DOMAIN DECOMPOSITION METHODS FOR THE DIFFUSION EQUATION WITH LOW-REGULARITY SOLUTION

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Abstract. We analyze matching and non-matching domain decomposition methods for the numerical approximation of the mixed diffusion equations. Special attention is paid to the case where the solution is of low regularity. Such a situation commonly arises in the presence of three or more intersecting material components with different characteristics. The domain decomposition method can be non-matching in the sense that the traces of the finite element spaces may not fit at the interface between subdomains. We prove well-posedness of the discrete problem, that is solvability of the corresponding linear system, provided two algebraic conditions are fulfilled. If moreover the conditions hold independently of the discretization, convergence is ensured.

1. Introduction

The diffusion equation can model different physical phenomena such as Darcy's law, Fick's law or the neutron diffusion. When formulated as a mixed system of equations, it allows to compute both the solution and its gradient. Hence, from a variational point of view, two approaches coexist. One uses either the primal variational formulation to focus on the solution; or the dual-mixed variational formulation to focus instead on the gradient of the solution.
The numerical analysis of domain decomposition methods for the mixed diffusion equation has already been studied for Darcy's law, see e.g. [1, 2, 3, 4], and also for Fick's law and the neutron diffusion equation, see e.g. [5, 6, 7]. In order to handle non-matching discretizations at the interface of the subdomains, a Lagrange multiplier can be introduced. This technique is known as the mortar finite element method [8], among its predecessors one can cite the hydrid finite element method (see [9] for the diffusion equation). From an algebraic point of view, the linear system obtained after discretization is similar to the one of the Schur complement method.
In this paper, we focus on matching and non-matching domain decompositions of the (dual) mixed formulation. We put special emphasis on the so-called *low-regularity solutions*. For the diffusion equation, we recall that the solution always belongs to the Sobolev space H^1. However, it may happen that the a priori regularity result (even for smooth, locally supported data) only guarantees that the solution is piecewise H^{1+r}, where $r > 0$ can be arbitrarily small: one says that the problem is H^{1+r}-regular. On the other hand, in the above mentioned references, when the regularity issue is explicitly taken into account, it is assumed that the solution is at least:

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- piecewise H^{1+r} with $r > 1/4$ in [7];
- piecewise H^{1+r} with $r > 1/2$ in [1, 2, 4];
- piecewise H^{1+r} with $r = 1$ in [3, 5].

So one aim of our paper is to devise a method that can be fully justified for a problem that is H^{1+r}-regular, with an exponent $r > 0$ that can be arbitrarily small. In other words, we address the "technicalities" needed for the design of a theory that handles low-regularity solutions. To reach that end, one has to modify the existing mathematical and variational frameworks. The other aim is to derive abstract (algebraic) conditions after discretization, to guarantee well-posedness of the discrete problem, and convergence.
The outline of the paper is as follows. In section 2, we introduce the notations, geometry and Hilbert spaces to define the problem setting. In particular, we will make use of vector-valued functions with L^2-jump of normal traces on the interface between subdomains. Then, in section 3, we write the continuous equations and the associated variational formulations of the mixed diffusion equations. We also define the *low-regularity case*. We next propose an equivalent multi-domain formulation, which fits into the category of domain decomposition methods. The well-posedness of the mixed, multi-domain formulation is studied in section 4 in the continuous case and in section 5 in the discrete case. In the discrete case, we exhibit two abstract algebraic conditions which imply the existence of a discrete inf-sup condition. This inf-sup condition ensures well-posedness of the discrete problem, and also convergence when it is uniform. In addition, these algebraic conditions drive the choice of the space of the Lagrange multipliers. We give numerical illustrations in section 6. Finally, we draw some conclusions and give perspectives in section 7.

2. Geometry, Hilbert spaces and notations

Throughout the paper, C is used to denote a generic positive constant which is independent of the meshsize, the triangulation and the quantities/fields of interest. We also use the shorthand notation $A \lesssim B$ for the inequality $A \leq CB$, where A and B are two scalar quantities, and C is a generic constant. Respectively, $A \approx B$ for the inequalities $A \lesssim B$ and $B \lesssim A$.
Vector-valued (resp. tensor-valued) function spaces are written in boldface character (resp. blackboard bold characters). Given an open set $O \subset \mathbb{R}^d$, $d = 1, 2, 3$, we use the notation $(\cdot|\cdot)_{0,O}$ (resp. $\|\cdot\|_{0,O}$) for the $L^2(O)$ and $\mathbf{L}^2(O) := (L^2(O))^d$ scalar products (resp. norms). More generally, $(\cdot|\cdot)_{s,O}$ and $\|\cdot\|_{s,O}$ (resp. $|\cdot|_{s,O}$) denote the scalar product and norm (resp. semi-norm) of the Sobolev spaces $H^s(O)$ and $\mathbf{H}^s(O) := (H^s(O))^d$, for $s \in \mathbb{R}$ (resp. for $s > 0$).
If moreover the boundary ∂O is Lipschitz, $\mathbf{n}$ denotes the unit outward normal vector field to ∂O. Finally, it is assumed that the reader is familiar with vector-valued function spaces related to the diffusion equation, such as $\mathbf{H}(\mathrm{div}\,; O)$, $\mathbf{H}_0(\mathrm{div}\,; O)$ etc.

We let $\mathcal{R}$ be a bounded, connected and open subset of $\mathbb{R}^d$, having a Lipschitz boundary which is piecewise smooth. We split $\mathcal{R}$ into N open disjoint parts, or subdomains, $(\mathcal{R}_i)_{i=1,N}$ with Lipschitz, piecewise smooth boundaries: $\overline{\mathcal{R}} = \cup_{i=1,N}\overline{\mathcal{R}_i}$ and the set $\{\mathcal{R}_i\}_{i=1,N}$ is called a partition of $\mathcal{R}$. For a field v defined over $\mathcal{R}$, we shall use the notations $v_i = v_{|\mathcal{R}_i}$, for $i = 1, N$.

Given a partition $\{\mathcal{R}_i\}_{i=1,N}$ of $\mathcal{R}$, let us introduce now function spaces with piecewise regular elements:

$$\begin{aligned}
\mathcal{P}H^s(\mathcal{R}) &= \left\{ \psi \in L^2(\mathcal{R}) \,|\, \psi_i \in H^s(\mathcal{R}_i),\, i = 1, N \right\},\ s > 0\,; \\
\mathcal{P}\mathbf{H}(\mathrm{div}\,, \mathcal{R}) &= \left\{\mathbf{q} \in \mathbf{L}^2(\mathcal{R}) \,|\, \mathbf{q}_i \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}_i),\, i = 1, N \right\}\,; \\
\mathcal{P}W^{1,\infty}(\mathcal{R}) &= \left\{ \psi \in L^\infty(\mathcal{R}) \,|\, \psi_i \in W^{1,\infty}(\mathcal{R}_i),\, i = 1, N\right\}.
\end{aligned}$$

Given a partition $\{\mathcal{R}_i\}_{i=1,N}$ of $\mathcal{R}$, we denote by Γ_{ij} the interface between two subdomains $\mathcal{R}_i$ and $\mathcal{R}_j$, for $i \neq j$: if the Hausdorff dimension of $\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j}$ is $d-1$, then $\Gamma_{ij} = int(\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j})$; otherwise, $\Gamma_{ij} = \emptyset$. By construction, $\Gamma_{ij} = \Gamma_{ji}$. We define the interface Γ_S, respectively the wirebasket $\partial\Gamma_W$ by

$$\Gamma_S = \bigcup_{i=1}^{N} \bigcup_{j=i+1}^{N} \overline{\Gamma_{ij}}, \quad \partial\Gamma_W = \bigcup_{i=1}^{N} \bigcup_{j=i+1}^{N} \partial\Gamma_{ij}\,.$$

When $d = 2$, the wirebasket consists of isolated crosspoints. When $d = 3$, the wirebasket consists of open edges and crosspoints. Introduce the subset of indices $\mathcal{I}_S := \{i \in \{1, ..., N\} \,|\, \partial\mathcal{R}_i \cap \Gamma_S = \partial\mathcal{R}_i\}$ and, for $i = 1, N$, the open set $\Gamma_i = \partial\mathcal{R}_i \backslash \overline{\Gamma_S}$. Let us define function spaces with zero boundary condition, for $i = 1, N$:

$$\begin{aligned}
H^1_{0,\Gamma_i}(\mathcal{R}_i) &= \left\{ \psi \in H^1(\mathcal{R}_i) \,|\, \psi_{|\Gamma_i} = 0 \right\}\,; \\
\mathcal{P}H^1_0(\mathcal{R}) &= \left\{ \psi \in L^2(\mathcal{R}) \,|\, \psi_i \in H^1_{0,\Gamma_i}(\mathcal{R}_i),\, i = 1, N \right\}\,.
\end{aligned}$$

When $\Gamma_{ij} \neq \emptyset$, let $H^{1/2}_{\Gamma_{ij}}$ be the set of $H^{1/2}(\Gamma_{ij})$ functions whose continuation by 0 to $\partial\mathcal{R}_i$ belongs to $H^{1/2}(\partial\mathcal{R}_i)$. One can prove that $H^{1/2}_{\Gamma_{ij}} = H^{1/2}_{\Gamma_{ji}}$.
For $\mathbf{p} \in \mathcal{P}\mathbf{H}(\mathrm{div}\,, \mathcal{R})$, let us set $[\mathbf{p} \cdot \mathbf{n}]_{ij} := \sum_{k=i,j} \mathbf{p}_k \cdot \mathbf{n}_{k|\Gamma_{ij}}$ the jump of the normal component of $\mathbf{p}$ on Γ_{ij}, when $\Gamma_{ij} \neq \emptyset$. $[\mathbf{p} \cdot \mathbf{n}]_{ij}$ is well defined in $(H^{1/2}_{\Gamma_{ij}})'$ the dual space of $H^{1/2}_{\Gamma_{ij}}$ (see e.g. [10]). The *global jump* $[\mathbf{p} \cdot \mathbf{n}]$ of the normal component on the interface is defined by:

$$[\mathbf{p} \cdot \mathbf{n}]_{|\Gamma_{ij}} := [\mathbf{p} \cdot \mathbf{n}]_{ij},\ \text{for } i, j = 1, N,\ i < j.$$

By definition, it holds $[\mathbf{p} \cdot \mathbf{n}] \in \prod_{i<j} (H^{1/2}_{\Gamma_{ij}})'$.

Lemma 1. *Let* $\mathbf{p} \in \mathbf{H}(\mathrm{div}\,, \mathcal{R})$. *Then its global jump vanishes:* $[\mathbf{p} \cdot \mathbf{n}] = 0$.

Proof. Let $\psi \in H^1_0(\mathcal{R})$, such that $\psi = 0$ in a neighbourhood of $\partial\Gamma_W$. Integrating by parts twice (cf. §A), we have:

$$\begin{aligned}
-\int_{\mathcal{R}} \mathrm{div}\,\mathbf{p}\,\psi &= \int_{\mathcal{R}} \mathbf{p} \cdot \mathbf{grad}\,\psi = \sum_{i=1}^{N} \int_{\mathcal{R}_i} \mathbf{p}_i \cdot \mathbf{grad}\,\psi_i \\
&= -\sum_{i=1}^{N} \int_{\mathcal{R}_i} \mathrm{div}\,\mathbf{p}_i\,\psi_i + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \langle [\mathbf{p} \cdot \mathbf{n}]_{ij}, \psi \rangle_{(H^{1/2}_{\Gamma_{ij}})', H^{1/2}_{\Gamma_{ij}}}\,.
\end{aligned}$$

By choosing ψ such that its support has a non-trivial intersection with a single Γ_{ij}, it follows that $[\mathbf{p} \cdot \mathbf{n}]_{ij} = 0$ for $i, j = 1, N$, $i < j$. $\square$

We introduce finally the following Hilbert spaces:

$$
\begin{aligned}
M &= \left\{ \psi_S \in \prod_{i<j} L^2(\Gamma_{ij}) \right\}, \|\psi_S\|_M = \left(\textstyle\sum_{i<j} \|\psi_{S|\Gamma_{ij}}\|^2_{0,\Gamma_{ij}} \right)^{1/2} ; \\
H^{1/2}_{-}(\Gamma_S) &= \left\{ \psi_S \in M \,|\, \psi_{S|\Gamma_{ij}} \in H^{1/2}(\Gamma_{ij}),\, \forall i<j \right\} , \text{ with graph norm}; \\
\mathbf{Q} &= \{ \mathbf{q} \in \mathcal{P}\mathbf{H}(\mathrm{div}\,, \mathcal{R}) \,|\, [\mathbf{q}\cdot\mathbf{n}] \in M \} , \\
& \quad \|\mathbf{q}\|_{\mathbf{Q}} := \left(\sum_{i=1}^{N} \|\mathbf{q}_i\|^2_{\mathbf{H}(\mathrm{div}\,,\mathcal{R}_i)} + || [\mathbf{q}\cdot\mathbf{n}] ||^2_M \right)^{1/2} ; \\
\mathbf{X} &= \left\{ \xi := (\mathbf{q}, \psi) \in \mathbf{Q} \times L^2(\mathcal{R}) \right\} , \|\xi\|_{\mathbf{X}} := \left(\|\mathbf{q}\|^2_{\mathbf{Q}} + \|\psi\|^2_{0,\mathcal{R}} \right)^{1/2} ; \\
\mathtt{W} &= \{ \mathtt{w} := (\xi, \psi_S) \in \mathbf{X} \times M \} , \|\mathtt{w}\|_{\mathtt{W}} := \left(\|\xi\|^2_{\mathbf{X}} + \|\psi_S\|^2_M \right)^{1/2} .
\end{aligned}
$$

By construction, one has $M \subset \prod_{i<j} (H^{1/2}_{\Gamma_{ij}})'$. In the definition of elements $\mathbf{q} = (\mathbf{q}_i)_{i=1,N}$ of $\mathbf{Q}$ it is important to note that one *does not require* $\mathbf{q}_i \cdot \mathbf{n}_{i|\Gamma_{ij}} \in L^2(\Gamma_{ij})$ for $i, j = 1, N$. This is based on the observation that in the case of a low-regularity solution, the normal trace on the interface of its gradient may not belong to M. On the other hand, the global jump of its (scaled) normal trace systematically belongs to M. We shall use this observation to define a variational formulation which is conforming in $\mathbf{Q}$. In this sense, our approach is different from the one of Yotov et al [1, 2, 3, 4].

3. Setting of the equations and the variational formulation

Given a source term $S_f \in L^2(\mathcal{R})$, we consider the following diffusion equation, written in its primal form:
Find $\phi \in H^1(\mathcal{R})$ such that:

$$
\left\{ \begin{array}{rcll} -\mathrm{div}\, D\, \mathbf{grad}\, \phi + \sigma\, \phi &=& S_f & \text{in } \mathcal{R} \\ \phi &=& 0 & \text{on } \partial\mathcal{R}, \end{array} \right. \tag{3.1}
$$

where ϕ is the primal unknown. The coefficients σ, respectively D, are a scalar field, resp. a tensor field. They satisfy the assumptions below:

$$
\left\{ \begin{array}{l} (\sigma, D) \in L^\infty(\mathcal{R}) \times \mathbb{L}^\infty(\mathcal{R}) , \\ \exists D_*, D^* > 0,\ \forall \boldsymbol{z} \in \mathbb{R}^d,\ D_* \|\boldsymbol{z}\|^2 \le (D\boldsymbol{z}, \boldsymbol{z}) \le D^* \|\boldsymbol{z}\|^2 \text{ a.e. in } \mathcal{R} , \\ \exists \sigma_*, \sigma^* > 0,\ 0 < \sigma_* \le \sigma \le \sigma^* \text{ a.e. in } \mathcal{R}. \end{array} \right. \tag{3.2}
$$

Remark 1. *All results in §3 still hold if $\sigma_* = 0$. In particular, the fact that $\sigma_* = 0$ has no influence on the regularity results of Proposition 1. On the other hand, handling the case $\sigma_* = 0$ requires extra care once we consider the multi-domain method. This is the reason why we make the assumption $\sigma_* > 0$ from the start.*

Classically, Problem (3.1) is equivalent to the following variational formulation:
Find $\phi \in H^1_0(\mathcal{R})$ such that $\forall \psi \in H^1_0(\mathcal{R})$:

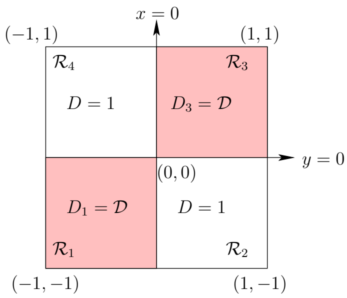
$$
\int_{\mathcal{R}} D\, \mathbf{grad}\, \phi \cdot \mathbf{grad}\, \psi + \int_{\mathcal{R}} \sigma \phi\, \psi = \int_{\mathcal{R}} S_f \psi. \tag{3.3}
$$

Under the assumptions on the coefficients, the primal problem (3.1) is well-posed, in the sense that for all $S_f \in L^2(\mathcal{R})$, there exists one and only one $\phi \in H^1(\mathcal{R})$ that solves (3.1), and in addition there holds $\|\phi\|_{1,\mathcal{R}} \lesssim \|S_f\|_{0,\mathcal{R}}$.

3.1. **Low-regularity solutions.** Under additional mild assumptions on the coefficients, the solution ϕ has some extra regularity (see [11], Theorem 4.1 and [12], Theorem 3.1): the diffusion equation is H^{1+r}-regular for some $r > 0$.

Proposition 1. *Let $\sigma \in \mathcal{P}W^{1,\infty}(\mathcal{R})$, $D \in \mathcal{P}\mathbb{W}^{1,\infty}(\mathcal{R})$ a symmetric tensor field, that fulfill (3.2). There exists $r_{\max} \in]0,1]$, called the regularity exponent, such that for all source terms $S_f \in L^2(\mathcal{R})$, the solution $\phi \in H^1(\mathcal{R})$ belongs to $\bigcap_{0\le r<r_{\max}} \mathcal{P}H^{1+r}(\mathcal{R})$ ($r_{\max} < 1$) or $\mathcal{P}H^2(\mathcal{R})$ ($r_{\max} = 1$) with continuous dependence: $\|\phi\|_{\bigcap_{0\le r<r_{\max}} \mathcal{P}H^{1+r}(\mathcal{R})} \lesssim \|S_f\|_{0,\mathcal{R}}$ ($r_{\max} < 1$) or $\|\phi\|_{\mathcal{P}H^2(\mathcal{R})} \lesssim \|S_f\|_{0,\mathcal{R}}$ ($r_{\max} = 1$).*

We define the *low-regularity case* as $r_{\max} < 1/2$ above. Since crosspoints ($d = 2, 3$) and edges ($d = 3$) are allowed in the wirebasket, this situation commonly arises in practice. As an illustration, let us mention the checkerboard setting. Set $\mathcal{R} =]-1,1[^2$, and divide it into four subsquares. Consider the problem: Find $\phi \in H^1(\mathcal{R})$ such that $-\mathrm{div}\, D\,\mathbf{grad}\,\phi + \phi = S_f$ with some boundary conditions. D is piecewise constant: $D = \mathcal{D}$ in the bottom left and top right subsquares, and 1 elsewhere, as shown on Fig. 1. A singular solution in the sense of [13] behaves like $\rho^{r_{\max}}$, where ρ



$\mathcal{D}$	5	7	10	15	30	100
r_{max}	0.54	0.46	0.39	0.32	0.23	0.13

FIGURE 1. Checkerboard setting; associated regularity exponent.

is the distance to the crosspoint $(0,0)$, whereas $r_{\max}$ depends on D. Some values of $r_{\max}$ against D are given on the right table of Fig. 1. For instance, when $\mathcal{D} \ge 7$, ϕ is of low-regularity. Identical conclusions can be drawn in $\mathcal{R} =]-1,1[^3$, partitioned into four sub-parallelepipeds with a wirebasket now equal to the edge $(0,0)\times]-1,1[$. As a matter of fact, high contrast between piecewise constant diffusion coefficient often appears in neutronics [7]. The notion of low-regularity solutions is used in section 6 devoted to numerics, where we study the convergence of non-conforming domain decompositions.

3.2. **Multi-domain formulation.** We now propose a multi-domain formulation equivalent to (3.1), which fits into the category of domain decomposition methods. Starting from the solution ϕ to (3.1), if one introduces the two auxiliary unknowns $\mathbf{p} := -D\,\mathbf{grad}\,\phi \in \mathbf{H}(\mathrm{div}\,, \mathcal{R})$, and $\phi_S := \phi_{|\Gamma_S} \in H^{1/2}_{-}(\Gamma_S) \subset M$, let us prove that one may write the diffusion problem in mixed, multi-domain, form as:

Find $(\mathbf{p}, \phi, \phi_S) \in \mathbf{Q} \times \mathcal{P}H^1_0(\mathcal{R}) \times M$ such that:

$$
(3.4) \qquad \left\{ \begin{array}{rcll} -D_i^{-1}\,\mathbf{p}_i\,-\,\mathbf{grad}\,\phi_i & = & 0 & \text{in } \mathcal{R}_i, \qquad \text{for } i = 1, N, \\ \operatorname{div}\mathbf{p}_i\,+\,\sigma_i\phi_i & = & S_{f,i} & \text{in } \mathcal{R}_i, \qquad \text{for } i = 1, N, \\ \phi_i & = & \phi_S & \text{on } \partial\mathcal{R}_i \cap \Gamma_S, \quad \text{for } i = 1, N, \\ [\mathbf{p}\cdot\mathbf{n}]_{ij} & = & 0 & \qquad \text{for } i, j = 1, N. \end{array} \right.
$$

Solving the mixed, multi-domain problem (3.4) is actually equivalent to solving (3.1), as the result below shows.

Theorem 1. *The triple $(\mathbf{p}, \phi, \phi_S) \in \mathbf{Q} \times \mathcal{P}H^1_0(\mathcal{R}) \times M$ solves (3.4) if, and only if, ϕ solves (3.1) with the same data.*

Proof. In order to prove that (3.1) $\Rightarrow$ (3.4), we define $\mathbf{p} = -D\,\mathbf{grad}\,\phi$ and $\phi_S = \phi_{|\Gamma_S}$. We just have to check that $\mathbf{p} \in \mathbf{Q}$. By construction, $\mathbf{p} \in \mathbf{H}(\operatorname{div}, \mathcal{R})$, so according to Lemma 1, $[\mathbf{p}\cdot\mathbf{n}] = 0 \in M$, and it follows that $\mathbf{p} \in \mathbf{Q}$.
Let us show that (3.4) $\Rightarrow$ (3.1). To prove that the distribution $\operatorname{div}\mathbf{p}$ belongs to $L^2(\mathcal{R})$, let $q \in H^1_0(\mathcal{R})$ such that $q = 0$ in a neighbourhood of $\partial\Gamma_W$:

$$
\begin{aligned}
-\langle \operatorname{div}\mathbf{p}, q\rangle_{(H^1_0(\mathcal{R}))', H^1_0(\mathcal{R})} &= \int_{\mathcal{R}} \mathbf{p}\cdot\mathbf{grad}\,q = \sum_{i=1}^{N} \int_{\mathcal{R}_i} \mathbf{p}_i\cdot\mathbf{grad}\,q_i \\
&= -\sum_{i=1}^{N} \left(\int_{\mathcal{R}_i} \operatorname{div}\mathbf{p}_i\,q_i - \sum_{j\neq i} \langle \mathbf{p}_i\cdot\mathbf{n}_i, q\rangle_{(H^{1/2}_{\Gamma_{ij}})', H^{1/2}_{\Gamma_{ij}}} \right) \\
&= \sum_{i=1}^{N} \int_{\mathcal{R}_i} (\sigma_i\phi_i - S_{f,i})q_i = \int_{\mathcal{R}} (\sigma\phi - S_f)q.
\end{aligned}
$$

Above, we removed the interface terms thanks to the last equation of (3.4), and we used the second equation of (3.4) to prove the penultimate equality. By density (see Theorem 5 (cf. §A)), we conclude that $\operatorname{div}\mathbf{p} = -\sigma\phi + S_f$ in $\mathcal{R}$, and $\operatorname{div}\mathbf{p}$ belongs to $L^2(\mathcal{R})$. Whereas, according to the third equation, we have $\phi \in H^1_0(\mathcal{R})$. Using finally the first equation, we conclude that $\phi \in H^1_0(\mathcal{R})$ is such that $-\operatorname{div}(D\,\mathbf{grad}\,\phi) + \sigma\,\phi = S_f$ in $\mathcal{R}$. $\square$

Remark 2. *In the low-regularity case, (see §3.1), one has $\mathbf{p}_i \in \mathbf{H}(\operatorname{div}, \mathcal{R}_i) \cap \mathbf{H}^r(\mathcal{R}_i)$ for $0 < r < r_{\max}$, for all i. Classically* [10, 14], *this implies that $\mathbf{p}_i \cdot \mathbf{n}_{i|\partial\mathcal{R}_i}$ belongs to $H^{-1/2+r}(\partial\mathcal{R}_i)$ for $0 < r < \min(r_{\max}, 1/2)$: hence $\mathbf{p}_i \cdot \mathbf{n}_{i|\Gamma_{ij}}$ belongs to $H^{-1/2+r}(\Gamma_{ij})$, for $i \neq j$. On the other hand, $\mathbf{p}$ does not belong to $\mathbf{H}^{1/2}(\mathcal{R})$ in general, and as a consequence $\mathbf{p}\cdot\mathbf{n}_{|\Gamma_{ij}}$ does not automatically belong to $L^2(\Gamma_{ij})$, for $i \neq j$.*

In practice, writing the diffusion equation in its mixed form allows to compute precisely both the solution and its gradient: it avoids the propagation of the numerical error from the solution to its gradient. On the other hand, using a domain decomposition method is interesting for many reasons: it is necessary when one wants to compute the solution on a parallel computer. It is also useful when, for some physical reason, one needs to capture rapidly oscillating phenomena in some, but not all, subregions. This may happen when D has large variations (as it is the case in neutronics). In this case, approximations with different scales can be used. Hence the choice of a mixed, multi-domain setting.
On each subdomain, the last equation of (3.4) can be seen as a constraint. In order

to obtain the variational formulation for the mixed, multi-domain problem (3.4), we consider any test functions $\mathbf{q} \in \mathbf{Q}$ and $\psi \in L^2(\mathcal{R})$, we multiply the first equation of (3.4) by $\mathbf{q}_i$, the second equation of (3.4) by $\psi_i \in L^2(\mathcal{R}_i)$, we integrate over $\mathcal{R}_i$, and we sum over $i = 1, N$. Then we multiply the last equation by a test function $\psi_S \in M$ and we integrate on Γ_S([1]). We sum the volume and the surface integrals to reach:

$$\begin{aligned}\sum_{i=1}^{N} \int_{\mathcal{R}_i} \big(-D_i^{-1}\, \mathbf{p}_i \cdot \mathbf{q}_i \,-\, \mathbf{grad}\, \phi_i \cdot \mathbf{q}_i \,+\, \psi_i \mathrm{div}\, \mathbf{p}_i \,+\, \sigma_i \phi_i \psi_i \big) \\ + \int_{\Gamma_S} [\mathbf{p} \cdot \mathbf{n}]\, \psi_S = \sum_{i=1}^{N} \int_{\mathcal{R}_i} S_{f,i} \psi_i.\end{aligned} \tag{3.5}$$

We recall that according to Theorem 1, we have globally $\phi \in H_0^1(\mathcal{R})$. One can integrate by parts (cf. §A) to remove the first order derivatives of ϕ_i:

$$-\sum_{i=1}^{N} \int_{\mathcal{R}_i} \mathbf{grad}\, \phi_i \cdot \mathbf{q}_i = \sum_{i=1}^{N} \int_{\mathcal{R}_i} \phi_i \,\mathrm{div}\, \mathbf{q}_i - \int_{\Gamma_S} [\mathbf{q} \cdot \mathbf{n}]\, \phi_S,$$

where we used the third equation of (3.4) on Γ_S. Hence, the regularity requirement on the solution can be lowered to $\phi_i \in L^2(\mathcal{R}_i)$ for all $i = 1, N$, ie. $\phi \in L^2(\mathcal{R})$. Finally we have that the solution to (3.4) also solves:
Find $((\mathbf{p}, \phi), \phi_S) \in \mathtt{W}$, such that $\forall ((\mathbf{q}, \psi), \psi_S) \in \mathtt{W}$:

$$\begin{aligned}\int_{\mathcal{R}} \big(-D^{-1}\, \mathbf{p} \cdot \mathbf{q} \,+\, \phi \,\mathrm{div}\, \mathbf{q} \,+\, \psi \,\mathrm{div}\, \mathbf{p} \,+\, \sigma\, \phi\, \psi \big) \\ + \int_{\Gamma_S} [\mathbf{p} \cdot \mathbf{n}]\, \psi_S \,-\, \int_{\Gamma_S} [\mathbf{q} \cdot \mathbf{n}]\, \phi_S = \int_{\mathcal{R}} S_f\, \psi.\end{aligned} \tag{3.6}$$

Above, ϕ_S, ψ_S play the role of Lagrange multipliers, with M the space of those Lagrange multipliers. We call the mixed, multi-domain variational formulation (3.6) the *domain decomposition+L^2-jumps method* (or DD+L^2-jumps method).

From now on, we use the notations:

- $\mathtt{u} = (\zeta, \phi_S)$, $\zeta = (\mathbf{p}, \phi)$, $\mathbf{p} = (\mathbf{p}_i)_{i=1,N}$ and $\phi = (\phi_i)_{i=1,N}$;
- $\mathtt{w} = (\xi, \psi_S)$, $\xi = (\mathbf{q}, \psi)$, $\mathbf{q} = (\mathbf{q}_i)_{i=1,N}$ and $\psi = (\psi_i)_{i=1,N}$;

and we define the bilinear forms:

$$c : \left\{ \begin{array}{lcl} \mathtt{W} \times \mathtt{W} & \rightarrow & \mathbb{R} \\ (\mathtt{u}, \mathtt{w}) & \mapsto & \displaystyle\int_{\mathcal{R}} \big(-D^{-1}\, \mathbf{p} \cdot \mathbf{q} + \phi \,\mathrm{div}\, \mathbf{q} + \psi \,\mathrm{div}\, \mathbf{p} + \sigma\, \phi\, \psi \big) \end{array} \right. , \tag{3.7}$$

$$\ell_S : \left\{ \begin{array}{lcl} \mathtt{W} \times \mathtt{W} & \rightarrow & \mathbb{R} \\ (\mathtt{u}, \mathtt{w}) & \mapsto & \displaystyle\int_{\Gamma_S} [\mathbf{p} \cdot \mathbf{n}]\, \psi_S \end{array} \right. , \tag{3.8}$$

and:

$$c_S : \left\{ \begin{array}{lcl} \mathtt{W} \times \mathtt{W} & \rightarrow & \mathbb{R} \\ (\mathtt{u}, \mathtt{w}) & \mapsto & c(\mathtt{u}, \mathtt{w}) \,+\, \ell_S(\mathtt{u}, \mathtt{w}) \,-\, \ell_S(\mathtt{w}, \mathtt{u}) \end{array} \right. . \tag{3.9}$$

[1]To be mathematically precise, we should be integrating on $\cup_{i<j} \Gamma_{ij}$ instead of Γ_S. We make this slight abuse of notations from now on.

We consider the linear form:

$$f : \left\{ \begin{array}{lcl} \mathtt{W} & \to & \mathbb{R} \\ \mathtt{w} & \mapsto & \displaystyle\int_{\mathcal{R}} S_f \psi \end{array} \right. . \tag{3.10}$$

We may rewrite the DD+L^2-jumps method (3.6) as:
Find $\mathtt{u} \in \mathtt{W}$ such that $\forall \mathtt{w} \in \mathtt{W}$:

$$c_S(\mathtt{u}, \mathtt{w}) = f(\mathtt{w}). \tag{3.11}$$

Theorem 2. *The solution to (3.11) satisfies (3.4).*

Proof. Recall that $\mathtt{u} = ((\mathbf{p}, \phi), \phi_S)$ and $\mathtt{w} = ((\mathbf{q}, \psi), \psi_S)$.
First, using $\mathbf{q} = \mathbf{0}$, $\psi = 0$, and $\psi_S = [\mathbf{p} \cdot \mathbf{n}]$, we obtain that $[\mathbf{p} \cdot \mathbf{n}] = 0$.
Second, choose $\mathbf{q} = \mathbf{0}$, ψ with $\psi_i \in \mathcal{D}(\mathcal{R}_i)$, $\psi_j = 0$ for $j \neq i$, and $\psi_S = 0$.
We have now:

$$\int_{\mathcal{R}_i} (\, \psi_i \mathrm{div}\, \mathbf{p}_i \,+\, \sigma_i \phi_i \psi_i) \,=\, \int_{\mathcal{R}_i} S_{f,i} \psi_i,$$

so that we recover $\mathrm{div}\, \mathbf{p}_i \,+\, \sigma_i \phi_i = S_{f,i}$ in $L^2(\mathcal{R}_i)$.
For the remainder of the proof, we consider $\psi = 0$, $\psi_S = 0$, i.e.

$$\forall \mathbf{q} \in \mathbf{Q},\, \int_{\mathcal{R}} D^{-1} \mathbf{p} \cdot \mathbf{q} \,+\, \phi \,\mathrm{div}\, \mathbf{q} \,-\, \int_{\Gamma_S} [\mathbf{q} \cdot \mathbf{n}]\, \phi_S \,=\, 0.$$

Consider $\mathbf{q}$ with $\mathbf{q}_i \in \mathcal{D}(\mathcal{R}_i)^d$ and $\mathbf{q}_j = 0$ for $j \neq i$. We have:

$$\begin{array}{rrcl} & \displaystyle\int_{\mathcal{R}_i} \left(-D_i^{-1} \mathbf{p}_i \cdot \mathbf{q}_i \,+\, \phi_i \mathrm{div}\, \mathbf{q}_i \right) & = & 0,\ \forall \mathbf{q}_i \in \mathcal{D}(\mathcal{R}_i)^d, \\ \Leftrightarrow & \displaystyle -\int_{\mathcal{R}_i} D_i^{-1} \mathbf{p}_i \cdot \mathbf{q}_i \,-\, \langle \mathbf{grad}\, \phi_i \,,\, \mathbf{q}_i \rangle & = & 0,\ \forall \mathbf{q}_i \in \mathcal{D}(\mathcal{R}_i)^d, \\ \Leftrightarrow & -D_i^{-1} \mathbf{p}_i - \mathbf{grad}\, \phi_i & = & 0,\ \text{in } \left(\mathcal{D}(\mathcal{R}_i)^d \right)', \\ \Leftrightarrow & -D_i^{-1} \mathbf{p}_i - \mathbf{grad}\, \phi_i & = & 0,\ \text{in } \mathbf{L}^2(\mathcal{R}_i). \end{array} \tag{3.12}$$

We reached the last line noting that $\mathbf{p}_i \in \mathbf{L}^2(\mathcal{R}_i)$. In particular $\phi_i \in H^1(\mathcal{R}_i)$.
Let us show next that $\phi_i = \phi_j$ on Γ_{ij}, when $\Gamma_{ij} \neq \emptyset$. Introduce $\mathcal{R}_{ij} := int(\overline{\mathcal{R}_i} \cup \overline{\mathcal{R}_j})$.
Consider $\mathbf{q} \in \mathbf{Q}$ such that $\mathbf{q}_{|\mathcal{R}_{ij}} \in (\mathcal{D}(\mathcal{R}_{ij}))^d$, and $\mathbf{q} = 0$ elsewhere. We have $[\mathbf{q} \cdot \mathbf{n}] = 0$ by construction. We obtain that:

$$\begin{array}{rrcl} & \displaystyle\sum_{k=i,j} \int_{\mathcal{R}_k} -D_k^{-1} \mathbf{p}_k \cdot \mathbf{q} \,+\, \phi_k \,\mathrm{div}\, \mathbf{q} & = & 0 \\ \Leftrightarrow & \displaystyle\sum_{k=i,j} \int_{\mathcal{R}_k} \mathbf{grad}\, \phi_k \cdot \mathbf{q} \,+\, \phi_k \,\mathrm{div}\, \mathbf{q} & = & 0,\ \text{from (3.12)}, \\ \Leftrightarrow & \displaystyle\sum_{k=i,j} \int_{\partial \mathcal{R}_k} \phi_k \, \mathbf{q} \,.\, \mathbf{n}_k & = & 0,\ \text{i.b.p. on } \mathcal{R}_i,\, \mathcal{R}_j \\ \Leftrightarrow & \displaystyle\int_{\Gamma_{ij}} (\, \phi_i \,-\, \phi_j \,)\, \mathbf{q} \,.\, \mathbf{n}_i & = & 0, \end{array}$$

so that $\phi_i \,=\, \phi_j$ on Γ_{ij}. It follows in particular that $\phi \in H^1(\mathcal{R})$.
We recover the homogeneous Dirichlet boundary condition on $\partial \mathcal{R}$ choosing $\mathbf{q} \in$

$(\,\mathcal{C}^\infty(\overline{\mathcal{R}})\,)^d$ since it holds: $\displaystyle\int_{\mathcal{R}} \mathbf{grad}\,\phi\cdot\mathbf{q} \,+\, \phi\,\mathrm{div}\,\mathbf{q} \,=\, 0$. We find $\phi \in H_0^1(\mathcal{R})$.
Finally, let us prove that $\phi_i = \phi_S$ on a given Γ_{ij}. We consider $\mathbf{q}\in\mathbf{Q}$ such that $\mathbf{q}_i \in (\,\mathcal{C}^\infty(\overline{\mathcal{R}_i})\,)^d$, $\mathbf{q}_i = 0$ in a neighbourhood of $\partial\mathcal{R}_i\setminus\overline{\Gamma_{ij}}$ and $\mathbf{q}_j = 0$ for $j\neq i$. We have:

$$\begin{array}{rrcl} & \displaystyle\int_{\mathcal{R}_i} \mathbf{grad}\,\phi_i\cdot\mathbf{q}_i \,+\, \phi_i\,\mathrm{div}\,\mathbf{q}_i \,-\, \int_{\Gamma_{ij}} \mathbf{q}_i\cdot\mathbf{n}_i\,\phi_S & = & 0, \\ \Leftrightarrow & \displaystyle\int_{\Gamma_{ij}} \mathbf{q}_i\cdot\mathbf{n}_i\,(\,\phi_i \,-\, \phi_S\,) & = & 0, \text{ i.b.p. on } \mathcal{R}_i, \end{array}$$

so that $\phi_i \,=\, \phi_S$ on Γ_{ij}. □

In order to prove that Problem (3.4) has a unique solution, one can use Theorem 1 directly. In our case however, we would like to discretize and solve numerically the mixed, multi-domain variational formulation (3.11), so we elect to show the so-called inf-sup condition for the variational formulation (3.11).

4. Well-posedness of the DD+L^2-jumps method

4.1. **Lifting of Lagrange multipliers.** Consider $\phi_S^\star \in M$ and, for $1\le i\le N$, $\phi_i^\star \in L^2(\mathcal{R}_i)$.
• If $i\in\mathcal{I}_S$ ($\partial\mathcal{R}_i\cap\Gamma_S = \partial\mathcal{R}_i$), we add a constant value δ_i to $\phi_i^\star$ so that $\widetilde{\phi_i} = \phi_i^\star + \delta_i$ has a mean value equal to $-\dfrac{1}{2}|\mathcal{R}_i|^{-1}\displaystyle\int_{\partial\mathcal{R}_i}\phi_S^\star$. We set $\widetilde{\phi_{i,S}} := \phi^\star_{S|\partial\mathcal{R}_i}$.
• If $i\notin\mathcal{I}_S$, we consider $\widetilde{\phi_{i,S}} \in L^2(\partial\mathcal{R}_i)$, the continuation of $\phi^\star_{S|\partial\mathcal{R}_i\cap\Gamma_S}$ by a constant value $\gamma_{i,S}$ to $\partial\mathcal{R}_i$. $\gamma_{i,S}$ is chosen so that the mean value of $\widetilde{\phi_{i,S}}$ is equal to $-2|\partial\mathcal{R}_i|^{-1}\displaystyle\int_{\mathcal{R}_i}\phi_i^\star$. We set $\widetilde{\phi_i} = \phi_i^\star$.
For $i = 1, N$, one has $\widetilde{\phi_i}\in L^2(\mathcal{R}_i)$ and $\widetilde{\phi_{i,S}}\in L^2(\partial\mathcal{R}_i)$ with:

$$\|\widetilde{\phi_i}\|_{0,\mathcal{R}_i} + \|\widetilde{\phi_{i,S}}\|_{0,\partial\mathcal{R}_i} \lesssim \|\phi_S^\star\|_{0,\partial\mathcal{R}_i\cap\Gamma_S} + \|\phi_i^\star\|_{0,\mathcal{R}_i}.$$

In primal form, the lifting reads: Find $u_i\in H^1(\mathcal{R}_i)$ such that

$$\left\{\begin{array}{rcll} -\Delta u_i & = & \widetilde{\phi_i} & \text{in } \mathcal{R}_i, \\ \dfrac{\partial u_i}{\partial n_i} & = & \dfrac{1}{2}\widetilde{\phi_{i,S}} & \text{on } \partial\mathcal{R}_i. \end{array}\right. \tag{4.1}$$

This is a pure Neumann problem for which there is constraint on the data. It is automatically fulfilled since $\displaystyle\int_{\mathcal{R}_i}\widetilde{\phi_i} + \frac{1}{2}\int_{\partial\mathcal{R}_i}\widetilde{\phi_{i,S}} = 0$, so well-posedness follows.
For all i, one has $\|u_i\|_{1,\mathcal{R}_i} + \|\Delta u_i\|_{0,\mathcal{R}_i} \lesssim \|\phi_S^\star\|_{0,\partial\mathcal{R}_i\cap\Gamma_S} + \|\phi_i^\star\|_{0,\mathcal{R}_i}$.
As far as the regularity of u_i is concerned, we recall that, according to [15], there holds $u_i\in H^{3/2}(\mathcal{R}_i)$, and $\|u_i\|_{3/2,\mathcal{R}_i} \lesssim \|\phi_S^\star\|_{0,\partial\mathcal{R}_i\cap\Gamma_S} + \|\phi_i^\star\|_{0,\mathcal{R}_i}$.
However, in our case, it does not fit our formulation of the diffusion equation as the mixed, multi-domain problem (3.4) or (3.11). Hence we choose to solve instead the equivalent problem:

Find $(\mathbf{v}_i, u_i) \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}_i) \times H^1(\mathcal{R}_i)$ such that

$$
\left\{
\begin{array}{rcll}
-\mathbf{v}_i + \mathbf{grad}\, u_i &=& 0, & \text{in } \mathcal{R}_i, \\
-\mathrm{div}\, \mathbf{v}_i &=& \widetilde{\phi}_i, & \text{in } \mathcal{R}_i, \\
\mathbf{v}_i \cdot \mathbf{n}_i &=& \dfrac{1}{2}\widetilde{\phi_{i,S}} & \text{on } \partial\mathcal{R}_i.
\end{array}
\right. \tag{4.2}
$$

Well-posedness and continuity of the lifting follow, indeed for the latter we have

$$
\|\mathbf{v}_i\|_{\mathbf{H}(\mathrm{div}\,, \mathcal{R}_i)} \leq C_{lift,i}\,(\,\|\phi_S^\star\|_{0,\partial\mathcal{R}_i\cap\Gamma_S} + \|\phi_i^\star\|_{0,\mathcal{R}_i}\,)\,. \tag{4.3}
$$

Regarding now the regularity of $\mathbf{v}_i$, there holds $\mathbf{v}_i \in \mathbf{H}^{1/2}(\mathcal{R}_i)$, and $\|\mathbf{v}_i\|_{1/2,\mathcal{R}_i} \lesssim \|\phi_S^\star\|_{0,\partial\mathcal{R}_i\cap\Gamma_S} + \|\phi_i^\star\|_{0,\mathcal{R}_i}$. According for instance to [16, §7], an equivalent, mixed variational formulation of problem (4.2) reads:
Find $(\mathbf{v}_i, u_i) \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}_i) \times L^2(\mathcal{R}_i)$ such that $\forall\,(\mathbf{q}_i^0, \psi_i) \in \mathbf{H}_0(\mathrm{div}\,, \mathcal{R}_i) \times L^2(\mathcal{R}_i)$:

$$
\left\{
\begin{array}{c}
\displaystyle\int_{\mathcal{R}_i} \mathbf{v}_i \cdot \mathbf{q}_i^0 + \int_{\mathcal{R}_i} \mathrm{div}\, \mathbf{v}_i\, \psi_i + \int_{\mathcal{R}_i} \mathrm{div}\, \mathbf{q}_i^0\, u_i = -\int_{\mathcal{R}_i} \widetilde{\phi}_i\, \psi_i, \\[2ex]
\mathbf{v}_i \cdot \mathbf{n}_i = \dfrac{1}{2}\widetilde{\phi_{i,S}} \text{ on } \partial\mathcal{R}_i.
\end{array}
\right. \tag{4.4}
$$

One actually solves the problem in the variable $\mathbf{v}_i^0 := \mathbf{v}_i - \mathbf{v}_i^{lift} \in \mathbf{H}_0(\mathrm{div}\,, \mathcal{R}_i)$, where $\mathbf{v}_i^{lift} \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}_i)$ is a continuous lifting of the normal trace $\dfrac{1}{2}\widetilde{\phi_{i,S}}$, and well-posedness is recovered with the help of an inf-sup condition.

4.2. **Inf-sup condition.** We can now proceed to obtain the well-posedness of (3.11) by proving in particular an inf-sup condition.

Theorem 3. *There exists a unique solution $\mathtt{u} \in \mathtt{W}$ to the mixed, multi-domain variational formulation (3.11).*

We give the proof of this result although existence and uniqueness of $\mathtt{u}$ is a consequence of the results of §3. As a matter of fact, the proof of the discrete inf-sup condition is obtained along the same lines.

Proof. To prove the claim, one looks for an inf-sup condition and a solvability condition [16, 17] to ensure well-posedness. The solvability condition writes

$$
\text{The set } \{\, \mathtt{w} \in \mathtt{W} \,|\, \forall \mathtt{u} \in \mathtt{W},\ c_S(\mathtt{u}, \mathtt{w}) = 0\,\} \text{ is equal to } \{0\}.
$$

Given an element $\mathtt{w} = (\mathbf{p}, \phi, \phi_S)$ of the set defined above, one checks that $(\mathbf{p}, \phi, \phi_S)$ solves (3.4) with zero data: this is analogous to the proof of Theorem 2. By uniqueness of the solution (cf. Theorem 1), it follows that $\mathtt{w} = 0$.
The inf-sup condition writes:

$$
\exists \eta > 0, \quad \inf_{\mathtt{u}\in\mathtt{W}} \sup_{\mathtt{w}\in\mathtt{W}} \frac{c_S(\mathtt{u}, \mathtt{w})}{\|\mathtt{u}\|_{\mathtt{W}}\, \|\mathtt{w}\|_{\mathtt{W}}} \geq \eta. \tag{4.5}
$$

• Assume first that $\mathrm{div}\, \mathbf{p}_i = 0$ in $\mathcal{R}_i$ for all $i = 1, N$, and $\phi_S = 0$. Then $c_S(\mathtt{u}, \mathtt{w})$ writes:

$$
c_S(\mathtt{u}, \mathtt{w}) = \int_{\mathcal{R}} \left(-D^{-1}\mathbf{p} \cdot \mathbf{q} + \phi\, \mathrm{div}\, \mathbf{q} + \sigma\, \phi\, \psi\right) + \int_{\Gamma_S} [\mathbf{p} \cdot \mathbf{n}]\, \psi_S. \tag{4.6}
$$

By choosing $\left\{ \begin{array}{rcll} (\mathbf{q},\psi) & = & (-\mathbf{p},\phi) & \in \mathbf{X}, \\ \psi_S & = & [\mathbf{p}\cdot\mathbf{n}] & \in M, \end{array}\right.$ one has: $\|\mathtt{u}\|_{\mathtt{W}} \geq \frac{1}{\sqrt{2}}\,\|\mathtt{w}\|_{\mathtt{W}}$, and moreover the expression of $c_S(\mathtt{u},\mathtt{w})$ involves only positive terms, which yields the bound:

$$
\begin{array}{rcl}
c_S(\mathtt{u},\mathtt{w}) & = & \displaystyle\int_{\mathcal{R}} \left(D^{-1}\mathbf{p}\cdot\mathbf{p} + \sigma\,\phi^2\right) + \int_{\Gamma_S} [\mathbf{p}\cdot\mathbf{n}]^2, \\
& \geq & \beta\,\left(\|\mathbf{p}\|^2_{\mathbf{Q}} + \|\phi\|^2_{0,\mathcal{R}}\right),\ \text{with } \beta := \min\left((D^*)^{-1}, 1, \sigma_*\right) > 0, \\
& = & \beta\,\|(\mathbf{p},\phi)\|^2_{\mathbf{X}} = \beta\,\|((\mathbf{p},\phi),\phi_S)\|^2_{\mathtt{W}}, \\
& \geq & \dfrac{\beta}{\sqrt{2}}\,\|\mathtt{u}\|_{\mathtt{W}}\|\mathtt{w}\|_{\mathtt{W}}.
\end{array}
\tag{4.7}
$$

• In the more general case $\operatorname{div}\mathbf{p}_i \neq 0$ for some i and $\phi_S = 0$, one can still proceed along the same lines, as long as the terms in $\phi_i \operatorname{div}\mathbf{p}_i$ cancel out. To fulfill the conditions, a possible choice to bound c_S is:

$$
\left\{ \begin{array}{rcll}
\mathbf{q} & = & -\mathbf{p} & \in \mathbf{Q}, \\
\psi_i & = & \dfrac{1}{2}\phi_i + \dfrac{1}{2}(\sigma_i)^{-1}\operatorname{div}\mathbf{p}_i & \in L^2(\mathcal{R}_i),\ \text{for } i = 1, N\,, \\
\psi_S & = & [\mathbf{p}\cdot\mathbf{n}] & \in M.
\end{array}\right.
$$

One can check that: $\|\mathtt{u}\|_{\mathtt{W}} \geq \lambda\,\|\mathtt{w}\|_{\mathtt{W}}$, with $\lambda := \min\left((1+\frac{1}{2}(\sigma_*)^{-2})^{-1/2}, 1/\sqrt{2}\right) > 0$. The bound on c_S now reads:

$$
\begin{array}{rcl}
c_S(\mathtt{u},\mathtt{w}) & = & \displaystyle\int_{\mathcal{R}} \left(D^{-1}\mathbf{p}\cdot\mathbf{p} + \frac{1}{2}\sigma^{-1}(\operatorname{div}\mathbf{p})^2 \,+\, \frac{1}{2}\sigma\,\phi^2\right) + \int_{\Gamma_S} [\mathbf{p}\cdot\mathbf{n}]^2, \\
& \geq & \gamma\,\left(\|\mathbf{p}\|^2_{\mathbf{Q}} + \|\phi\|^2_{0,\mathcal{R}}\right),\ \text{with } \gamma := \min\left((D^*)^{-1}, \dfrac{1}{2}\sigma_*, \dfrac{1}{2}(\sigma^*)^{-1}, 1\right) > 0, \\
& = & \gamma\,\|(\mathbf{p},\phi)\|^2_{\mathbf{X}} \,=\, \gamma\,\|((\mathbf{p},\phi),\phi_S)\|^2_{\mathtt{W}}, \\
& \geq & \gamma\,\lambda\,\|\mathtt{u}\|_{\mathtt{W}}\|\mathtt{w}\|_{\mathtt{W}},
\end{array}
$$

• In the general case, $\operatorname{div}\mathbf{p}_i \neq 0$ for some i and $\phi_S \neq 0$, we now choose:

$$
\left\{ \begin{array}{rclll}
\mathbf{q}_i & = & -\mathbf{p}_i - \alpha\mathbf{v}_i & \in \mathbf{H}(\operatorname{div}, \mathcal{R}_i), & \text{for } i = 1, N\,, \\
\psi_i & = & \dfrac{1}{2}\phi_i + \dfrac{1}{2}(\sigma_i)^{-1}\operatorname{div}\mathbf{p}_i & \in L^2(\mathcal{R}_i), & \text{for } i = 1, N\,, \\
\psi_S & = & -\phi_S + [\mathbf{p}\cdot\mathbf{n}] & \in M, &
\end{array}\right.
$$

where $\alpha > 0$ is to be fixed and $\mathbf{v}_i$, $i = 1, N$ are defined by (4.2) or (4.4) with $\phi^\star_S = \phi_S$ and $\phi^\star_i = 0$.
We have:

$$
\begin{array}{rcl}
c_S(\mathtt{u},\mathtt{w}) & = & \displaystyle\int_{\mathcal{R}} \left(D^{-1}\mathbf{p}\cdot\mathbf{p} + \frac{1}{2}\sigma^{-1}(\operatorname{div}\mathbf{p})^2 + \frac{1}{2}\sigma\,\phi^2\right) \\
& & \displaystyle + \int_{\Gamma_S} \left([\mathbf{p}\cdot\mathbf{n}]^2 + \alpha\,\phi_S^2\right) \,+\, \int_{\mathcal{R}} \alpha\,D^{-1}\mathbf{p}\cdot\mathbf{v} \\
& & \displaystyle -\frac{\alpha}{2}\sum_{i\in\mathcal{I}_S} |\mathcal{R}_i|^{-1}\int_{\partial\mathcal{R}_i}\phi_S\int_{\mathcal{R}_i}\phi_i.
\end{array}
\tag{4.8}
$$

Using successively Young's and Cauchy-Schwarz' inequalities, one can bound the last term. We get that $\forall i \in \mathcal{I}_S$, $\forall \eta > 0$:

$$-\frac{\alpha}{2}\,|\mathcal{R}_i|^{-1}\int_{\partial\mathcal{R}_i}\phi_S\int_{\mathcal{R}_i}\phi \geq -\frac{\alpha}{4}\left(\,|\partial\mathcal{R}_i|\,|\mathcal{R}_i|^{-1}\eta\,\|\phi_S\|^2_{0,\partial\mathcal{R}_i} + \eta^{-1}\|\phi\|^2_{0,\mathcal{R}_i}\,\right).$$

Let us bound the last-but-one term of (4.8). We have for $i = 1, N$:

$$(4.9)\qquad \begin{aligned} \int_{\mathcal{R}_i}\alpha\,D_i^{-1}\mathbf{p}_i\cdot\mathbf{v}_i \quad &\geq \quad -\frac{1}{2}\int_{\mathcal{R}_i}\left(D_i^{-1}\mathbf{p}_i\cdot\mathbf{p}_i + \alpha^2\,D_i^{-1}\mathbf{v}_i\cdot\mathbf{v}_i\right), \\ &\geq \quad -\frac{1}{2}\int_{\mathcal{R}_i}D_i^{-1}\mathbf{p}_i\cdot\mathbf{p}_i - \frac{\alpha^2}{2}\,(D_*)^{-1}\,C^2\,\|\phi_S\|^2_{0,\partial\mathcal{R}_i\cap\Gamma_S}, \end{aligned}$$

according to (4.3), with $C = \max_{i=1,N} C_{lift,i}$.
Summing for $i = 1, N$, we deduce that:

$$(4.10)\qquad \begin{aligned} c_S(\mathtt{u},\mathtt{w}) \quad &\geq \quad \frac{1}{2}\int_{\mathcal{R}}\left(D^{-1}\mathbf{p}\cdot\mathbf{p} + \sigma^{-1}(\operatorname{div}\mathbf{p})^2 + \left(\sigma - \frac{\alpha}{2\,\eta}\right)\phi^2\right) \\ &\quad + \int_{\Gamma_S}[\mathbf{p}\cdot\mathbf{n}]^2 + \alpha\left(1 - \frac{1}{2}\alpha\,C^2\,(D_*)^{-1} - \eta\,C'\right)\int_{\Gamma_S}\phi_S^2, \end{aligned}$$

where $C' = \dfrac{1}{4}\max_{i\in\mathcal{I}_S}(\,|\partial\mathcal{R}_i|\,|\mathcal{R}_i|^{-1}\,) > 0$. Set $\eta = \dfrac{1}{2}(C')^{-1}$.
Choose $\alpha > 0$ so that $\alpha^* := \min\left(\alpha(1 - \alpha\,C^2\,(D_*)^{-1}), \sigma_* - \alpha C'\right) > 0$. We have:

$$(4.11)\qquad c_S(\mathtt{u},\mathtt{w}) \geq \eta_0\,\|\mathtt{u}\|^2_{\mathtt{W}},$$

with $\eta_0 := \min\left(\frac{1}{2}(D^*)^{-1}, \frac{1}{2}(\sigma^*)^{-1}, \frac{1}{2}\alpha^*, 1\right) > 0$. Finally one gets that:

$$(4.12)\qquad \|\mathtt{u}\|_{\mathtt{W}} \geq \eta_1\,\|\mathtt{w}\|_{\mathtt{W}},$$

with $\eta_1 := \eta_1\,(\sigma_*, C, \alpha) > 0$.
We obtain thus the inf-sup condition (4.5) with $\eta = \eta_0\,\eta_1$. □

5. Discretization

We study abstract, conforming, discretization of the variational formulation (3.11). To that aim, we introduce discrete, finite-dimensional, spaces indexed by a (small) parameter h as follows:

$$\mathbf{Q}_{i,h} \subset \mathbf{H}(\operatorname{div}, \mathcal{R}_i), \text{ and } L_{i,h} \subset L^2(\mathcal{R}_i), \text{ for } i = 1, N.$$

For approximation purposes, and following Definition 2.14 in [17], we assume for every i that $(\mathbf{Q}_{i,h})_h$, resp. $(L_{i,h})_h$ have the *approximability property* in the sense that

$$(5.1)\qquad \begin{aligned} &\forall \mathbf{q}_i \in \mathbf{H}(\operatorname{div}, \mathcal{R}_i),\ \lim_{h\to 0}\left(\inf_{\mathbf{q}_{i,h}\in\mathbf{Q}_{i,h}}\|\mathbf{q}_i - \mathbf{q}_{i,h}\|_{\mathbf{H}(\operatorname{div},\mathcal{R}_i)}\right) = 0, \\ &\forall \psi_i \in L^2(\mathcal{R}_i),\ \lim_{h\to 0}\left(\inf_{\psi_{i,h}\in L_{i,h}}\|\psi_i - \psi_{i,h}\|_{0,\mathcal{R}_i}\right) = 0, \end{aligned}$$

and also that $L_{i,h}$ includes piecewise constant fields.
We impose the following requirements:

- $\mathbf{q}_{i,h}\cdot\mathbf{n}_{|\partial\mathcal{R}_i} \in L^2(\partial\mathcal{R}_i)$ for all $\mathbf{q}_{i,h} \in \mathbf{Q}_{i,h}$, $i = 1, N$;
- $\operatorname{div}\mathbf{Q}_{i,h} \subset L_{i,h}$, $i = 1, N$.

Then, let

$$\mathbf{Q}_h = \prod_{i=1,N} \mathbf{Q}_{i,h} \quad \text{and} \quad L_h = \prod_{i=1,N} L_{i,h}.$$

In particular, the discretization $\mathbf{Q}_h \times L_h$ is globally conforming in $\mathbf{Q} \times L^2(\mathcal{R})$. We endow $\mathbf{Q}_h$ with the norm $\|\cdot\|_{\mathbf{Q}}$, while L_h is endowed with $\|\cdot\|_{0,\mathcal{R}}$.
We then define $T_{i,h}$ as the space of the normal traces of vectors of $\mathbf{Q}_{i,h}$ on $\partial\mathcal{R}_i \cap \Gamma_S$:

$$T_{i,h} := \left\{ q_{i,h} \in L^2(\partial\mathcal{R}_i \cap \Gamma_S) \,|\, \exists\, \mathbf{q}_{i,h} \in \mathbf{Q}_{i,h} \,|\, q_{i,h} = \mathbf{q}_{i,h} \cdot \mathbf{n}_{i|\partial\mathcal{R}_i \cap \Gamma_S} \right\}. \tag{5.2}$$

Several situations[2] can occur on a given interface Γ_{ij}:

- *non-nested meshes*: $T_{i,h} \not\subset T_{j,h}$ and $T_{j,h} \not\subset T_{i,h}$;
- *nested meshes*: $T_{i,h} \subset T_{j,h}$ or $T_{j,h} \subset T_{i,h}$;
- *matching meshes*: nested meshes with $T_{i,h} = T_{j,h}$.

Remark 3. *Matching domain decompositions correspond to matching meshes ; resp., non-matching domain decompositions correspond either to non-nested, or nested but non-matching, meshes.*

Let us denote by $M_h \subset M$ the discrete space of the Lagrange multipliers, and by $M_h^{ij} \subset L^2(\Gamma_{ij})$ the discrete space of the Lagrange multipliers restricted to the interface Γ_{ij}. We introduce the discrete projection operators from the spaces of normal traces $T_{i,h}$ to M_h, and vice versa:

$$\left\{ \begin{array}{rcl} \Pi_i : T_{i,h} & \to & M_h \\ q_{i,h} & \mapsto & \Pi_i(q_{i,h}) \end{array} \right. , \qquad \left\{ \begin{array}{rcl} \pi_i : M_h & \to & T_{i,h} \\ \psi_{S,h} & \mapsto & \pi_i(\psi_{S,h}) \end{array} \right. .$$

These projections are defined by:

$$\forall q_{i,h} \in T_{i,h}, \forall \psi_{S,h} \in M_h \quad \left\{ \begin{array}{l} \displaystyle\int_{\partial\mathcal{R}_i \cap \Gamma_S} (\Pi_i(q_{i,h}) - q_{i,h})\ \psi_{S,h} = 0 \\ \displaystyle\int_{\partial\mathcal{R}_i \cap \Gamma_S} (\pi_i(\psi_{S,h}) - \psi_{S,h})\ q_{i,h} = 0 \end{array} \right. . \tag{5.3}$$

As the operators Π_i and π_i are orthogonal projections, they are continuous, with a continuity modulus equal to 1.

Proposition 2. *The following inequalities hold:*

$$\forall q_{i,h} \in T_{i,h}, \forall \psi_{S,h} \in M_h \quad \left\{ \begin{array}{l} \|\pi_i(\psi_{S,h})\|_{0,\partial\mathcal{R}_i\cap\Gamma_S} \leq \|\psi_{S,h}\|_{0,\partial\mathcal{R}_i\cap\Gamma_S} \textit{ and} \\ \|\Pi_i(q_{i,h})\|_{0,\partial\mathcal{R}_i\cap\Gamma_S} \leq \|q_{i,h}\|_{0,\partial\mathcal{R}_i\cap\Gamma_S}. \end{array} \right. \tag{5.4}$$

Next, let $\mathbf{p}_h \in \mathbf{Q}_h$. We define the discrete jump of the normal component of $\mathbf{p}_h$ on the interface Γ_{ij} as $[\mathbf{p}_h \cdot \mathbf{n}]_{h,ij} := \sum_{k=i,j} \Pi_k(\mathbf{p}_{k,h} \cdot \mathbf{n}_{k|\Gamma_{ij}})$. The discrete global jump of the normal component, $[\mathbf{p}_h \cdot \mathbf{n}_h]_h \in M_h$, is defined by:

$$[\mathbf{p}_h \cdot \mathbf{n}]_{h|\Gamma_{ij}} := [\mathbf{p}_h \cdot \mathbf{n}]_{h,ij}, \text{ for } i, j = 1, N.$$

We finally define:

$$\begin{array}{rcl} \mathbf{X}_h & = & \{\, \xi_h := (\mathbf{q}_h, \psi_h) \in \mathbf{Q}_h \times L_h \} \text{ , endowed with } \|\cdot\|_{\mathbf{X}} \,, \\ \mathtt{W}_h & = & \{\, \mathtt{w}_h := (\xi_h, \psi_{S,h}) \in \mathbf{X}_h \times M_h \} \text{ , endowed with } \|\cdot\|_{\mathtt{W}} \,. \end{array}$$

[2]Non-nested meshes are also called sliding meshes in the literature, see [18].

In the remainder of this section, we develop the numerical analysis for the DD+L^2-jumps method. In our setting, the conforming discretization of the variational formulation (3.11) reads:
Find $((\mathbf{p}_h, \phi_h), \phi_{S,h}) \in \mathtt{W}_h$, such that $\forall((\mathbf{q}_h, \psi_h), \psi_{S,h}) \in \mathtt{W}_h$:

$$\begin{aligned} &\int_{\mathcal{R}} \left(-D^{-1}\,\mathbf{p}_h \cdot \mathbf{q}_h \;+\; \phi_h \operatorname{div} \mathbf{q}_h \;+\; \psi_h \operatorname{div} \mathbf{p}_h \;+\; \sigma\, \phi_h\, \psi_h \right) \\ &\qquad + \int_{\Gamma_S} [\mathbf{p}_h \cdot \mathbf{n}]\, \psi_{S,h} - \int_{\Gamma_S} [\mathbf{q}_h \cdot \mathbf{n}]\, \phi_{S,h} = \int_{\mathcal{R}} S_f\, \psi_h. \end{aligned} \tag{5.5}$$

Or equivalently:

$$\text{Find } \mathtt{u}_h \in \mathtt{W}_h \text{ such that } \forall \mathtt{w}_h \in \mathtt{W}_h,\; c_S(\mathtt{u}_h, \mathtt{w}_h) = f(\mathtt{w}_h). \tag{5.6}$$

For the study of the well-posedness in the discrete case, we have to lift continuously the discrete Lagrange multipliers. To that aim, one simply discretizes the variational formulation in §4.1. So we assume that, given $\phi_h \in L_h$ and $\phi_{S,h} \in M_h$, there exists for all $i = 1, N$ a discrete lifting $\mathbf{v}_{i,h} \in \mathbf{Q}_{i,h}$ that fulfills the discrete counterpart of (4.3) and of (4.4) (the constant is denoted by C'_{lift}). We label those two discrete conditions (4.3h)-(4.4h).

5.1. **Discrete inf-sup condition.** The discrete inf-sup condition to be found writes:

$$\exists\, \eta_h > 0, \quad \inf_{\mathtt{u}_h \in \mathtt{W}_h} \sup_{\mathtt{w}_h \in \mathtt{W}_h} \frac{c_S(\mathtt{u}_h, \mathtt{w}_h)}{\|\mathtt{u}_h\|_{\mathtt{W}}\, \|\mathtt{w}_h\|_{\mathtt{W}}} \geq \eta_h. \tag{5.7}$$

Once (5.7) is achieved, one obtains existence and uniqueness of the discrete solution $\mathtt{u}_h$, hence the corresponding linear system is well-posed. So conditions ensuring that (5.7) holds can be viewed as *algebraic conditions*. More generally, our aim is to obtain that $(\eta_h)_h$ is uniformly bounded away from 0. In this sense, one has a *uniform discrete inf-sup condition*, from which the error analysis can be derived.

Theorem 4. *Assume that the conditions*

$$\exists \beta_h > 0,\, \forall \mathbf{q}_h \in \mathbf{Q}_h,\; \int_{\Gamma_S} [\mathbf{q}_h \cdot \mathbf{n}]_h\, [\mathbf{q}_h \cdot \mathbf{n}] \geq \beta_h \int_{\Gamma_S} [\mathbf{q}_h \cdot \mathbf{n}]^2 \tag{5.8}$$

and

$$\begin{aligned} &\exists \gamma_h > 0,\, \forall \psi_{S,h} \in M_h, \\ &\sum_{i=1}^{N} \sum_{j=i+1}^{N} \int_{\Gamma_{ij}} \left(\pi_i(\psi_{S,h})^2 \;+\; \pi_j(\psi_{S,h})^2 \right) \geq \gamma_h \|\psi_{S,h}\|_M^2, \end{aligned} \tag{5.9}$$

hold, then the discrete inf-sup condition (5.7) is fulfilled. If in addition β_h and γ_h can be chosen independently of h in these two inequalities, (5.7) is a uniform discrete inf-sup condition.

Remark 4. *In other words, (5.8)-(5.9) can be seen as the* algebraic conditions. *We discuss these conditions in the next subsection.*

Proof. In order to show the inf-sup condition (5.7), we consider $\mathtt{u}_h := ((\mathbf{p}_h, \phi_h), \phi_{S,h}) \in \mathtt{W}_h$, and we define $\mathtt{w}_h := ((\mathbf{q}_h, \psi_h), \psi_{S,h}) \in \mathtt{W}_h$ as

$$\tag{5.10} \left\{ \begin{array}{lcll} \mathbf{q}_{i,h} & = & -\mathbf{p}_{i,h} - \alpha' \mathbf{v}_{i,h}, & \text{for } i = 1, N, \\ \\ \psi_{i,h} & = & \dfrac{1}{2}\phi_{i,h} + \dfrac{1}{2}(\sigma_i)^{-1} \mathrm{div}\, \mathbf{p}_{i,h}, & \text{for } i = 1, N, \\ \\ \psi_{S,h} & = & -\phi_{S,h} + [\mathbf{p}_h \cdot \mathbf{n}]_h, & \end{array} \right.$$

where $\alpha' > 0$ is to be determined, whereas the discrete lifting $\mathbf{v}_{i,h}$ of $\phi_{S,h}$ is governed by (4.3h)-(4.4h) with data $\phi_S^\star = \pi_i(\phi_{S,h})$ and $\phi_i^\star = 0$. To have $\psi_h \in L_h$ in (5.10), we assume for the moment that σ_i is constant for $i = 1, N$. The general case will be considered hereafter. With this choice, we have:

$$\tag{5.11} \begin{array}{rcl} c_S(\mathtt{u}_h, \mathtt{w}_h) & = & \displaystyle\int_{\mathcal{R}} \left(D^{-1}\mathbf{p}_h \cdot \mathbf{p}_h + \frac{1}{2}\sigma^{-1} (\mathrm{div}\, \mathbf{p}_h)^2 + \frac{1}{2} \sigma\, \phi_h^2 \right) \\ & & \displaystyle + \int_{\Gamma_S} [\mathbf{p}_h \cdot \mathbf{n}]_h \, [\mathbf{p}_h \cdot \mathbf{n}] \\ & & \displaystyle + \frac{\alpha'}{2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \int_{\Gamma_{ij}} \phi_{S,h} \; (\, \pi_i(\phi_{S,h}) \; + \; \pi_j(\phi_{S,h})\,) \\ & & \displaystyle + \alpha' \int_{\mathcal{R}} D^{-1} \mathbf{p}_h \cdot \mathbf{v}_h \\ & & \displaystyle - \frac{\alpha'}{2} \sum_{i \in \mathcal{I}_S} |\mathcal{R}_i|^{-1} \int_{\partial \mathcal{R}_i} \pi_i(\phi_{S,h}) \int_{\mathcal{R}_i} \phi_{i,h}. \end{array}$$

According to (4.3h), (5.4), and following (4.9) we can bound the last but one term like:

$$\int_{\mathcal{R}} \alpha' D^{-1} \mathbf{p}_h \cdot \mathbf{v}_h \geq -\frac{1}{2} \int_{\mathcal{R}} D^{-1} \mathbf{p}_h \cdot \mathbf{p}_h - \frac{1}{2} \alpha'^2 (D_*)^{-1} (C'_{lift})^2 \|\phi_{S,h}\|_M^2.$$

Moreover using the definition (5.3) of the projections π_i and π_j, we have:

$$\tag{5.12} \int_{\Gamma_{ij}} \phi_{S,h} \; (\pi_i(\phi_{S,h}) \; + \; \pi_j(\phi_{S,h})) \; = \; \int_{\Gamma_{ij}} \left(\pi_i(\phi_{S,h})^2 \; + \; \pi_j(\phi_{S,h})^2 \right).$$

Then using Young's inequality for the last term in (5.11), it follows that $\forall \eta > 0$:

$$\tag{5.13} \begin{array}{rcl} c_S(\mathtt{u}_h, \mathtt{w}_h) & \geq & \displaystyle \frac{1}{2} \int_{\mathcal{R}} \left(D^{-1}\mathbf{p}_h \cdot \mathbf{p}_h + \sigma^{-1} (\mathrm{div}\, \mathbf{p}_h)^2 + \left(\sigma \; - \; \frac{\alpha'}{2\eta} \right) \phi_h^2 \right) \\ & & \displaystyle + \int_{\Gamma_S} [\mathbf{p}_h \cdot \mathbf{n}]_h \, [\mathbf{p}_h \cdot \mathbf{n}] \\ & & \displaystyle + \frac{1}{2}\, \alpha' (1 \; - \; \eta\, C') \sum_{i=1}^{N} \sum_{j=i+1}^{N} \int_{\Gamma_{ij}} \left(\pi_i(\phi_{S,h})^2 \; + \; \pi_j(\phi_{S,h})^2 \right) \\ & & \displaystyle - \frac{1}{2} \alpha'^2 (D_*)^{-1} (C'_{lift})^2 \|\phi_{S,h}\|_M^2, \end{array}$$

where $C' = \dfrac{1}{2}\max_{i\in\mathcal{I}_S}(\,|\partial\mathcal{R}_i|\,|\mathcal{R}_i|^{-1}\,) > 0$. Set $\eta = \dfrac{1}{2}(C')^{-1}$ to obtain:

$$\begin{aligned} c_S(\mathtt{u}_h,\mathtt{w}_h) \quad \geq \quad & \frac{1}{2}\int_{\mathcal{R}} \left(D^{-1}\mathbf{p}_h\cdot\mathbf{p}_h + \sigma^{-1}(\operatorname{div}\mathbf{p}_h)^2 + (\sigma\,-\,\alpha'\,C')\,\phi_h^2\right) \\ & + \int_{\Gamma_S}[\mathbf{p}_h\cdot\mathbf{n}]_h\,[\mathbf{p}_h\cdot\mathbf{n}] \\ & + \frac{\alpha'}{4}\sum_{i=1}^{N}\sum_{j=i+1}^{N}\int_{\Gamma_{ij}}\left(\pi_i(\phi_{S,h})^2\,+\,\pi_j(\phi_{S,h})^2\right) \\ & - \frac{1}{2}\alpha'^2(D_*)^{-1}(C'_{lift})^2\|\phi_{S,h}\|_M^2, \end{aligned} \tag{5.14}$$

Suppose that the condition (5.8) holds, then

$$\int_{\Gamma_S}[\mathbf{p}_h\cdot\mathbf{n}]_h\,[\mathbf{p}_h\cdot\mathbf{n}] \geq \beta_h\int_{\Gamma_S}[\mathbf{p}_h\cdot\mathbf{n}]^2\,,$$

with $\beta_h > 0$. Then, provided that $\alpha' \in]0,\sigma_*\,(C')^{-1}[$, it remains to bound from below the last two terms by $\|\phi_{S,h}\|_M^2$ times a strictly positive constant, to achieve (5.7). For this to hold, one uses condition (5.9). Indeed, it implies

$$\begin{aligned} & \frac{\alpha'}{4}\sum_{i=1}^{N}\sum_{j=i+1}^{N}\int_{\Gamma_{ij}}\left(\pi_i(\phi_{S,h})^2\,+\,\pi_j(\phi_{S,h})^2\right) \\ & \qquad -\frac{1}{2}\alpha'^2(D_*)^{-1}(C'_{lift})^2\|\phi_{S,h}\|_M^2 \quad \geq \quad \frac{1}{2}\alpha'\left(\frac{\gamma_h}{2}-\alpha'(D_*)^{-1}(C'_{lift})^2\right)\|\phi_{S,h}\|_M^2. \end{aligned}$$

So taking α' "sufficiently small" leads to the claim when σ is piecewise-constant. When $\sigma \in \mathcal{P}W^{1,\infty}(\mathcal{R})$ is not piecewise-constant, one modifies (5.10) by choosing

$$\psi_{i,h} = \frac{1}{2}\phi_{i,h} + \frac{1}{2}\pi_i^0((\sigma_i)^{-1})\operatorname{div}\mathbf{p}_{i,h} \text{ for } i = 1, N,$$

where π_i^0 is the orthogonal projection on piecewise constant fields in $L^2(\mathcal{R}_i)$. According to [17], Proposition 1.135,

$$\|(\sigma_i)^{-1} - \pi_i^0((\sigma_i)^{-1})\|_{\infty,\mathcal{R}} \leq \epsilon_\sigma(h), \text{ with } \lim_{h\to 0}\epsilon_\sigma(h) = 0. \tag{5.15}$$

With this choice, there are extra terms in the expression of $c_S(\mathtt{u}_h,\mathtt{w}_h)$ compared to (5.11), namely

$$\frac{1}{2}\int_{\mathcal{R}}(\pi_i^0((\sigma_i)^{-1}) - (\sigma_i)^{-1})(\operatorname{div}\mathbf{p}_h)^2 + (\pi_i^0((\sigma_i)^{-1}) - (\sigma_i)^{-1})\sigma_i\phi_h\operatorname{div}\mathbf{p}_h.$$

Thanks to (5.15), the first term has no influence, because one has the bound

$$\int_{\mathcal{R}}(\pi_i^0((\sigma_i)^{-1}) - (\sigma_i)^{-1})(\operatorname{div}\mathbf{p}_h)^2 \geq -\epsilon_\sigma(h)\,\|\operatorname{div}\mathbf{p}_h\|_{0,\mathcal{R}}^2.$$

Likewise for the second term. Indeed, using the Cauchy-Schwarz inequality with (5.15) yields

$$\int_{\mathcal{R}}(\pi_i^0((\sigma_i)^{-1}) - (\sigma_i)^{-1})\sigma_i\phi_h\operatorname{div}\mathbf{p}_h \geq -\frac{\sigma^*}{2}\,\epsilon_\sigma(h)\,(\|\phi_h\|_{0,\mathcal{R}}^2 + \|\operatorname{div}\mathbf{p}_h\|_{0,\mathcal{R}}^2).$$

Hence the claim still holds. □

A straightforward consequence of (5.8) is obtained by taking $(\mathbf{q}_h, \psi_h, \psi_{S,h}) = (0, 0, [\mathbf{p}_h \cdot \mathbf{n}]_h) \in \mathtt{W}_h$ as a test-function in (5.5), which leads to

$$0 = \int_{\Gamma_S} [\mathbf{p}_h \cdot \mathbf{n}]\,[\mathbf{p}_h \cdot \mathbf{n}]_h \geq \beta_h \int_{\Gamma_S} [\mathbf{p}_h \cdot \mathbf{n}]^2.$$

Hence, $[\mathbf{p}_h \cdot \mathbf{n}] = 0$ and so $\mathbf{p}_h \in \mathbf{Q}_h \cap \mathbf{H}(\mathrm{div}\,, \mathcal{R})$.
One infers the classical convergence result below [17]. More precise convergence rates are given in §6.

Corollary 1. *Assume that the algebraic conditions (5.8)-(5.9) are fulfilled uniformly in h, that the discretization $\mathbf{Q}_h \times L_h$ satisfies the approximability properties (5.1) and that it is globally conforming in $\mathbf{Q} \times L^2(\mathbb{R})$. Then the solutions $(\mathtt{u}_h)_h$ to the* DD+L^2-jumps method *converge to the solution* $\mathtt{u}$ *to (3.11). In other words:*

$$\lim_{h\to 0} \left(||\mathbf{p} - \mathbf{p}_h||_{\mathbf{H}(\mathrm{div}\,,\mathcal{R})} \,+\, ||\phi - \phi_h||_{0,\mathcal{R}} \,+\, ||\phi_S - \phi_{S,h}||_M \right) = 0.$$

5.2. **A study of the conditions (5.8)-(5.9).** Let us begin by some observations regarding the two conditions.
First, we noted that if (5.8) holds, then $[\mathbf{p}_h \cdot \mathbf{n}] = 0$ and $\mathbf{p}_h \in \mathbf{Q}_h \cap \mathbf{H}(\mathrm{div}\,, \mathcal{R})$.
Second, if (5.9) holds, then $\psi_{S,h} \mapsto \left(\sum_{i=1}^N \sum_{j=i+1}^N \int_{\Gamma_{ij}} (\pi_i(\psi_{S,h})^2 \,+\, \pi_j(\psi_{S,h})^2 \right)^{1/2}$ is a norm on M_h. Indeed, we conclude from (5.4) that $\forall \psi_{S,h} \in M_h$

$$\gamma_h \, \|\psi_{S,h}\|_M^2 \leq \sum_{i=1}^N \sum_{j=i+1}^N \int_{\Gamma_{ij}} \left(\pi_i(\psi_{S,h})^2 \,+\, \pi_j(\psi_{S,h})^2 \right) \leq 2\, \|\psi_{S,h}\|_M^2.$$

To realize the conditions (5.8)-(5.9), one can think of two alternatives depending on whether the discrete space of the Lagrange multipliers M_h is build/inferred directly with the help of the discrete spaces of normal traces $(T_{i,h})_{i=1,N}$. That is, whether $\sum_{i=1}^N T_{i,h} \subset M_h$ or not.

5.2.1. *The case* $\sum_{i=1}^N T_{i,h} \subset M_h$. One can disregard the situation where $\sum_{i=1}^N T_{i,h}$ is a *strict* subset of M_h. In this case, there exists $\psi_{S,h}^\perp \neq 0$ that belongs to the orthogonal subspace of $\sum_{i=1}^N T_{i,h}$ in M_h, and if $(\mathbf{p}_h, \phi_h, \phi_{S,h})$ is a solution to (5.5), then so is $(\mathbf{p}_h, \phi_h, \phi_{S,h} + \psi_{S,h}^\perp)$. This is a straightforward consequence of the fact for all test-functions $(\mathbf{q}_h, \psi_h, \psi_{S,h})$, it holds $[\mathbf{q}_h \cdot \mathbf{n}] \in \sum_{i=1}^N T_{i,h}$ and so $\int_{\Gamma_S} [\mathbf{q}_h \cdot \mathbf{n}]\, \psi_{S,h}^\perp = 0$: the discrete solution is not unique. Hence in order to obtain a well-posed problem, M_h must be chosen as:

$$M_h \,=\, \sum_{i=1}^N T_{i,h}. \tag{5.16}$$

In this setting, conditions (5.8)-(5.9) follow readily with constants β_h and γ_h independent of h. In particular, all projections Π_i are equal to the identity so that :

$$\forall \mathbf{q}_h \in \mathbf{Q}_h, \int_{\Gamma_S} [\mathbf{q}_h \cdot \mathbf{n}]_h \, [\mathbf{q}_h \cdot \mathbf{n}] = \int_{\Gamma_S} [\mathbf{q}_h \cdot \mathbf{n}]^2.$$

When M_h is defined by (5.16), there are two alternatives. For nested meshes on Γ_{ij}, the definition is straightforward, M_h^{ij} is the larger of the two spaces $T_{i,h}$ and $T_{j,h}$. On the other hand for non-nested meshes, the explicit construction of M_h^{ij} can be delicate. We refer to Gander et al [19, 20] for an implicit approach that allows to build the space M_h and the action of its test-fields in the variational formulation via an algorithm that is only used when the computations are performed.

5.2.2. *Other constructions.* To our knowledge, the other alternative has been first studied by Yotov [21] (in case of RTN finite elements). This is a situation where *nested* meshes are considered, but where $T_{i,h}+T_{j,h}$ and M_h^{ij} can be different on all interfaces Γ_{ij}. Here, the space M_h^{ij} must be taylored with care to obtain conditions (5.8)-(5.9). As an example, in [21], a $2D$ domain $\mathcal{R}$ was considered, with P_0 discrete normal traces with given meshsize $h_S = h_i$ on Γ_{ij}, P_1 discrete Lagrange multipliers with meshsize $2h_S$ and where the triangulation used for the normal traces is a subdivision of the triangulation used for the Lagrange multipliers. The best constant in (5.8) can be computed on the reference element, with a value of the parameter $\hat\beta = \frac{3}{4}$, so that $\beta_h = \frac{3}{4}$. Other examples can be constructed for $3D$ geometries. For instance, with Q_0 normal traces with given meshsize $h_S = h_i$ on Γ_{ij}, Q_1 discrete Lagrange multipliers with meshsize $2h_S$ and the same rule governing the triangulations as above. In this case one finds that $\beta_h = \hat\beta = \frac{17}{32}$. Also, one readily checks that $\hat\pi\hat\psi = 0$ implies $\hat\psi = 0$ on the reference element, so condition (5.9) is fulfilled too with a constant γ_h that is independent of h. We postulate that these results could be extended as follows. Let $m \geq 2$: in $2D$ (resp. $3D$), P_0 (resp. Q_0) discrete normal traces with meshsize $h_S = h_i$, in conjunction with P_m (resp. Q_m) discrete Lagrange multipliers with meshsize $m\,h_S$ and again the same rule governing the triangulations as above.

6. Numerical illustrations

As soon as the discrete inf-sup condition is satisfied uniformly, one can use Strang's lemma [17] to derive the error estimates: one must compute the error between the solution $(\phi, \mathbf{p}, \phi_S)$ and its discrete interpolant. The convergence rate is driven by the error on $\mathbf{p}$. Let us give some details in the case of RTN finite elements [22, 23], under the assumptions of Proposition 1 on the coefficients σ, D. We refer to [24, 25] to state that for a low-regularity solution, $||\mathbf{p}_i - \mathbf{p}_{i,R}||_{\mathbf{H}(\mathrm{div}\,,\mathcal{R}_i)} \lesssim h^{r_{\max}-\epsilon}$ for all $\epsilon > 0$, where $\mathbf{p}_{i,R}$ is the RTN interpolant on subdomain $\mathcal{R}_i$, and $r_{\max}$ is characterized in Proposition 1. While using a domain decomposition with matching meshes, one has $[(\mathbf{p} - \mathbf{p}_R)\cdot\mathbf{n}] = 0$ with $\mathbf{p}_R = (\mathbf{p}_{i,R})_{i=1,N}$. In the other cases, $[(\mathbf{p} - \mathbf{p}_R)\cdot\mathbf{n}] \neq 0$. Nevertheless, one can prove that if the meshes remain nested and the triangulation are quasi-uniform on the non-matching interfaces, then $||[(\mathbf{p} - \mathbf{p}_R)\cdot\mathbf{n}]||_M \lesssim h^{1/2}$. For non-nested meshes, error estimates are difficult to obtain [26], unless there is some measure of structure in the non-nestedness (see below for an illustration). This allows to recover the expected convergence rate of order $r_{\max}$ in the presence of low-regularity solutions:

$$||\mathbf{p} - \mathbf{p}_h||_{\mathbf{H}(\mathrm{div}\,,\mathcal{R})} + ||\phi - \phi_h||_{0,\mathcal{R}} + ||\phi_S - \phi_{S,h}||_M \lesssim h^{r_{\max}-\epsilon}, \qquad \forall\epsilon > 0.$$

As a matter of fact, up to now this expected convergence rate was *extrapolated* from the known results for regular solutions. Following the methodology of [27], one can also recover the Aubin-Nitsche estimates:

$$||\phi - \phi_h||_{0,\mathcal{R}} \lesssim h^{2r_{\max}-\epsilon}, \qquad \forall\epsilon > 0.$$

Let us consider the checkerboard setting as shown on Fig. 1. We consider singular solutions $\phi_{sing} \in H^1(\mathcal{R})$ in the sense of [13]. In polar coordinates (ρ, θ) centered at

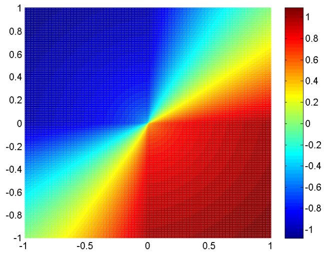


FIGURE 2. Map of ϕ_{sing} when $r_{\max} = 0.20$.

$(0,0)$, it writes $\phi_{sing}(\rho, \theta) = \rho^{r_{\max}} f(\theta)$, where f is piecewise smooth. By construction, $-\mathrm{div}\, D\, \mathbf{grad}\, \phi_{sing} = 0$. We solve:
Find $\phi \in H^1(\mathcal{R})$ such that:

$$\begin{cases} -\mathrm{div}\, D\, \mathbf{grad}\, \phi + \phi &= \phi_{sing} \quad \text{in } \mathcal{R} \\ \phi &= \phi_{sing} \quad \text{on } \partial\mathcal{R}, \end{cases}$$

whose solution is $\phi = \phi_{sing}$. We study the error on two different sets of domain decomposition meshes, calling $\mathtt{h}_i$ the meshsize in $\mathcal{R}_i$, for $1 \le i \le 4$. The parameter is $h = \mathtt{h}_2$ and we choose:

- a DD with nested meshes: $\mathtt{h}_1 = \mathtt{h}_3$, $\mathtt{h}_2 = \mathtt{h}_4$ and $\mathtt{h}_1 = \frac{1}{2}\mathtt{h}_2$;
- a DD with non-nested meshes: $\mathtt{h}_1 = \mathtt{h}_3$, $\mathtt{h}_2 = \mathtt{h}_4$ and $\mathtt{h}_1 = \frac{2}{3}\mathtt{h}_2$.

In the latter case, we remark that there is some structure since the meshsizes are locked. We consider the singular solution ϕ_{sing} with either $r_{\max} = 0.45$ or $r_{\max} = 0.20$, see Fig. 2. The relative errors and corresponding convergence rates are given in Tab. 1 and 2.

	$r_{\max} = 0.45$		$r_{\max} = 0.20$	
$1/h$	$\|\|\phi - \phi_h\|\|_{0,\mathcal{R}}$	$\|\|\mathbf{p} - \mathbf{p}_h\|\|_{0,\mathcal{R}}$	$\|\|\phi - \phi_h\|\|_{0,\mathcal{R}}$	$\|\|\mathbf{p} - \mathbf{p}_h\|\|_{0,\mathcal{R}}$
25	$5.13\, e^{-3}$	$5.90\, e^{-2}$	$3.00\, e^{-2}$	$3.80\, e^{-1}$
50	$2.73\, e^{-3}$	$4.26\, e^{-2}$	$2.31\, e^{-2}$	$3.28\, e^{-1}$
100	$1.46\, e^{-3}$	$3.09\, e^{-2}$	$1.78\, e^{-2}$	$2.82\, e^{-1}$
rate	$h^{0.90}$	$h^{0.46}$	$h^{0.37}$	$h^{0.21}$

TABLE 1. Nested meshes.

7. CONCLUSION

The *DD+L^2-jumps method* is a general framework which allows to show the well-posedness of the mixed, multi-domain diffusion variational formulation, especially in the case of a *low-regularity solution*. In the discrete case, the well-posedness depends on two algebraic conditions which will drive the choice of the discretized space of the Lagrange multipliers. One can choose two *different* discretizations in two adjacent subdomains, such as RTN, BDM, BDFM or ABF mixed finite elements [16]. The critical issue is in the choice of the discrete space of the Lagrange

	$r_{\max} = 0.45$		$r_{\max} = 0.20$	
$1/h$	$\|\|\phi - \phi_h\|\|_{0,\mathcal{R}}$	$\|\|\mathbf{p} - \mathbf{p}_h\|\|_{0,\mathcal{R}}$	$\|\|\phi - \phi_h\|\|_{0,\mathcal{R}}$	$\|\|\mathbf{p} - \mathbf{p}_h\|\|_{0,\mathcal{R}}$
24	$9.76\,e^{-3}$	$1.33\,e^{-1}$	$3.98\,e^{-2}$	$5.05\,e^{-1}$
48	$5.11\,e^{-3}$	$9.63\,e^{-2}$	$3.04\,e^{-2}$	$4.39\,e^{-1}$
96	$2.71\,e^{-3}$	$7.02\,e^{-2}$	$2.34\,e^{-2}$	$3.80\,e^{-1}$
192	$1.44\,e^{-3}$	$5.12\,e^{-2}$	$1.80\,e^{-2}$	$3.29\,e^{-1}$
rate	$h^{0.92}$	$h^{0.46}$	$h^{0.38}$	$h^{0.20}$

TABLE 2. Non-nested meshes.

multipliers to enforce conditions (5.8)-(5.9). Note also that one can use Hilbert spaces including L^2-jumps to model fractures (see the recent paper [28]).
Obviously, the numerical results can be improved in different manners: one can use graded meshes [29], an hp-finite element method [30], or an enriched finite element method, such as XFEM [31] or the singular complement method [32]. If one knows where the singularities occur, these methods can be used a priori. On the other hand, using a posteriori error estimates enable adaptive mesh refinement strategies [3, 33].
A possible continuation of this paper is the study of the steady-state neutron diffusion problem. Mathematically, it can be expressed as a generalized eigenvalue problem, where the physical solution is the fundamental mode, that is the eigenfunction associated to the smallest eigenvalue [7]. The numerical analysis of this problem is carried out in [34].

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Appendix A. Complements: Integrating by parts

We recall below some integration by parts formulas that are used to build the variational formulations. We follow first [10, Section 3]. Introducing the trace space $H^{1/2}(\partial\mathcal{R}_i)$ of elements of $H^1(\mathcal{R}_i)$ on $\partial\mathcal{R}_i$, $i = 1, N$, we recall Green's first identity:

$$\forall(\mathbf{q}_i,\, \phi_i) \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}_i) \times H^1(\mathcal{R}_i),$$

$$\int_{\mathcal{R}_i} \mathbf{grad}\,\phi_i \cdot \mathbf{q}_i \,+\, \phi_i \,\mathrm{div}\,\mathbf{q}_i \,=\, \langle \mathbf{q}_i \cdot \mathbf{n}_i\,,\, \phi_i\rangle_{\left(H^{1/2}(\partial\mathcal{R}_i)\right)',\, H^{1/2}(\partial\mathcal{R}_i)}.$$

Next, we recall some density results from [11] below. Let us introduce, for $H \in \{H^1(\mathcal{R}), H^1_0(\mathcal{R}), (H^1_{0,\Gamma_i}(\mathcal{R}_i))_{i=1,N}\}$:

$$H_- := \{\phi \in H \,|\, \phi = 0 \text{ in a neighbourhood of } \partial\Gamma_W\}.$$

Theorem 5. *Let $H \in \{H^1(\mathcal{R}), H^1_0(\mathcal{R}), (H^1_{0,\Gamma_i}(\mathcal{R}_i))_{i=1,N}\}$: H_- is dense in H.*

Finally, for $i = 1, N$ Green's first identity in $\mathcal{R}_i$ becomes with the help of the trace spaces $H^{1/2}_{\Gamma_{ij}}$:

$$\forall(\mathbf{q}_i, \phi_i) \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}_i) \times \left(H^1_{0,\Gamma_i}(\mathcal{R}_i)\right)_-,$$

$$\int_{\mathcal{R}_i} \mathbf{grad}\,\phi_i \cdot \mathbf{q}_i \,+\, \phi_i \,\mathrm{div}\,\mathbf{q}_i \,=\, \sum_{j=1}^{N} \langle \mathbf{q}_i \cdot \mathbf{n}_i, \phi_i\rangle_{\left(H^{1/2}_{\Gamma_{ij}}\right)',\, H^{1/2}_{\Gamma_{ij}}}.$$

As a consequence, let $\phi \in \left(H^1_0(\mathcal{R})\right)_-$ and $\mathbf{q} \in \mathbf{Q}$. Denoting by $\phi_S = \phi_{|\Gamma_S}$, we may write:

$$\begin{aligned}
\int_{\mathcal{R}} \mathbf{grad}\,\phi \cdot \mathbf{q} + \phi\,\mathrm{div}\,\mathbf{q} &= \sum_{i=1}^{N} \int_{\mathcal{R}_i} \mathbf{grad}\,\phi_i \cdot \mathbf{q}_i \,+\, \phi_i\,\mathrm{div}\,\mathbf{q}_i \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \langle \mathbf{q}_i \cdot \mathbf{n}_i, \phi_i\rangle_{\left(H^{1/2}_{\Gamma_{ij}}\right)',\, H^{1/2}_{\Gamma_{ij}}}, \\
&= \sum_{i=1}^{N}\sum_{j=i+1}^{N} \langle [\mathbf{q} \cdot \mathbf{n}]_{ij}, \phi_S\rangle_{\left(H^{1/2}_{\Gamma_{ij}}\right)',\, H^{1/2}_{\Gamma_{ij}}}, \\
&= \int_{\Gamma_S} [\mathbf{q} \cdot \mathbf{n}]\,\phi_S.
\end{aligned}$$

According to Theorem 5, the result still holds for $\phi \in H^1_0(\mathcal{R})$ and $\mathbf{q} \in \mathbf{Q}$.