Two-dimensional Maxwell's equations with sign-changing coefficients

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Abstract

We consider the theoretical study of time harmonic Maxwell's equations in presence of sign-changing coefficients, in a two-dimensional configuration. Classically, the problems for both the Transverse Magnetic and the Tranverse Electric polarizations reduce to an equivalent scalar Helmholtz type equation. For this scalar equation, we have already studied consequences of the presence of sign-changing coefficients in previous papers, and we summarize here the main results. Then we focus on the alternative approach which relies on the two-dimensional vectorial formulations of the TM or TE problems, and we exhibit some unexpected effects of the sign-change of the coefficients. In the process, we provide new results on the scalar equations.

Keywords: Maxwell's equation, transverse polarization, metamaterial, plasmonics, time harmonic regime, Fredholm alternative, compact embedding

Introduction

The recent and promising developments of photonic metamaterials [15, 16] and of plasmonics [14, 1] raised new issues in the theoretical and numerical study of time harmonic Maxwell's equations: we are concerned here with the possible sign-change of the dielectric permittivity and/or of the magnetic

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permeability. This sign-change occurs for instance at the interface between a metal and a classical medium (vacuum or dielectric) at optical frequencies, when the metal presents a dielectric permittivity with a negligible imaginary part and a negative real part. This property is essential for the existence of plasmonic surface waves. Sign-change of the coefficients also takes place at the interface between a dielectric and a so-called left-handed metamaterial, for which both the dielectric permittivity and the magnetic permeability take negative real values (with again a small imaginary part that we neglect from now on).

We have already obtained several results related to this topic [5, 6, 7, 2, 9] for both the theoretical and the numerical aspects. In particular, we have carried out a rather thorough analysis of the corresponding scalar problem in [2]. More precisely, we have proved that the equation

$$-\mathrm{div}\,(\mu^{-1}\nabla\varphi) - \omega^2\varepsilon\varphi = f \tag{1}$$

in a bounded domain Ω, with $f \in \mathrm{L}^2(\Omega)$ and with Dirichlet boundary conditions, may be strongly ill-posed in the usual H^1 framework for a sign-changing function μ, and we have derived conditions on μ which guarantee that the problem is of Fredholm type. The main ingredient in [2] is the so-called $\mathtt{T}$-coercivity concept, which consists in finding an isomorphism $\mathtt{T}$ of $\mathrm{H}^1_0(\Omega)$ such that the bilinear form

$$(\varphi, \psi) \mapsto \int_\Omega \mu^{-1}\nabla\varphi\cdot\nabla(\mathtt{T}\psi)$$

is coercive on $\mathrm{H}^1_0(\Omega)$. The method is powerful although quite simple, since the operators $\mathtt{T}$ are built by elementary geometrical arguments.

These results have direct counterparts for Maxwell's equations in 2D configurations. In this case, it is well-known that these equations give rise to two systems of equations, without any coupling, corresponding respectively to the so-called Transverse Electric (TE) and Transverse Magnetic (TM) polarizations. Moreover, each of them reduces to a scalar equation similar to (1), where φ is the component of the electric or magnetic field parallel to the direction of invariance of the medium and of the data. Even so, the study of two-dimensional Maxwell's equations with sign-changing coefficients is interesting in its own right. Indeed, it is a preliminary step to solving two-and-a-half-dimensional electromagnetic problems which are not reducible to scalar problems, such as for instance plasmonic waveguide problems.

There is an additional motivation for choosing a vectorial formulation instead of a scalar one. As a matter of fact, numerical resolution of the TM problem using the scalar formulation provides a good approximation of the electric field, which is the unknown of the scalar problem, whereas it provides a poor approximation of the magnetic field, as it corresponds to the derivatives of the scalar unknown. On the contrary, discretizing the vectorial formulation gives an accurate approximation of the magnetic field in the appropriate norms to measure it.

Finally, from the mathematical point of view, one has to address new issues, such as compact embedding of some sets of vectorial fields. Also the proofs of the new results collected in sections 3 and 4 are original: they use again the $\mathtt{T}$-coercivity concept but the operators $\mathtt{T}$ are no longer built from geometrical transformations as in [2]. Instead, we define them in a more abstract way, using the well-posedness of other problems. This way, we exhibit strong connections between scalar problems (1) with Dirichlet and Neumann conditions. As a consequence, a complete description of the results for the Neumann problem can be directly deduced from [2], where only the Dirichlet problem has been considered. Let us mention that parts of these results can be extended to three-dimensional Maxwell problems (see [3]).

The outline of the paper is as follows. In the next section, we briefly derive the TE and TM systems of equations and their equivalent scalar and vectorial formulations. On the sequel of the paper, we focus on the TE problem. The approach based on the scalar equation is discussed in section 2, where we recall the main results concerning scalar transmission problems with sign-changing coefficients. Section 3 is devoted to the vectorial formulation: we introduce two hypotheses, respectively related to the Dirichlet and Neumann static scalar problems. Moreover, these two hypotheses are proved to be equivalent one to the other and, in addition, they imply a Fredholm property for the vectorial formulation. At this stage, a question remains: what happens for the vectorial approach when this hypothesis is not satisfied? This point is addressed in section 4, where we define an appropriate variational setting.

1. Mathematical formulations for the Transverse Electric and Magnetic problems

1.1. The equations for the TE and TM polarizations

We consider a domain which is invariant in one direction and bounded in the transverse ones. More precisely, we introduce a domain Ω of $\mathbb{R}^2$, that is an open bounded connected set, with a connected Lipschitz boundary; then, we define $D := \{(x, y, z) \in \Omega \times \mathbb{R}\}$ and we suppose that the dielectric permittivity ε and the magnetic permeability μ in D are real-valued functions of $(x, y) \in \Omega$, which means that they are independent from z. Since we are interested by sign-changing coefficients, we suppose only that $\varepsilon \in \mathrm{L}^\infty(\Omega)$, $\mu \in \mathrm{L}^\infty(\Omega)$, $\varepsilon^{-1} \in \mathrm{L}^\infty(\Omega)$ and $\mu^{-1} \in \mathrm{L}^\infty(\Omega)$. Notice in particular that vanishing ε or μ are forbidden.

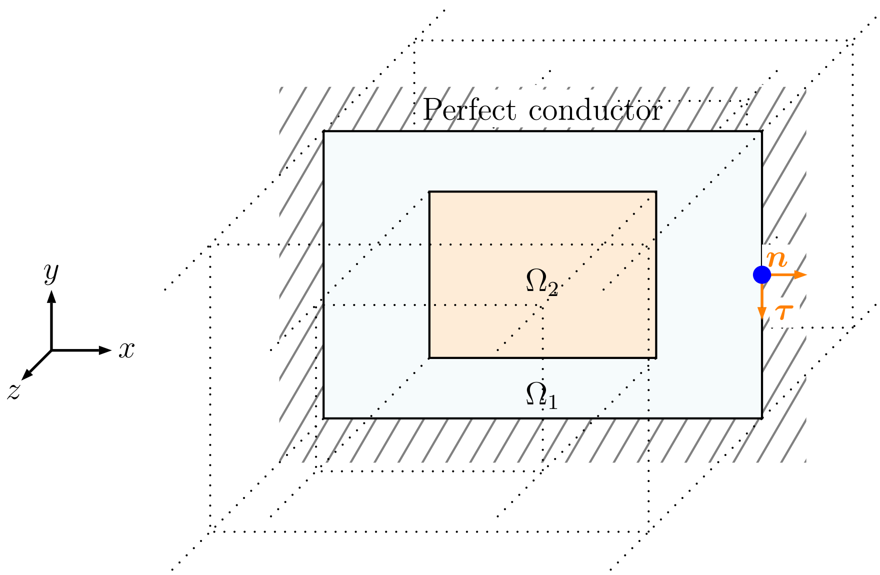

Figure 1: A model problem: Ω is the union of the two subdomains Ω_1 and Ω_2 and $\varepsilon|_{\Omega_1} > 0$, $\mu|_{\Omega_1} > 0$, $\varepsilon|_{\Omega_2} < 0$ and/or $\mu|_{\Omega_2} < 0$

In presence of a current density $\boldsymbol{J}$, the time-harmonic electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$ is solution of Maxwell's equations:

$$i\omega\varepsilon\boldsymbol{E} + \mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J} \quad \text{and} \quad -i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = 0 \quad \text{in } D, \tag{2}$$

where a time behavior in $e^{-i\omega t}$ is assumed, $\omega > 0$. We suppose moreover that D is bounded by a perfect conductor, so that the tangential trace of $\boldsymbol{E}$, and therefore the normal trace of $\mu\boldsymbol{H}$, both vanish on ∂D:

$$\boldsymbol{E} \times \boldsymbol{\nu} = 0 \quad \text{and} \quad \mu\boldsymbol{H} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial D \tag{3}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal vector field to ∂D.

Remark 1.1. *Using (2) and (3), one finds easily that* $\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{H}-\boldsymbol{J})\times\boldsymbol{\nu} = 0$ *on* ∂D.

Assuming finally that the current density $\boldsymbol{J}$ is independent from z, the problem becomes completely independent from z and the simplification $\frac{\partial \cdot}{\partial z} = 0$ leads to the following expanded equations where we have set $\boldsymbol{J} := (J_x, J_y, J_z)^t$, $\boldsymbol{E} := (E_x, E_y, E_z)^t$ and $\boldsymbol{H} := (H_x, H_y, H_z)^t$:

$$\begin{aligned} &i\omega\varepsilon E_x + \frac{\partial H_z}{\partial y} = J_x, \quad i\omega\varepsilon E_y - \frac{\partial H_z}{\partial x} = J_y, \quad i\omega\varepsilon E_z + \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z, \\ &-i\omega\mu H_x + \frac{\partial E_z}{\partial y} = 0, \ -i\omega\mu H_y - \frac{\partial E_z}{\partial x} = 0, \ -i\omega\mu H_z + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \end{aligned} \tag{4}$$

Denoting by $\boldsymbol{n} = (n_x, n_y)^t$ the unit outward normal vector field to $\partial\Omega$, and by $\boldsymbol{\tau} = (\tau_x, \tau_y)^t = (n_y, -n_x)^t$ the vector such that $(\boldsymbol{\tau}, \boldsymbol{n})$ is an orthonormal basis, boundary conditions can be recast as

$$\begin{aligned} &\mu(H_x n_x + H_y n_y) = 0, \quad E_z = 0, \quad E_x n_y - E_y n_x = 0, \\ &\varepsilon^{-1}\left[\left(\frac{\partial H_z}{\partial y} - J_x\right) n_y - \left(\frac{\partial H_z}{\partial x} - J_y\right) n_x\right] = 0. \end{aligned} \tag{5}$$

As far as the functional framework is concerned, we suppose classically that $\boldsymbol{J} := (J_x, J_y, J_z)^t \in \mathrm{L}^2(\Omega)^3$ and, for the sake of simplicity, we suppose in addition that $\operatorname{div}\boldsymbol{J} = 0$. Then, extending what is done for usual materials, we look for a square-integrable electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$.

Classically, it appears that equations (4)-(5) can be equivalently written as two decoupled systems for $(\boldsymbol{H}_\perp, E_z)$ and $(\boldsymbol{E}_\perp, H_z)$, where we have introduced the transverse fields $\boldsymbol{E}_\perp := (E_x, E_y)^t$ and $\boldsymbol{H}_\perp := (H_x, H_y)^t$. We also introduce, for the sake of conciseness, the following 2D differential operators:

$$u \mapsto \mathbf{curl}\,u = \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right)^t \quad \text{and} \quad \boldsymbol{u} = (u_x, u_y)^t \mapsto \operatorname{curl}\boldsymbol{u} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}.$$

The first problem involves the unknowns $(\boldsymbol{H}_\perp, E_z)$ and is called the TE problem (where TE stands for Transverse Electric):

$$
\begin{array}{rcll}
\multicolumn{4}{l}{\text{Find } E_z \in \mathrm{L}^2(\Omega) \text{ and } \boldsymbol{H}_\perp \in \mathbf{L}^2(\Omega) \text{ such that:}} \\
i\omega\varepsilon E_z + \operatorname{curl} \boldsymbol{H}_\perp &=& J_z & \text{in } \Omega, \\
-i\omega\mu \boldsymbol{H}_\perp + \mathbf{curl}\, E_z &=& 0 & \text{in } \Omega, \\
E_z &=& 0 & \text{on } \partial\Omega, \\
\mu \boldsymbol{H}_\perp \cdot \boldsymbol{n} &=& 0 & \text{on } \partial\Omega,
\end{array}
\tag{6}
$$

where $\mathbf{L}^2(\Omega) := \mathrm{L}^2(\Omega)^2$.

The second one involves the unknowns $(\boldsymbol{E}_\perp, H_z)$ and is called the TM problem (for Transverse Magnetic):

$$
\begin{array}{rcll}
\multicolumn{4}{l}{\text{Find } H_z \in \mathrm{L}^2(\Omega) \text{ and } \boldsymbol{E}_\perp \in \mathbf{L}^2(\Omega) \text{ such that:}} \\
-i\omega\mu H_z + \operatorname{curl} \boldsymbol{E}_\perp &=& 0 & \text{in } \Omega, \\
i\omega\varepsilon \boldsymbol{E}_\perp + \mathbf{curl}\, H_z &=& \boldsymbol{J}_\perp & \text{in } \Omega, \\
\varepsilon^{-1}(\mathbf{curl}\, H_z - \boldsymbol{J}_\perp) \cdot \boldsymbol{\tau} &=& 0 & \text{on } \partial\Omega, \\
\boldsymbol{E}_\perp \cdot \boldsymbol{\tau} &=& 0 & \text{on } \partial\Omega.
\end{array}
\tag{7}
$$

1.2. Functional spaces and Green formulas

In what follows, we use classical functional spaces whose definitions are recalled below:

$$
\begin{array}{ll}
\mathbf{H}(\operatorname{curl}; \Omega) &:= \left\{ \boldsymbol{u} \in \mathbf{L}^2(\Omega) \,|\, \operatorname{curl} \boldsymbol{u} \in \mathrm{L}^2(\Omega) \right\}; \\
\mathbf{H}(\operatorname{div}; \Omega) &:= \left\{ \boldsymbol{u} \in \mathbf{L}^2(\Omega) \,|\, \operatorname{div} \boldsymbol{u} \in \mathrm{L}^2(\Omega) \right\}; \\
\mathbf{H}_N(\operatorname{curl}; \Omega) &:= \{ \boldsymbol{u} \in \mathbf{H}(\operatorname{curl}; \Omega) \,|\, \boldsymbol{u} \cdot \boldsymbol{\tau} = 0 \text{ on } \partial\Omega \}; \\
\mathbf{V}_N(\varepsilon; \Omega) &:= \{ \boldsymbol{u} \in \mathbf{H}(\operatorname{curl}; \Omega) \,|\, \operatorname{div}(\varepsilon\, \boldsymbol{u}) = 0,\ \boldsymbol{u} \cdot \boldsymbol{\tau} = 0 \text{ on } \partial\Omega \}; \\
\mathbf{V}_T(\mu; \Omega) &:= \{ \boldsymbol{u} \in \mathbf{H}(\operatorname{curl}; \Omega) \,|\, \operatorname{div}(\mu\, \boldsymbol{u}) = 0,\ \mu \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \}.
\end{array}
\tag{8}
$$

We denote $(\cdot,\cdot)_\Omega$, resp. $\|\cdot\|_\Omega$, the scalar product, resp. the norm, of both spaces $\mathrm{L}^2(\Omega)$ and $\mathbf{L}^2(\Omega)$. Some classical results from [13] will be used in the sequel:

Theorem 1.2. *The application $\boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n}$ (resp. $\boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{\tau}$) defined on $\mathscr{C}^\infty(\overline{\Omega})^2$ can be extended by continuity to a surjective linear application from $\mathbf{H}(\operatorname{div}; \Omega)$ (resp. $\mathbf{H}(\operatorname{curl}; \Omega)$) in $\mathrm{H}^{-1/2}(\partial\Omega)$. Moreover, the following Green formulas hold:*

$$
(\boldsymbol{v}, \nabla\varphi)_\Omega + (\operatorname{div} \boldsymbol{v}, \varphi)_\Omega = \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_{\partial\Omega},\ \forall (\varphi, \boldsymbol{v}) \in \mathrm{H}^1(\Omega) \times \mathbf{H}(\operatorname{div}; \Omega), \tag{9}
$$

$$
(\boldsymbol{v}, \mathbf{curl}\, \varphi)_\Omega - (\operatorname{curl} \boldsymbol{v}, \varphi)_\Omega = \langle \boldsymbol{v} \cdot \boldsymbol{\tau}, \varphi \rangle_{\partial\Omega},\ \forall (\varphi, \boldsymbol{v}) \in \mathrm{H}^1(\Omega) \times \mathbf{H}(\operatorname{curl}; \Omega). \tag{10}
$$

As a consequence, one has:

Corollary 1.3.

$$\mathbf{V}_T(\mu;\,\Omega) = \left\{\boldsymbol{u} \in \mathbf{H}(\mathrm{curl}\,;\,\Omega) \,|\, (\mu\boldsymbol{u}, \nabla\varphi)_\Omega = 0, \quad \forall\varphi \in \mathrm{H}^1(\Omega)\right\},$$

$$\mathbf{V}_N(\varepsilon;\,\Omega) = \left\{\boldsymbol{u} \in \mathbf{H}_N(\mathrm{curl}\,;\,\Omega) \,|\, (\varepsilon\,\boldsymbol{u}, \nabla\varphi)_\Omega = 0, \quad \forall\varphi \in \mathrm{H}^1_0(\Omega)\right\}.$$

1.3. Equivalent scalar and vectorial formulations of the TE problem

From now on, we focus on the TE problem (6), but a very similar study can be made for the TM problem (7).

Classically, one can eliminate the unknown $\boldsymbol{H}_\perp$ and derive an equivalent scalar formulation for E_z. Alternatively, one can eliminate E_z to derive a two-dimensional vectorial formulation for $\boldsymbol{H}_\perp$. This is explained below.

Theorem 1.4. *Second order scalar formulation.*
1) If $(\boldsymbol{H}_\perp, E_z)$ satisfies problem (6) then E_z is a solution of

$$\begin{aligned}&\textit{Find } E_z \in \mathrm{H}^1_0(\Omega) \ \ \textit{such that:}\\ &\int_\Omega \mu^{-1}\nabla E_z\cdot\nabla v - \omega^2\varepsilon E_z v = \int_\Omega i\omega J_z v, \quad \forall v \in \mathrm{H}^1_0(\Omega).\end{aligned} \tag{11}$$

2) Conversely if E_z satisfies problem (11) then $(\boldsymbol{H}_\perp, E_z) = ((i\omega\mu)^{-1}\mathbf{curl}\,E_z, E_z)$ is a solution of (6).

Proof. 1) If $(\boldsymbol{H}_\perp, E_z)$ satisfies (6) then $E_z \in \mathrm{H}^1_0(\Omega)$ and

$$\mathrm{div}\,(\mu^{-1}\nabla E_z) = -\mathrm{curl}\,(\mu^{-1}\mathbf{curl}\,E_z) = -i\omega\,\mathrm{curl}\,\boldsymbol{H}_\perp = -\omega^2\varepsilon E_z - i\omega J_z.$$

This proves that E_z satisfies (11).
2) Suppose conversely that E_z is a solution of (11). Then, $\mathrm{div}\,(\mu^{-1}\nabla E_z) + \omega^2\varepsilon E_z = -i\omega J_z$. Defining $\boldsymbol{H}_\perp := (i\omega\mu)^{-1}\mathbf{curl}\,E_z \in \mathbf{L}^2(\Omega)$, one clearly has $-i\omega\mu\boldsymbol{H}_\perp + \mathbf{curl}\,E_z = 0$. Moreover, on $\partial\Omega$, $\mu\boldsymbol{H}_\perp\cdot\boldsymbol{n} = (i\omega)^{-1}\,\mathbf{curl}\,E_z\cdot\boldsymbol{n} = (i\omega)^{-1}\,\nabla E_z\cdot\boldsymbol{\tau} = 0$ since $E_z \in \mathrm{H}^1_0(\Omega)$. Finally

$$\mathrm{curl}\,\boldsymbol{H}_\perp = (i\omega)^{-1}\mathrm{curl}\,(\mu^{-1}\mathbf{curl}\,E_z) = -(i\omega)^{-1}\mathrm{div}\,(\mu^{-1}\nabla E_z) = J_z - i\omega\varepsilon E_z,$$

which ends the proof. ∎

Let us now consider the vectorial formulation for $\boldsymbol{H}_\perp$:

Proposition 1.5. *Second order vectorial formulation.*
1) If $(\boldsymbol{H}_\perp, E_z)$ satisfies problem (6) then $\boldsymbol{H}_\perp$ is a solution of

$$\begin{array}{l} \textit{Find } \boldsymbol{H}_\perp \in \mathbf{H}(\mathrm{curl}\,;\,\Omega) \ \ \textit{such that:} \\ \displaystyle\int_\Omega \varepsilon^{-1}\mathrm{curl}\,\boldsymbol{H}_\perp \mathrm{curl}\,\boldsymbol{v} - \omega^2\mu\boldsymbol{H}_\perp\cdot\boldsymbol{v} = \int_\Omega \varepsilon^{-1}J_z \mathrm{curl}\,\boldsymbol{v},\ \forall \boldsymbol{v}\in\mathbf{H}(\mathrm{curl}\,;\,\Omega). \end{array} \tag{12}$$

2) Conversely, if $\boldsymbol{H}_\perp$ satisfies problem (12) then

$$(\boldsymbol{H}_\perp, E_z) = (\boldsymbol{H}_\perp, i(\omega\varepsilon)^{-1}(\mathrm{curl}\,\boldsymbol{H}_\perp - J_z))$$

is a solution of (6).

Proof. 1) Suppose $(\boldsymbol{H}_\perp, E_z)$ is a solution of (6). Then, one has: $i\omega E_z + \varepsilon^{-1}\mathrm{curl}\,\boldsymbol{H}_\perp - \varepsilon^{-1}J_z = 0$. Multiplying by $\mathrm{curl}\,\boldsymbol{v}$ for $\boldsymbol{v}\in\mathbf{H}(\mathrm{curl}\,;\,\Omega)$, integrating by parts and using $E_z \in \mathrm{H}^1_0(\Omega)$ with $\mathbf{curl}\,E_z = i\omega\mu\boldsymbol{H}_\perp$, we get (12).
2) Conversely, if $\boldsymbol{H}_\perp$ satisfies (12), let us set $E_z = i(\omega\varepsilon)^{-1}(\mathrm{curl}\,\boldsymbol{H}_\perp - J_z) \in \mathrm{L}^2(\Omega)$. In this case, the following equation is clearly satisfied: $i\omega\varepsilon E_z + \mathrm{curl}\,\boldsymbol{H}_\perp = J_z$. On the other hand, from (12), we get $\mathbf{curl}\,(\varepsilon^{-1}(\mathrm{curl}\,\boldsymbol{H}_\perp - J_z)) - \omega^2\mu\boldsymbol{H}_\perp = 0$ which leads to

$$-i\omega\mu\boldsymbol{H}_\perp + \mathbf{curl}\,E_z = 0. \tag{13}$$

In particular, $E_z \in \mathrm{H}^1(\Omega)$. Then using (12), we obtain $(E_z, \mathrm{curl}\,\boldsymbol{v})_\Omega = i\omega(\mu\boldsymbol{H}_\perp, \boldsymbol{v})_\Omega$ for all $\boldsymbol{v}\in\mathbf{H}(\mathrm{curl}\,;\,\Omega)$. Using next (13) and the surjectivity of the tangential trace from $\mathbf{H}(\mathrm{curl}\,;\,\Omega)$ in $\mathrm{H}^{-1/2}(\partial\Omega)$ (Theorem 1.2), we deduce that $E_z = 0$ on $\partial\Omega$. Finally $\mu\boldsymbol{H}_\perp\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ results from (13). ■

2. Mathematical study of the scalar formulation for E_z

From Theorem 1.4, it results that the well-posedness of the TE problem (6) is equivalent to the well-posedness of the scalar problem (11). This leads us now to focus on the theoretical properties of formulation (11). We consider first the classical case of a positive μ and then the case of a sign-changing μ. We say that μ is sign-changing if $\inf_\Omega \mu < 0$ and $\sup_\Omega \mu > 0$ where inf and sup stand for essential infimum and supremum. Remind that, by hypothesis, $\mu^{-1} \in \mathrm{L}^\infty(\Omega)$ so that such a function μ is bounded away from zero.

Let us first introduce some notations which will be useful in the sequel.

2.1. Notations and preliminary results

Let $u \in \mathrm{H}_0^1(\Omega)$. By Riesz representation Theorem, there exists a unique $f \in \mathrm{H}_0^1(\Omega)$ such that

$$(\nabla f, \nabla v)_\Omega = (\mu^{-1}\nabla u, \nabla v)_\Omega, \qquad \forall v \in \mathrm{H}_0^1(\Omega).$$

If we set $A_D^{1/\mu}u = f$ (where the index D stands for the Dirichlet condition), it is straightforward that $A_D^{1/\mu} \in \mathcal{L}(\mathrm{H}_0^1(\Omega))$, the set of the linear bounded operators of $\mathrm{H}_0^1(\Omega)$, and we have:

$$(\nabla(A_D^{1/\mu}u), \nabla v)_\Omega = (\mu^{-1}\nabla u, \nabla v)_\Omega, \qquad \forall (u,v) \in \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega). \tag{14}$$

Notice, again by Riesz representation Theorem, that the action of any continuous linear form ℓ on $\mathrm{H}_0^1(\Omega)$ can be equivalently written $\ell(v) = (\nabla f, \nabla v)_\Omega$ for all $v \in \mathrm{H}_0^1(\Omega)$, for some $f \in \mathrm{H}_0^1(\Omega)$. As a consequence, proving the well-posedness of the problem

$$\text{Find } u \in \mathrm{H}_0^1(\Omega) \text{ such that:} \int_\Omega \mu^{-1}\nabla u \cdot \nabla v = \ell(v), \quad \forall v \in \mathrm{H}_0^1(\Omega),$$

amounts to proving that $A_D^{1/\mu}$ is an isomorphism of $\mathrm{H}_0^1(\Omega)$.

In the same way, we define for all $\omega \in \mathbb{C}$ the operator $A_D^{1/\mu}(\omega) \in \mathcal{L}(\mathrm{H}_0^1(\Omega))$ such that:

$$\begin{aligned}&(\nabla(A_D^{1/\mu}(\omega)u), \nabla v)_\Omega = (\mu^{-1}\nabla u, \nabla v)_\Omega - \omega^2(\varepsilon u, v)_\Omega,\\ &\forall (u,v) \in \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega),\end{aligned} \tag{15}$$

and the well-posedness of problem (11) is a consequence of the invertibility of $A_D^{1/\mu}(\omega)$. We will use extensively the following result which is a direct consequence of the compact embedding of $\mathrm{H}_0^1(\Omega)$ into $\mathrm{L}^2(\Omega)$ (cf. [10, Theorem 8.26]).

Lemma 2.1. *Suppose that $A_D^{1/\mu} = A_D^{1/\mu}(0)$ is an isomorphism of $\mathrm{H}_0^1(\Omega)$. Then for all $\omega \in \mathbb{C}$, $A_D^{1/\mu}(\omega)$ is a Fredholm operator. Moreover, $A_D^{1/\mu}(\omega)$ is an isomorphism for all $\omega^2 \in \mathbb{C}\backslash\mathscr{S}$, where $\mathscr{S}$ is at most a countable set with no accumulation points.*

When $A_D^{1/\mu}(\omega)$ is a Fredholm operator, existence of a solution to problem (11) is equivalent to its uniqueness: in this case, we will say that problem (11) is of Fredholm type.

If μ is a positive real-valued function, the mathematical analysis of (11) is very classical.

Lemma 2.2. *Suppose*

$$\inf_{\Omega} \mu > 0.$$

Then problem (11) is of Fredholm type and it is well-posed for all $\omega^2 \in \mathbb{C}\backslash\mathscr{S}$*, where* $\mathscr{S} = \{\omega_1^2, \omega_2^2 \cdots\}$ *is a countable set of real numbers. Moreover, the following alternative holds:*

1. *either* $\inf_{\Omega} \varepsilon > 0$ *and then* $\mathscr{S} \subset \mathbb{R}^{+*}$ *and* $\lim \omega_n^2 = +\infty$ *;*
2. *or* ε *is sign-changing and then the sequence* (ω_n^2) *accumulates at* $+\infty$ *and* $-\infty$*.*

Proof. Using Lax-Milgram Theorem and the positivity of μ, it is clear that $A_D^{1/\mu}$ is an isomorphism. Then we can apply Lemma 2.1. Finally, the alternative comes from classical results of spectral theory of selfadjoint compact operators. We refer for instance to [12] and Refs. therein for the case of a sign-changing coefficient ε. ∎

2.2. The results for a sign-changing μ

Let us consider now the more difficult case where μ is sign-changing, which has been extensively studied in some of our previous papers [7, 2]. We give here a review of the main results of [2]. We suppose that Ω is the union of two (sub)domains Ω_1 and Ω_2, where $\mu|_{\Omega_1} > 0$ and $\mu|_{\Omega_2} < 0$. The interface between the two subdomains is denoted by Σ:

$$\Sigma = int(\partial\Omega_1 \cap \partial\Omega_2) = \partial\Omega_1 \setminus \partial\Omega = \partial\Omega_2 \setminus \partial\Omega.$$

Some fundamental results are established for particular geometries and are the main tools for the general result. Let us mention that our results generalize those obtained for piecewise constant μ by a boundary integral equation approach [11].

The symmetric case. The simplest case is that of a domain which is symmetrical with respect to the interface.

Lemma 2.3. *Suppose that* Σ *is included in a straight line* $\Delta \subset \mathbb{R}^2$ *and that* Ω_2 *is the image of* Ω_1 *by the mirror symmetry of axis* Δ*. Then, condition*

$$\frac{\inf_{\Omega_1} \mu}{\sup_{\Omega_2} |\mu|} > 1 \quad \textit{or} \quad \frac{\inf_{\Omega_2} |\mu|}{\sup_{\Omega_1} \mu} > 1, \tag{16}$$

implies that problem (11) is of Fredholm type and is well-posed for all $\omega^2 \in \mathbb{C}\backslash\mathscr{S}$*, where* $\mathscr{S}$ *is at most a countable set with no accumulation points.*

Proof. It is proved in [2], Theorem 3.1, that $A_D^{1/\mu}$ is an isomorphism as soon as (16) is fulfilled. The Lemma follows, proceeding like in the proof of the previous Lemma. ∎

Remark 2.4.

1. *Notice that if μ takes constant values μ_i in Ω_i for $i = 1$ and 2, the condition on μ reduces to $\mu_1/\mu_2 \neq -1$. This condition is sharp: indeed, $A_D^{1/\mu}$ has an infinite dimensional kernel when $\mu_1/\mu_2 = -1$ (cf. [2, Theorem 6.1]), so that in this case, for all ω, $A_D^{1/\mu}(\omega)$ is not a Fredholm operator.*
2. *If only μ has a sign-change ($\inf_\Omega \varepsilon > 0$), a more precise description of the structure of $\mathscr{S}$ can be given: $\mathscr{S} = \{\omega_1^2, \omega_2^2 \cdots\}$ is a countable set of real numbers and the sequence (ω_n^2) accumulates at $+\infty$ and $-\infty$. On the other hand, if the coefficients μ and ε are both sign-changing, this result is no longer true. In particular, complex eigenfrequencies can appear.*

The corner case. Let us describe now what happens when the interface Σ presents a corner:

Lemma 2.5. *Suppose that Ω is a disk of radius $R > 0$ and that, for some $\alpha \in]0, 2\pi[$, we have, expressed in polar coordinates:*

$$\Omega_1 := \{(r\cos\theta, r\sin\theta) \,|\, 0 < r < R,\ 0 < \theta < \alpha\},$$
$$\Omega_2 := \{(r\cos\theta, r\sin\theta) \,|\, 0 < r < R,\ \alpha < \theta < 2\pi\}.$$

Then, condition

$$\frac{\inf_{\Omega_1}\mu}{\sup_{\Omega_2}|\mu|} > I_\alpha \quad or \quad \frac{\inf_{\Omega_2}|\mu|}{\sup_{\Omega_1}\mu} > I_\alpha, \tag{17}$$

where

$$I_\alpha = \max\left(\frac{2\pi-\alpha}{\alpha}, \frac{\alpha}{2\pi-\alpha}\right),$$

implies that problem (11) is of Fredholm type and is well-posed for all $\omega^2 \in \mathbb{C}\backslash\mathscr{S}$, where $\mathscr{S}$ is at most a countable set with no accumulation points.

Proof. It is proved in [2], Theorem 3.3, that $A_D^{1/\mu}$ is an isomorphism as soon as (17) is fulfilled. Again, the Lemma follows, proceeding like in the proof of Lemma 2.2. ∎

Remark 2.6. 1. *If μ takes constant values μ_i in Ω_i for $i = 1$ and 2, the condition on μ reduces to $\mu_1/\mu_2 \notin [-I_\alpha, -1/I_\alpha]$, which is an interval which always contains -1, and reduces to $\{-1\}$ if and only if $\alpha = \pi$.*

2. *Item 2. of Remark 2.4 is also true for this configuration.*
3. *We have proved in [4] that for $\mu_1/\mu_2 \in]-I_\alpha, -1/I_\alpha[$, problem (11) is indeed ill-posed, due to a strange black-hole phenomenon at the corner.*

A general result. Let us consider now a more general geometry: we suppose that Ω_2 is an inclusion inside Ω, so that $\Sigma = \partial\Omega_2$ and $\partial\Omega_1 = \partial\Omega \cup \Sigma$. We suppose moreover that Σ is a polygonal (closed) curve. Then, combining a localization technique with the previous Lemmas 2.3 and 2.5 as in [2], we obtain the following

Lemma 2.7. *Let $\alpha_1, \alpha_2, \cdots, \alpha_N$ denote the angles at the N vertices of Σ. Then, condition*

$$\frac{\inf_{\Omega_1} \mu}{\sup_{\Omega_2} |\mu|} > I_\alpha \quad or \quad \frac{\inf_{\Omega_2} |\mu|}{\sup_{\Omega_1} \mu} > I_\alpha, \tag{18}$$

where

$$I_\alpha = \max\left(I_{\alpha_1}, I_{\alpha_1}, \cdots, I_{\alpha_N}\right),$$

implies that problem (11) is of Fredholm type.

In the particular geometry of figure 1, one has for instance $I_\alpha = 3$.

In fact, condition (18) can be weakened. Indeed, one can prove that it is sufficient to consider the values of μ near the interface, and even their limits on the interface when μ_1 and μ_2 are continuous functions. Moreover, it is not necessary to impose a global condition on the whole interface Σ. For instance, on the straight parts of Σ, it suffices to impose $(\mu_1/\mu_2)(\mathbf{x}) \neq -1$. Finally, the case of a curvilinear polygon can be treated with similar arguments. We refer the reader to a more complete description of the results in [2].

Remark 2.8. *The result of Lemma 2.7 is weaker than those of Lemmas 2.3 and 2.5. By virtue of analytic Fredholm theorem, to prove that problem (11) is well-posed for all $\omega^2 \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is discrete, we need uniqueness for one value of ω. This can be obtained when only μ has a sign-change and* $\inf_\Omega \varepsilon > 0$, *and again in this case, $\mathscr{S}$ is a set of real numbers accumulating at $+\infty$ and $-\infty$. However, in the case of both sign-changing coefficients, we have not been able to extend the theory.*

3. Mathematical study of the vectorial formulation for $\boldsymbol{H}_\perp$

Let us turn now to the analysis of the vectorial formulation of the TE problem. We refer the reader to the introduction for the motivations of this second approach.

3.1. A formulation in a space of divergence free fields

Like in the classical case of positive coefficients, the mathematical analysis of problem (12) is not straightforward: indeed, Fredholm theory cannot be applied because the embedding of $\mathbf{H}(\mathrm{curl}\,;\Omega)$ into $\mathbf{L}^2(\Omega)$ is not compact. As usual, the solution consists in taking into account the equation $\mathrm{div}\,(\mu\boldsymbol{H}_\perp) = 0$ which is a direct consequence of (12) and Corollary 1.3. First we have the

Proposition 3.1. *If $(\boldsymbol{H}_\perp, E_z)$ satisfies problem (6), then $\boldsymbol{H}_\perp$ is a solution of*

$$\begin{array}{l} \textit{Find } \boldsymbol{H}_\perp \in \mathbf{V}_T(\mu;\,\Omega) \quad \textit{such that:} \\ \displaystyle\int_\Omega \varepsilon^{-1}\mathrm{curl}\,\boldsymbol{H}_\perp\mathrm{curl}\,\boldsymbol{v} - \omega^2\mu\boldsymbol{H}_\perp\cdot\boldsymbol{v} = \int_\Omega \varepsilon^{-1}J_z\mathrm{curl}\,\boldsymbol{v}, \quad \boldsymbol{v}\in\mathbf{V}_T(\mu;\,\Omega). \end{array} \tag{19}$$

The study of this formulation for sign-changing coefficients ε and μ raises several questions which are addressed in the next section. But first of all, we need some reciprocal statement to Proposition 3.1. And surprisingly, for a sign-changing μ, the equivalence between (19) and (6) may fail. To illustrate this phenomenon, let us define the space of functions with zero mean-value:

$$\mathrm{H}^1_\#(\Omega) := \{v \in \mathrm{H}^1(\Omega)\,|\,\int_\Omega v = 0\}.$$

Classically, the mapping $(u, v) \mapsto (\nabla u, \nabla v)_\Omega$ defines a scalar product on $\mathrm{H}^1_\#(\Omega)$. Let us introduce now, using Riesz representation Theorem, the bounded operator $A^\mu_N : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ such that

$$(\nabla(A^\mu_N u), \nabla v)_\Omega = (\mu\nabla u, \nabla v)_\Omega, \qquad \forall (u, v) \in \mathrm{H}^1_\#(\Omega)\times\mathrm{H}^1_\#(\Omega), \tag{20}$$

where the subscript N stands for the Neumann boundary condition.

When μ is strictly positive, Lax-Milgram Theorem shows that A^μ_N is an isomorphism of $\mathrm{H}^1_\#(\Omega)$. But for a sign-changing μ, A^μ_N may not be invertible. In particular, it can happen that there exists $\lambda_N \in \mathrm{H}^1_\#(\Omega)\setminus\{0\}$ such that (cf. [2])

$$(\mu\nabla\lambda_N, \nabla v)_\Omega = 0, \qquad \forall v \in \mathrm{H}^1_\#(\Omega).$$

In other words, it can happen that there exists $\lambda_N \in \ker A_N^\mu$ with $\lambda_N \neq 0$. Then $\boldsymbol{H}_\perp = \nabla\lambda_N$ is a solution of (19) with $J_z = 0$. On the other hand, there is no $E_z \in \mathrm{H}_0^1(\Omega)$ such that $(E_z, \nabla\lambda_N)$ is a solution of (6) with $J_z = 0$. As a consequence, (19) and (6) are not equivalent in this case. That is why the next Proposition requires that A_N^μ is injective:

Proposition 3.2. *Suppose that the following hypothesis holds:*

$$(\mathcal{H}_N(\mu)) \quad A_N^\mu \textit{ is an isomorphism.}$$

If $\boldsymbol{H}_\perp$ satisfies (19), then the pair $(\boldsymbol{H}_\perp, E_z) = (\boldsymbol{H}_\perp, i(\omega\varepsilon)^{-1}(\operatorname{curl}\boldsymbol{H}_\perp - J_z))$ is a solution of problem (6).

Proof. Suppose that $\boldsymbol{H}_\perp$ satisfies (19). To prove that $(\boldsymbol{H}_\perp, E_z)$ is a solution of (6), it suffices to prove that $\boldsymbol{H}_\perp$ satisfies (12) and then to use Proposition 1.5. Since A_N^μ is an isomorphism, we can define for $\boldsymbol{v} \in \mathbf{H}(\operatorname{curl}; \Omega)$ the unique $\psi \in \mathrm{H}_\#^1(\Omega)$ such that $(\mu\nabla\psi, \nabla\psi')_\Omega = (\mu\boldsymbol{v}, \nabla\psi')_\Omega$ for all $\psi' \in \mathrm{H}_\#^1(\Omega)$. By Corollary 1.3, $\boldsymbol{v} - \nabla\psi \in \mathbf{V}_T(\mu; \Omega)$. Plugging $\boldsymbol{v} - \nabla\psi$ in (19), we get $(\varepsilon^{-1}\operatorname{curl}\boldsymbol{H}_\perp, \operatorname{curl}\boldsymbol{v})_\Omega - \omega^2(\mu\boldsymbol{H}_\perp, \boldsymbol{v})_\Omega = (\varepsilon^{-1}J_z, \operatorname{curl}\boldsymbol{v})_\Omega$, which proves that $\boldsymbol{H}_\perp$ satisfies (12). ■

3.2. A result of compact embedding

To prove the well-posedness of (19) by using Fredholm theory, we need a result of compactness of the embedding of $\mathbf{V}_T(\mu; \Omega)$ in $\mathbf{L}^2(\Omega)$. Such a result can fail for sign-changing μ (see Proposition 7.1 of [8]). We give here a general abstract proof of this result under an appropriate hypothesis on μ:

Theorem 3.3. *Suppose that the following hypothesis holds:*

$$(\mathcal{H}_D(1/\mu)) \quad A_D^{1/\mu} \textit{ is an isomorphism.}$$

Then the embedding of $\mathbf{V}_T(\mu; \Omega)$ into $\mathbf{L}^2(\Omega)$ is compact.

Remark 3.4. *Notice that hypothesis $(\mathcal{H}_D(1/\mu))$ already appeared in Lemma 2.1. Then in Lemmas 2.3, 2.5 and 2.7,we have established more explicit conditions on μ to ensure that $(\mathcal{H}_D(1/\mu))$ is satisfied.*

Proof. Let us consider a bounded sequence $(\boldsymbol{u}_n)$ of $\mathbf{V}_T(\mu;\,\Omega)$. For $n \in \mathbb{N}$, we set $f_n := \operatorname{curl} \boldsymbol{u}_n$. The sequence (f_n) is bounded in $\mathrm{L}^2(\Omega)$. Since $\operatorname{div} \mu\boldsymbol{u}_n = 0$ in Ω and $\mu\boldsymbol{u}_n \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, and since $\partial\Omega$ is a connected set, there exists, by Theorem 3.12 of [13], a bounded sequence (φ_n) of $\mathrm{H}^1_0(\Omega)$ such that $\mathbf{curl}\, \varphi_n = \mu\boldsymbol{u}_n$. To prove the Theorem, we just have to prove that we can extract from $\mathbf{curl}\, \varphi_n$, or equivalently from $\nabla\varphi_n$, a subsequence which converges in $\mathbf{L}^2(\Omega)$. Using Green formula (10), we deduce that for all $\varphi' \in \mathrm{H}^1_0(\Omega)$,

$$(\operatorname{curl} \boldsymbol{u}_n, \varphi')_\Omega = (\mu^{-1}\mathbf{curl}\, \varphi_n, \mathbf{curl}\, \varphi')_\Omega = (\mu^{-1}\nabla\varphi_n, \nabla\varphi')_\Omega,$$

which can be written, using the operator $A_D^{1/\mu}$ defined by (14):

$$(f_n, \varphi')_\Omega = (\nabla(A_D^{1/\mu}\varphi_n), \nabla\varphi')_\Omega.$$

Then setting $\varphi_{mn} := \varphi_m - \varphi_n$ and $f_{mn} := f_m - f_n$, and finally choosing $\varphi' = A_D^{1/\mu}\varphi_{mn}$ above, we obtain

$$(\nabla(A_D^{1/\mu}\varphi_{mn}), \nabla(A_D^{1/\mu}\varphi_{mn}))_\Omega = (f_{mn}, A_D^{1/\mu}\varphi_{mn})_\Omega.$$

Since $(A_D^{1/\mu}\varphi_n)$ is a bounded sequence of $\mathrm{H}^1_0(\Omega)$, it admits by Rellich Theorem a subsequence, still denoted $(\varphi_n)_n$, strongly convergent in $\mathrm{L}^2(\Omega)$. Then from the previous equality, it results that $(A_D^{1/\mu}\varphi_n)$ is a Cauchy sequence in $\mathrm{H}^1_0(\Omega)$. Since $A_D^{1/\mu}$ is an isomorphism, the sequence (φ_n) is also a Cauchy sequence in $\mathrm{H}^1_0(\Omega)$, which ends the proof of the Theorem. ∎

Corollary 3.5. *Suppose hypotheses* $(\mathcal{H}_N(\mu))$ *and* $(\mathcal{H}_D(1/\mu))$ *hold. Then the application* $(\boldsymbol{u}, \boldsymbol{v}) \mapsto (\operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{v})_\Omega$ *defines a scalar product on* $\mathbf{V}_T(\mu;\,\Omega)$.

Proof. Let us prove the existence of a positive constant C such that

$$\|\boldsymbol{u}\|_\Omega \le C\ \|\operatorname{curl} \boldsymbol{u}\|_\Omega\,, \qquad \forall \boldsymbol{u} \in \mathbf{V}_T(\mu;\,\Omega). \tag{21}$$

By contradiction, consider a sequence $(\boldsymbol{u}_n)$ of $\mathbf{V}_T(\mu;\,\Omega)$ such that

$$\forall n \in \mathbb{N}, \quad \|\boldsymbol{u}_n\|_\Omega = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \|\operatorname{curl} \boldsymbol{u}_n\|_\Omega = 0.$$

By the compact embedding result we have just proved, there exists a subsequence, still denoted $(\boldsymbol{u}_n)$, which converges to $\boldsymbol{u}$ in $\mathbf{L}^2(\Omega)$. By construction, $\|\boldsymbol{u}\|_\Omega = 1$ and $\operatorname{curl} \boldsymbol{u} = 0$ in Ω. Since $\partial\Omega$ is a connected set, we deduce from Theorem 3.2 of [13] the existence of $\varphi \in \mathrm{H}^1_\#(\Omega)$ such that $\boldsymbol{u} = \nabla\varphi$. Since $\boldsymbol{u} \in \mathbf{V}_T(\mu;\,\Omega)$, we have $\varphi \in \ker A_N^\mu$, which implies $\varphi = 0$ because by hypothesis, A_N^μ is an isomorphism. This implies $\boldsymbol{u} = 0$, which contradicts $\|\boldsymbol{u}\|_\Omega = 1$. ∎

3.3. Equivalence between hypotheses ($\mathcal{H}_D(1/\mu)$) and ($\mathcal{H}_N(\mu)$)

Two hypotheses on μ emerged naturally above, hypotheses ($\mathcal{H}_N(\mu)$) and ($\mathcal{H}_D(1/\mu)$). We prove in the next theorem that these two hypotheses are in fact equivalent.

Theorem 3.6. *Hypotheses ($\mathcal{H}_D(1/\mu)$) and ($\mathcal{H}_N(\mu)$) are equivalent. In other words, the operator $A_N^\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$, defined by (20), is an isomorphism if and only if the operator $A_D^{1/\mu} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$, defined by (14), is an isomorphism.*

Proof. Suppose A_N^μ is an isomorphism. For $u \in \mathrm{H}^1_0(\Omega)$, there exists a unique $\varphi \in \mathrm{H}^1_\#(\Omega)$ such that $(\mu\nabla\varphi, \nabla\varphi')_\Omega = (\mu\nabla u, \mathbf{curl}\,\varphi')_\Omega$ for all $\varphi' \in \mathrm{H}^1_\#(\Omega)$. Since $(\mu\nabla\varphi, \nabla\varphi')_\Omega = (\mu\,\mathbf{curl}\,\varphi, \mathbf{curl}\,\varphi')_\Omega$, we have, $\mathrm{curl}\,(\mu(\nabla u - \mathbf{curl}\,\varphi)) = 0$ and $\mu(\nabla u - \mathbf{curl}\,\varphi)\cdot\boldsymbol{\tau} = 0$ on $\partial\Omega$. But $\partial\Omega$ is a connected set, so there exists a unique $\psi \in \mathrm{H}^1_0(\Omega)$ such that $\mu(\nabla u - \mathbf{curl}\,\varphi) = \nabla\psi$ (cf. Theorem 3.2 in [13]). Let us denote by $\mathtt{T} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ the bounded operator defined by $\mathtt{T}u = \psi$.
For all $v \in \mathrm{H}^1_0(\Omega)$, we get

$$\begin{aligned}(\nabla(A_D^{1/\mu}(\mathtt{T}u)), \nabla v)_\Omega &= (\mu^{-1}\nabla(\mathtt{T}u), \nabla v)_\Omega \\ &= (\nabla u - \mathbf{curl}\,\varphi, \nabla v)_\Omega \\ &= (\nabla u, \nabla v)_\Omega.\end{aligned}$$

In other words, we have proved that $A_D^{1/\mu} \circ \mathtt{T}$ is equal to the identity in $\mathrm{H}^1_0(\Omega)$, which proves the invertibility of $A_D^{1/\mu}$, since $A_D^{1/\mu}$ is self-adjoint.
Conversely, suppose A_D^μ is an isomorphism. We can proceed in a similar manner as in the first part of the proof. For $u \in \mathrm{H}^1_\#(\Omega)$, let $\varphi \in \mathrm{H}^1_0(\Omega)$ be the unique solution of $(\mu^{-1}\nabla\varphi, \nabla\varphi')_\Omega = (\mu^{-1}\nabla u, \mathbf{curl}\,\varphi')_\Omega,\ \forall\varphi' \in \mathrm{H}^1_0(\Omega)$. Noting that $(\mu^{-1}\nabla\varphi, \nabla\varphi')_\Omega = (\mu^{-1}\,\mathbf{curl}\,\varphi, \mathbf{curl}\,\varphi')_\Omega$, we get, $\mathrm{curl}\,(\mu^{-1}(\nabla u - \mathbf{curl}\,\varphi)) = 0$ and consequently (see again Theorem 3.2 in [13]), there exists a unique $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mu^{-1}(\nabla u - \mathbf{curl}\,\varphi) = \nabla\psi$. With the help of the bounded operator $\mathtt{T} : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ such that $\mathtt{T}u = \psi$, we obtain for all $v \in \mathrm{H}^1_\#(\Omega)$

$$\begin{aligned}(\nabla(A_N^\mu(\mathtt{T}u)), \nabla v)_\Omega &= (\mu\nabla(\mathtt{T}u), \nabla v)_\Omega \\ &= (\nabla u - \mathbf{curl}\,\varphi, \nabla v)_\Omega \\ &= (\nabla u, \nabla v)_\Omega.\end{aligned}$$

We conclude that A_N^μ is an isomorphism of $\mathrm{H}^1_\#(\Omega)$ with $(A_N^\mu)^{-1} = \mathtt{T}$. ∎

3.4. Mathematical study of problem (19)

Now, we have all the ingredients to study problem (19), if we assume that hypothesis ($\mathcal{H}_D(1/\mu)$) (or equivalently hypothesis ($\mathcal{H}_N(\mu)$)) holds.

Theorem 3.7. *Suppose the equivalent hypotheses ($\mathcal{H}_D(1/\mu)$) and ($\mathcal{H}_N(\mu)$) hold. Then problem (19) is of Fredholm type and is well-posed for all $\omega^2 \in \mathbb{C}\backslash\mathscr{S}$, where $\mathscr{S}$ is at most a countable set with no accumulation points.*

Remark 3.8. *In other words, the sign-change of ε which may appear in the bilinear form $(\varepsilon^{-1}\mathrm{curl}\,\boldsymbol{u}, \mathrm{curl}\,\boldsymbol{v})_\Omega$ has no influence on the 2D Maxwell problem.*

Proof. Let us define, using Riesz representation Theorem and Corollary 3.5, the bounded operator $\mathscr{A}_T(\omega) : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)$ such that, for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbf{V}_T(\mu;\,\Omega) \times \mathbf{V}_T(\mu;\,\Omega)$, there holds

$$(\mathrm{curl}\,(\mathscr{A}_T(\omega)\boldsymbol{u}), \mathrm{curl}\,\boldsymbol{v})_\Omega = (\varepsilon^{-1}\mathrm{curl}\,\boldsymbol{u}, \mathrm{curl}\,\boldsymbol{v})_\Omega - \omega^2(\mu\boldsymbol{u}, \boldsymbol{v})_\Omega. \tag{22}$$

Let us prove first that $\mathscr{A}_T(0)$ is an isomorphism.

Given $\boldsymbol{u} \in \mathbf{V}_T(\mu;\,\Omega)$, there exists a unique $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $-\Delta\varphi = \varepsilon\,\mathrm{curl}\,\boldsymbol{u}$ and then, thanks to hypothesis ($\mathcal{H}_N(\mu)$), a unique $\psi \in \mathrm{H}^1_\#(\Omega)$ such that

$$(\mu\nabla\psi, \nabla\psi')_\Omega = (\mu\,\mathbf{curl}\,\varphi, \nabla\psi')_\Omega \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

We denote by $\mathbb{T} : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)$ the bounded operator which maps $\boldsymbol{u} \in \mathbf{V}_T(\mu;\,\Omega)$ to $\mathbb{T}\boldsymbol{u} = \mathbf{curl}\,\varphi - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$.

For all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbf{V}_T(\mu;\,\Omega) \times \mathbf{V}_T(\mu;\,\Omega)$, we get

$$\begin{aligned}
(\mathrm{curl}\,(\mathscr{A}_T(0)(\mathbb{T}\boldsymbol{u})), \mathrm{curl}\,\boldsymbol{v})_\Omega &= (\varepsilon^{-1}\mathrm{curl}\,(\mathbb{T}\boldsymbol{u}), \mathrm{curl}\,\boldsymbol{v})_\Omega \\
&= (\varepsilon^{-1}\mathrm{curl}\,(\mathbf{curl}\,\varphi - \nabla\psi), \mathrm{curl}\,\boldsymbol{v})_\Omega \\
&= (-\varepsilon^{-1}\Delta\varphi, \mathrm{curl}\,\boldsymbol{v})_\Omega \\
&= (\mathrm{curl}\,\boldsymbol{u}, \mathrm{curl}\,\boldsymbol{v})_\Omega.
\end{aligned}$$

Consequently $\mathscr{A}_T(0) \circ \mathbb{T}$ is equal to the identity of $\mathbf{V}_T(\mu;\,\Omega)$, which proves *a fortiori*, since $\mathscr{A}_T(0)$ is selfadjoint, that $\mathscr{A}_T(0)^{-1} = \mathbb{T}$.

But, from Theorem 3.3, we deduce that $\mathscr{A}_T(\omega) - \mathscr{A}_T(0)$ is a compact operator for all $\omega \in \mathbb{C}$. The application of the analytic Fredholm theorem allows us to conclude. ∎

4. Relaxing hypothesis ($\mathcal{H}_D(1/\mu)$)

The results we have obtained above are satisfactory when the equivalent hypotheses ($\mathcal{H}_D(1/\mu)$) and ($\mathcal{H}_N(\mu)$) hold. Indeed, the previous analysis provides a coherent description of the two approaches, relying respectively on the scalar and on the vectorial formulation of the TE problem.

But what happens if ($\mathcal{H}_D(1/\mu)$) does not hold ? More precisely, suppose that the scalar problem (11) in E_z is well-posed. This means that the operator $A_D^{1/\mu}(\omega)$ is an isomorphism, which does not imply in general that $A_D^{1/\mu} = A_D^{1/\mu}(0)$ is an isomorphism. It implies only that $A_D^{1/\mu}$ is a Fredholm operator.

In what follows, we consider the case where $A_D^{1/\mu}(\omega)$ is an isomorphism and $A_D^{1/\mu}$ is a Fredholm operator with a kernel which is not reduced to $\{0\}$ (see [8] for some examples). Then, by Theorem 3.6, A_N^{μ} is not an isomorphism. We will see more precisely that A_N^{μ} is also a Fredholm operator with a kernel which is not reduced to $\{0\}$. Then, as explained just before Proposition 3.2, the vectorial formulation (19) is in this case not equivalent to the initial TE problem. The relevant question is then, how to study the vectorial problem (12) ?

4.1. An additional result on the operators $A_D^{1/\mu}$ and A_N^{μ}

Theorem 4.1. *The operator $A_N^{\mu} : \mathrm{H}^1_{\#}(\Omega) \to \mathrm{H}^1_{\#}(\Omega)$ is a Fredholm operator if and only if $A_D^{1/\mu} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is a Fredholm operator. Moreover, if A_N^{μ} and $A_D^{1/\mu}$ are Fredholm operators, then $\dim\ker A_D^{1/\mu} = \dim\ker A_N^{\mu}$.*

Proof. Suppose A_N^{μ} is a Fredholm operator. If A_N^{μ} is injective, then A_N^{μ} is an isomorphism and then, by Theorem 3.6, $A_D^{1/\mu}$ is also an isomorphism. Let us consider here the case where A_N^{μ} has a non-trivial kernel $\mathrm{vect}(\lambda_N^1, \cdots, \lambda_N^{n_N})$ where the λ_N^i are such that $(\nabla\lambda_N^i, \nabla\lambda_N^j)_\Omega = \delta_{ij}$.

To prove that $A_D^{1/\mu}$ is a Fredholm operator, we build a right parametrix for $A_D^{1/\mu}$, i.e. a right inverse to $A_D^{1/\mu}$ modulo a compact operator. So, we are looking for a bounded operator $\mathtt{T}$ and a compact operator $\mathtt{K}$ of $\mathrm{H}^1_0(\Omega)$ such that $A_D^{1/\mu} \circ \mathtt{T} + \mathtt{K}$ is equal to the identity operator of $\mathrm{H}^1_0(\Omega)$.
We proceed by generalizing the proof of Theorem 3.6 and by using the following result, which is a straightforward consequence of the Fredholm alternative: given a continuous linear form ℓ on $\mathrm{H}^1_{\#}(\Omega)$, the problem

$$\text{Find } \varphi \in \mathrm{H}^1_{\#}(\Omega) \text{ such that } (\mu\nabla\varphi, \nabla\varphi')_\Omega = \ell(\varphi'), \quad \forall\varphi' \in \mathrm{H}^1_{\#}(\Omega),$$

admits solutions if and only if

$$\ell(\lambda_N^i) = 0, \quad i = 1, \cdots, n_N. \tag{23}$$

Moreover, there is a unique solution $\varphi \in \mathrm{S}_N$ where

$$\mathrm{S}_N = \{\psi \in \mathrm{H}^1_{\#}(\Omega) \text{ such that } (\nabla\psi, \nabla\lambda_N^i)_\Omega = 0, \quad i = 1, \cdots, n_N\}. \tag{24}$$

In particular, for all $u \in \mathrm{H}^1_0(\Omega)$, there exists a unique $\varphi \in \mathrm{S}_N$ satisfying

$$(\mu\nabla\varphi, \nabla\varphi')_\Omega = (\mu(\nabla u - \sum_{i=1}^{n_N} \beta^i \mathbf{\Gamma}_N^i), \mathbf{curl}\,\varphi')_\Omega, \qquad \forall \varphi' \in \mathrm{H}^1_{\#}(\Omega), \tag{25}$$

where we have set $\beta^i := (\mu\nabla u, \mathbf{curl}\,\lambda_N^i)$ and $\mathbf{\Gamma}_N^i := \mu^{-1}\mathbf{curl}\,\lambda_N^i$, $i = 1 \dots n_N$.

Note that (25) can be rewritten as $(\mu(\nabla u - \sum_{i=1}^{n_N} \beta^i \mathbf{\Gamma}_N^i - \mathbf{curl}\,\varphi), \mathbf{curl}\,\varphi')_\Omega = 0, \quad \forall \varphi' \in \mathrm{H}^1_{\#}(\Omega)$. As a consequence, $\mathrm{curl}\,(\mu(\nabla u - \sum_{i=1}^{n_N} \beta^i \mathbf{\Gamma}_N^i - \mathbf{curl}\,\varphi)) = 0$, and $\mu(\nabla u - \sum_{i=1}^{n_N} \beta^i \mathbf{\Gamma}_N^i - \mathbf{curl}\,\varphi) \cdot \boldsymbol{\tau} = 0$ on $\partial\Omega$. By Theorem 3.2 of [13], there exists a unique $\psi \in \mathrm{H}^1_0(\Omega)$ such that $\mu(\nabla u - \sum_{i=1}^{n_N} \beta^i \mathbf{\Gamma}_N^i - \mathbf{curl}\,\varphi) = \nabla\psi$. Then defining $\mathtt{T} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ by $\mathtt{T}u = \psi$ and $\mathtt{K} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ by

$$(\nabla(\mathtt{K}u), \nabla v)_\Omega = \sum_{i=1}^{n_N} (\mu\nabla u, \mathbf{curl}\,\lambda_N^i)(\mathbf{\Gamma}_N^i, \nabla v), \quad \forall v \in \mathrm{H}^1_0(\Omega),$$

one can check that $(\nabla(A_D^{1/\mu}(\mathtt{T}u)), \nabla v)_\Omega = (\nabla u, \nabla v)_\Omega - (\nabla(\mathtt{K}u), \nabla v)_\Omega$ for all $v \in \mathrm{H}^1_0(\Omega)$. Consequently, $A_D^{1/\mu} \circ \mathtt{T} + \mathtt{K} = \mathtt{Id}$. Moreover, $\mathtt{K}$ is a compact operator because its range is finite dimensional, of dimension less than or equal to n_N. Since $A_D^{1/\mu}$ is selfadjoint, this proves that $A_D^{1/\mu}$ is a Fredholm operator whose kernel has a dimension less than or equal to n_N. This can be rewritten: $\dim\ker A_D^{1/\mu} \le \dim\ker A_N^\mu$.
We can prove conversely that if $A_D^{1/\mu}$ is a Fredholm operator, then A_N^μ is a Fredholm operator and $\dim\ker A_N^\mu \le \dim\ker A_D^{1/\mu}$, which ends the proof. ■

4.2. Enriched vectorial formulation

In this subsection, we suppose that A_N^μ has a non-trivial kernel and we use the notations of the proof of Theorem 4.1. By the definition (24) of S_N, we have $\mathrm{H}^1_{\#}(\Omega) = \ker A_N^\mu \overset{\perp}{\oplus} \mathrm{S}_N$. Let us introduce now the space

$$\tilde{\mathbf{V}}_T(\mu; \Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathrm{curl}\,; \Omega) \,|\, (\mu\boldsymbol{u}, \nabla\varphi)_\Omega = 0, \ \forall \varphi \in \mathrm{S}_N\}\,.$$

Since $S_N \subset H^1_{\#}(\Omega)$, by Corollary 1.3, we have $\mathbf{V}_T(\mu;\,\Omega) \subset \tilde{\mathbf{V}}_T(\mu;\,\Omega)$. Moreover, the embedding is strict and by simple arguments, one can prove that there exists an antidual basis of $(\nabla\lambda^i_N)$, denoted by $\boldsymbol{\Lambda}^i_N \in \tilde{\mathbf{V}}_T(\mu;\,\Omega)$, $i = 1\ldots n_N$, such that $(\mu\boldsymbol{\Lambda}^i_N, \nabla\lambda^j_N)_\Omega = \delta_{ij}$, for $j = 1\ldots n_N$. As a consequence, there holds

$$\tilde{\mathbf{V}}_T(\mu;\,\Omega) = \mathbf{V}_T(\mu;\,\Omega) \oplus \mathrm{vect}(\boldsymbol{\Lambda}^i_N)_{i=1}^{n_N}.$$

Proposition 4.2. *Second order vectorial formulation (enriched).*
1)If $(\boldsymbol{H}_\perp, E_z)$ is a solution of (6) then $\boldsymbol{H}_\perp$ is a solution of

$$\begin{cases} \textit{Find } \boldsymbol{H}_\perp \in \tilde{\mathbf{V}}_T(\mu;\,\Omega) \ \ \textit{such that:} \\ \displaystyle\int_\Omega \varepsilon^{-1}\mathrm{curl}\,\boldsymbol{H}_\perp \mathrm{curl}\,\boldsymbol{v} - \omega^2\mu\boldsymbol{H}_\perp\cdot\boldsymbol{v} = \int_\Omega \varepsilon^{-1}J_z\mathrm{curl}\,\boldsymbol{v},\ \forall\boldsymbol{v}\in\tilde{\mathbf{V}}_T(\mu;\,\Omega). \end{cases} \tag{26}$$

2) If $\boldsymbol{H}_\perp$ satisfies (26), the pair $(\boldsymbol{H}_\perp, E_z) = (\boldsymbol{H}_\perp, i(\omega\varepsilon)^{-1}(\mathrm{curl}\,\boldsymbol{H}_\perp - J_z))$ is a solution of problem (6).

Proof. The first implication is straightforward. Conversely, suppose $\boldsymbol{H}_\perp$ satisfies (26). To prove that $(\boldsymbol{H}_\perp, E_z) = (\boldsymbol{H}_\perp, i(\omega\varepsilon)^{-1}(\mathrm{curl}\,\boldsymbol{H}_\perp - J_z))$ is a solution of (6), it suffices to prove that $\boldsymbol{H}_\perp$ satisfies (12) and then use Proposition 1.5.

For $\boldsymbol{v} \in \mathbf{H}(\mathrm{curl}\,;\,\Omega)$, there is a unique $\psi \in S_N$ such that $(\mu\nabla\psi, \nabla\psi')_\Omega = (\mu\boldsymbol{v}, \nabla\psi')_\Omega$ for all $\psi' \in S_N$, so that the field $\boldsymbol{v} - \nabla\psi$ belongs to $\tilde{\mathbf{V}}_T(\mu;\,\Omega)$. Injecting $\boldsymbol{v} - \nabla\psi$ in (26), we get $(\varepsilon^{-1}\mathrm{curl}\,\boldsymbol{H}_\perp, \mathrm{curl}\,\boldsymbol{v})_\Omega - \omega^2(\mu\boldsymbol{H}_\perp, \boldsymbol{v})_\Omega = (\varepsilon^{-1}J_z, \mathrm{curl}\,\boldsymbol{v})_\Omega$ which ends the proof. ∎

Again, to study the enriched formulation (26), we need a result of compact embedding which is given by the

Theorem 4.3. *The embedding of $\tilde{\mathbf{V}}_T(\mu;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact and the application $\boldsymbol{u} \mapsto \|\mathrm{curl}\,\boldsymbol{v}\|_\Omega + \sum_{i=1}^{n_N}|\alpha^i|$, where $\boldsymbol{u} = \boldsymbol{v} + \sum_{i=1}^{n_N}\alpha^i\boldsymbol{\Lambda}^i_N$ with $\boldsymbol{v} \in \mathbf{V}_T(\mu;\,\Omega)$ and $(\alpha^1,\ldots,\alpha^{n_N}) \in \mathbb{R}^{n_N}$, defines a norm on $\tilde{\mathbf{V}}_T(\mu;\,\Omega)$, equivalent to the norm $(\|\cdot\|^2_\Omega + \|\mathrm{curl}\,\cdot\|^2_\Omega)^{1/2}$.*

Proof. With obvious notations, consider $(\boldsymbol{u}_n)$, with $\boldsymbol{u}_n = \boldsymbol{v}_n + \sum_{i=1}^{n_N}\alpha^i_n\boldsymbol{\Lambda}^i_N$, a bounded sequence of $\tilde{\mathbf{V}}_T(\mu;\,\Omega)$. To prove the Theorem, it suffices to prove that $(\boldsymbol{v}_n)$ has a subsequence converging in $\mathbf{L}^2(\Omega)$. Following the proof of Theorem 3.3, up to the extraction of a subsequence, we introduce the bounded sequence (φ_n) of $H^1_0(\Omega)$, strongly converging in $L^2(\Omega)$, such that $\mathbf{curl}\,\varphi_n = \mu\boldsymbol{v}_n$

and we have for all $\varphi' \in \mathrm{H}^1_0(\Omega)$:

$$\begin{aligned}(\mu^{-1}\mathbf{curl}\,\varphi_n, \mathbf{curl}\,\varphi')_\Omega &= (\mu^{-1}\nabla\varphi_n, \nabla\varphi')_\Omega = (\nabla\varphi_n, \nabla(A_D^{1/\mu}\varphi'))_\Omega \\ &= (\mathrm{curl}\,\boldsymbol{v}_n, \varphi')_\Omega.\end{aligned}$$

On the other hand, we have established in the proof of Theorem 4.1 that $A_D^{1/\mu} \circ \mathtt{T} + \mathtt{K} = \mathtt{Id}$ where $\mathtt{T}$ and $\mathtt{K}$ are respectively a bounded and a compact operator of $\mathrm{H}^1_0(\Omega)$. Then setting $\varphi_{mn} = \varphi_m - \varphi_n$ and $\mathrm{curl}\,\boldsymbol{v}_{mn} = \mathrm{curl}\,\boldsymbol{v}_m - \mathrm{curl}\,\boldsymbol{v}_n$, and taking $\varphi' = \mathtt{T}\varphi_{mn}$, we obtain

$$\|\nabla\varphi_{mn}\|^2_\Omega = (\mathrm{curl}\,\boldsymbol{v}_{mn}, \mathtt{T}\varphi_{mn})_\Omega + (\nabla\varphi_{mn}, \nabla(\mathtt{K}\varphi_{mn}))_\Omega.$$

By the boundedness of $\mathtt{T}$ and the compactness of $\mathtt{K}$, one can easily deduce that $(\nabla\varphi_n)$ is a Cauchy sequence of $\mathbf{L}^2(\Omega)$, which ends the proof. ■

We have now all necessary ingredients to prove the

Theorem 4.4. *Suppose that $A_D^{1/\mu}(\omega)$ is an isomorphism, then (26) is well-posed, and problem (6) has one, and only one, solution.*

Proof. The idea is again to build an operator $\tilde{\mathbb{T}} : \tilde{\mathbf{V}}_T(\mu;\Omega) \to \tilde{\mathbf{V}}_T(\mu;\Omega)$ such that for all $\boldsymbol{u} \in \tilde{\mathbf{V}}_T(\mu;\Omega)$,

$$(\varepsilon^{-1}\mathrm{curl}\,\boldsymbol{u}, \mathrm{curl}\,\tilde{\mathbb{T}}\boldsymbol{u})_\Omega = (\mathrm{curl}\,\boldsymbol{u}, \mathrm{curl}\,\boldsymbol{u})_\Omega.$$

The Theorem follows by using the compactness result proved in Theorem 4.3 together with Proposition 4.2.

To build an operator $\tilde{\mathbb{T}}$, consider for $\boldsymbol{u} \in \tilde{\mathbf{V}}_T(\mu;\Omega)$ the unique $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $-\Delta\varphi = \varepsilon\,\mathrm{curl}\,\boldsymbol{u}$ and the unique $\psi \in \mathrm{S}_N$ such that $(\mu\nabla\psi, \nabla\psi')_\Omega = (\mu\,\mathbf{curl}\,\varphi, \nabla\psi')_\Omega$ for all $\psi' \in \mathrm{S}_N$. Then we set $\tilde{\mathbb{T}}\boldsymbol{u} = \mathbf{curl}\,\varphi - \nabla\psi$. ■

This last result allows us to provide, in a systematic way, a vectorial formulation for $\boldsymbol{H}_\perp$ which is both well-posed and equivalent to problem (6), as soon as this last problem has a unique solution. Either $A_D^{1/\mu}$ is an isomorphism and the classical formulation (19) in $\mathbf{V}_T(\mu;\Omega)$ can be used, or $A_D^{1/\mu}$ has a non trivial kernel, and one should work in $\tilde{\mathbf{V}}_T(\mu;\Omega)$ with formulation (26).

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References

[1] A. Aubry, D.Y. Lei, A.I. Fernàndez-Domínguez, Y. Sonnefraud, S.A. Maier, J.B. Pendry, Plasmonic light-harvesting devices over the whole visible spectrum, Nano Lett. 10 (2010) 2574–2579.

[2] A.S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., T-coercivity for scalar interface problems between dielectrics and metamaterials, Math. Mod. Num. Anal. 46 (2012) 1363–1387.

[3] A.S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., T-coercivity for the Maxwell problem with sign-changing coefficients, HAL, http://hal.inria.fr/hal-00762275/ (2012).

[4] A.S. Bonnet-Ben Dhia, L. Chesnel, X. Claeys, Radiation condition for a non-smooth interface between a dielectric and a metamaterial, Math. Models Meth. App. Sci. (to appear).

[5] A.S. Bonnet-Ben Dhia, P. Ciarlet Jr., C. Zwölf, Two-and three-field formulations for wave transmission between media with opposite sign dielectric constants, J. Comput. Appl. Math. 204 (2007) 408–417.

[6] A.S. Bonnet-Ben Dhia, P. Ciarlet Jr., C. Zwölf, A new compactness result for electromagnetic waves. Application to the transmission problem between dielectrics and metamaterials, Math. Models Meth. App. Sci. 18 (2008) 1605–1631.

[7] A.S. Bonnet-Ben Dhia, P. Ciarlet Jr., C. Zwölf, Time harmonic wave diffraction problems in materials with sign-shifting coefficients, J. Comput. Appl. Math. 234 (2010) 1912–1919. *Corrigendum* J. Comput. Appl. Math. 234 (2010) 2616.

[8] L. Chesnel, P. Ciarlet Jr., Compact imbeddings in electromagnetism with interfaces between classical materials and metamaterials, SIAM J. Math. Anal. 43 (2011) 2150–2169.

[9] L. Chesnel, P. Ciarlet Jr., $\mathtt{T}$-coercivity and continuous Galerkin methods: application to transmission problems with sign changing coefficients, Numer. Math. (to appear).

[10] D. Colton, R. Kress, Inverse acoustic and electromagnetic scattering theory (2nd Edition), volume 93 of *Applied Mathematical Sciences*, Springer Verlag, 1998.

[11] M. Costabel, E. Stephan, A direct boundary integral method for transmission problems, J. of Math. Anal. and Appl. 106 (1985) 367–413.

[12] J. Fleckinger, M. Lapidus, Eigenvalues of elliptic boundary value problems with an indefinite weight function, Transactions of the American Mathematical Society 295 (1986) 305–324.

[13] V. Girault, P.A. Raviart, Finite element methods for Navier-Stokes equations, volume 5 of *Springer Series in Computational Mathematics*, Springer-Verlag, Berlin, 1986.

[14] S. Maier, Plasmonics: Fundamentals and Applications, Springer, 2007.

[15] D. Maystre, S. Enoch, Perfect lenses made with left-handed materials: Alice's mirror?, JOSA A 21 (2004) 122–131.

[16] J. Pendry, Negative refraction makes a perfect lens, Physical Review Letters 85 (2000) 3966–3969.