4.5}$$

The vector $\underline{s}(\underline{x}) = \nabla\tau(\underline{x}) = \underline{n}/c$ is called the **wave-slowness vector**.

The evolution of $s(\underline{x}, t)$ with time is given by :

$$\frac{ds}{dt} = \cancel{\frac{\partial s}{\partial t}} + (\nabla s).\frac{d\underline{x}}{dt} = -\frac{1}{c}\nabla c.\underline{n} \tag{4.6}$$

As $\underline{s}$ is parallel to $\underline{n}$:

$$\frac{d\underline{s}}{dt} = -\frac{1}{c}(\nabla c) \tag{4.7}$$

The ray-tracing equations in a medium at rest are thus:

$$\frac{d\underline{x}}{dt} = c^2\underline{s} \quad \text{or} \quad \frac{dx_i}{dt} = c^2 s_i, \tag{4.8}$$

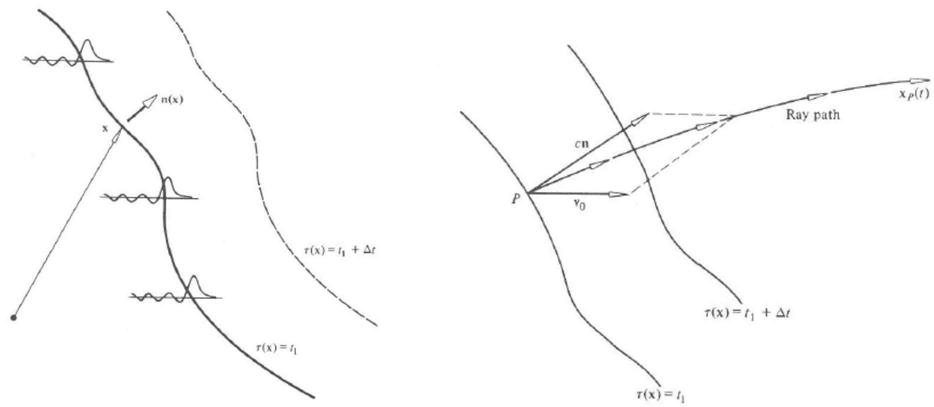
$$\frac{d\underline{s}}{dt} = -\frac{1}{c}\nabla c \quad \text{or} \quad \frac{ds_i}{dt} = -\frac{1}{c}\frac{\partial c}{\partial x_i}. \tag{4.9}$$

These equations can be numerically integrated in time to determine $\underline{x}$ and $\underline{s}$.

### 4.1.2 In a moving medium

In a moving medium, the ray velocity $\underline{v}_{\text{ray}}$ is generally not aligned with the normal to the wavefront $\underline{n}$ anymore. The equation for the ray trajectory at a point $\underline{x}(t)$ is:

$$\underline{v}_{\text{ray}} = \frac{d\underline{x}(t)}{dt} = \underline{v}_0(\underline{x}, t) + c(\underline{x})\underline{n}(\underline{x}, t). \tag{4.10}$$




(a) (b)

Figure 4.1: (a) Wavefront defined by the function $\tau(\underline{x})$ and the normal $\underline{x}$, and (b) wavefront in a moving medium as a function of $c(\underline{x})\underline{n}$ and $\underline{v}_0$. Taken from Pierce (1989, Chapter 8).

Using Equations (4.2) and (**??**), the wave-slowness vector $\underline{s} = \nabla\tau$ (normal to the wavefront) must now follow the relationship:

$$\nabla\tau.\frac{d\underline{x}(t)}{dt} = 1 \Leftrightarrow \underline{s}.(c\underline{n} + \underline{v}_0) = 1 \quad \text{or} \quad c\underline{s}.\underline{n} = 1 - \underline{v}_0.\underline{s}.$$

Since $\underline{s} = (\underline{s}.\underline{n})\underline{n}$, we obtain:

$$\underline{n} = \frac{\underline{s}}{(\underline{s}.\underline{n})} = \frac{c\underline{s}}{\Omega}, \tag{4.11}$$

with $\Omega = 1 - \underline{v}_0.\underline{s}$. The Eikonal equation in a moving medium is thus:

$$\boxed{s^2 = |\nabla\tau(\underline{x})|^2 = \frac{\Omega(\underline{x})^2}{c^2(\underline{x})}}\,. \tag{4.12}$$

After some vector manipulations, Pierce (1989, Chapter 8) obtains the ray-tracing equations in a moving medium:

$$\frac{d\underline{x}}{dt} = \frac{c^2\underline{s}}{\Omega} + \underline{v}_0 \quad \text{or} \quad \frac{dx_i}{dt} = \frac{c^2 s_i}{\Omega} + v_{0i}, \tag{4.13}$$

$$\frac{d\underline{s}}{dt} = -\frac{\Omega}{c}\nabla c - \underline{s} \times (\nabla \times \underline{v}_0) - (\underline{s}.\nabla)\underline{v}_0 \quad \text{or} \quad \frac{ds_i}{dt} = -\frac{\Omega}{c}\frac{\partial c}{\partial x_i} - \sum_{j=1}^{3} s_j \frac{\partial v_{0j}}{\partial x_i}. \tag{4.14}$$

These equations can be numerically integrated in time to determine $\underline{x}$ and $\underline{s}$.

## 4.2 Equations of geometrical acoustics

We will now formally derive the equations of geometrical acoustics, not only for the ray trajectories but also for the associated wave amplitudes. This derivation is done for a homogeneous medium at rest.

Let us start from the Helmholtz equation ($e^{-i\omega t}$ convention):

$$\nabla^2 p_c(\underline{x}) + k_0^2 p_c(\underline{x}) = 0. \tag{4.15}$$

We introduce the following notation: $p_c(\underline{x}) = \hat{P}(\underline{x}, \omega) e^{i\omega\tau(\underline{x})}$, where the amplitude is $\hat{P}(\underline{x}, \omega)$ and the phase is written $\omega\tau(\underline{x})$. Since $\nabla\left(e^{i\omega\tau(\underline{x})}\right) = i\omega\nabla\tau e^{i\omega\tau(\underline{x})}$, the Laplacian term is written:

$$\nabla^2 p_c = \left(\nabla^2\hat{P} + 2i\omega\nabla\tau.\nabla\hat{P} + i\omega\nabla^2\tau\hat{P} - \omega^2(\nabla\tau)^2\hat{P}\right) e^{i\omega\tau(\underline{x})}. \tag{4.16}$$

As a result, the Helmholtz equation is rewritten:

$$\nabla^2\hat{P} + i\omega\left(2\nabla\tau.\nabla\hat{P} + \nabla^2\tau\hat{P}\right) - \omega^2\hat{P}\left[(\nabla\tau)^2 - \frac{1}{c_0^2}\right] = 0. \tag{4.17}$$

We are looking for an asymptotic expression in the high frequency limit:

$$\hat{P}(\underline{x}, \omega) = \hat{P}_0(\underline{x}) + \frac{1}{\omega}\hat{P}_1(\underline{x}) + \frac{1}{\omega^2}\hat{P}_2(\underline{x}) + ... \tag{4.18}$$

Substituting this expression into Equation (4.17) and equating each power of $\omega$:

$$\omega^0 : \quad \hat{P}_0\left[(\nabla\tau)^2 - \frac{1}{c_0^2}\right] = 0 \tag{4.19}$$

$$\omega^1 : \quad i\left(2\nabla\tau.\nabla\hat{P}_0 + \nabla^2\tau\hat{P}_0\right) - \hat{P}_1\cancel{\left[(\nabla\tau)^2 - \frac{1}{c_0^2}\right]} = 0 \tag{4.20}$$

$$\omega^2 : \quad ... \tag{4.21}$$

The first equation corresponds to the Eikonal equation:

$$|\nabla\tau|^2 = \frac{1}{c_0^2}. \tag{4.22}$$

The solution to this equation corresponds to the **rays** connecting the wavefront surfaces (straight lines for a homogeneous medium). The second equation corresponds to the transport equation where we suppose $\hat{P} \approx \hat{P}_0$ at first order:

$$2\nabla\tau.\nabla\hat{P} + \nabla^2\tau\hat{P} \quad \text{or} \quad \nabla.(\hat{P}^2\nabla\tau) = 0. \tag{4.23}$$

The transport equation gives the **variation of amplitude** along ray paths.

To find the domain of validity of the geometrical acoustics approximation, we notice that we basically neglected the first term in Equation (4.17) in order to obtain the Eikonal and transport equations. As a result, the geometrical acoustics is valid when:

$$\left|\frac{\nabla^2\hat{P}}{\hat{P}}\right| \ll \left(\frac{\omega}{c}\right)^2. \tag{4.24}$$

Thus the higher the frequency the better the approximation.

Ostashev (1997) extends this analysis to inhomogeneous moving media. He shows that geometrical acoustics is valid if $\ell k \gg 1$ or $\ell \gg \lambda$, where $\ell$ the characteristic scale of variation of the ambient quantities in space. This means in particular that the concept of refraction makes sense only in the geometrical acoustics approximation; Snell-Descartes law does not depend on acoustic frequency and is applicable if $\lambda \ll \ell$. For $\lambda \gg \ell$ or $\lambda \sim \ell$, scattering and/or diffraction effects will occur.

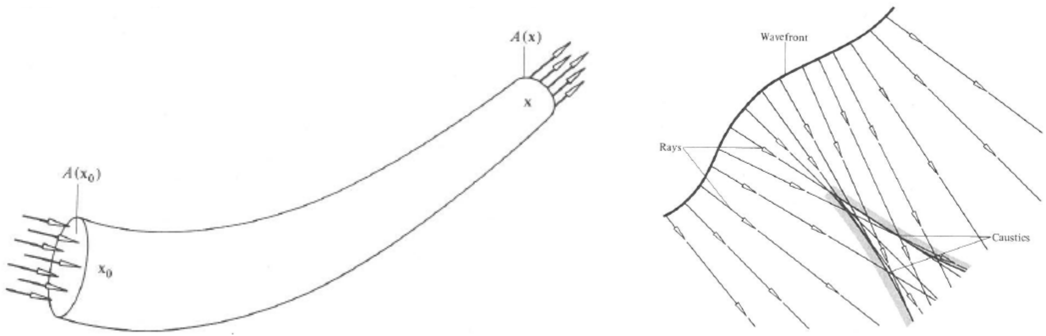
## 4.3 Wave amplitude along rays and caustics

Let us consider a ray tube, that corresponds to the envelope of all the rays passing through a tiny area $A(\underline{x}_0)$, as shown in Figure 4.2(a). Applying Gauss's theorem to the transport equation (4.23):

$$\begin{aligned}\int_{V_{\text{ray tube}}} \nabla.(\hat{P}^2\nabla\tau)dV &= \int_{A_{\text{ray tube}}} \hat{P}^2\nabla\tau.\underline{n}dA \\ &= \hat{P}^2(\underline{x})A(\underline{x})(\nabla\tau.\underline{n})_{\underline{x}} - \hat{P}^2(\underline{x}_0)A(\underline{x}_0)(\nabla\tau.\underline{n})_{\underline{x}_0} = 0.\end{aligned} \tag{4.25}$$

This yields the following amplitude variation along rays in a homogeneous medium at rest:

$$\hat{P}(\underline{x}) = \hat{P}(\underline{x}_0)\sqrt{\frac{A(\underline{x}_0)}{A(\underline{x})}} \quad \text{or} \quad \hat{P}^2(\underline{x})A(\underline{x}) = \text{constant} \tag{4.26}$$



(a) (b)

Figure 4.2: (a) Schematics of a ray tube, and (b) example of a caustic. Taken from Pierce (1989, Chapter 8).

The amplitude is inversely proportional to the square root of ray tube area.

In an inhomogeneous medium, the rays are now curved because of sound speed and/or velocity gradients. The previous result has been generalized by Pierce (1989, Section 8-6) that shows the **conservation of energy along rays**. The amplitude variation along rays in an inhomogeneous medium at rest is written:

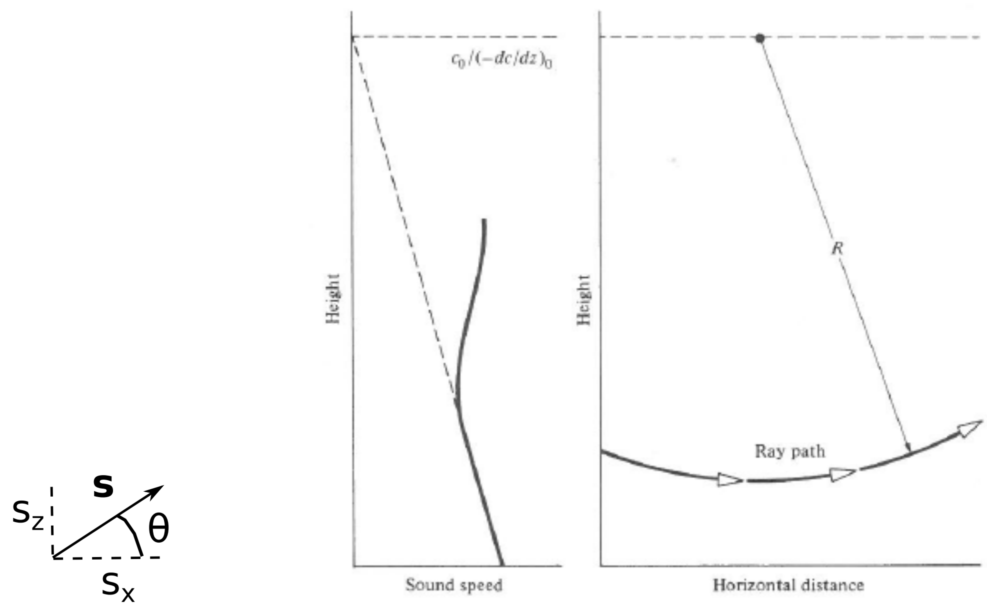
$$\frac{\hat{P}^2 A}{\rho_0 c} = \text{constant} \qquad (4.27)$$

The amplitude variation along rays in an inhomogeneous moving medium corresponds to the Blokhintzev invariant (Pierce, 1989, Equation (8-6.13)):

$$\frac{\hat{P}^2 |\underline{v}_{ray}| A}{(1 - \underline{v}_0 . \nabla\tau)\rho_0 c^2} = \text{constant}, \qquad (4.28)$$

with $|\underline{v}_{ray}| = |\underline{v}_0 + c\underline{n}|$.

When two rays intersect, the ray tube areas go to zero ($A(\underline{x}) = 0$), thus the pressure amplitude goes to infinity from the previous expressions. The envelope formed by a family of intersecting rays is called a **caustics**. An example is shown in Figure 4.2(b). When this happens, there is a need to think in terms of waveforms instead of rays. This is the goal of the geometrical theory of diffraction which is an extension of geometrical acoustics. Salomons (2001, Appendix L) presents a ray model that employs the theory of Ludwig and Kravtsov for the effects of caustics.




(a) (b)

Figure 4.3: (a) Ray direction, and (b) ray path considering a linear sound speed profile. Taken from Pierce (1989, Chapter 8).

## 4.4 Solutions to the ray-tracing equations

It is possible to calculate analytically the solutions to the ray-tracing equations in some simple cases. For instance in a medium at rest, $\underline{v}_0 = 0$ and $\Omega = 1$, so and the ray-tracing equations are simply:

$$\frac{d\underline{x}}{dt} = c^2\underline{s} \quad \text{or} \quad \frac{dx_i}{dt} = c^2 s_i, \tag{4.29}$$

$$\frac{d\underline{s}}{dt} = -\frac{1}{c}\nabla c \quad \text{or} \quad \frac{ds_i}{dt} = -\frac{1}{c}\frac{\partial c}{\partial x_i}. \tag{4.30}$$

These equations can be integrated in time for a given initial direction: $\underline{s} = s_x\underline{e}_x + s_z\underline{e}_z$, as shown in Figure 4.3(a). From the Eikonal equation:

$$|s|^2 = s_x^2 + s_z^2 = \frac{1}{c^2} \Rightarrow s_x = \frac{\cos\theta}{c} \quad \text{and} \quad s_z = \frac{\sin\theta}{c}. \tag{4.31}$$

In a layered atmosphere at rest, such that $c = c(z)$, the equations can be further simplified. We see that $\frac{ds_x}{dt} = 0$ which means $s_x = s_{x0}$. As a result

Snell-Descartes law is retrieved:

$$s_{x0} = \frac{\cos\theta(z)}{c(z)} = \text{constant}. \tag{4.32}$$

If a simple linear sound speed profile of the form $c(z) = c_0 + az$ is chosen, then $\nabla c = a\underline{e}_z$. In this specific case, it is possible to show that the ray trajectories are **arcs of circle** of radius $R = 1/(as_{x0}) = c/(a\cos\theta_0)$, where $s_{x0} = s_x(t = 0)$ and $\theta_0 = \theta(t = 0)$, centered at height $c_0/a$ (Pierce, 1989, Section 8-3). Figure 4.3(b) illustrates the shape of the ray paths for a linear sound speed profile. The greater the sound speed gradient $a$ is, the smaller the radius of curvature $R$ will be.

Pierce (1989, Section 8-3) extends the analysis to a moving medium, and shows that the rays bend with a radius of curvature:

$$R = c/\left(\frac{dc}{dz}\cos\theta + \frac{dv_x}{dz}\right). \tag{4.33}$$

For rays propagating in nearly horizontal directions, $\cos\theta \approx 1$ and $R \approx c/\left(\frac{dc_{eff}}{dz}\right)$, where $c_{eff}$ is the effective sound speed defined as:

$$c_{eff}(z) = c(z) + v_x(z). \tag{4.34}$$

If $\theta$ is greater than approximately $30^o$, the influence of a wind speed gradient is greater than the influence of a sound speed gradient of same amplitude, and the effective sound speed approximation becomes less accurate.

# Bibliography


Ainslie, M. and McColm, J. (**1998**). "A simplified formula for viscous and chemical absorption in sea water", Journal of the Acoustical Society of America **103**, 1671–1672.

Attenborough, K., Li, K., and Horoshenkov, K. (**2007**). *Predicting outdoor sound* (Taylor & Francis).

Bass, H., Sutherland, L., Zuckerwar, A., Blackstock, D., and Hester, D. (**1995**). "Atmospheric absorption of sound: Further developments", Journal of the Acoustical Society of America **97**, 680–683.

Bass, H., Sutherland, L., Zuckerwar, A., Blackstock, D., and Hester, D. (**1996**). "Erratim: Atmospheric absorption of sound: Further developments", Journal of the Acoustical Society of America **99**, 1259.

Blanc-Benon, P., Dallois, L., and Juvé, D. (**2001**). "Long range sound propagation in a turbulent atmosphere within the parabolic approximation", Acta Acustica united with Acustica **87**, 659–669.

Boutillon, X. (**2017**). "éléments d'acoustique", Cours de l'École Polytechnique.

Brown, M. G., Godin, O. A., Zang, X., Ball, J. S., Zabotin, N. A., Zabotina, L. Y., and Williams, N. J. (**2016**). "Ocean acoustic remote sensing using ambient noise: results from the Florida Straits", Geophysical Journal International **206**, 574–589.

Dallois, L., Blanc-Benon, P., and Juvé, D. (**2001**). "A wide-angle parabolic equation for acoustic waves in inhomogeneous moving media: Applications to atmospheric sound propagation", Journal of Computational Acoustics **9**, 477–494.

Di, X. and Gilbert, K. (**1993**). "An exact laplace transform formulation for a point source above a ground surface", Journal of the Acoustical Society of America **93**, 714–720.

Dragna, D., Attenborough, K., and Blanc-Benon, P. (**2015**). "On the inadvisability of using single parameter impedance models for representing the acoustical properties of ground surfaces", Journal of the Acoustical Society of America **138**, 2399–2413.

Embleton, T. (**1996**). "Tutorial on sound propagation outdoors", Journal of the Acoustical Society of America **100**, 31–48.

Evans, L., Bass, H., and Sutherland, L. (**1972**). "Atmospheric absorption of sound: theoretical predictions", Journal of the Acoustical Society of America **51**, 1565–1575.

Francois, R. and Garrison, G. (**1982**a). "Sound absorption based on ocean measurements: Part i: Pure water and magnesium sulfate contributions", Journal of the Acoustical Society of America **72**, 896–907.

Francois, R. and Garrison, G. (**1982**b). "Sound absorption based on ocean measurements: Part ii: Boric acid contribution and equation for total absorption", Journal of the Acoustical Society of America **72**, 1879–1890.

Lamancusa, J. S. (**2009**). "Engineering noise control", Course notes ME 458, Penn State University.

Ostashev, V. (**1997**). *Acoustics in Moving Inhomogeneous Media* (E and FN SPON).

Ostashev, V., Wilson, D., Liu, L., Aldridge, D., Symons, N., and Marlin, D. (**2005**). "Equations for finite-difference, time-domain simulation of sound propagation in moving inhomogeneous media and numerical implementation", Journal of the Acoustical Society of America **117**, 503–517.

Ostashev, V. E., Vecherin, S. N., Wilson, D. K., Ziemann, A., and Goedecke, G. H. (**2009**). "Recent progress in acoustic travel-time tomography of the atmospheric surface layer", Meteorologische Zeitschrift **18**, 125–133.

Panton, R., Kheirandish, S., and Wagner, M. (**2007**). *Springer Handbook of Experimental Fluid Mechanics*, chapter 1. Experiment as a Boundary-Value Problem (Springer, edited by C. Tropea, A.L. Yarin and J.F. Foss).

Pierce, A. (**1989**). *Acoustics: An Introduction to its Physical Principles and Applications* (Acoustical Society of America).

Rienstra, S. and Hirschberg, A. (**2014**). "An introduction to acoustics", Eindhoven University of Technology.

Salomons, E. M. (**2001**). *Computational Atmospheric Acoustics* (Kluwer Academic Publishers).