Stability Of The $\mathbf{P}^1_{nc} - (P^0 + P^1)$ Element

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Abstract We solve the Stokes problem numerically. We analyse the $\mathbf{P}^1_{nc} - (P^0 + P^1)$ mixed finite element method which exhibits interesting numerical features. However, only an incomplete proof of the inf-sup condition is available. We prove here this condition and the stability of the method.

1 Stokes Problem

A popular low-order method for solving the Stokes problem is the nonconforming Crouzeix-Raviart mixed finite element method [1, Example 4] (cf. [2, 3] for higher orders). The discrete velocity is piecewise affine, continuous in the barycentre of the interfaces between elements, but not globally continuous, $\mathbf{P}^1_{nc}$; and the discrete pressure is piecewise constant, P^0. This method introduces some consistency error on the discrete velocity, which may lead to an unphysical solution when solving the Navier-Stokes equations, especially when the source field is a strong gradient [4, 5]. To recover consistency, the discrete pressure space can be generated by P^1-continuous plus P^0 basis functions, cf. the TrioCFD code [6, 7]. This leads to the $\mathbf{P}^1_{nc} - (P^0 + P^1)$ mixed finite element method, which exhibits interesting numerical features. However, only an incomplete proof of the stability is available in [8, 9]. Below, we prove the inf-sup condition. Consider the Stokes problem set in an open,

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bounded, connected subset Ω of $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz boundary $\partial\Omega$:

$$\text{Find } (\mathbf{u}, p) \text{ such that } -\nu\,\boldsymbol{\Delta}\,\mathbf{u} + \nabla p = \mathbf{f} \text{ and } \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \tag{1}$$

with homogeneous boundary conditions $\mathbf{u} = 0$ on $\partial\Omega$ and a normalisation condition: $\int_\Omega p = 0$. The case of inhomogeneous Dirichlet boundary conditions, that is $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ with $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ such that $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}_{|\partial\Omega} = 0$ can be studied similarly. Given D an open subset of $\mathbb{R}^d$, let us set $\mathbf{L}^2(D) = (L^2(D))^d$, and $L^2_{zmv}(D) = \{q \in L^2(D) \,|\, \int_D q = 0\}$. We denote by $(\cdot,\cdot)_D$ the $L^2(D)$ or $\mathbf{L}^2(D)$ inner product, and $\|\cdot\|_D$ the associated norm. Due to Poincaré-Steklov inequality [10, Lemma 3.24], the semi-norm in $H^1_0(\Omega)$ is equivalent to the natural norm, so that $(v, w)_{H^1_0(\Omega)} = (\nabla v, \nabla w)_\Omega$ and $\|v\|_{H^1_0(\Omega)} = \|\nabla v\|_\Omega$. Let us set $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1_0(\Omega) \,|\, \operatorname{div}\mathbf{v} = 0\}$ which is a closed subset of $\mathbf{H}^1_0(\Omega)$. We denote by $\mathbf{V}^\perp$ the orthogonal of $\mathbf{V}$ in $\mathbf{H}^1_0(\Omega)$.

Proposition 1 *[13, Theorem 3.1 (e)]. The operator* $\operatorname{div} : \mathbf{H}^1_0(\Omega) \to L^2(\Omega)$ *is an isomorphism of* $\mathbf{V}^\perp$ *onto* $L^2_{zmv}(\Omega)$*. We call* $C_{\text{div}} > 1$ *the constant such that:* $\forall p \in L^2_{zmv}(\Omega)$, $\exists! \mathbf{v}_p \in \mathbf{V}^\perp \,|\, \operatorname{div}\mathbf{v}_p = p$ *and* $\|\mathbf{v}_p\|_{\mathbf{H}^1_0(\Omega)} \le C_{\text{div}}\|p\|_\Omega$.

Let $\mathbf{f} \in (\mathbf{H}^1_0(\Omega))'$. The variational formulation of Problem (1) reads:
Find $(\mathbf{u}, p) \in \mathbf{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$ such that $\forall(\mathbf{v}, q) \in \mathbf{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$:

$$\nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_0(\Omega)} - (p, \operatorname{div}\mathbf{v})_\Omega - (q, \operatorname{div}\mathbf{u})_\Omega = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}^1_0(\Omega)}. \tag{2}$$

One can prove that Problem (2) is well-posed with the T-coercivity theory [14, 15, 16], using Poincaré-Steklov inequality [10, Lemma 3.24] and Proposition 1.

2 Discretization And Stability Of The $\mathrm{P}^1_{nc} - (P^0 + P^1)$ Scheme

Consider $(\mathcal{T}_h)_h$ a family of simplicial triangulations of Ω. For a given $\mathcal{T}_h := \{K\}$, we call $\mathcal{F}_h := \{F\}$ its set of $(d-1)$-simplices, called facets (faces if $d = 3$, edges if $d = 2$), and $\mathcal{V}_h := \{S\}$ its set of vertices. In the bounds, we will use $\lesssim$ and $\gtrsim$ for (hidden) constants independent of $\mathcal{T}_h$.

- For $K \in \mathcal{T}_h$, we call h_K and ρ_K the diameters of K and its inscribed sphere respectively, and we let: $\sigma_K = \frac{h_K}{\rho_K}$. When the $(\mathcal{T}_h)_h$ is a shape-regular triangulation family (see e.g. [10, def. 11.2]), there exists a constant $\sigma > 1$, called the shape regularity parameter, such that for all h, for all $K \in \mathcal{T}_h$, $\sigma_K \le \sigma$.
- Let $\mathcal{F}_h = \mathcal{F}^i_h \cup \mathcal{F}^b_h$, where $\mathcal{F}^b_h = \{F \in \mathcal{F}_h \,|\, F \subset \partial\Omega\}$, $\mathcal{F}^i_h = \mathcal{F}_h \setminus \mathcal{F}^b_h$.
For $K \in \mathcal{T}_h$, we set $\mathcal{F}^i_K = \{F \in \mathcal{F}^i_h \,|\, F \subset \partial K\}$.

For $F \in \mathcal{F}_h$, M_F denotes the barycentre of F, and $\mathbf{n}_F$ its unit normal (outward oriented if $F \subset \partial\Omega$). For $K \in \mathcal{T}_h$ and $F \subset \partial K$, we denote by $\mathbf{n}_{F,K}$ the normal vector of F outgoing from K. Let $\hat{K}$ be the reference simplex. For $K \in \mathcal{T}_h$, we denote by $T_K : \hat{K} \to K$ the geometric mapping such that $\forall \hat{\mathbf{x}} \in \hat{K}$, $\mathbf{x}_{|K} = T_K(\hat{\mathbf{x}}) = \mathbb{B}_K\hat{\mathbf{x}} + \mathbf{b}_K$, where $\mathbb{B}_K \in \mathbb{R}^{d\times d}$ and $\mathbf{b}_K \in \mathbb{R}^d$. Let us set $J_K = \det(\mathbb{B}_K)$. Recall that [10, Lemma 11.1]: $|J_K| = |K|/|\hat{K}|$, $\|\mathbb{B}_K\| \le h_K/\rho_{\hat{K}}$, $\|\mathbb{B}_K{}^{-1}\| \le h_{\hat{K}}/\rho_K$. For $v \in L^2(\Omega)$, we set: $v_K = v_{|K}$, $\hat{v}_K = v \circ T_K$. By changing the variable, we get: $\|v\|_K^2 = |J_K|\,\|\hat{v}_K\|_{\hat{K}}^2$. Let $v \in H^1(K)$, then $\nabla v_{|K} = (\mathbb{B}_K{}^{-1})^T \nabla_{\hat{\mathbf{x}}}\hat{v}$, and $\|\nabla v\|_K^2 \le \|\mathbb{B}_K{}^{-1}\|^2\,|J_K|\,\|\nabla_{\hat{\mathbf{x}}}\hat{v}\|_{\hat{K}}^2$, $\|\nabla_{\hat{\mathbf{x}}}\hat{v}\|_{\hat{K}}^2 \le \|\mathbb{B}_K\|^2\,|J_K|^{-1}\,\|\nabla v\|_K^2$. Let $K \in \mathcal{T}_h$ and $F \subset \partial K$, we denote by $T_{F,K}$ the geometric mapping such that $\forall \mathbf{x} \in F$, $T_{F,K}(T_K^{-1}(\mathbf{x})) = \mathbf{x}$. We let $\hat{F}_{F,K} = T_K^{-1}(F)$. According to [1, Eq. (3.17)]:

$$\forall K \in \mathcal{T}_h,\ \forall F \subset \partial K, \quad |F|\,|K|^{-1} \lesssim \rho_K^{-1}. \tag{3}$$

We set $\mathcal{P}_h H^1 = \{v \in L^2(\Omega \,|\, \forall K \in \mathcal{T}_h,\ v_{|K} \in H^1(K)\}$, endowed with: $(v,w)_h := \sum_{K\in\mathcal{T}_h} (\nabla v, \nabla w)_K$, $\|v\|_h^2 = \sum_{K\in\mathcal{T}_h} \|\nabla v\|_K^2$. Let $F \in \mathcal{F}_h^i$ such that $F = \partial K_L \cap \partial K_R$ and $\mathbf{n}_F = \mathbf{n}_{F,K_L}$. The jump of $v \in \mathcal{P}_h H^1$ across F is defined by: $[v]_F := v_{|K_L} - v_{|K_R}$. For $F \in \mathcal{F}_h^b$, we set: $[v]_F := v_{|F}$.

For a subset D of $\mathbb{R}^d$, for $k \in \mathbb{N}$, we call $P^k(D)$ the set of degree k polynomials on D, $\mathbf{P}^k(D) = (P^k(D))^d$, and $\mathcal{P}^k_{disc}(\mathcal{T}_h) = \left\{q \in L^2(\Omega) \,|\, \forall K \in \mathcal{T}_h,\ q_{|K} \in P^k(K)\right\}$. For $k > 0$ we define $\mathcal{P}^k(\mathcal{T}_h) = C^0(\overline{\Omega}) \cap \mathcal{P}^k_{disc}(\mathcal{T}_h)$. We call $\pi_0 : L^2(\Omega) \mapsto \mathcal{P}^0_{disc}(\mathcal{T}_h)$ the L^2-orthogonal projection onto $\mathcal{P}^0_{disc}(\mathcal{T}_h)$, such that $\forall q \in L^2(\Omega)$, $\pi_0(q) = \sum_{K\in\mathcal{T}_h} \underline{q}_K \mathbb{1}_K$ where $\forall K \in \mathcal{T}_h$, $\underline{q}_K = \int_K q/|K|$.

For $K \in \mathcal{T}_h$ and a vertex S of K, $\lambda_{S,K}$ denotes the barycentric coordinate related to S. We can endow $\mathcal{P}^1(\mathcal{T}_h)$ with the basis $(\phi_S)_{S\in\mathcal{V}_h}$ such that: $\forall K \in \mathcal{T}_h$, $\phi_{S|K} = \lambda_{S,K}$ if $S \in K$, zero otherwise, so that $\mathcal{P}^1(\mathcal{T}_h) = \text{vect}\left((\phi_S)_{S\in\mathcal{V}_h}\right)$.

Using [12, Thm. 5] and Eq. (3), one can prove that:

$$\forall w \in P^k(K),\ \forall F \subset \partial K, \quad \|w\|_F \lesssim (|F|\,|K|^{-1})^{\frac{1}{2}}\,\|w\|_K \lesssim (\rho_K)^{-\frac{1}{2}}\,\|w\|_K. \tag{4}$$

Using Eq. (4) with $w = v - \underline{v}_K$ and Poincaré-Steklov inequality [10, Lemma 12.11] in K, it holds for (with below, $\underline{v}_F = \int_F v/|F|$):

$$\forall v \in P^k(K),\ \forall F \subset \partial K, \quad \|v - \underline{v}_F\|_F \lesssim \|v - \underline{v}_K\|_F \lesssim (\sigma_K\, h_K)^{1/2}\,\|\nabla v\|_K. \tag{5}$$

Using the proof of [17, Lemma 2] and [12, Thm. 5], we can prove that:

$$\forall v \in P^1(K), \quad \|\nabla v\|_K \lesssim (\rho_K)^{-1}\|v - \underline{v}_K\|_K. \tag{6}$$

Let $\mathbf{X}_{0,h}$ be the space of the discrete velocity of the nonconforming Crouzeix-Raviart mixed finite element method, defined by:

$$\begin{aligned} &\mathbf{X}_h = (X_h)^d, \qquad X_h = \{v_h \in \mathcal{P}^1_{disc}(\mathcal{T}_h) \,|\, \forall F \in \mathcal{F}_h^i,\ \textstyle\int_F [v_h] = 0\}\,; \\ &\mathbf{X}_{0,h} = (X_{0,h})^d,\ X_{0,h} = \{v_h \in X_h \,|\, \forall F \in \mathcal{F}_h^b,\ \textstyle\int_F v_h = 0\}. \end{aligned} \tag{7}$$

The condition on the jumps of v_h on the inner facets is often called the patch-test condition. Due to this condition, one can prove that :

Proposition 2 *[1, Lemma 2]. The broken norm $v_h \longrightarrow \|v_h\|_h$ is a norm over $X_{0,h}$.*

For $K \in \mathcal{T}_h$ and $F \subset \partial K$, $\lambda_{F,K}$ denotes the barycentric coordinate related to the vertex $S_{F,K}$ opposite to facet F in K. We can endow X_h with the basis $(\psi_F)_{F\in\mathcal{F}_h}$ such that for all $K \in \mathcal{T}_h$, $\psi_{F|K} = 1 - d\lambda_{F,K}$ if $F \subset \partial K$ and zero otherwise. We then have $\psi_{F|F} = 1$, so that $[\psi_F]_F = 0$ if $F \in \mathcal{F}_h^i$, and for all $F' \neq F$, $\int_{F'} \psi_F = 0$. We have: $X_h = \mathrm{vect}\left((\psi_F)_{F\in\mathcal{F}_h}\right)$ and $X_{0,h} = \mathrm{vect}((\psi_F)_{F\in\mathcal{F}_h^i})$.
Let $\Pi_h : \mathbf{H}^1(\Omega) \mapsto \mathbf{X}_h$ be the Crouzeix-Raviart interpolation operator, such that: $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\Pi_h(\mathbf{v}) = \sum_{F\in\mathcal{F}_h} \underline{\mathbf{v}}_F\, \psi_F$, $\underline{\mathbf{v}}_F = \int_F \mathbf{v}/|F|$.

Proposition 3 *[17, Lemma 2]. The interpolation operator Π_h is such that: $\forall \mathbf{v}_h \in \mathbf{H}^1_0(\Omega)$, $\|\Pi_h(\mathbf{v}_h)\|_h \leq \|\mathbf{v}_h\|_{\mathbf{H}^1_0(\Omega)}$.*

However, due to the consistency error on the discrete velocity, it is well-known [18, 19] that choosing the space of the discrete pressures equal to $\mathcal{P}^0_{disc}(\mathcal{T}_h)\cap L^2_{zmv}(\Omega)$ yields inaccurate results when solving Navier-Stokes equations. Starting from [20], a new element is proposed in [8]. The space of the discrete pressures is then: $Q_h = Q_{0,h}+Q_{1,h}$, where $Q_{0,h} := \mathcal{P}^0_{disc}(\mathcal{T}_h)\cap L^2_{zmv}$ and for $k > 0$, $Q_{k,h} := \mathcal{P}^k(\mathcal{T}_h)\cap L^2_{zmv}(\Omega)$. One can build an L^2-orthogonal sum letting $\tilde{Q}_{1,h} = \{q_h \in Q_h \,|\, \forall K \in \mathcal{T}_h,\ \int_K q_h = 0\}$.

Proposition 4 *We have: $Q_h = Q_{0,h} \overset{\perp}{\oplus} \tilde{Q}_{1,h}$. It holds for all $q_h \in Q_h$: $\|q_h\|^2_\Omega = \|\pi_0(q_h)\|^2_\Omega + \|q_h - \pi_0(q_h)\|^2_\Omega$.*

To prove the discrete inf-sup condition of $\mathbf{X}_{0,h} \times Q_h$, we need some Lemmas.

Lemma 1 *Let $k > 0$, $(\mathbf{v}, q) \in \mathbf{X}_{0,h} \times Q_{k,h}$. It holds:*

$$|\textstyle\sum_{K\in\mathcal{T}_h}\sum_{F\subset\partial K} (\mathbf{v}\cdot\mathbf{n}_{F,K}\,,\, q)_F| \lesssim \sigma\, \|\mathbf{v}\|_h\, \|q - \pi_0(q)\|_\Omega. \tag{8}$$

Proof. We start with the proof of [20, Lemma 3.1]. Using the patch-test condition, one obtains:

$$\textstyle\sum_{K\in\mathcal{T}_h}\sum_{F\subset\partial K} (\mathbf{v}\cdot\mathbf{n}_{F,K}, q)_F = \sum_{K\in\mathcal{T}_h}\sum_{F\subset\partial K} \left((\mathbf{v}_K - \underline{\mathbf{v}}_F)\cdot\mathbf{n}_{F,K}, q_K - \underline{q}_K\right)_F.$$

We have, using Eq. (4) with $w = q_K - \underline{q}_K$ and Eq. (5) for the components of $\mathbf{v}$:

$$\left|\left((\mathbf{v}_K - \underline{\mathbf{v}}_F)\cdot\mathbf{n}_{F,K}, q_K - \underline{q}_K\right)_F\right| \lesssim \sigma_K\, \|\underline{\nabla}\mathbf{v}_K\|_K\, \|q_K - \underline{q}_K\|_K.$$

Summing and using the discrete Cauchy-Schwarz inequality, we obtain (8). □

Introducing the element-wise divergence operator div_h, we prove next that one can build an explicit right inverse of div_h from Q_h to $\mathbf{X}_{0,h}$.

Lemma 2 *For all $q_h \in Q_h$, there exists $\mathbf{v}_{0,h} \in \mathbf{X}_{0,h}$ such that:*

$$-(\mathrm{div}_h\, \mathbf{v}_{0,h}, \pi_0(q_h))_\Omega = \nu^{-1}\, \|\pi_0(q_h)\|^2_\Omega, \tag{9}$$

$$\|\mathbf{v}_{0,h}\|_h \leq C_{\mathrm{div}}\, \nu^{-1}\, \|\pi_0(q_h)\|_\Omega. \tag{10}$$

Proof. Let $q_h \in Q_h$. From Proposition 1, $\exists \mathbf{v}_{\pi_0(q_h)} \in \mathbf{H}^1_0(\Omega) \mid \operatorname{div} \mathbf{v}_{\pi_0(q_h)} = \pi_0(q_h)$ and $\|\mathbf{v}_{\pi_0(q_h)}\|_{\mathbf{H}^1_0(\Omega)} \leq C_{\mathrm{div}} \|\pi_0(q_h)\|_\Omega$. Let $\mathbf{v}_{0,h} = -\nu^{-1}\, \Pi_h(\mathbf{v}_{\pi_0(q_h)})$. Integrating by part twice, we get: $\forall K \in \mathcal{T}_h$, $\int_K \operatorname{div} \mathbf{v}_{0,h} = -\nu^{-1} \int_K \operatorname{div} \mathbf{v}_{\pi_0(q_h)}$, hence (9). Using Proposition 3, we get (10). □

We can also build an explicit right inverse of the discrete divergence operator from $Q_{1,h}$ to $\mathbf{X}_{T,h} = \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall F \in \mathcal{F}^i_h, \quad \mathbf{v}_h \cdot \mathbf{n}_F = 0\}$, if [20, Hyp. 4.1] holds:

Hypothesis 1. *We suppose that $\mathcal{T}_h$ is such that each element $K \in \mathcal{T}_h$ has at most $d-1$ facets that lie on $\partial\Omega$.*

Lemma 3 *Assuming Hypothesis 1, $\forall q_{1,h} \in Q_{1,h}$, $\exists \mathbf{v}_{1,h} \in \mathbf{X}_{T,h}$ such that:*

$$(\nabla q_{1,h}\,,\, \mathbf{v}_{1,h})_\Omega \gtrsim \nu^{-1}\, \sigma^{-2(d-2)}\, \|q_{1,h} - \pi_0(q_{1,h})\|^2_\Omega, \tag{11}$$

$$(\operatorname{div} \mathbf{v}_{1,h}, q_{0,h})_K = 0 \quad \forall q_{0,h} \in Q_{0,h}, \tag{12}$$

$$\|\mathbf{v}_{1,h}\|_h \lesssim \nu^{-1}\, \sigma^2\, \|q_{1,h} - \pi_0(q_{1,h})\|_\Omega. \tag{13}$$

Proof. Let us consider $q_{1,h} \in Q_{1,h}$. For all $K \in \mathcal{T}_h$, we let $q_K := q_{1,h|K}$ and $\hat{q}_K = q_K \circ T_K$. Since $q_{1,h} \in H^1(\Omega)$ we have (see e.g. [21, Prop. 2.2.10]): $\forall F \in \mathcal{F}^i_h \mid F = K \cap K'$, $\nabla q_K \times \mathbf{n}_F = \nabla q_{K'} \times \mathbf{n}_F$. We construct $\mathbf{v}_{1,h} \in \mathbf{X}_{T,h} \mid \forall F \in \mathcal{F}^i_h$, $\mathbf{v}_{1,h}(M_F) = \nu^{-1}\, h_F^2\, \mathbf{n}_F \times (\nabla q_{1,h} \times \mathbf{n}_F)$, where h_F is the diameter of F. We have:

$$\forall K \in \mathcal{T}_h \quad (\nabla q_{1,h}, \mathbf{v}_{1,h})_K = \nu^{-1}\,(d+1)^{-1}\,|K| \textstyle\sum_{F \in \mathcal{F}^i_K} h_F^2\, |\nabla q_K \times \mathbf{n}_F|^2. \tag{14}$$

Let $\mathbf{n}_{\hat{F}_{F,K}}$ be the unit normal vector of the facet $\hat{F}_{F,K}$ outgoing from $\hat{K}$. We have: $\mathbf{n}_{F,K} = |K|\,(\mathbb{B}_K{}^{-1})^T\, |\hat{F}_{F,K}|\, |F|^{-1}\, \mathbf{n}_{\hat{F}_{F,K}}$.
Consider $d = 2$. Changing the variable, assuming Hypothesis 1, we get:

$$\textstyle\sum_{F \in \mathcal{F}^i_K} h_F^2\, |\nabla q_K \times \mathbf{n}_F|^2 = \sum_{F \in \mathcal{F}^i_K} |\hat{F}_{F,K}|^2 |\nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|^2 \gtrsim |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2. \tag{15}$$

Consider $d = 3$. Changing the variable, one now finds that $|\nabla q_K \times \mathbf{n}_F| = |\hat{F}_{F,K}|\, |F|^{-1}\, |\mathbb{B}_K\, \nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|$. Hence: $|\nabla q_K \times \mathbf{n}_F| \gtrsim |F|^{-1}\, \|\mathbb{B}_K{}^{-1}\|^{-1}\, |\nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|$. Notice that $h_F\, |F|^{-1}\, \|\mathbb{B}_K{}^{-1}\|^{-1} \gtrsim \sigma_K^{-1}$. Assuming Hypothesis 1, we then have:

$$\textstyle\sum_{F \in \mathcal{F}^i_K} h_F^2\, |\nabla q_K \times \mathbf{n}_F|^2 \gtrsim \sigma_K^{-2} |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2. \tag{16}$$

Using Poincaré-Steklov inequality [10, Lemma 12.11] in $\hat{K}$, we have: $|K|\, |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2 \gtrsim \|\hat{q}_K - \underline{q}_K\|^2_{\hat{K}}$ with $\underline{q}_K = \int_K q_K / |K|$. Using (15) and (16) in (14), it holds:

$$(\nabla q_{1,h}, \mathbf{v}_{1,h})_K \gtrsim \nu^{-1}\, \sigma_K^{-2\,(d-2)} |K|\, |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2 \gtrsim \nu^{-1}\, \sigma_K^{-2\,(d-2)} \|q_K - \underline{q}_K\|^2_K. \tag{17}$$

We obtain (11) by summation. Remark that $\nabla \psi_{F|K} = |K|^{-1}\, |F|\, \mathbf{n}_{F,K}$. Let us prove (12). Recall the orthogonality $\nabla \psi_{F|K}(M_F) \perp \mathbf{v}_{1,h}(M_F)$, so that $(\operatorname{div} \mathbf{v}_{1,h}, q_{0,h})_K = |K|\, q_{0,h}(K) \sum_{F \in \mathcal{F}^i_K} \nabla \psi_{F|K}(M_F) \cdot \mathbf{v}_{1,h}(M_F) = 0$. Let us prove (13). For all $K \in \mathcal{T}_h$, $\underline{\underline{\nabla}} \mathbf{v}_{1,h|K} = \sum_{F \in \mathcal{F}^i_K} \mathbf{v}_{1,h}(M_F) \otimes \nabla \psi_{F|K}$. Hence: $\|\underline{\underline{\nabla}} \mathbf{v}_{1,h}\|^2_K \lesssim \nu^{-2} \sum_{F \in \mathcal{F}^i_K} |F|^2\, |K|^{-1}\, h_F^4\, |\nabla q_K \times$

$\mathbf{n}_F|^2$. Using (3), and assuming Hypothesis 1, we get: $\|\underline{\nabla}\mathbf{v}_{1,h}\|_K^2 \lesssim \nu^{-2}\,\sigma_K^2\,h_K^2\,|K|\,\sum_{F\in\mathcal{F}_K^i}|\nabla q_K\times \mathbf{n}_F|^2 \lesssim \nu^{-2}\,\sigma_K^2\,h_K^2\,\|\nabla q_K\|_K^2$. From Eq. (6): $\|\underline{\nabla}\mathbf{v}_{1,h}\|_K \lesssim \nu^{-1}\,\sigma_K^2\,\|q_K - \underline{q}_K\|_K$. □

Let $b_h : \mathbf{X}_h \times Q_h \to \mathbb{R}$ be defined by $b_h(\mathbf{v}_h, q_h) = -\left(\mathrm{div}_h\,\mathbf{v}_h, q_{0,h}\right)_\Omega + (\mathbf{v}_h, \nabla q_{1,h})_\Omega$ with $q_h = q_{0,h} + q_{1,h}$ such that $(q_{0,h}, q_{1,h}) \in Q_{0,h}\times Q_{1,h}$.

Theorem 1 (Stability) *The following continuity property holds:*

$$\forall(\mathbf{v}_h, q_h) \in \mathbf{X}_{0,h}\times Q_h, \quad |b_h(\mathbf{v}_h, q_h)| \lesssim \sigma\,\|\mathbf{v}_h\|_h\,\|q_h\|_\Omega. \tag{18}$$

Assuming Hypothesis 1, the following inf-sup condition holds:

$$\forall q_h \in Q_h,\ \exists \mathbf{v}_h \in \mathbf{X}_{0,h}, \quad b_h(\mathbf{v}_h, q_h) \gtrsim \tfrac{1}{\sqrt{2}}\,C_{\mathrm{div}}^{-2}\,\sigma^{-2\,d}\,\|\mathbf{v}_h\|_h\,\|q_h\|_\Omega. \tag{19}$$

Proof. Let us prove (18). Let $(\mathbf{v}_h, q_h) \in \mathbf{X}_{0,h}\times Q_h$. Integrating by parts, we have:

$$b_h(\mathbf{v}_h, q_h) = -(\mathrm{div}_h\,\mathbf{v}_h, \pi_0(q_h))_\Omega + \textstyle\sum_{K\in\mathcal{T}_h}\sum_{F\subset\partial K}(\mathbf{v}_h\cdot\mathbf{n}_{F,K}, q_{1,h})_F. \tag{20}$$

Using Cauchy-Schwarz, we have: $|(\mathrm{div}_h\,\mathbf{v}_h, \pi_0(q_h))_\Omega| \le \sqrt{d}\|\mathbf{v}_h\|_h\,\|\pi_0(q_h)\|_\Omega$. Using (8) in (20), we obtain (18) from Proposition 4. Let us prove (19), starting from the proof of [20, Lemma 4.2]. Let $q_h \in Q_h\backslash\{0\}$, where $q_h = q_{0,h} + q_{1,h}$ is such that $(q_{0,h}, q_{1,h}) \in Q_{0,h}\times Q_{1,h}$. Let $\mathbf{v}_h := \mathbf{v}_{0,h}$ be like in Lemma 2. Using (20) with $\mathbf{v}_h = \mathbf{v}_{0,h}$, (8) and (9), letting $\tilde{q}_{1,h} = q_h - \pi_0(q_h)$, we have:

$$b_h(\mathbf{v}_{0,h}, q_h) \gtrsim \nu^{-1}\,\|\pi_0(q_h)\|_\Omega^2 - \sigma\,\|\mathbf{v}_{0,h}\|_h\,\|\tilde{q}_{1,h}\|_\Omega. \tag{21}$$

Using (10) and Young inequality, we have for all $\varepsilon > 0$:

$$-\|\mathbf{v}_{0,h}\|_h\,\|\tilde{q}_{1,h}\|_\Omega \ge -\tfrac{1}{2}C_{\mathrm{div}}\,\nu^{-1}(\varepsilon\,\|\pi_0(q_h)\|_\Omega^2 + \varepsilon^{-1}\,\|\tilde{q}_{1,h}\|_\Omega^2). \tag{22}$$

We now insert the bound (22) in (21) to get:

$$b_h(\mathbf{v}_{0,h}, q_h) \gtrsim \nu^{-1}\Big((1 - \frac{\varepsilon}{2}\,C_{\mathrm{div}}\,\sigma)\,\|\pi_0(q_h)\|_\Omega^2 - \frac{C_{\mathrm{div}}}{2\,\varepsilon}\,\sigma\,\|\tilde{q}_{1,h}\|_\Omega^2\Big). \tag{23}$$

Let $\mathbf{v}_{1,h}$ be like in Lemma 3. Using (11) and (12), we get:

$$b_h(\mathbf{v}_{1,h}, q_h) = (\nabla q_{1,h}, \mathbf{v}_{1,h})_\Omega \gtrsim \nu^{-1}\,\sigma^{-2\,(d-2)}\,\|\tilde{q}_{1,h}\|_\Omega^2 \tag{24}$$

Last, we set $\mathbf{v}_h^\star = \mu\mathbf{v}_{0,h} + \mathbf{v}_{1,h}$, where $\mu > 0$. Using (23) and (24), we have:

$$\begin{aligned} b_h(\mathbf{v}_h^\star, q_h) &= \mu\,b_h(\mathbf{v}_{0,h}, q_h) + b_h(\mathbf{v}_{1,h}, q_h), \\ &\gtrsim \nu^{-1}\Big(\mu\,\big(1 - \tfrac{\varepsilon}{2}\,C_{\mathrm{div}}\,\sigma\big)\,\|\pi_0(q_h)\|_\Omega^2 + \big(\sigma^{-2\,(d-2)} - \tfrac{\mu\,C_{\mathrm{div}}}{2\,\varepsilon}\,\sigma\big)\,\|\tilde{q}_{1,h}\|_\Omega^2\Big). \end{aligned}$$

Let us choose $\varepsilon = (C_{\mathrm{div}}\,\sigma)^{-1} < 1$ and $\mu = \sigma^{-2\,(d-2)}\,\varepsilon^2$. We obtain that:

$$b_h(\mathbf{v}_h^\star, q_h) \gtrsim C_{\min}\nu^{-1}\,\|q_h\|_\Omega^2 \text{ with } C_{\min} = \varepsilon^2\,\sigma^{-2\,(d-2)} = C_{\mathrm{div}}^{-2}\sigma^{-2\,(d-1)}. \tag{25}$$

Let us bound $\|\mathbf{v}^\star_h\|_h$ by $\|q_h\|_\Omega$. We have: $\|\mathbf{v}^\star_h\|^2_h \lesssim \left(\mu^2 \|\mathbf{v}_{0,h}\|^2_h + \|\mathbf{v}_{1,h}\|^2_h\right)$. Using (10) and (13) $\|\mathbf{v}^\star_h\|^2_h \lesssim \nu^{-2} (\mu^2 C^2_{\text{div}} \|\pi_0(q_h)\|^2_\Omega + \sigma^4 \|\tilde{q}_{1,h}\|^2_\Omega)$, hence:

$$\|\mathbf{v}^\star_h\|^2_h \leq C^2_{\max} \nu^{-2} \|q_h\|^2_\Omega \text{ with } C_{\max} = \sqrt{2}\,\sigma^2. \tag{26}$$

Using (26) in (25), we get (19) since $C_{\min} C^{-1}_{\max} = \frac{1}{\sqrt{2}} C^{-2}_{\text{div}} \sigma^{-2\,d}$. □

Let $\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_{0,h} \,|\, \forall q_h \in Q_h,\ b_h(\mathbf{v}_h, q_h) = 0\}$. We know from [22, Lemma 3.1] that $\mathbf{V}_h \subset \{\mathbf{v}_h \in \mathbf{X}_{0,h} \,|\, \forall K \in \mathcal{T}_h, \operatorname{div} \mathbf{v}_{h|K} = 0\}$. Using $P^0 + P^1$ discrete pressures improves consistency. Indeed, integrating by parts, we have that for all $(\mathbf{v}_h, q_{1,h}) \in \mathbf{V}_h \times Q_{1,h}$, $0 = b_h(\mathbf{v}_h, q_{1,h}) = -(\operatorname{div}_h \mathbf{v}, \pi_0(q_{1,h}))_\Omega + \sum_{F \in \mathcal{F}_h} \int_F q_{1,h} [\mathbf{v}_h] \cdot \mathbf{n}_F = 0 + \sum_{F \in \mathcal{F}_h} \int_F q_{1,h} [\mathbf{v}_h] \cdot \mathbf{n}_F$. Hence:

Lemma 4 *For all $(\mathbf{v}_h, q_{1,h}) \in \mathbf{V}_h \times Q_{1,h}$, it holds: $\sum_{F \in \mathcal{F}_h} \int_F q_{1,h} [\mathbf{v}_h] \cdot \mathbf{n}_F = 0$.*

Let $\ell_{\mathbf{f}} \in \mathcal{L}(\mathbf{X}_h, \mathbb{R})$ be such that $\forall \mathbf{v}_h \in \mathbf{X}_h$, $\ell_{\mathbf{f}}(\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_\Omega$ if $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\ell_{\mathbf{f}}(\mathbf{v}_h) = \langle \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h) \rangle_{\mathbf{H}^1_0(\Omega)}$ if $\mathbf{f} \notin \mathbf{L}^2(\Omega)$, where $\mathcal{I}_h : \mathbf{X}_{0,h} \longrightarrow \mathbf{Y}_{0,h}$ is for instance an averaging operator [10, §22.4.1], with $\mathbf{Y}_{0,h} = \{\mathbf{v}_h \in \mathbf{H}^1_0(\Omega) \,|\, \forall K \in \mathcal{T}_h, \mathbf{v}_{h|K} \in \mathbf{P}^k(K)\}$. The discretization of Problem (1) with the $\mathbf{P}^1_{nc} - (P^0 + P^1)$ element reads:
Find $(\mathbf{u}_h, p_h) \in \mathbf{X}_{0,h} \times Q_h$ such that $\forall (\mathbf{u}_h, p_h) \in \mathbf{X}_{0,h} \times Q_h$

$$\nu(\mathbf{u}_h, \mathbf{v}_h)_h + b_h(\mathbf{v}_h, p_h) + b_h(\mathbf{u}_h, q_h) = \ell_{\mathbf{f}}(\mathbf{v}_h). \tag{27}$$

Due to Proposition 2 and Theorem 1, Problem (27) is well posed.

3 Convergence Of The $\mathbf{P}^1_{nc} - (P^0 + P^1)$ Element

Following the proofs of [1, Theorems 3, 4, 6], we can prove

Theorem 2 *Suppose that Hypothesis 1 holds and that the solution of Problem (1) is such that $(\mathbf{u}, p) \in (\mathbf{V} \cap \mathbf{H}^2(\Omega)) \times (H^1(\Omega) \cap L^2_{zmv}(\Omega))$. It holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim \sigma\, h\, \nu^{-1} \|\mathbf{f}\|_\Omega. \tag{28}$$

$$\text{If } \Omega \text{ is convex} \quad \|\mathbf{u} - \mathbf{u}_h\|_\Omega \lesssim \sigma^2\, h^2\, \nu^{-1} \|\mathbf{f}\|_\Omega. \tag{29}$$

$$\|(p - p_h) - \pi_0(p - p_h)\|_\Omega \lesssim \nu\, \sigma^{(2\,d-1)}\, h \|\mathbf{f}\|_\Omega. \tag{30}$$

$$\|\pi_0(p - p_h)\|_\Omega \lesssim \nu\, C_{\text{div}}\, \sigma^{(2d+1)}\, h \|\mathbf{f}\|_\Omega. \tag{31}$$

In the case of inhomogeneous Dirichlet boundary conditions, one can also recover (28) and (29), following [1, §7], where a lifting of $\mathbf{g}$ is used.

When $\Omega \subset \mathbb{R}^2$, we have moreover the two Lemmas below.

Lemma 5 *For all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_{2,h}$, $(\nabla q_h, \mathbf{v}_h)_\Omega = 0$.*

Proof. Let $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_{2,h}$. Integrating by parts and using Lemma 4, we get that $\forall q_{1,h} \in Q_{1,h}$: $(\nabla q_h, \mathbf{v}_h)_\Omega = -\sum_{F\in\mathcal{F}_h} \int_F q_h[\mathbf{v}_h]\cdot\mathbf{n}_F = -\sum_{F\in\mathcal{F}_h}\int_F (q_h - q_{1,h})[\mathbf{v}_h]\cdot\mathbf{n}_F = \sum_{F\in\mathcal{F}_h}\int_F (q_h - q_{1,h})[\mathbf{v}_h - \mathbf{v}_h(M_F)]\cdot\mathbf{n}_F$. Hence, we obtain: $(\nabla q_h, \mathbf{v}_h)_\Omega = -\sum_{K\in\mathcal{T}_h}\sum_{F\subset\partial K}\int_F (q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F))\cdot\mathbf{n}_{F,K}$. Choose $q_{1,h}$ such that $\forall S \in \mathcal{V}_h$, $q_{1,h}(S) = q_{2,h}(S)$. Then, $\forall F \in \mathcal{F}_h$, the degree 3 polynomial $(q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F))$ vanishes at the quadrature points of Simpson's rule. Since it is exact for degree 3 polynomials, $\forall F \in \mathcal{F}_h$, $\int_F (q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F))\cdot\mathbf{n}_{F,K} = 0$ and $(\nabla q_h, \mathbf{v}_h)_\Omega = 0$. □

Lemma 6 *Let $\phi \in H^3(\Omega)\cap L^2_{zmv}(\Omega)$ such that $|\phi|_{H^3(\Omega)} \neq 0$.*
Then, $\forall \mathbf{v}_h \in \mathbf{V}_h$, $(\nabla\phi, \mathbf{v}_h)_\Omega \lesssim \sigma\, h^2\, |\phi|_{H^3(\Omega)}\, \|\mathbf{v}_h\|_\Omega$.

Proof. Let $\mathbf{v}_h \in \mathbf{V}_h$. Using Lemma 5 and Cauchy-Schwarz, it holds: $|(\nabla\phi, \mathbf{v}_h)_\Omega| = |(\,\nabla(\phi - q_h), \mathbf{v}_h\,)_\Omega|$ for all $q_h \in Q_{2,h}$. We then use [10, Lemma 11.9] to conclude. □

Theorem 3 *Suppose that Ω is convex. Let $p \in H^3(\Omega)\cap L^2_{zmv}(\Omega)$ such that $|p|_{H^3(\Omega)} \neq 0$. Consider Problem* (1) *with $\mathbf{f} = \nabla p$. Then $(\mathbf{u}_h, p_h)$, the solution of Problem* (27) *is such that (where $\delta p = p - p_h$):*

$$\nu\,\|\mathbf{u}_h\|_h \lesssim \sigma^2\, h^3\, |p|_{H^3(\Omega)}, \qquad \nu\,\|\mathbf{u}_h\|_\Omega \lesssim \sigma^3\, h^4\, |p|_{H^3(\Omega)}. \tag{32}$$

$$\|\delta p - \pi_0(\delta p)\|_\Omega \lesssim \sigma^4\, h^2\, |p|_{H^2(\Omega)},\ \|\pi_0(\delta p)\|_\Omega \lesssim C_{\rm div}\,\sigma^6\, h^2\, |p|_{H^2(\Omega)}. \tag{33}$$

Proof. Setting $\mathbf{v}_h = \mathbf{u}_h$ as test-function in Problem (27) and using Lemma 6, it holds: $\nu\,\|\mathbf{u}_h\|_h^2 = (\nabla\phi, \mathbf{u}_h)_\Omega \lesssim \sigma\, h^2\, |\phi|_{H^3(\Omega)}\, \|\mathbf{u}_h\|_\Omega$. From [1, Theorem 4], we have: $\|\mathbf{u}_h\|_\Omega \lesssim \sigma\, h\, \|\mathbf{u}_h\|_h$. We deduce that $\nu\,\|\mathbf{u}_h\|_h \lesssim \sigma^2\, h^3\, |\phi|_{H^3(\Omega)}$. Using this estimate, we get that $\nu\,\|\mathbf{u}_h\|_\Omega \lesssim \sigma^3\, h^4\, |\phi|_{H^3(\Omega)}$. The proof of (33) is similar to that of Eqs (30)-(31), using (32) and noticing that $\forall K \in \mathcal{T}_h$, $\|\nabla(\phi - \pi_1\phi)\|_K \lesssim \sigma_K\, h_K\, |\phi|_{H^2(K)}$. □

4 Numerical Results

To get $p_h \in L^2_{zmv}(\Omega)$, we eliminate a degree of freedom in $\mathcal{P}^0_{disc}(\mathcal{T}_h)$ and in $\mathcal{P}^1(\mathcal{T}_h)$ and post-process by subtracting the mean-value. Let us set: $\|(\mathbf{u}, p)\|_\nu^2 := \|\nabla\mathbf{u}\|_\Omega^2 + \nu^{-2}\|p\|_\Omega^2$. Let $\delta_h\mathbf{u} = \Pi_h\mathbf{u} - \mathbf{u}_h$, $\delta p = p - p_h$ and π^1_{disc} be the L^2-orthogonal projection onto $\mathcal{P}^1_{disc}(\mathcal{T}_h)$. We compute the discrete errors values: $\varepsilon_0^\nu(\mathbf{u}_h) := \|\delta_h\mathbf{u}\|_\Omega/\|(\mathbf{u}, p)\|_\nu$, $\varepsilon_1^\nu(\mathbf{u}_h) := \|\delta_h\mathbf{u}\|_h/\|(\mathbf{u}, p)\|_\nu$ and $\varepsilon_0^\nu(p_h) := \nu^{-1}(\|\pi_0(\delta p)\|_\Omega^2 + \|\pi^1_{disc}(\delta p) - \pi_0(\delta p)\|_\Omega^2)^{\frac{1}{2}}/\|(\mathbf{u}, p)\|_\nu$. Let $\tau_{0,\mathbf{u}}$, $\tau_{1,\mathbf{u}}$ and τ_p be the averaged convergence rates of $\varepsilon_0^\nu(\mathbf{u})$, $\varepsilon_1^\nu(\mathbf{u})$ and $\varepsilon_0^\nu(p)$.
We first consider a 2D case illustrating Thm. 3, with the prescribed solution $\mathbf{u} = 0$, $\mathbf{f} = \nabla\psi$ with $\psi(x, y) = \exp(-10(1 - x + 2y))$. Fig. 1 shows $\varepsilon_0^\nu(\mathbf{u}_h)$, $\varepsilon_1^\nu(\mathbf{u}_h)$ and $\varepsilon_0^\nu(p_h)$ against h, for $\nu = 10^{-2}$, with P^0 (left) or $P^0 + P^1$ (right) discrete pressures, comparing the $\mathbf{P}^1_{nc} - P^0$ [1, Example 4] and $\mathbf{P}^1_{nc} - (P^0 + P^1)$ schemes. Tab. 1 reports observed convergence rates, which are better for the $\mathbf{P}^1_{nc} - (P^0 + P^1)$ scheme.

Let us consider stationary Navier-Stokes equations set in $\Omega = (0, 1)^3$: Find $(\mathbf{u}, p)$

Fig. 1 $2D$ case. Plots of $\varepsilon_0^\nu(\mathbf{u}_h)$, $\varepsilon_1^\nu(\mathbf{u}_h)$ and $\varepsilon_0^\nu(p_h)$ for $\nu = 10^{-2}$.

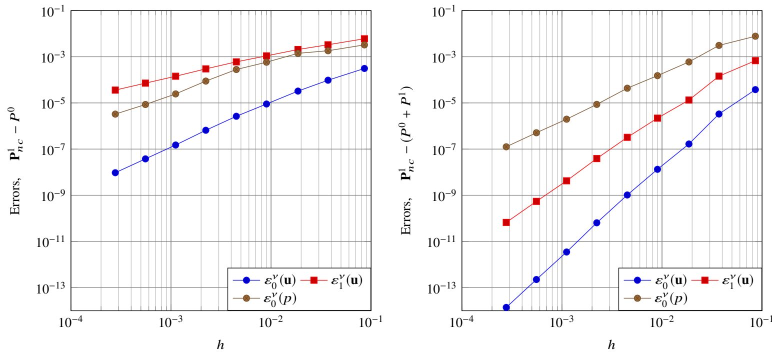

Table 1 $2D$ case, averaged convergence rates on the last five meshes.

	P^0	$P^0 + P^1$
$\tau_{0,\mathbf{u}}$	2.02	4.03
$\tau_{1,\mathbf{u}}$	1.01	3.07
τ_p	1.60	2.10

such that $-\nu\,\mathbf{\Delta}\,\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}$ and $\operatorname{div}\mathbf{u} = 0$ in Ω, with inhomogeneous Dirichlet boundary conditions. Computations are done with the TrioCFD code [6, 7] with the $\mathbf{P}^1_{nc} - (P^0 + P^1)$ scheme. The convection term is discretized using the MUSCL scheme [6]. We study the prescribed solution $\mathbf{u} = (y - z, z - x, x - y)^T$, $p = \frac{1}{2}(x^2 + y^2 + z^2) - xy - xz - yz - 1/4$ so that $\mathbf{f} = 0$. Fig. 2 shows $\varepsilon_0^\nu(\mathbf{u}_h)$ (left) and $\varepsilon_0^\nu(p_h)$ (right) against h, for $\nu = 10^{-1}$; 10^{-2}; 10^{-3}.

Fig. 2 $3D$ case. Plots of $\varepsilon_0^\nu(\mathbf{u}_h)$ and $\varepsilon_0^\nu(p_h)$ for $\nu = 10^{-1}$, $\nu = 10^{-2}$ or $\nu = 10^{-3}$.

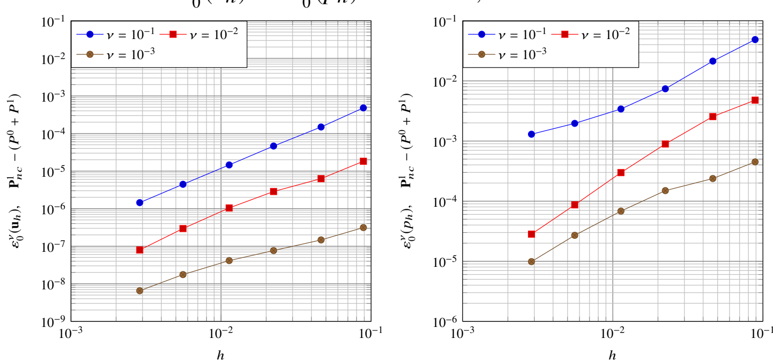

Table 2 reports the convergence rates $\tau_{0,\mathbf{u}}$ (left), and τ_p (right). More numerical results are available in [8, 9, 6, 23]. The $\mathbf{P}^1_{nc} - (P^0 + P^1)$ scheme shows good approximation properties and should therefore be suitable for solving the Navier-Stokes equations in the context of industrial simulation. More generally, it seems interesting

Table 2 $3D$ case, averaged convergence rates on the last three meshes.

ν	10^{-1}	10^{-2}	10^{-3}
$\tau_{0,\mathbf{u}}$	1.68	1.88	1.41

ν	10^{-1}	10^{-2}	10^{-3}
τ_p	0.70	1.72	1.35

to design and study discrete pressure enrichment for higher order nonconforming Crouzeix-Raviart mixed FEM [2, 3].

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