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Analysis of the Scott-Zhang interpolation in the fractional order Sobolev spaces

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Abstract. Since it was originally designed, the Scott-Zhang interpolation operator has been very popular. Indeed, it possesses two keys features: it can be applied to fields without pointwise values and it preserves the boundary condition. However, no approximability properties seem to be available in the literature when the regularity of the field is weak. In this Note, we provide some estimates for such weakly regular fields, measured in Sobolev spaces with fractional order between 0 and 1.

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1 Introduction

Let Ω be a (polyhedral) domain, that is a bounded, open, connected subset of $\mathbb{R}^d$ with (polyhedral) Lipschitz boundary $\partial\Omega$. Here, $d = 1, 2, 3$. The unit outward normal vector field to the boundary is called $\boldsymbol{n}$.
Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$, made up of (closed) simplices, with meshsize defined by $h := \max_{K\in\mathcal{T}_h} h_K$. The family of triangulations $(\mathcal{T}_h)_h$ is chosen to be shape regular, in the sense that

$$\exists\sigma > 0,\ \forall h,\ \forall K \in \mathcal{T}_h,\ \frac{h_K}{\rho_K} \le \sigma.$$

Above, h_K and ρ_K are respectively the diameter of the simplex K, and the diameter of the largest ball contained in K. For all simplices K, we define its neighborhood $S_K := int(\cup_{K_i,\,K_i\cap K\neq\emptyset} K_i)$. We use the notation $L^p(K)$, respectively $H^s(K)$, and $\int_K\,dx$ etc. in lieu of $L^p(int(K))$, resp. $H^s(int(K))$, and $\int_{int(K)}\,dx$ etc.
Now we introduce the H^1-conforming finite element spaces consisting of piecewise polynomials:

$$V_h := \{v \in C^0(\overline{\Omega})\ :\ v_{|K} \in P_r(K),\ \forall K \in \mathcal{T}_h\},\quad V_h^0 := V_h \cap H_0^1(\Omega).$$

Above, P_r is the vector space of polynomials of degree at most r (in d variables). Given $K \in \mathcal{T}_h$, we denote by (K, Σ_K, P_K) the associated Lagrange finite element,

and by $(\hat{K}, \hat{\Sigma}, \hat{P})$ the reference element.
In this Note, we assume that the field to be approximated belongs to $H^{\mathtt{s}}(\Omega)$, for some order $\mathtt{s} > \frac{1}{2}$, which allows one to consider traces on the boundary of simplices[1]. On the other hand, if the order is greater than or equal to 1, the theory developed by Scott and Zhang [6] applies, so we focus here on the case $\mathtt{s} \in]\frac{1}{2}, 1[$. We also introduce an order $\mathtt{t} \in [0, \mathtt{s}]$: the approximation properties of the interpolation operator shall be measured in $H^{\mathtt{t}}(\Omega)$-norm. We follow below the structure of [6]: definition of the interpolation operator, stability estimates and approximability results. Finally, for $1 < \mathtt{p} < \infty$ and order $\mathtt{s} \in]\frac{1}{p}, 1[$, we briefly outline how similar results can be obtained for fields of $W^{\mathtt{s},\mathtt{p}}(\Omega)$ in $W^{\mathtt{t},\mathtt{p}}(\Omega)$-norm, where $\mathtt{t} \in [0, \mathtt{s}]$.
We use the notation $A \lesssim B$ for $A \leq CB$, where A and B are scalar quantities, and C is a generic positive constant which is independent of the meshsize, the simplices and the quantities of interest.

2 Definition of the Scott-Zhang interpolation operator

Let us briefly recall how the interpolation operator is defined in [6], and check that it is indeed well-defined from $H^{\mathtt{s}}(\Omega)$ to V_h. Classically, an order $\mathtt{s}$ larger than $\frac{1}{2}$ is required to define the trace on the boundary. Moreover, cf. [1, §7], in a domain ω of $\mathbb{R}^d$, the trace mapping is continuous from $H^{\mathtt{s}}(\omega)$ to $L^1(\partial\omega)$. Let $(M_i)_{i=1,\cdots,N}$ be the set of interpolation nodes of $\mathcal{T}_h$, and $(\phi_i)_{i=1,\cdots,N}$ a basis of V_h such that $\phi_i(M_j) = \delta_{ij}$, with δ_{ij} the Kronecker symbol. Let us now define the interpolation operator Π_h.
For each node M_i, choose a control simplex K_i as follows: either there exists $K' \in \mathcal{T}_h$ such that $M_i \in int(K')$; otherwise, there exists a $(d-1)$-simplex K' such that $M_i \in int(K')$, or M_i belongs to a $(d-2)$-simplex, so that there exists a $(d-1)$-simplex K' such that $M_i \in \partial K'$; then, set $K_i = K'$. In the latter cases, K' is the "face" of a d-simplex K, and $(K', \Sigma_{K'}, P_{K'}) := (K, \Sigma_K, P_K)_{|K'}$ defines a finite element. Then, one introduces an $L^2(K_i)$-dual basis $(\psi^i_\ell)_\ell$ of the nodal basis of the finite element $(K_i, \Sigma_{K_i}, P_{K_i})$; the nodal basis includes in particular $\phi_{i|K_i}$. One chooses the element of the dual basis $\psi_i \in \{\psi^i_\ell,\ \ell\}$ such that $\int_{K_i} \psi_i(y)\,\phi_i(y)\,dy = 1$. By construction, for all j, one has $\int_{K_i} \psi_i(y)\,\phi_j(y)\,dy = \delta_{ij}$.
Given v such that for all i, $v_{|K_i} \in L^1(K_i)$, one defines $\Pi_h v \in V_h$ by

$$\Pi_h v = \sum_i \left(\int_{K_i} \psi_i(y)\, v(y)\, dy \right) \phi_i \text{ in } \overline{\Omega}. \tag{1}$$

[1] There exist other interpolation operators in the literature, which do not require the use of traces (see for instance [2, 4]).

From the above, we conclude that the operator Π_h is well-defined for elements of $H^{\mathtt{s}}(\Omega)$. Finally, to enforce the boundary condition, one chooses a control simplex $K_i \subset \partial\Omega$ when $M_i \in \partial\Omega$.

Proposition 2.1. *For* $\mathtt{s} \in]\frac{1}{2}, 1[$ *and* $v \in H^{\mathtt{s}}(\Omega)$*,* $\Pi_h v \in V_h$ *is well defined. If* $v \in H^{\mathtt{s}}_0(\Omega)$*, then* $\Pi_h v \in V_h^0$*. Finally, for all* $v_h \in V_h$*, one has* $\Pi_h v_h = v_h$*.*

3 Stability estimates and approximability results

To obtain stability estimates, one has first to provide bounds on $\int_{K_i} \psi_i(y)\, v(y)\, dy$ in (1). Note that one has $|\int_{K_i} \psi_i(y)\, v(y)\, dy| \leq \|\psi_i\|_{L^\infty(K_i)}\, \|v\|_{L^1(K_i)}$.
First, estimates have been provided in [6, Lemma 3.1], namely $\|\psi_i\|_{L^\infty(K_i)} \lesssim h_K^{-dim(K_i)}$, where $dim(K_i)$ is the dimension of the simplex K_i (either d or $d-1$), and $K \in \mathcal{T}_h$ is such that $K_i \subset K$.
Second, to bound $\|v\|_{L^1(K_i)}$ for v with appropriate smoothness, one proceeds as follows.
If $K_i = K$ for some $K \in \mathcal{T}_h$, then one has obviously $\|v\|_{L^1(K_i)} \leq h_K^{d/2}\|v\|_{L^2(K)}$.
If K_i is the face of some $K \in \mathcal{T}_h$, one uses the trace theorem on the reference element, see below.
Let us recall classical finite element estimates, which can be found for instance in [3]. If we introduce the affine mappings $\hat{x} \mapsto F_K(\hat{x}) = B_K\hat{x} + b_K$, which maps the reference element $\hat{K}$ on K, resp. $\hat{x} \mapsto B_F\hat{x} + b_F$, which maps the reference face $\hat{F}$ on a face F (for instance K_i) of K, there holds

$$|B_K| \lesssim h_K^d,\ \|B_K\| \lesssim h_K,\ |B_K^{-1}| \lesssim \rho_K^{-d},\ \|B_K^{-1}\| \lesssim \rho_K^{-1} \text{ and } |B_F| \lesssim h_K^{d-1}.$$

Due to the shape regularity assumption, one can replace ρ_K^{-d} (resp. ρ_K^{-1}) by h_K^{-d} (resp. h_K^{-1}).
Then, given $v \in L^2(K)$, resp. $v \in H^{\mathtt{s}}(K)$, let $\hat{v} \in L^2(\hat{K})$, resp. $\hat{v} \in H^{\mathtt{s}}(\hat{K})$ be defined by $\hat{v}(\hat{x}) = v \circ F_K(\hat{x})$, $\hat{x} \in \hat{K}$. Starting from the expressions of the L^2-norm or the $H^{\mathtt{t}}$-semi-norm ($\mathtt{t} \in]0, \mathtt{s}]$), one finds easily by direct computations the useful bounds

$$\|v\|_{L^2(K)} \leq |B_K|^{1/2}\|\hat{v}\|_{L^2(\hat{K})} \lesssim h_K^{d/2}\|\hat{v}\|_{L^2(\hat{K})},$$
$$|v|_{H^{\mathtt{t}}(K)} \leq \|B_K^{-1}\|^{d/2+\mathtt{t}}\,|B_K||\hat{v}|_{H^{\mathtt{t}}(\hat{K})} \lesssim h_K^{d/2-\mathtt{t}}|\hat{v}|_{H^{\mathtt{t}}(\hat{K})};$$
$$\|\hat{v}\|_{L^2(\hat{K})} \leq |B_K^{-1}|^{1/2}\|v\|_{L^2(K)} \lesssim h_K^{-d/2}\|v\|_{L^2(K)},$$
$$|\hat{v}|_{H^{\mathtt{t}}(\hat{K})} \leq \|B_K\|^{d/2+\mathtt{t}}\,|B_K^{-1}||v|_{H^{\mathtt{t}}(K)} \lesssim h_K^{\mathtt{t}-d/2}|v|_{H^{\mathtt{t}}(K)}.$$

Note that these estimates can be aggregated for $\mathtt{t} \in [0, \mathtt{s}]$, with the notation $|\cdot|_{H^0(\cdot)}$ in lieu of $\|\cdot\|_{L^2(\cdot)}$.

Proposition 3.1. *For given* $\mathtt{s} \in]\frac{1}{2}, 1[$*, let* $v \in H^{\mathtt{s}}(K)$*: if* K_i *is a face of* K*, then*

$$\|v\|_{L^1(K_i)} \lesssim \{h_K^{d/2-1}\|v\|_{L^2(K)} + h_K^{d/2+\mathtt{s}-1}|v|_{H^{\mathtt{s}}(K)}\}.$$

Proof. Using first a change of variables and then the trace theorem on the reference element $\hat{K}$ and finally the bounds that we recalled above with $F = K_i$, one finds successively

$$\begin{array}{rcl} \|v\|^2_{L^1(K_i)} & \leq & |B_F|^2\|\hat{v}\|^2_{L^1(\hat{F})} \lesssim |B_F|^2\|\hat{v}\|^2_{H^{\mathtt{s}}(\hat{K})} \lesssim h_K^{2d-2}(\|\hat{v}\|^2_{L^2(\hat{K})} + |\hat{v}|^2_{H^{\mathtt{s}}(\hat{K})}) \\ & \lesssim & h_K^{d-2}\|v\|^2_{L^2(K)} + h_K^{d+2\mathtt{s}-2}|v|^2_{H^{\mathtt{s}}(K)}. \end{array}$$

It is now possible to derive the stability estimates. Let us call Π_K the local interpolation operator, restricted to a simplex K. According to (1), $\Pi_K v = \sum_{i\, s.t.\, M_i \in K}(\int_{K_i} \psi_i(y)\, v(y)\, dy)\phi_i$; in the summation, the number of interpolation nodes that belong to K is uniformly bounded.

Theorem 3.2. *For given* $\mathtt{s} \in]\frac{1}{2}, 1[$ *and* $\mathtt{t} \in [0, \mathtt{s}]$*, one has the stability estimate:*

$$\forall h,\ \forall K \in \mathcal{T}_h,\ \forall v \in H^{\mathtt{s}}(S_K), \quad |\Pi_K v|_{H^{\mathtt{t}}(K)} \lesssim h_K^{-\mathtt{t}}\|v\|_{L^2(S_K)} + h_K^{\mathtt{s}-\mathtt{t}}|v|_{H^{\mathtt{s}}(S_K)}.$$

Proof. Given ϕ_i a basis function, $\hat{\phi}_i(\hat{x}) = \phi_i \circ F_K(\hat{x})$, $\hat{x} \in \hat{K}$, defines an element of the generic finite element basis on $\hat{K}$. Hence, $|\hat{\phi}_i|_{H^{\mathtt{t}}(\hat{K})} \lesssim 1$. Then, one finds

$$\begin{array}{rcl} |\Pi_K v|_{H^{\mathtt{t}}(K)} & \leq & \displaystyle\sum_{i\, s.t.\, M_i \in K} \|\psi_i\|_{L^\infty(K_i)}\|v\|_{L^1(K_i)}\, |\phi_i|_{H^{\mathtt{t}}(K)} \\ & \lesssim & h_K^{d/2-\mathtt{t}} \displaystyle\sum_{i\, s.t.\, M_i \in K} \|\psi_i\|_{L^\infty(K_i)}\|v\|_{L^1(K_i)}. \end{array}$$

Two instances may occur, depending on whether each K_i is a d-simplex or a $(d-1)$-simplex.
If K_i is a d-simplex then by definition $K = K_i$: $\|\psi_i\|_{L^\infty(K_i)} \lesssim h_K^{-d}$ and, by the Cauchy-Schwarz inequality,

$$h_K^{d/2-\mathtt{t}}\, \|\psi_i\|_{L^\infty(K_i)}\|v\|_{L^1(K_i)} \lesssim h_K^{-d/2-\mathtt{t}}\, h_K^{d/2}\|v\|_{L^2(K)} = h_K^{-\mathtt{t}}\|v\|_{L^2(K)}.$$

If K_i is a $(d-1)$-simplex, then $\|\psi_i\|_{L^\infty(K_i)} \lesssim h_K^{1-d}$. Moreover, by definition, K_i is a face of either $K' = K$, or of one of its neighbors $K' \in S_K$. With the help of Proposition 3.1, it follows that

$$h_K^{d/2-\mathtt{t}}\, \|\psi_i\|_{L^\infty(K_i)}\|v\|_{L^1(K_i)} \lesssim h_K^{1-d/2-\mathtt{t}}\{h_{K'}^{d/2-1}\|v\|_{L^2(K')} + h_{K'}^{d/2+\mathtt{s}-1}|v|_{H^{\mathtt{s}}(K')}\}.$$

Now, a regular family of triangulations is locally quasi-uniform (cf. [3]), i.e. for $K' \in S_K$, one has $h_{K'} \lesssim h_K \lesssim h_{K'}$. We end the proof by noting that $card(\{i\, s.t.\, M_i \in K\}) \lesssim 1$. $\square$
Let us conclude by the approximability results.

Theorem 3.3. *For given* $\mathtt{s} \in]\frac{1}{2}, 1[$ *and* $\mathtt{t} \in [0, \mathtt{s}]$, *the following local approximability result holds:*

$$\forall h \lesssim 1,\ \forall K \in \mathcal{T}_h,\ \forall v \in H^{\mathtt{s}}(S_K), \quad \|v - \Pi_h v\|_{H^{\mathtt{t}}(K)} \lesssim h_K^{\mathtt{s}-\mathtt{t}} |v|_{H^{\mathtt{s}}(S_K)}. \quad (2)$$

Proof. We consider first the case $\mathtt{t} = \mathtt{s}$, then the case $\mathtt{t} = 0$, and we conclude by interpolation.
Let $p \in P_0$: in this case, one has $|v - p|_{H^{\mathtt{s}}(S_K)} = |v|_{H^{\mathtt{s}}(S_K)}$. Now, thanks to Proposition 2.1 and Theorem 3.2 ($\mathtt{t} = 0\,; \mathtt{t} = \mathtt{s}$), one obtains

$$\begin{aligned}
\|v - \Pi_h v\|_{H^{\mathtt{s}}(K)} &\leq \|v - p\|_{H^{\mathtt{s}}(K)} + \|\Pi_h(v - p)\|_{H^{\mathtt{s}}(K)} \\
&\leq \|v - p\|_{L^2(K)} + |v|_{H^{\mathtt{s}}(K)} \\
&\qquad + \|\Pi_h(v - p)\|_{L^2(K)} + |\Pi_h(v - p)|_{H^{\mathtt{s}}(K)} \\
&\lesssim (1 + h_K^{-\mathtt{s}})\|v - p\|_{L^2(S_K)} + (1 + h_K^{\mathtt{s}})|v|_{H^{\mathtt{s}}(S_K)}.
\end{aligned}$$

This is true for all $p \in P_0$, so one can consider the infimum over P_0. On the other hand, we recall that the Bramble-Hilbert lemma (similarly to [6, §4], for fractional $\mathtt{s}$) yields $\inf_{p \in P_0} \|v - p\|_{L^2(S_K)} \lesssim h_K^{\mathtt{s}} |v|_{H^{\mathtt{s}}(S_K)}$. Hence, (2) holds for $\mathtt{t} = \mathtt{s}$.
Now, using again the Bramble-Hilbert lemma and Theorem 3.2 ($\mathtt{t} = 0$), one finds

$$\|v - \Pi_h v\|_{L^2(K)} \lesssim h_K^{\mathtt{s}} |v|_{H^{\mathtt{s}}(S_K)},$$

i.e. (2) holds for $\mathtt{t} = 0$.
To end the proof, we recall that, thanks to interpolation theory in Hilbert spaces [5, Ch. 1, §2], one has the uniform bound

$$\forall \hat{w} \in H^{\mathtt{s}}(\hat{K})/\mathbb{R},\ |\hat{w}|_{H^{\mathtt{t}}(\hat{K})} \lesssim (|\hat{w}|_{H^{\mathtt{s}}(\hat{K})})^{\mathtt{t}/\mathtt{s}} (\|\hat{w}\|_{L^2(\hat{K})})^{1-\mathtt{t}/\mathtt{s}}.$$

Applied to $w = v - \Pi_h v$, we find that (2) holds for $\mathtt{t} \in]0, \mathtt{s}[$. Indeed, $\hat{w}$ belongs to $H^{\mathtt{s}}(\hat{K})/\mathbb{R}$, so

$$\begin{aligned}
\|w\|_{H^{\mathtt{t}}(K)} &\lesssim h_K^{d/2-\mathtt{t}} (|\hat{w}|_{H^{\mathtt{s}}(\hat{K})})^{\mathtt{t}/\mathtt{s}} (\|\hat{w}\|_{L^2(\hat{K})})^{1-\mathtt{t}/\mathtt{s}} \\
&\lesssim h_K^{d/2-\mathtt{t}} (h_K^{\mathtt{s}-d/2}\, |w|_{H^{\mathtt{s}}(K)})^{\mathtt{t}/\mathtt{s}} (h_K^{-d/2}\, \|w\|_{L^2(K)})^{1-\mathtt{t}/\mathtt{s}} \\
&\lesssim (|v - \Pi_h v|_{H^{\mathtt{s}}(K)})^{\mathtt{t}/\mathtt{s}} (\|v - \Pi_h v\|_{L^2(K)})^{1-\mathtt{t}/\mathtt{s}} \lesssim h_K^{\mathtt{s}-\mathtt{t}}\, |v|_{H^{\mathtt{s}}(K)}.
\end{aligned}$$

To reach the last inequality, we applied (2) for $\mathtt{t} = \mathtt{s}$ and $\mathtt{t} = 0$. $\square$

As a by-product, one obtains the result below. Indeed, one can check that the shape regularity assumption implies that the minimal value of the solid angles at the vertices of the simplices is strictly positive, and similarly for the minimal value of the dihedral angles at the edges of the simplicies. This implies in turn that $\max_{K' \in \mathcal{T}_h} card(\{K \; s.t. \, K' \subset S_K\}) \lesssim 1$.

Corollary 3.4. *For given* $\mathtt{s} \in]\frac{1}{2}, 1[$ *and* $\mathtt{t} \in [0, \mathtt{s}]$, *the following global approximability result holds:*

$$\forall h \lesssim 1,\ \forall v \in H^{\mathtt{s}}(\Omega), \quad \|v - \Pi_h v\|_{H^{\mathtt{t}}(\Omega)} \lesssim h^{\mathtt{s}-\mathtt{t}} |v|_{H^{\mathtt{s}}(\Omega)}.$$

4 Extension

Let $1 < \mathtt{p} < \infty$ ($\mathtt{p} \neq 2$) and $\mathtt{s} \in]\frac{1}{\mathtt{p}}, 1[$ be given. For fields in $W^{\mathtt{s},\mathtt{p}}(\Omega)$, let us briefly outline how one can obtain stability estimates and approximability results in $W^{\mathtt{t},\mathtt{p}}(\Omega)$, with order $\mathtt{t} \in [0, \mathtt{s}]$. The main differences with the previous sections are twofold: obviously, exponents change from 2 to $\mathtt{p}$; $W^{\mathtt{s},\mathtt{p}}$ are Banach spaces, whereas $H^{\mathtt{s}}$ are Hilbert spaces. We keep the same notations as before.

First, one can define the Scott-Zhang interpolation operator, the trace mapping being continuous from $W^{\mathtt{s},\mathtt{p}}(\omega)$ to $L^1(\partial\omega)$ [1, §7]. As a consequence, the results of Proposition 2.1 hold, for fields $v \in W^{\mathtt{s},\mathtt{p}}(\Omega)$, respectively $v \in W_0^{\mathtt{s},\mathtt{p}}(\Omega)$.

Then, if one recalls that the $W^{\mathtt{t},\mathtt{p}}$-semi-norm reads

$$|v|_{W^{\mathtt{t},\mathtt{p}}(\omega)} = \left(\int_\omega \int_\omega \frac{|v(x) - v(y)|^{\mathtt{p}}}{|x - y|^{d+\mathtt{tp}}} \, dx \, dy \right)^{1/\mathtt{p}},$$

one finds by direct computations the bounds

$$\|v\|_{L^{\mathtt{p}}(K)} \lesssim h_K^{d/\mathtt{p}} \|\hat{v}\|_{L^{\mathtt{p}}(\hat{K})}, \quad |v|_{W^{\mathtt{t},\mathtt{p}}(K)} \lesssim h_K^{d/\mathtt{p}-\mathtt{t}} |\hat{v}|_{W^{\mathtt{t},\mathtt{p}}(\hat{K})};$$
$$\|\hat{v}\|_{L^{\mathtt{p}}(\hat{K})} \lesssim h_K^{-d/\mathtt{p}} \|v\|_{L^{\mathtt{p}}(K)}, \quad |\hat{v}|_{W^{\mathtt{t},\mathtt{p}}(\hat{K})} \lesssim h_K^{\mathtt{t}-d/\mathtt{p}} |v|_{W^{\mathtt{t},\mathtt{p}}(K)}.$$

Given $v \in W^{\mathtt{s},\mathtt{p}}(K)$ and K_i a face of K, the face-to-simplex estimate becomes

$$\|v\|_{L^1(K_i)} \lesssim \{ h_K^{d(\frac{\mathtt{p}-1}{\mathtt{p}})-1} \|v\|_{L^{\mathtt{p}}(K)} + h_K^{d(\frac{\mathtt{p}-1}{\mathtt{p}})+\mathtt{s}-1} |v|_{W^{\mathtt{s},\mathtt{p}}(K)} \}.$$

Hence, given $v \in W^{\mathtt{s},\mathtt{p}}(S_K)$, one derives a stability estimate:

$$|\Pi_K v|_{W^{\mathtt{t},\mathtt{p}}(K)} \lesssim h_K^{-\mathtt{t}} \|v\|_{L^{\mathtt{p}}(S_K)} + h_K^{\mathtt{s}-\mathtt{t}} |v|_{W^{\mathtt{s},\mathtt{p}}(S_K)}.$$

Using now interpolation theory for Banach spaces [1, §7], one finds the bound

$$\forall \hat{w} \in W^{\mathtt{s},\mathtt{p}}(\hat{K})/\mathbb{R},\ |\hat{w}|_{W^{\mathtt{t},\mathtt{p}}(\hat{K})} \lesssim (|\hat{w}|_{W^{\mathtt{s},\mathtt{p}}(\hat{K})})^{\mathtt{t}/\mathtt{s}}(\|\hat{w}\|_{L^{\mathtt{p}}(\hat{K})})^{1-\mathtt{t}/\mathtt{s}}.$$

This allows us to generalize the local approximability result:

$$\forall h \lesssim 1,\ \forall K \in \mathcal{T}_h,\ \forall v \in W^{\mathtt{s},\mathtt{p}}(S_K), \quad \|v - \Pi_h v\|_{W^{\mathtt{t},\mathtt{p}}(K)} \lesssim h_K^{\mathtt{s}-\mathtt{t}}|v|_{W^{\mathtt{s},\mathtt{p}}(S_K)}.$$

Finally, one obtains a global approximability result:

$$\forall h \lesssim 1,\ \forall v \in W^{\mathtt{s},\mathtt{p}}(\Omega), \quad \|v - \Pi_h v\|_{W^{\mathtt{t},\mathtt{p}}(\Omega)} \lesssim h^{\mathtt{s}-\mathtt{t}}|v|_{W^{\mathtt{s},\mathtt{p}}(\Omega)}.$$

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