MF207: ACOUSTICS IN FLUID MEDIA

Course notes 2023

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Contents

Chapter 1

Wave equation and acoustic plane waves

In this chapter, the linearized acoustic equations are derived from the equations of fluid mechanics. The wave equation is obtained in the time and in the frequency domain, and plane wave solutions are described. Finally, the acoustic intensity and power are introduced, and the sound pressure level is defined. This chapter is based mostly on the books of Pierce [1] and Chaigne [2].

1.1 Equations of fluid mechanics

We start here from the equations of fluid mechanics written for the pressure $p_t(\underline{x}, t)$, the velocity $\underline{v_t}(\underline{x}, t)$ the density $\rho_t(\underline{x}, t)$ and the entropy $S_t(\underline{x}, t)$ that are functions of space $\underline{x}$ and time t. The subscript t means that these are the total values in the fluid medium.

First, the conservation of mass or equation of continuity is written:

$$\frac{\partial \rho_t}{\partial t} + \nabla.(\rho_t \underline{v_t}) = \frac{D\rho_t}{Dt} + \rho_t \nabla.\underline{v_t} = 0, \tag{1.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v_t}.\nabla$$

is called the **material or total derivative**.

Second, law of momentum conservation is given by:

$$\rho_t \left[\frac{\partial \underline{v_t}}{\partial t} + (\underline{v_t}.\nabla)\underline{v_t} \right] = -\nabla p_t + \nabla.\tau + \rho_t g \underline{e}_z, \tag{1.2}$$

where $\rho_t g \underline{e}_z$ corresponds to the gravitational force, that will be neglected in the following, and τ is the viscous stress tensor. The viscous stress tensor is important to model aeroacoustic source generation [1], but will be neglected here in the context of acoustic propagation. The equation of momentum conservation is called **Euler equation** for an inviscid fluid.

The last equation is given by the equation of state of the fluid, that can be written as a general law linking pressure to density:

$$p_t = f(\rho_t). \tag{1.3}$$

1.2 Linear acoustics approximation and wave equation

We now linearize the equations of fluid mechanics in a homogeneous medium at rest, such that the mean velocity $\underline{v_0} = 0$, and the mean density ρ_0 and pressure p_0 are independent of space and time. Let us introduce the variables associated to acoustic waves in a fluid medium:

- acoustic pressure [Pa]: $p(\underline{x}, t) = p_t(\underline{x}, t) - p_0$;
- particle velocity [m/s]: $\underline{v}(\underline{x}, t) = \underline{v_t}(\underline{x}, t)$;
- density associated to acoustic fluctuations [kg/m^3]: $\rho(\underline{x}, t) = \rho_t(\underline{x}, t) - \rho_0$.

Keeping only terms of order 1, the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla.\underline{v} = 0. \tag{1.4}$$

Similarly, the linearized Euler equation is given by:

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = -\nabla p. \tag{1.5}$$

[1] This will be studied in MF208 Aeroacoustics and propagation in inhomogeneous moving medium

Finally, a first-order Taylor's expansion of the equation of state yields:

$$p_t = p_0 + \left(\frac{\partial p_t}{\partial \rho_t}\right)_{\rho_t=\rho_0} (\rho - \rho_0) \Leftrightarrow p = c^2\rho \quad \text{with} \quad c^2 = \left(\frac{\partial p_t}{\partial \rho_t}\right)_{\rho_t=\rho_0}. \tag{1.6}$$

Equations (1.4), (1.5) and (1.6) can be combined to obtain a single equation on the acoustic pressure p. We first replace ρ by p/c^2 in Equation (1.4) using (1.6):

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \rho_0 \nabla . \underline{v} = 0. \tag{1.7}$$

Then, we substract the time derivative of Equation (1.7) and the divergence of Equation (1.5) in order to eliminate the particle velocity:

$$\boxed{\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0}\,. \tag{1.8}$$

The operator $\nabla^2 = \Delta$ is called Laplacian and is written in cartesian coordinates:

$$\nabla^2 p = \Delta p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}. \tag{1.9}$$

The equations derived in this section have been obtained in the linear acoustics approximation, that are valid for:

$$|p| \ll p_0, \quad |v| \ll c, \quad |\rho| \ll \rho_0. \tag{1.10}$$

The linear acoustics approximation is valid in many applications. For instance, the amplitude of acoustic pressure corresponding to the threshold of pain is around 90 Pa (about 130 dB re. 20 μPa), which is still two orders of magnitude smaller than the atmospheric pressure that is close to 10^5 Pa.

1.3 Sound speed

As can be seen in the wave equation (1.8), and as will be seen in Section 1.5, c is a velocity that is commonly called sound speed, that is defined as:

$$c^2 = \left(\frac{\partial p_t}{\partial \rho_t}\right)_{\rho_t=\rho_0}. \tag{1.11}$$

Let us assume that we consider a perfect gas, for which the pressure ρ_t can be related to the density ρ_t and the absolute temperature T (in Kelvins):

$$p = \rho r T, \tag{1.12}$$

where r is the specific gas constant: $r \approx 287\,\mathrm{J/kg/K}$ for air. As acoustic processes are nearly isentropic (adiabatic and reversible), the Laplace's law can be used that is written for a perfect gas:

$$p_t/\rho_t^\gamma = \text{constant} \quad \Rightarrow \quad \frac{dp_t}{p_t} = \gamma \frac{d\rho_t}{\rho_t}, \tag{1.13}$$

with γ the ratio of specific heats: $\gamma = 1.4$ for air. Thus:

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}} = \sqrt{\gamma r T_0}, \tag{1.14}$$

with T_0 the mean temperature. The sound speed is seen to increase with air temperature. It is approximately $340\,\mathrm{m/s}$ at $20°\mathrm{C}$ ($293\,\mathrm{K}$).

The behavior of a liquid such as water is more complex and cannot be modeled as a perfect gas. The sound speed in fresh water is about $1481\,\mathrm{m/s}$ at $20°\mathrm{C}$ and close to $1500\,\mathrm{m/s}$ in seawater.

1.4 Waves in the frequency domain

For a harmonic wave at angular frequency $\omega = 2\pi f$, the pressure can be written $p(\underline{x}, t) = A(\underline{x}) \cos(\omega t + \phi(\underline{x}))$, where A is the amplitude and ϕ is the phase that are both functions of space. It is useful to introduce the following complex notation:

$$p(\underline{x}, t) = \mathrm{Re}\left[P(\underline{x}) e^{-i\omega t}\right], \tag{1.15}$$

where Re denotes the real part and the $P(\underline{x}) = A(\underline{x}) e^{-i\phi(\underline{x})}$ is the complex pressure amplitude. Introducing $P(\underline{x}) e^{-i\omega t}$ into the wave equation:

$$\boxed{\Delta P + k^2 P = 0}\,, \tag{1.16}$$

where $k = \omega/c = 2\pi/\lambda$ is the acoustic wave number, and λ is the wavelength. Equation (1.16) is called the **Helmholtz equation**. Many computational

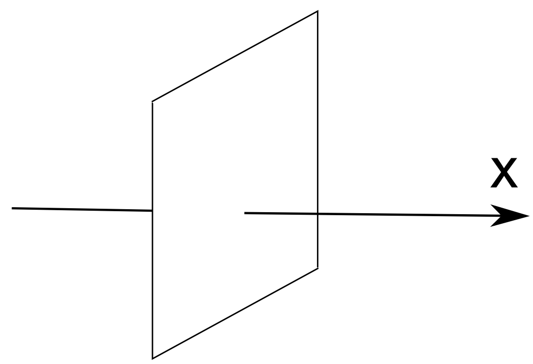


Figure 1.1: Plane wave traveling along the x-direction.

methods assume a harmonic sound field as any sound signal can be decomposed into harmonic components using the Fourier transform (spectral decomposition), and it is easier to solve in the frequency domain as there is no time derivative to evaluate.

Remark: it is also possible to use the $e^{j\omega t}$ convention instead of the $e^{-i\omega t}$ convention. In this case, we would have:

$$p(\underline{x}, t) = \mathrm{Re}\left[P(\underline{x})e^{j\omega t}\right], \tag{1.17}$$

with $P(\underline{x}) = A(\underline{x})e^{j\phi(\underline{x})}$. The Helmholtz equation remains the same with both notations!

1.5 Plane waves

Plane waves correspond to specific solutions to the wave equation where the wavefronts are planar, as seen in Figure 1.1. As a result, the wave equation is simply:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0, \tag{1.18}$$

and the general solution, also called d'Alembert solution, is given by:

$$p(x, t) = F_+\left(t - \frac{x}{c}\right) + F_-\left(t + \frac{x}{c}\right), \tag{1.19}$$

where the function F_+ describes the wave propagation in the positive x direction, and F_- describes the wave propagation in the negative x direction. The associated particle velocity field can be obtained from the linearized Euler equation (1.5):

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = -\nabla p = -\frac{\partial p}{\partial x}\underline{e_x} = \frac{1}{c}\left[F_+\left(t - \frac{x}{c}\right) - F_-\left(t + \frac{x}{c}\right)\right]\underline{e_x}. \tag{1.20}$$

As a result, $\underline{v} = v_x \underline{e_x}$ with:

$$v_x(x,t) = \frac{1}{\rho_0 c}\left[F_+\left(t - \frac{x}{c}\right) - F_-\left(t + \frac{x}{c}\right)\right]\underline{e_x}. \tag{1.21}$$

Let us consider a special case of interest, that is a harmonic plane wave traveling along the positive x axis, with $p(\underline{x}, t) = Re\{P(x)e^{-i\omega t}\}$ and $\underline{v}(x,t) = Re\{\underline{V}(x)e^{-i\omega t}\}$:

$$P(x) = P_0 e^{ikx}, \tag{1.22}$$

$$\underline{V}(x,t) = \frac{P(x,t)}{\rho_0 c}\underline{e_x}. \tag{1.23}$$

With this type of waves the amplitude remains constant with distance. As a result, the ratio of pressure to velocity is constant for a plane wave and equal to $Z_{c,fluid} = \rho_0 c$. The quantity $Z_{c,fluid}$ is called the **characteristic acoustic impedance** of the fluid.

1.6 Acoustic intensity and power

The time-averaged acoustic power of a source is defined as:

$$\langle W_a \rangle = \int_S \langle \underline{I} \rangle . \underline{n} dS = \int_S \langle p\underline{v} \rangle . \underline{n} dS, \tag{1.24}$$

where $\underline{n}$ is the normal to the surface S and $\underline{I}\rangle$ is the time-averaged acoustic intensity given by:

$$\langle \underline{I} \rangle = \frac{1}{T}\int_{t_0}^{t_0+T} \underline{I}(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)\underline{v}(t)dt. \tag{1.25}$$

For harmonic waves, let $p(\underline{x}, t) = Re\{P(\underline{x})e^{-i\omega t}\}$ and $\underline{v}(\underline{x}, t) = Re\{\underline{V}(\underline{x})e^{-i\omega t}\}$. The time-averaged acoustic intensity for sinusoidal waves becomes:

$$\langle \underline{I} \rangle = \frac{1}{2}Re\{P\underline{V}^*\}. \tag{1.26}$$

1.7 Sound pressure level and sound power level

The sound pressure level (SPL) is defined as:

$$L_p = 10 \log_{10} \left(\frac{p_{rms}^2}{p_{ref}^2} \right) = 20 \log_{10} \left(\frac{p_{rms}}{p_{ref}} \right), \tag{1.27}$$

where p_{ref} is a reference pressure and p_{rms} is the time-averaged or rms pressure:

$$p_{rms}^2 = \langle p^2 \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} p^2(t) dt. \tag{1.28}$$

For a harmonic wave, $p_{rms} = \max |p| / \sqrt{2}$. Similarly, the sound power level (SWL) is defined as:

$$L_W = 10 \log_{10} \left(\frac{\langle W_a \rangle}{W_{ref}} \right), \tag{1.29}$$

with W_{ref} a reference power. The reference pressure p_{ref} is typically 2×10^{-5} Pa in air (threshold of hearing at 1 kHz) and 10^{-6} Pa in water.

Chapter 2

Acoustic elementary sources

2.1 Velocity potential

To solve acoustic problems, it is sometimes convenient to introduce an acoustic velocity potential Φ associated with the particle velocity $\underline{v}$. Taking the curl of Equation (1.5):

$$\frac{\partial}{\partial t}\nabla \times \underline{v} = 0, \tag{2.1}$$

since $\nabla \times \nabla p = 0$. This means that the rotational of particle velocity is independent of time. If the acoustic field is irrotational ($\nabla \times \underline{v} = 0$), then the particle velocity derives from a potential Φ: $\underline{v} = \nabla\Phi$. The relationship between p and Φ is obtained from Equation (1.5):

$$p(r,t) = -\rho_0 \frac{\partial \Phi}{\partial t}. \tag{2.2}$$

Replacing this expression into the wave equation (1.8), we see that Φ satisfies the same equation as p:

$$\boxed{\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0}\;. \tag{2.3}$$

It is convenient to solve for the velocity potential because acoustic pressure and particle velocity can be deduced by taking the temporal or spatial derivative of Φ, as will be seen in Section 2.2.

2.2 Spherical waves

We now consider waves with spherical symmetry, which means that the variables do not depend on the spherical coordinates θ and ϕ: $p = p(r,t)$ and $\underline{v} = v(r,t)\underline{e_r}$. The wavefronts are spheres, and the acoustic intensity vector is along along the r direction: $\underline{I} = I_r\underline{e_r}$. This solution corresponds to the case of a point source with spherical symmetry.

Rewriting the homogeneous wave equation (2.3) for the velocity potential in spherical coordinates:

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{r}\frac{\partial^2(r\Phi)}{\partial r^2} = 0 \Leftrightarrow \frac{1}{c^2}\frac{\partial^2(r\Phi)}{\partial t^2} - \frac{\partial^2(r\Phi)}{\partial r^2} = 0.$$

This means that $r\Phi$ can be written as a sum of a function of $t - r/c$ and a function of $t + r/c$, as done in Section 1.5 for plane waves. If we keep only the outward-going wave:

$$\Phi(r,t) = \frac{1}{r}F\left(t - \frac{r}{c}\right), \tag{2.4}$$

and thus:

$$p(r,t) = -\rho_0\frac{\partial\Phi}{\partial t} = -\frac{\rho_0}{r}F'\left(t - \frac{r}{c}\right), \tag{2.5}$$

$$v(r,t) = \frac{\partial\Phi}{\partial r} = \frac{p(r,t)}{\rho_0 c} - \frac{1}{r^2}F\left(t - \frac{r}{c}\right). \tag{2.6}$$

It appears that the pressure amplitude decreases as $1/r$. Also, the particle velocity is composed of two terms. Since the second term decreases as $1/r^2$, it becomes negligible if r is sufficiently large (far-field) and $v(r,t) \approx \frac{p(r,t)}{\rho_0 c}$, which corresponds to the relationship for plane waves.

It is possible to calculate the acoustic power of this wave by integrating over a sphere of radius r. From Equation (1.24), considering that the acoustic intensity is constant on the sphere and that $\underline{n} = \underline{e}_r$:

$$\langle W_a\rangle = \int_S \langle\underline{I}\rangle.\underline{e}_r dS = 4\pi r^2\langle I(r)\rangle. \tag{2.7}$$

If we consider a harmonic spherical wave of the form $p(\underline{x},t) = Re\{P(r)e^{j\omega t}\}$, with

$$P(r) = \frac{A}{r}e^{-jkr}, \tag{2.8}$$

the following time-averaged acoustic intensity is obtained from Equation (1.26):

$$\langle I(r)\rangle = \frac{|P|^2}{2\rho_0 c} = \frac{\langle p^2\rangle}{\rho_0 c}. \tag{2.9}$$

From Equations (2.7) and (2.9), the acoustic power is thus:

$$\langle W_a\rangle = 4\pi r^2 \frac{\langle p^2\rangle}{\rho_0 c} = \frac{2\pi|A|^2}{\rho_0 c}. \tag{2.10}$$

It appears clearly that the acoustic power is independent of the distance r since A is a constant; the acoustic power $< W_a >$ is a characteristics of the source(s) inside the sphere S.

From the previous expression, it is possible to derive a simple relationship between the sound pressure level and the sound power level:

$$\boxed{L_p = L_W - 10\log_{10}(4\pi r^2)}\ , \tag{2.11}$$

where $W_{ref} = p_{ref}^2/(\rho_0 c)$. In air, we consider typically $p_{ref} = 20 \times 10^{-6}$ Pa and $\rho_0 c \approx 415\,\mathrm{kg/m^2/s}$, thus $W_{ref} \approx 10^{-12}$ W. The term $10\log_{10}(4\pi r^2)$ is called **geometrical spreading**. This means that there is an attenuation of $10\log_{10}(4) \approx 6$ dB of the sound pressure level L_p when the distance r is doubled (6 dB attenuation per doubling distance).

2.3 Acoustic field radiated by a pulsating sphere and a monopole

In order to introduce, let us consider a pulsating sphere of radius a whose velocity is v_a, as schematically shown in Figure 2.1. In the harmonic regime, $v_a = V_a e^{j\omega t}$. Since the sphere velocity is independent on θ and ϕ, the acoustic pressure is given by Equation (2.8), and the particle velocity can be obtained from Euler's equation:

$$V(r) = \frac{A}{\rho_0 c r}\left(1 + \frac{1}{jkr}\right) e^{-jkr}. \tag{2.12}$$

The normal particle velocity at $r = a$ must be equal to the sphere velocity: $V(a) = V_a$. As a result:

$$A = jk\rho_0 c V_a \frac{a^2}{1 + jka} e^{jka}, \tag{2.13}$$

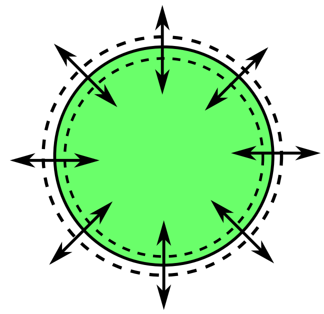


Figure 2.1: Schematics for a pulsating sphere of radius a.

and the acoustic pressure field can be written:

$$P(r) = \frac{a}{r}\frac{\rho_0 c}{4\pi a^2}Q\frac{jka}{1+jka}e^{-jk(r-a)}, \tag{2.14}$$

where $Q = 4\pi a^2 V_a$ is the volume velocity of the pulsating sphere. Using Equations (2.9) and (2.7), the expression for the time-averaged acoustic intensity and power are given by:

$$\langle I(r)\rangle = \frac{1}{2}Re\{P(r)\underline{V}(r)^*\} = \frac{\rho_0 c}{2}V_a^2\frac{a^2}{r^2}\frac{(ka)^2}{1+(ka)^2}, \tag{2.15}$$

$$\langle W_a\rangle = 4\pi r^2\langle I(r)\rangle = \rho_0 c\frac{Q^2}{8\pi a^2}\frac{(ka)^2}{1+(ka)^2}. \tag{2.16}$$

When the radius of the pulsating sphere is much smaller than the acoustic wavelength ($ka \ll 1$) and the receiver is sufficiently far from the sphere ($r \gg a$), the acoustic pressure becomes:

$$P(r) = \frac{j\omega\rho_0 Q}{4\pi r}e^{-jkr}. \tag{2.17}$$

This expression corresponds to an omnidirectional point source called **monopole**. The time-averaged acoustic intensity and power become for the monopole:

$$\langle I(r)\rangle = \frac{\rho_0 c}{32\pi^2}\frac{k^2Q^2}{r^2}, \tag{2.18}$$

$$\langle W_a\rangle = \frac{\rho_0 c}{8\pi}k^2Q^2. \tag{2.19}$$

2.4 Acoustic field radiated by a dipole

Another important elementary source called **dipole** is obtained by combining two monopoles whose volumes velocities are out of phase ($\pm Q$), as shown in Figure 2.2(a). In the far-field ($r \gg d$), the distances r_1 and r_2 can be written: $r_1 \approx r - d\cos\theta/2$ and $r_2 \approx r + d\cos\theta/2$. As a result:

$$P(r,\theta) = \frac{j\omega\rho_0 Q}{4\pi}\left(\frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2}\right) \approx \frac{-\omega\rho_0 Q e^{-jkr}}{2\pi r}\sin\left(\frac{kd}{2}\cos\theta\right). \quad (2.20)$$

When the separation d between the monopoles is small compared to the acoustic wavelength ($kd \ll 1$), this expression can be simplified:

$$P(r,\theta) \approx \frac{-\omega\rho_0 kdQe^{-jkr}}{4\pi r}\cos\theta. \quad (2.21)$$

One important feature of the dipole that is apparent in Equation (2.21) is that the acoustic pressure radiation depends on the angle θ. It is maximal in the axis of the dipole ($\theta_n = n\pi$) and minimal in the perpendicular direction ($\theta_n = (2n+1)\pi/2$). The associated directivity factor is given by:

$$D(\theta) = \frac{|P(r,\theta)|}{\max_\theta |P(r,\theta)|} = |\cos\theta|, \quad (2.22)$$

and is plotted in Figure 2.2(b).

In the far-field, it can be shown that the particle velocity is along r, with $V(r,\theta) \approx P(r,\theta)/(\rho_0 c)$. As a result, we obtain:

$$\langle I_r\rangle = \frac{1}{2}Re[PV_r^*] = \frac{|P(r,\theta)|^2}{2\rho_0 c} = \frac{\rho_0 ck^4 d^2|Q|^2}{32\pi^2 r^2}\cos^2\theta. \quad (2.23)$$

The acoustic power radiated by the dipole is given by:

$$\langle W_a\rangle = \int_S \langle I(r,\theta)\rangle r^2 \sin\theta d\theta d\phi = 2\pi r^2 \int_0^\pi \langle I_r\rangle \sin\theta d\theta. \quad (2.24)$$

Since

$$\int_0^\pi \cos^2\theta \sin\theta d\theta = \left[-\frac{1}{3}\cos^3\theta\right]_0^\pi = \frac{2}{3},$$

we obtain finally:

$$\langle W_a\rangle = \frac{\rho_0 ck^4 d^2|Q|^2}{24\pi}. \quad (2.25)$$

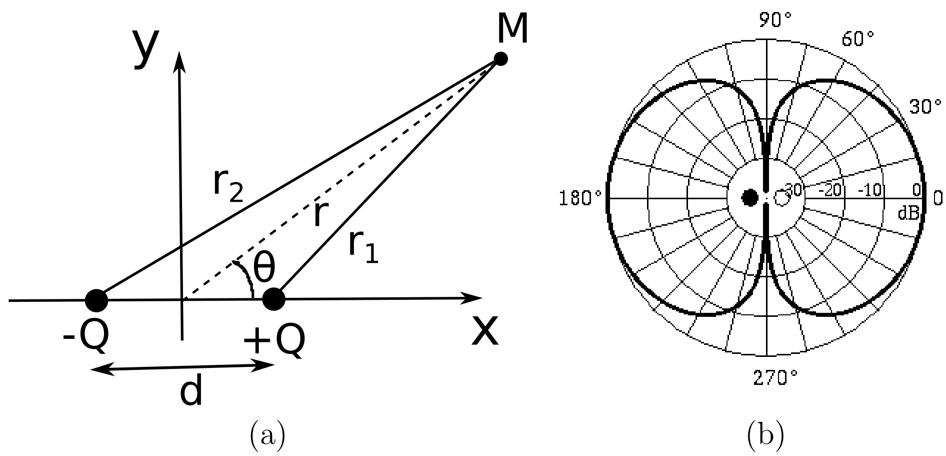


Figure 2.2: (a) Acoustic dipole seen as the combination of two monopoles of volume velocities $\pm Q$, and (b) associated directivity for $kd \ll 1$.

2.5 Inhomogenous wave equation with source terms

Another way to introduce acoustic sources is to include source terms on the right hand-side of the linearized equations of Section 1.2. In presence of a source of mass, the linearized continuity equation becomes:

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \rho_0 \nabla . \underline{v} = \dot{m}_V, \tag{2.26}$$

with $\dot{m}_V$ the injected mass per unit volume and per unit time (in kg/m^3/s). An example of such a source is a air bubble oscillating in a liquid or a loudspeaker inserted in a baffle. Similarly, in presence of external forces in the fluid, the linearized Euler equation is written:

$$\rho_0 \frac{\partial \underline{v}}{\partial t} + \nabla p = \underline{f}_V, \tag{2.27}$$

with $\underline{f}_V$ the exterior forces imposed to the fluid per unit volume. This corresponds to oscillating sources, or to a loudspeaker without baffle.

To obtain the wave equation, we substract the time derivative of Equation (2.26) and the divergence of Equation (2.27) as done in Section 1.2, and we obtain:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = \ddot{m}_V - \nabla . \underline{f}_V. \tag{2.28}$$

We obtain two additional source terms on the right-hand side. The first term corresponds to an acceleration term, and is thus similar to the pulsating sphere or the monopole. The second term depends on the orientation of the exterior force, and is thus similar to the dipole.

2.6 Green's function

2.6.1 Definition

The Green's function is the solution of the wave equation with a unit point impulsive source term:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) g(\underline{x}, t|\underline{x}_S, \tau) = \delta(\underline{x} - \underline{x}_S)\delta(t - \tau), \tag{2.29}$$

with τ the source time, t the receiver time, $\underline{x}_S$ the source position, and $\underline{x}$ the receiver position. The Green's function should be zero for $t < \tau$ due to causality considerations. Let $G(\omega, \underline{x}|\underline{x}_S)$ be the Fourier transform of the Green's function:

$$G(\omega, \underline{x}|\underline{x}_S) = \int_{-\infty}^{+\infty} g(\underline{x}, t|\underline{x}_S, \tau)e^{-j\omega t}dt. \tag{2.30}$$

Equation (2.29) becomes:

$$\left(-\frac{\omega^2}{c^2} - \Delta\right) G(\omega, \underline{x}|\underline{x}_S) = \delta(\underline{x} - \underline{x}_S)\int_{-\infty}^{+\infty} \delta(t - \tau)e^{-j\omega t}dt = \delta(\underline{x} - \underline{x}_S)e^{-j\omega\tau}, \tag{2.31}$$

using the sifting property of the delta function:

$$\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)dx = f(x_0). \tag{2.32}$$

The phase term $e^{-j\omega\tau}$ is equal to 1 if the origin of the impulse is chosen at $\tau = 0$. As a result, the frequency-domain Green's function simply written $G(\underline{x}|\underline{x}_S)$ in the following is solution of the Helmholtz equation with a point source term:

$$\Delta G(\underline{x}|\underline{x}_S) + k^2 G(\underline{x}|\underline{x}_S) = -\delta(\underline{x} - \underline{x}_S), \tag{2.33}$$

where $k = \omega/c$.

2.6.2 Green's function in free field

In order to derive the expression for the Green's function in free field, let us start from the inhomogeneous wave equation (2.28) with a point source term $\dot{m}_V = \rho_0 q(t)\delta(\underline{x})$ and $f_V = 0$. For a harmonic volume velocity $q(t) = Qe^{j\omega t}$, the acoustic pressure is written $p(\underline{x},t) = P(\underline{x})e^{j\omega t}$ and is solution of the following Helmholtz equation:

$$\Delta P + k^2 P = -j\omega\rho_0 Q\delta(\underline{r}). \tag{2.34}$$

The solution has spherical symmetry such that $P(r) = \frac{A}{r}e^{-jkr}$. Let us now integrate this equation over a sphere of radius ϵ and volume V_ϵ:

$$A\int_{V_\epsilon}\Delta\left(\frac{e^{-jkr}}{r}\right)dV + Ak^2\int_{V_\epsilon}\frac{e^{-jkr}}{r}dV = -j\omega\rho_0 Q\int_{V_\epsilon}\delta(\underline{x})dV = -j\omega\rho_0 Q. \tag{2.35}$$

The idea is then to let ϵ tend to zero. The first integral can be evaluated using the divergence theorem:

$$\begin{aligned} A\int_{V_\epsilon}\Delta\left(\frac{e^{-jkr}}{r}\right)dV = A\int_S \nabla\left(\frac{e^{-jkr}}{r}\right).\underline{n}dS &= A\int_S\left(-jk-\frac{1}{r}\right)\frac{e^{-jkr}}{r}dS \\ &= -4\pi\epsilon^2 A\left[\frac{jk}{\epsilon}+\frac{1}{\epsilon^2}\right]e^{-jk\epsilon} \\ &\underset{\epsilon\to 0}{\longrightarrow} -4\pi A. \end{aligned}$$

An upper bound can be found for the second integral:

$$Ak^2\int_{V_\epsilon}\frac{e^{-jkr}}{r}dV = Ak^2\int_V\frac{e^{-jkr}}{r}4\pi r^2 dr \leq 4\pi\int_0^\epsilon r dr = 2\pi\epsilon^2 \underset{\epsilon\to 0}{\longrightarrow} 0. \tag{2.36}$$

As a result, $A = \frac{j\omega\rho_0 Q}{4\pi}$ and we finally obtain:

$$P(r) = \frac{j\omega\rho_0 Q}{4\pi r}e^{-jkr}, \tag{2.37}$$

which corresponds to the monopole solution given by Equation (2.17). Comparing Equations (2.33) and (2.34), it appears that $P(r) = j\omega\rho_0 G(r)$. Generalizing this result to a point source at $\underline{x}_S$ instead of 0, the free-field Green's function is obtained:

$$G(\underline{x}|\underline{x}_S) = \frac{e^{-jk|\underline{x}-\underline{x}_S|}}{4\pi|\underline{x}-\underline{x}_S|} = \frac{e^{-jkr}}{4\pi r}, \tag{2.38}$$

with $r = |\underline{x} - \underline{x}_S|$ the source-receiver distance.

Let us now take the inverse Fourier transform of this expression:

$$\int_{-\infty}^{+\infty} G(\omega, \underline{x}|\underline{x}_S) e^{j\omega t} d\omega = \int_{-\infty}^{+\infty} \frac{e^{-jkr}}{4\pi r} e^{j\omega t} d\omega = \frac{1}{4\pi r} \delta\left(t - \frac{r}{c}\right).$$

To obtain the time-domain Green's function, a time shift τ must be added:

$$g(\underline{x}, t|\underline{x}_S, \tau) = \frac{1}{4\pi r} \delta\left(t - \tau - \frac{r}{c}\right). \tag{2.39}$$

This simply means that the receiver time t is equal to the sum of the source time τ and the propagation time r/c.

2.6.3 Properties

The Green's function in free field meets the reciprocity property, which means that the Green's function remains the same if source and receiver positions are interchanged in a medium at rest:

$$\text{time domain:} \quad G(\underline{x}, t|\underline{x}_S, \tau) = \frac{1}{4\pi r} \delta\left(t - \tau - \frac{r}{c}\right) = G(\underline{x}_S, -\tau|\underline{x}, -t)$$

$$\text{frequency domain:} \quad G(\underline{x}|\underline{x}_S) = \frac{e^{-jkr}}{4\pi r} = G(\underline{x}_S|\underline{x}).$$

The superposition principle also applies. For N point sources of amplitudes S_n:

$$p_c(\underline{x}) = \sum_{n=1}^{N} S_n G(\underline{x}|\underline{x}_{Sn}). \tag{2.40}$$

Similarly, for a continuous distribution of monopole over a source volume V_S:

$$p_c(\underline{x}) = \int_{V_S} S(\underline{x}_S) G(\underline{x}|\underline{x}_S) dV_s, \tag{2.41}$$

with $S(\underline{x}_S)$ the monopole-amplitude distribution per unit volume.

Remark: using the $e^{-i\omega t}$ convention, the Green's function in free field is given by:

$$G(\underline{x}|\underline{x}_S) = \frac{e^{ik|\underline{x} - \underline{x}_S|}}{4\pi|\underline{x} - \underline{x}_S|} = \frac{e^{ikr}}{4\pi r}. \tag{2.42}$$

Chapter 3

Guided waves

An acoustical waveguide is a particular slender, hollow and rigid structure (typically a pipe) allowing the confinement and propagation of acoustic waves in a well-defined direction inside the structure.

Waveguides can be found in many applications such as wind instruments where pipes are tuned to a chosen resonant frequency, exhaust pipes that are designed to trap high level engine noise, pipes in industrial buildings that transmit unwanted noises over great distances, or even the streets of large cities that also transmit sound from one end to the other.

Another important category of waveguides, very useful to our societies, concerns electromagnetic waves. But, although their modeling and analysis is very similar to the case of acoustic waves, they are obviously not the subject of this course.

In this chapter we will first revisit the equations of acoustic propagation in this particular geometry that is the waveguide. Then we will introduce modeling concepts such as the impedance of the waveguide, the transmission matrix, the notion of impedance. These concepts will allow us to study the main properties and uses of acoustic waveguides.

3.1 Acoustic wave propagation in a duct

3.1.1 Propagation in a rectangular duct

Let us consider the system sketched on figure 3.1. It consists of a duct of rectangular section (dimensions $H \times L$) in which we aim at analyzing the

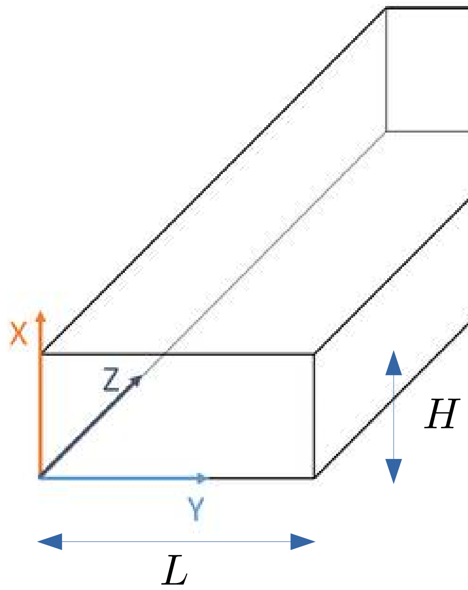


Figure 3.1: Sketch view of a duct of rectangular section filled by a compressible fluid.

acoustic waves propagation.

The following hypotheses are made:

- The duct is infinite in $z-$direction.
- The walls are perfectly rigid.
- There are no visco-thermal losses in the fluid (as considered up to now in the course).

As demonstrated in Chapter 1, the pressure in the fluid obeys the linear wave equation for acoustics (1.8), which writes in cartesian coordinates:

$$\frac{\partial^2 p}{\partial t^2} - c^2 \left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right] = 0. \tag{3.1}$$

The fluid is inviscid and the walls are perfectly rigid. The velocity hence satisfy the following boundary conditions on these walls:

$$\underline{v} \, . \, \underline{n} = 0. \tag{3.2}$$

It can be rewritten in terms of the pressure by making use of the linearized Euler equation (1.5),

$$\underline{\nabla}\, p \,.\, \underline{n} = 0. \tag{3.3}$$

Let us look for a solution of these equations in the form:

$$p(x, y, z, t) = P(z, t) f(x) g(y). \tag{3.4}$$

Introducing Equation (3.4) into (3.1) and dividing the resulting expression by Pfg, we obtain:

$$\frac{1}{c^2}\frac{\ddot{P}}{P} - \frac{P''}{P} - \frac{f''}{f} - \frac{g''}{g} = 0. \tag{3.5}$$

Putting the term depending on x on the right,

$$\begin{aligned} -\frac{f''}{f} &= -\frac{1}{c^2}\frac{\ddot{P}}{P} + \frac{P''}{P} + \frac{g''}{g} \\ &= \text{constant} \\ &= k_x^2 \end{aligned}$$

The solution for f takes hence the form of an harmonic functions, $f(x) = A\cos(k_x x) + B\sin(k_x x)$, and making use of the boundary conditions on rigid walls at $x = 0$ and $x = L$, we show that there is an infinite and discrete set of solutions:

$$f(x) = A\cos k_x x \quad \text{with } k_x = \frac{m\pi}{H}\ ,\ m \in \mathbb{N}. \tag{3.6}$$

The same can be done for the term that depends on y only:

$$\begin{aligned} -\frac{g''}{g} &= -\frac{1}{c^2}\frac{\ddot{P}}{P} + \frac{P''}{P} - k_x^2 \\ &= \text{constant} \\ &= k_y^2. \end{aligned}$$

And considering the boundary conditions,

$$g(y) = B\cos k_y y \quad \text{with } k_y = \frac{n\pi}{L}\ ,\ n \in \mathbb{N}. \tag{3.7}$$

Finally, we show that the pressure consists in a sum of solutions,

$$p(x, y, z, t) = \sum_m^\infty \sum_n^\infty P_{mn}(z, t) f_m(x) g_n(y). \tag{3.8}$$

Each component $P_m n(z,t)$ is governed by a one-dimensional propagation equation (one for each pair m, n):

$$\frac{\partial^2 P_{mn}}{\partial t^2} - c^2 \frac{\partial^2 P_{mn}}{\partial z^2} + c^2 b_{mn} P_{mn} = 0, \tag{3.9}$$

with

$$b_{mn} = \left(\frac{m^2 \pi^2}{H^2} + \frac{n^2 \pi^2}{L^2} \right). \tag{3.10}$$

Let us now look for solutions of these wave equations in the form of a propagative wave,

$$P_{mn} = \mathrm{e}^{\mathrm{j}(kz - \omega t)}. \tag{3.11}$$

Introducing this in (3.9), we obtain the dispersion relation for each family of waves (m, n):

$$k^2 = \frac{\omega^2}{c^2} - b_{mn}. \tag{3.12}$$

Depending on the sign on the right-hand side, we have then two possibilities:

- If $\omega^2 < c^2 b_{mn}$, k is real and the waves are neutral (propagating) waves. The waves propagate along the duct without any loss.
- If $\omega^2 < c^2 b_{mn}$, k is purely imaginary and the waves are evanescent. There is no energy transport by these waves in this case.

We have hence identified the so-called cutoff frequency of each mode (m, n),

$$f_{mn} = \frac{c}{2} \sqrt{\frac{m^2}{H^2} + \frac{n^2}{L^2}}. \tag{3.13}$$

For $(m, n) = (0, 0)$, the cutoff frequency $f_{00} = 0$, indicating that there are propagating waves at any frequencies. Moreover for this particular propagation mode, $f(x)g(y) = 1$ and the pressure is homogeneous in each section. Consequently, these waves are plane waves. For any other values of m and n, there is a frequency below which no energy propagates.

Finally, we introduce the cutoff frequency of the duct,

$$f_c = \min_{m \neq 0, n \neq 0} f_{mn}. \tag{3.14}$$

For any frequency smaller than f_c only plane waves are propagating in the duct. This is the general regime in which waveguides are used.

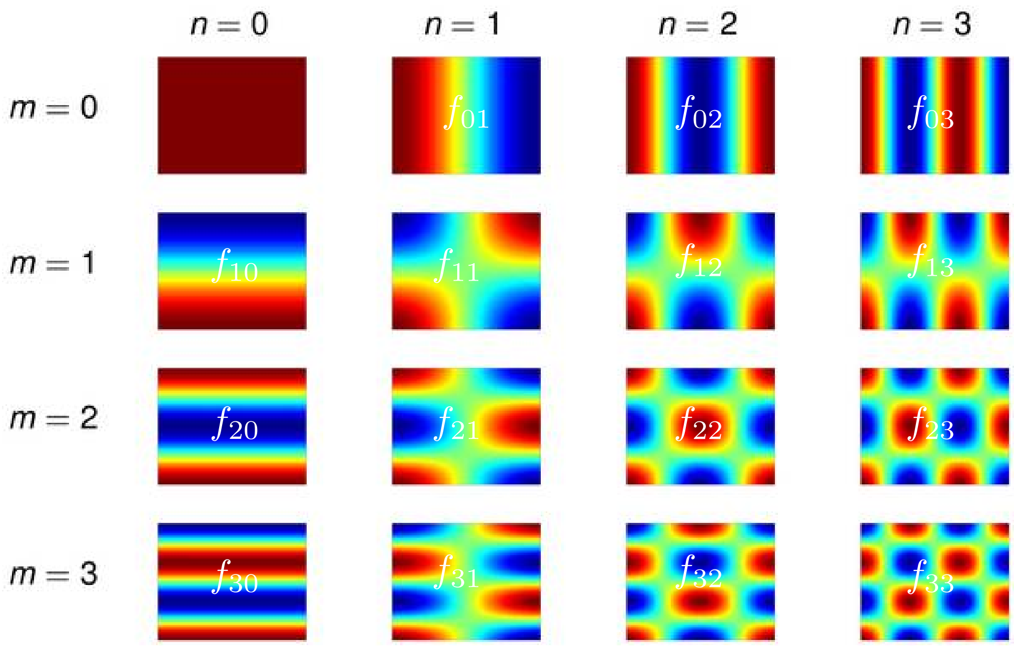


Figure 3.2: First pressure modes in the cross-section of the duct. Levels are arbitrary with blue being the minimum, negative value, red being the maximum positive value. Pressure is homogeneous in the case $(m, n) = (0, 0)$, indicating a plane wave propagation.

The general solution for the pressure in the duct can be written as the sum of all these propagation modes,

$$p(x,y,z,t) = \sum_{m}^{\infty}\sum_{n}^{\infty} \cos\left(\frac{m\pi x}{H}\right) \cos\left(\frac{n\pi y}{L}\right) \left(A_{mn}\mathrm{e}^{-jkz} + B_{mn}\mathrm{e}^{jkz}\right) \mathrm{e}^{j\omega t}. \tag{3.15}$$

The cross-section pressure distribution $\cos(k_{xm}x)\cos(k_{yn}y)$ of the 16 first modes is plotted on Figure 3.2. All the pressure modes, except $(0,0)$ give rise to evanescent waves below their corresponding cutoff frequency.

3.1.2 The duct of circular cross-section

Considering a duct of circular cross-section of radius a lead to similar calculations as those done above in the rectangular case. We show only the main results here.

The linear acoustics Equation (1.8) is now written in cylindrical coordinates, and looking for solution in the form $p(r,\theta,z,t) = P(z,t)f(r)g(\theta)$ that satisfy the zero normal velocity condition on $r = a$, we finally obtain a series of Bessel's equations leading to the following general solution for the pressure:

$$P(r,\theta,z,t) = \sum_{m}^{\infty}\sum_{n}^{\infty} \cos(m\theta) J_m\left(\chi_{mn}\frac{r}{a}\right) \left(A_{mn}\mathrm{e}^{-\mathrm{j}kz} + B_{mn}\mathrm{e}^{-\mathrm{j}kz}\right) \mathrm{e}^{\mathrm{j}\omega t} \tag{3.16}$$

The 16 first cross-sectional pressure modes are plotted on Figure 3.3. For this geometry, the cutoff frequency is given by f_{10} and reads,

$$f_{10} \sim \frac{1.84c}{2\pi a}. \tag{3.17}$$

Hence, the smaller is the radius a, the higher is the cutoff frequency.

3.2 Plane wave propagation analysis in ducts

In this section we restrict the analysis to frequencies below the cutoff frequency of the duct, so that only plane waves actually propagate in the system. In this context, we present some classical tools and methods to characterize the properties of acoustical waveguides.

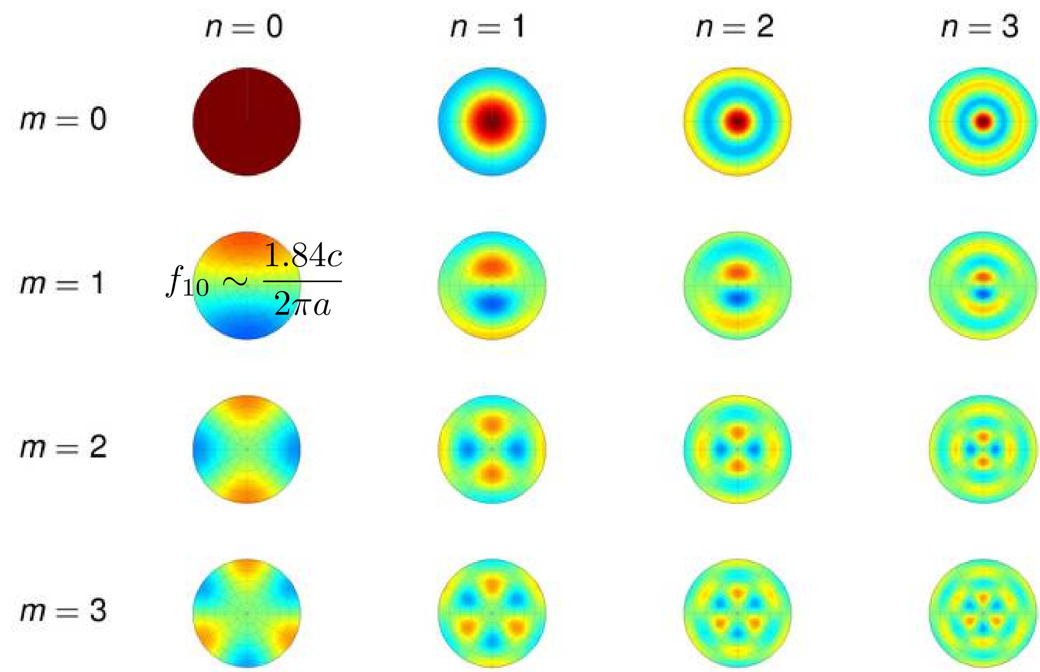


Figure 3.3: First pressure modes in the cross-section of the duct. Levels are arbitrary with blue being the minimum, negative value, red being the maximum positive value. Pressure is homogeneous in the case $(m, n) = (0, 0)$, indicating a plane wave propagation. The minimum cutoff frequency f_{10} is indicated on the figure for reference.

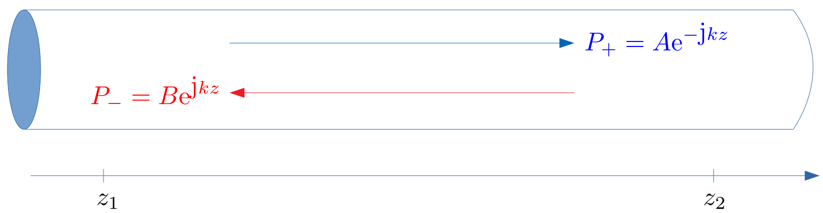


Figure 3.4: Schematic view of a portion of a duct between z_1 and z_2 with two waves propagating at a given frequency.

3.2.1 Characteristic impedance of the waveguide

Consider a plane wave propagating in the positive $x-$direction. Pressure and velocity read,

$$p(z,t) = Ae^{j(\omega t - kz)} = P(z)e^{j(\omega t)},$$
$$\underline{v}(z,t) = \frac{p(z,t)}{\rho_0 c}\underline{e}_z = V(z)e^{j(\omega t)}\underline{e}_z. \tag{3.18}$$

In Chapter 1, the characteristic acoustic impedance of the fluid has been introduced as the ratio between the pressure and the velocity for a plane wave,

$$Z_{c,fluid} = \frac{P}{V} = \rho_0 c_0, \tag{3.19}$$

which is the same for the guided plane wave in the duct.

Let us now introduce the acoustic flow-rate for a duct of cross section S:

$$U(z) = V(z)S. \tag{3.20}$$

We define the characteristic impedance of the waveguide as the ratio between the pressure and the flow rate:

$$Z_c = \frac{P}{U} = \frac{\rho_0 c_0}{S}. \tag{3.21}$$

3.2.2 Transfer matrix modelling

The characteristic impedances given in equations (3.19) and (3.21) refer to one particular wave propagating in one direction and do not depend on space.

Let us now consider a part of the waveguide between z_1 and z_2. For a given frequency, the duct carries a wave propagating in the positive $z-$direction, the other propagating in the negative $z-$direction, with respective amplitudes A and B. The system is sketched on figure 3.4. The pressure and flow rate read,

$$\begin{aligned} P(z) &= A\mathrm{e}^{-\mathrm{j}kz} + B\mathrm{e}^{\mathrm{j}kz}, \\ U(z) &= \mathrm{S}\rho_0 c \left(A\mathrm{e}^{-\mathrm{j}kz} - B\mathrm{e}^{\mathrm{j}kz} \right). \end{aligned} \tag{3.22}$$

The pressure and velocity at z_1 can be expressed as function as the pressure and velocity at z_2. This writes in matrix form:

$$\begin{bmatrix} P(z_1) \\ U(z_1) \end{bmatrix} = \begin{bmatrix} \cos k(z_2 - z_1) & \mathrm{j}Z_c \sin k(z_2 - z_1) \\ \mathrm{j}Z_c^{-1} \sin k(z_2 - z_1) & \cos k(z_2 - z_1) \end{bmatrix} \begin{bmatrix} P(z_2) \\ U(z_2) \end{bmatrix}. \tag{3.23}$$

The matrix in equation (3.23) is referred to as the *transfer matrix*.

Transfer matrix modelling is particularly useful for predicting the acoustic transmission, or resonance frequencies of complex piping systems consisting of chained elements. Each element can be modelled by a transfer matrix, so that the whole system model can be deduced from a product of the transfer matrices of all its individual elements.

3.2.3 Impedance propagation modelling

Let us introduce a local impedance $Z(z)$, defined as the ratio between pressure and acoustic flow rate,

$$Z(z) = \frac{P(z)}{U(z)}. \tag{3.24}$$

It is then possible to show from equation (3.23) that

$$Z(z_1) = \frac{P(z_1)}{U(z_1)} = Z_c \frac{\mathrm{j} \tan k(z_2 - z_1) + \frac{Z(z_2)}{Z_c}}{1 + \mathrm{j}\frac{Z(z_2)}{Z_c} \tan k(z_2 - z_1)}. \tag{3.25}$$

Impedance propagation modelling can be even more convenient than transfer matrix modelling, as we are now dealing with scalar functions, instead of matrices.

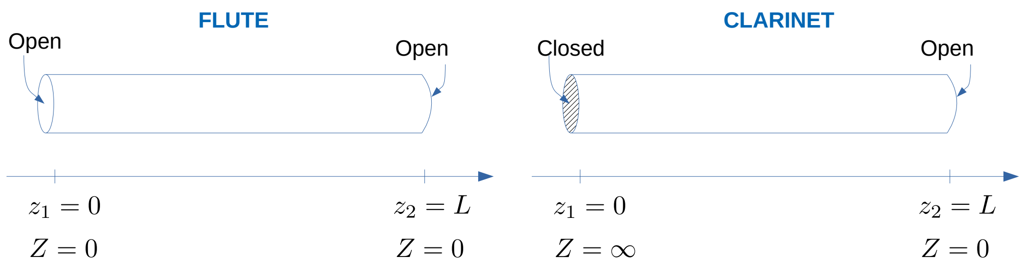


Figure 3.5: Two pipes of length L representative of the acoustic properties of a flute (left) and a clarinet (right).

Application to the eigenfrequency calculation of a finite pipe

As a first example, consider a pipe of length L, whose extremities can be open or closed. If the pipe is open at both extremities, this models a flute, as the mouthpiece and the end are open. The mouthpiece of a clarinet has a reed and when played by the musician, this end is closed. These systems are sketched on figure 3.5. As an approximation, we can consider that at an open end, the pressure is always equal to the atmospheric pressure, hence the pressure fluctuation vanishes. As a consequence, the impedance is zero. Conversely, at a closed end the velocity (or flow rate) vanishes, and the impedance is infinite. For both systems, the impedance $Z(L) = 0$. Using equation (3.25), we calculate the impedance at $z = 0$:

$$Z(0) = \mathrm{j} \tan kL. \tag{3.26}$$

Hence, for a flute, we have:

$$\text{Flute:} \quad Z(0) = 0 \quad \Rightarrow \quad k = \frac{n\pi}{L}, \quad f = \frac{nc}{2L}, \tag{3.27}$$

and for a clarinet:

$$\text{Clarinet:} \quad Z(0) = \infty \quad \Rightarrow \quad k = \frac{(2n+1)\pi}{L}, \quad f = \frac{(2n+1)c}{4L}. \tag{3.28}$$

The fundamental frequency is halved compared to the open-open case, and one would hear a change in the timber of the produced sound.

Cross section change

Using the impedance formalism, it is possible to model a change of cross-section, sketched on figure 3.6. As the flow rate is maintained and the is

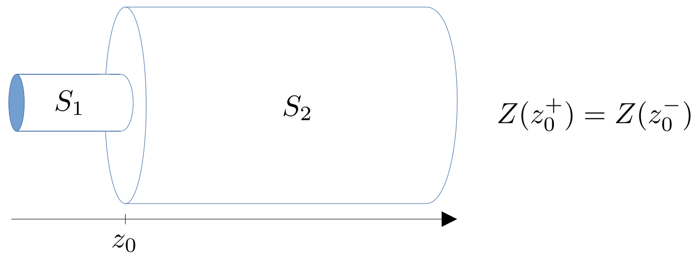


Figure 3.6: Change of cross-section.

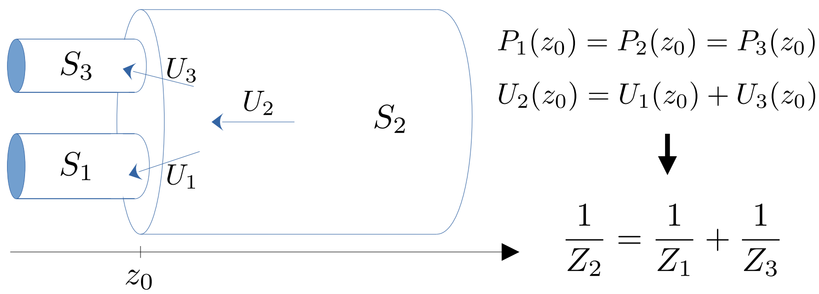


Figure 3.7: Derivation.

no pressure jump at this change of cross-section, the impedance is preserved at the passage of this discontinuity. When propagating the impedance in each duct to solve the problem, one has to remind that the characteristic impedances Z_{c1} and Z_{c2} are different, because of the change of cross-section.

Derivation

Another classical acoustical waveguide element is the derivation, of which an example is sketched on figure 3.7. At such a discontinuity, the volume velocity conservation and the uniqueness of the pressure can be invoked, to show that the impedance in one duct at the discontinuity is equal to the sum of the impedances in all other ducts. Note that this law necessitates to make a convention choice for the sign of the flow rate, hence the impedance as well. This kind of discontinuity allows us to characterize the influence of holes in wind instruments, or to model sound propagation in complex duct networks in the industry.

Radiation impedance of an open end

To use the impedance formalism in the presence of an open end, we considered zero pressure fluctuations (sec. 3.2.3), hence zero impedance. Although this gives a rough acceptable approximation of the resonant frequencies, the way the plane wave in the tube progressively adapts to a spherical wave in the outside 3D world induces a more complicated impedance behaviour. To take into account this phenomenon, a radiation impedance is introduced:

$$Z_{\mathrm{r}} = R_{\mathrm{r}} + \mathrm{j}X_{\mathrm{r}}. \tag{3.29}$$

The real part of Z characterizes here the acoustic power transmitted outside the duct, while the imaginary part characterizes the phase shift induced by the wave reflection that is not occurring exactly at the duct termination.

The low frequency limit of the radiation impedance was demonstrated by [3]:

$$Z_r = Z_c \left(\frac{(ka)^2}{4} + \mathrm{j}k\Delta L \right), \tag{3.30}$$

with $\Delta L \sim 0.613a$ for a non baffled pipe termination.

Horns

It was said without demonstration that the real part of the radiation impedance quantifies the amount of power radiated to the outside and its low frequency limit scales as the squared radius of the pipe. Thus it can improve the radiation efficiency to increase the radius. That is the aim of horns of figure 3.8

A simple and efficient way to model the acoustic propagation in horns is to perform a discretization of the duct considering chained ducts with progressive cross-section changes.

3.3 Lumped acoustic elements

If the characteristic size of an acoustic element in a waveguide network is small compared to the considered wavelength, an even simpler approximate model can be considered. In the case of the duct part of figure 3.4 is short compared to the wavelength, a Taylor expansion at first order can be made

Figure 3.8: Left: A large horn loudspeaker (Source: Wikimedia Commons). Middle: *Dog looking at and listening to a phonograph* by Francis Barraud, 1998 (Source: Wikimedia Commons). Right: Schematic view of a discretized horn.

for $kL \ll 1$:

$$\begin{bmatrix} P(z_1) \\ U(z_1) \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{j}\frac{\rho_0 \omega L}{S} \\ \mathrm{j}\frac{j\omega LS}{\rho_0 c^2} & 1 \end{bmatrix} \begin{bmatrix} P(z_2) \\ U(z_2) \end{bmatrix} \tag{3.31}$$

This approximation can be convenient to analyse the acoustic behaviour of small elements. As an example, consider the system schematized in figure 3.9. It consists of two pipes. The system is open at $z = z_1$ and closed at $z = z_3$.

Using successive transfer matrices approaches from z_3, where the flow-rate is zero, to the opening at z_1, we can express the ratio between the pressure at the opening, $P(z_1)$ and that inside the volume, $P(z_2) = P(z_3)$:

$$\frac{P(z_2)}{P(z_1)} = \frac{1}{1 - \frac{\omega^2}{\omega_c^2}}, \tag{3.32}$$

with,

$$\omega_c = c\sqrt{\frac{S_1}{L_1 V_2}}, \tag{3.33}$$

where V_2 is the volume of the large duct portion. We hence show that a resonance occurs inside this volume if the inlet pressure contains fluctuations at the frequency ω_c. This system is referred to as an *Helmholtz resonator*. It must be noted that in the limit of lumped elements, the resonance frequency

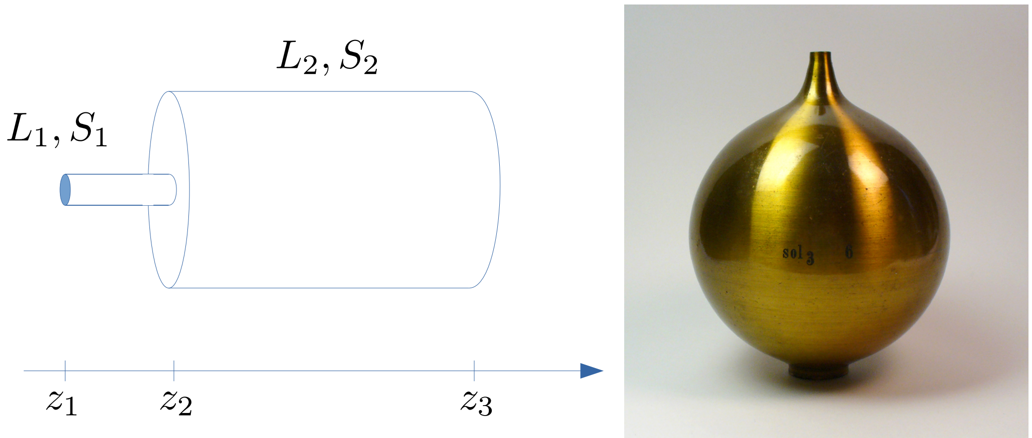


Figure 3.9: Left: Schematic view of an Helmholtz resonator modelled as a lumped element. Right: Photograph of an Helmholtz resonator. Source: Wikimedia commons.

do not depend on the shape of the large duct portion, but only on its volume. A photograph of an Helmholtz resonator is shown on the right of figure 3.9. When put near the ear, we can clearly ear a distinct note (here a G, called Sol in french). This note come from the resonance at a specific frequency, which the ambient noise is sufficient to create.

Chapter 4

Room acoustics

4.1 Modal theory of room acoustics

4.1.1 Modes in a rectangular room

We are looking for the solution of the wave equation in a rectangular room of dimensions $L_x \times L_y \times L_z$ with rigid walls, as shown in Figure 4.1. Looking for the solutions under the form $p(x, y, z, t) = P(x, y, z)e^{j\omega t}$, with $P(x, y, z) = P_x(x)P_y(y)P_z(z)$, we obtain the following Helmholtz equation:

$$\Delta P + k^2 P = 0 \Leftrightarrow P_x'' P_y P_z + P_x P_y'' P_z + P_x P_y P_z'' + k^2 P_x P_y P_z = 0, \tag{4.1}$$

with $k = \omega/c = 2\pi/\lambda$. This equation can be rewritten:

$$\frac{P_x''}{P_x} + \frac{P_y''}{P_y} + \frac{P_z''}{P_z} + k^2 = 0. \tag{4.2}$$

In this equation, only the term P_x''/P_x depends on x hence

$$\frac{P_x''}{P_x} = -C_x \Leftrightarrow P_x'' + C_x P_x = 0, \tag{4.3}$$

thus

$$P_x(x) = \begin{cases} A_x \cosh(\sqrt{-C_x}x) + B_x \sinh(-\sqrt{-C_x}x) & \text{if} \quad C_x < 0 \\ A_x \cos(\sqrt{C_x}x) + B_x \sin(\sqrt{C_x}x) & \text{if} \quad C_x > 0. \end{cases} \tag{4.4}$$

The same result can be obtained along y and z.

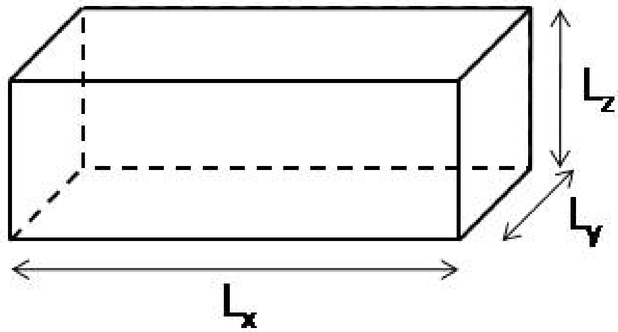


Figure 4.1: Rectangular room of dimensions $L_x \times L_y \times L_z$.

Le us now use the boundary conditions on the wall. The walls are rigid which means that the particle velocity must be zero on the walls. If we project the linearized Euler's equation on the normal $\underline{n}$ to the wall we obtain:

$$\rho_0 V_n = -\frac{\partial P}{\partial n} = 0. \tag{4.5}$$

Applying these boundary conditions on the walls located at $x = 0$ and $x = L$:

$$P_x'(0) = 0 \quad \text{and} \quad P_x'(L) = 0. \tag{4.6}$$

The condition $P_x'(0) = 0$ imposes $B_x = 0$. If $C_x < 0$, the condition $P_x'(L)$ yields $A_x \sinh(\sqrt{-C_x}L) = 0$ that is only possible for $A_x = 0$. To obtain a non-trivial solution, we must impose $C_x = k_x^2 > 0$. In this case, the condition $P_x'(L)$ yields $\sin(k_x L) = 0$ which imposes $k_x = k_m = m\pi/L_x$, $m \geq 0$. As a result:

$$P_x(x) = A_x \cos(k_m x) \quad \text{with} \quad k_m = \frac{m\pi}{L_x}, m \geq 0. \tag{4.7}$$

The same type of solution is obtained along y and z hence the modal shapes have the following form:

$$\Psi_{mnp}(x, y, z) = A \cos\left(\frac{m\pi}{L_x}x\right) \cos\left(\frac{n\pi}{L_y}y\right) \cos\left(\frac{p\pi}{L_z}z\right). \tag{4.8}$$

The acoustic wavenumber is obtained from Equation (4.2) :

$$k^2 = k_x^2 + k_y^2 + k_z^2 \Leftrightarrow k_{mnp}^2 = \left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2 + \left(\frac{p\pi}{L_z}\right)^2, \tag{4.9}$$

and the eigenfrequency associated with mode (m,n,p) is :

$$f_{mnp} = \frac{ck_{mnp}}{2\pi} = \frac{c}{2}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2 + \left(\frac{p}{L_z}\right)^2}. \tag{4.10}$$

As the functions Ψ_{mnp} form an orthogonal modal basis, the frequency-domain acoustic pressure can be written as a linear combination of the modal shapes:

$$P(x, y, z, f) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} Q_{mnp}(f)\Psi_{mnp}(x, y, z), \tag{4.11}$$

with $Q_{mnp}(f)$ the modal coordinates.

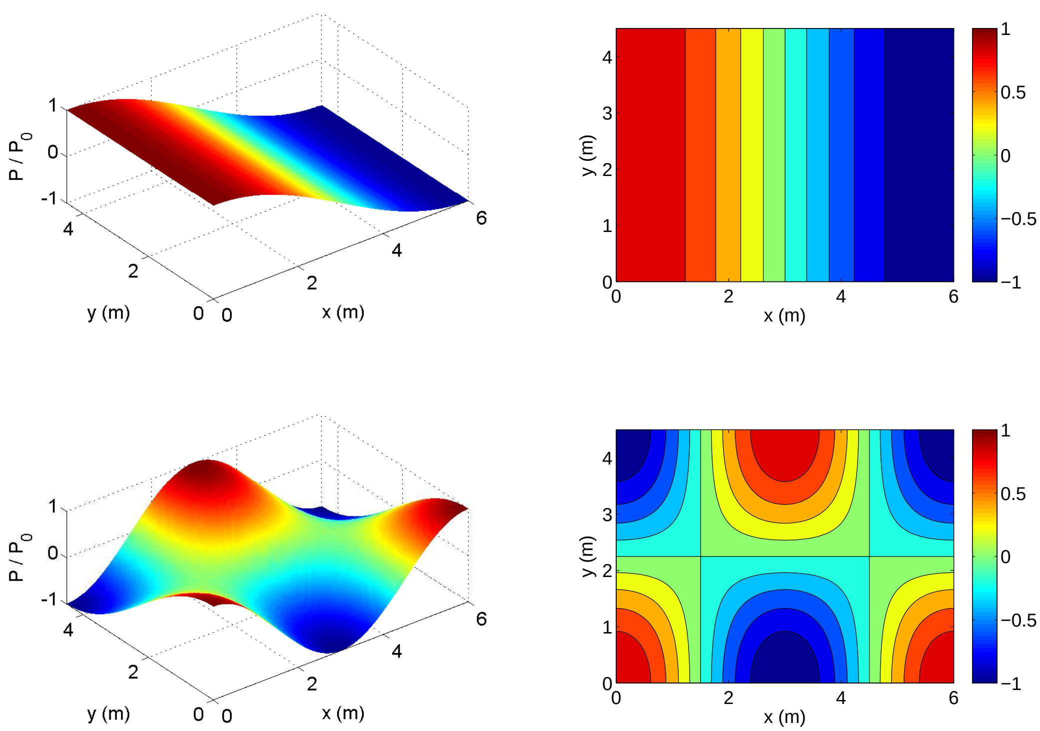


Figure 4.2: Modal shapes for the axial mode (1,0,0) (top plots) and the tangential model (2,1,0) (bottom plots).

Different types of modes can be obtained depending on the value of the triplet (m, n, p):

- Axial modes: modes for which only one of the three indices is nonzero, such as $(m, 0, 0)$. The lines of constant pressure are thus parallel to the walls, with maximum values on the walls, as can be seen in Figure 4.2(a) for the mode $(1, 0, 0)$;

- Tangential modes: modes for which only one of the three indices is zero, such as $(m, n, 0)$. The acoustic pressure is now maximum in the corners, as can be seen in Figure 4.2(b) for the mode $(2, 1, 0)$;

- Oblique modes: modes for which all three indices are nonzero.

4.1.2 Modal density

The number of modes of frequency below a given value f_{max} is difficult to evaluate exactly. However, at high frequency, the number of oblique modes is much greater than the number of axial and tangential modes, and a useful approximation can be obtained. Indeed, in the (k_x, k_y, k_z) wavenumber domain, each mode corresponds to an elementary volume $\frac{\pi}{L_x}\frac{\pi}{L_y}\frac{\pi}{L_z}$. The volume corresponding to $k < k_{max} = 2\pi f_{max}/c$ is $1/8^{\text{th}}$ of a sphere of radius k_{max}, as only positive values of k_x, k_y and k_z are acceptable. As a result, the number of modes such that $k < k_{max}$ or $f < f_{max}$ is given by:

$$N(f) \approx \frac{V_{\text{sphere}}/8}{V_{\text{mode}}} = \frac{\frac{1}{8}\frac{4}{3}\pi k_{max}^3}{\frac{\pi^3}{L_x L_y L_z}} = \frac{L_x L_y L_z}{6\pi^2}\left(\frac{2\pi f_{max}}{c}\right)^3 = \frac{4\pi}{3} V \left(\frac{f}{c}\right)^3, \tag{4.12}$$

with $V = L_x L_y L_z$ the volume of the room.

The room modal density is given by the derivative of $N(f)$ with respect to f:

$$D(f) = \frac{dN}{df} = \frac{1}{(\Delta f)_{\text{mode}}} \approx \frac{4\pi V f^2}{c^3}, \tag{4.13}$$

where $(\Delta f)_{\text{mode}}$ is the mean spacing between two consecutive modes in Hertz. The modal density $D(f)$ is expressed in modes/Hz.

4.2 Sabine statistical theory of room acoustics

4.2.1 Assumptions

The statistical theory or reverberation theory presented here has been developed by Wallace C. Sabine in the beginning of the 20^{th} century. It relies on the following assumptions:

1. the energy density is uniform (independent on space $\underline{x}$):

$$w(\underline{x}, t) = \frac{1}{2}\rho_0 v^2 + \frac{1}{2}\frac{p^2}{\rho_0 c^2} \approx w(t); \tag{4.14}$$

2. we assume that a diffuse acoustic field is present, which means that it is a superposition of many elementary plane waves whose directions are uniformly distributed over a solid angle of 4π;

3. the energy per unit surface absorbed by a wall is proportional to the incident energy per unit surface.

These assumptions are valid at high frequencies, when we can neglect the modal behavior of the room.

In the following, we will derive an equation on the energy density averaged over a period T greater than the acoustic period $1/f$:

$$\overline{w}(t) = \frac{1}{T}\int_{t_0}^{t_0+T} w(\tau) d\tau.$$

Following assumption 2 given above, the energy density w is written as:

$$w = \sum_i w_i = \sum_i \frac{p_i^2}{\rho_0 c^2}, \tag{4.15}$$

because $v_i = p_i/\rho_0 c$ for a plane wave thus $w_i = p_i^2/\rho_0 c^2$ following Equation (4.14). The time-averaged energy density is thus:

$$\overline{w} = \frac{1}{\rho_0 c^2}\sum_i \overline{p_i^2}. \tag{4.16}$$

If the plane waves are uncorrelated then:

$$\overline{p^2} = \overline{\left(\sum_i p_i\right)^2} = \sum_i \overline{p_i^2} + \sum_{i\neq j} \overline{p_i p_j} \approx \sum_i \overline{p_i^2}, \tag{4.17}$$

thus we obtain finally the following relationship between the time-averaged energy density and the mean-squared pressure:

$$\overline{w} \approx \frac{\overline{p^2}}{\rho_0 c^2}. \tag{4.18}$$

4.2.2 Equation of energy conservation in rooms

Let us consider a room where the energy is brought by a sound source of power W_s. The energy is dissipated due to the wall absorption, thus the equation of energy conservation can be written:

$$\frac{d}{dt}\int_V \overline{w} dV = W_s - W_a, \tag{4.19}$$

with W_a the acoustic power absorbed by the wall. As $\overline{w}$ is independent on $\underline{x}$, and W_a is proportional to the incident power W_i:

$$V\frac{d\overline{w}}{dt} = W_s - \alpha W_i, \tag{4.20}$$

where α is the absorption coefficient that is between 0 and 1.

In order to calculate W_i, let us consider an an elementary wall surface ΔS, as shown in Figure 4.3. For a plane wave of acoustic intensity I_i and direction $\underline{n}_i$, the acoustic power over the elementary surface ΔS is given by:

$$dW_i = \overline{dI}_i \underline{n}_i . \underline{n}_s \Delta S = \overline{dI}_i \cos\beta \Delta S, \tag{4.21}$$

with $\underline{n}_s$ the normal to the wall. For a plane wave, $\overline{dI}_i = c\overline{dw}_i$. As we assume a diffuse field, the energy is distributed equally over all directions (4π steradians), thus $\overline{dw}_i = \overline{w} d\Omega/(4\pi)$. As can be seen in Figure 4.3, $d\Omega = 2\pi \sin\beta d\beta$, with $0 \leq \beta \leq \pi/2$, thus the incident acoustic power is given by:

$$W_i = \frac{c\overline{w}\Delta S}{4\pi}\int_0^{\pi/2} 2\pi \sin\beta\cos\beta d\beta = \frac{c\overline{w}\Delta S}{4}. \tag{4.22}$$

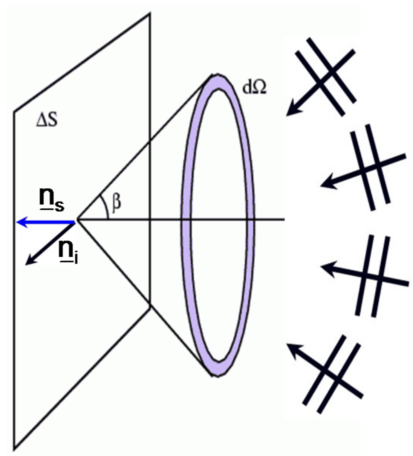


Figure 4.3: Elementary surface ΔS and solid angle $d\Omega$.

Injecting this expression in Equation (4.20), and considering that each wall surface S_i may have a different absorption coefficient α_i, we obtain the equation of evolution of the time-averaged energy density:

$$V\frac{d\overline{w}}{dt} + \frac{c\overline{w}}{4}\sum_i \alpha_i S_i = W_s. \tag{4.23}$$

4.2.3 Reverberation time

In the permanent regime, Equation (4.23) yields simply:

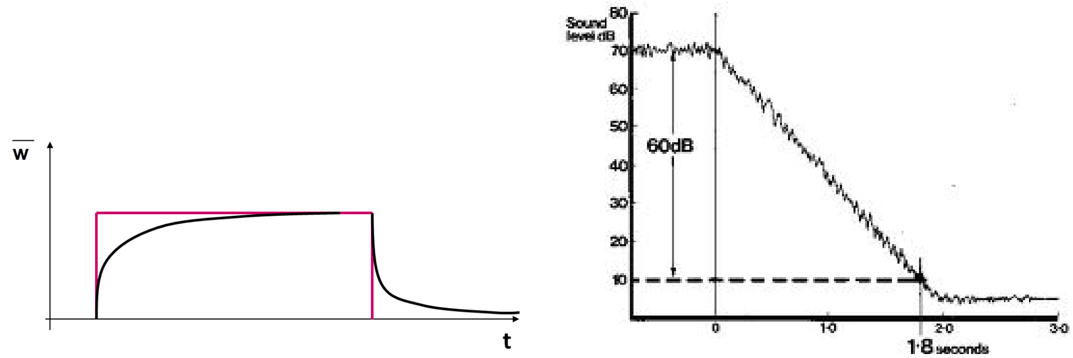
$$\overline{w} = \frac{4W_s}{cA_s}, \tag{4.24}$$

where A_s is the equivalent area of open windows given by:

$$A_s = \sum_i \alpha_i S_i = \overline{\alpha} S, \tag{4.25}$$

with $\overline{\alpha}$ the mean absorption coefficient in the room. For an open window, the sound is indeed completely absorbed (or transmitted outside the room) by the surface element S_i ($\alpha_i = 1$) such that $(A_s)_i = S_i$. If the source is stopped at $t = 0$ from the permanent regime, the energy density follows a decreasing exponential law:

$$\overline{w} = \frac{4W_s}{cA_s}\exp\left(-\frac{t}{\tau}\right), \tag{4.26}$$



(a) (b)

Figure 4.4: (a) Energy density increase or decrease following Sabine statistical theory, (b) illustration of the reverberation time TR_{60}.

with $\tau = \frac{4V}{cA_s}$. See Figure 4.4(a).

The reverberation time is defined as the time required for the energy density, or equivalently the mean-squared pressure as $\overline{p^2} \approx \rho_0 c^2 \overline{w}$, to decrease by a factor of 60 dB: $\overline{w}(t = TR_{60}) = 10^{-6}\overline{w}(t = 0)$. See Figure 4.4(b). This yields Sabine's equation:

$$TR_{60} = \frac{24V\ln(10)}{c\sum_i \alpha_i S_i} = \frac{24V\ln(10)}{c\overline{\alpha}S} \approx 0,16\frac{V(\mathrm{m}^3)}{A_s(\mathrm{m}^2)},$$

considering $c \approx 343\,\mathrm{m/s}$ at 20°C. When $\overline{\alpha} \to 0$, $TR_{60} \to \infty$ (no absorption in the room). When $\overline{\alpha} \to 1$, $TR_{60} \to 0,16V/S$, whereas we could expect a value of 0 (total absorption in the room). In this case Sabine's theory is not valid, as the assumption of diffuse field requires a relatively small absorption coefficient.

4.2.4 Validity of statistical theory: Schroeder frequency

To determine the validity of the statistical theory, where the modal behavior of the room is neglected, we need to determine the frequency range over which there is a large modal overlap, which means that many modes will contribute at a given frequency. We assume here that the mode damping is well approximated by Sabine theory, which yields a -3 dB bandwith $(\Delta\omega)_{res} = 1/\tau$, that is:

$$(\Delta f)_{res} = \frac{(\Delta\omega)_{res}}{2\pi} = \frac{cA_s}{8\pi V}.$$

We consider that a high modal overlap is reached when at least three modes are found within the bandwidth $(\Delta f)_{res}$:

$$3(\Delta f)_{mode} \leq (\Delta f)_{res}, \tag{4.27}$$

where $(\Delta f)_{mode}$ is the mean spacing between two consecutive modes given by Equation (4.13):

$$(\Delta f)_{\mathrm{mode}} = \frac{1}{D(f)} \approx \frac{c^3}{4\pi V f^2}. \tag{4.28}$$

The condition (4.27) corresponds to assumption 1 of the statistical theory that states that the energy density must be uniform. We deduce from the above question that this assumption is met for:

$$f \geq f_{Schroeder} = c\sqrt{\frac{6}{\sum_i \alpha_i S_i}} \approx 2000\sqrt{\frac{TR_{60}}{V}}, \tag{4.29}$$

where $f_{Schroeder}$ is the Schroeder frequency. Generally speaking, Sabine statistical theory is valid only at high frequencies for small to medium rooms, and over the whole frequency range for large reverberant rooms. For instance, for a concert hall or reverberation time 2 s and volume 10000 m^3, $f_{Schroeder} \approx 28$ Hz so the assumption of uniformity of energy density is valid for all frequencies above 28 Hz.

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