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Mathematical Problems in Mechanics

A Lagrangian approach to intrinsic linearized elasticity

Une approche lagrangienne de l'élasticité linéarisée intrinsèque

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ABSTRACT

We consider the pure traction problem and the pure displacement problem of three-dimensional linearized elasticity. We show that, in each case, the intrinsic approach leads to a quadratic minimization problem constrained by Donati-like relations. Using the Babuška–Brezzi inf–sup condition, we then show that, in each case, the minimizer of the constrained minimization problem found in an intrinsic approach is the first argument of the saddle-point of an *ad hoc* Lagrangian, so that the second argument of this saddle-point is the Lagrange multiplier associated with the corresponding constraints.

RÉSUMÉ

On considère le problème en déplacement pur et le problème en traction pure de l'élasticité linéarisée tri-dimensionnelle. On montre que, dans chaque cas, l'approche intrinsèque conduit à un problème de minimisation quadratique avec des contraintes semblables à celles de Donati. Utilisant la condition inf–sup de Babuška–Brezzi, on montre ensuite que, dans chaque cas, le minimiseur du problème de minimisation avec contraintes trouvé dans une approche intrinsèque est le premier argument du point-selle d'un lagrangien approprié, ce qui fait que le second argument de ce point-selle est le multiplicateur de Lagrange associé aux contraintes correspondantes.

1. Notations and preliminaries

Latin indices vary in the set $\{1, 2, 3\}$ and the summation convention with respect to repeated indices is systematically used in conjunction with this rule.

Spaces of functions, vector fields in $\mathbb{R}^3$, and 3×3 matrix fields, are respectively denoted by italic capitals, boldface Roman capitals, and special Roman capitals. The space of all symmetric matrices of order 3 is denoted $\mathbb{S}^3$. The subscript s appended to a special Roman capital denotes a space of symmetric matrix fields. Let Ω be an open subset of $\mathbb{R}^3$. The

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notation $D'(\Omega)$ denotes the space of distributions defined over Ω. The notations $H^m(\Omega)$ and $H_0^m(\Omega)$ designate the usual Sobolev spaces. Combining the above rules, we shall thus encounter spaces such as $\boldsymbol{D}'(\Omega), \mathbb{D}'(\Omega), \mathbb{L}_s^2(\Omega)$, etc.

A *domain in* $\mathbb{R}^3$ is a bounded, connected, open subset of $\mathbb{R}^3$ whose boundary is Lipschitz-continuous, the set Ω being locally on a single side of its boundary.

Complete proofs of the results stated here are found in [6].

2. An intrinsic approach to the pure traction problem

Let a domain Ω in $\mathbb{R}^3$, with boundary Γ, be the reference configuration of a linearly elastic body, characterized by its elasticity tensor field $\mathbf{A} = (A_{ijk\ell})$ with components $A_{ijk\ell} \in L^\infty(\Omega)$. It is assumed as usual that these components satisfy the symmetry relations $A_{ijk\ell} = A_{jik\ell} = A_{k\ell ij}$, and that there exists a constant $\alpha > 0$ such that

$$\mathbf{A}(x)\boldsymbol{t} : \boldsymbol{t} \geqslant \alpha \boldsymbol{t} : \boldsymbol{t} \quad \text{for almost all } x \in \Omega \text{ and all matrices } \boldsymbol{t} = (t_{ij}) \in \mathbb{S}^3,$$

where $(\mathbf{A}(x)\boldsymbol{t})_{ij} := A_{ijk\ell}(x)t_{k\ell}$. The body is subjected to applied body forces with density $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$ in its interior and to applied surface forces of density $\boldsymbol{g} \in \boldsymbol{L}^{4/3}(\Gamma)$ on its boundary. Finally, it is assumed that the linear form $L \in \mathcal{L}(\boldsymbol{H}^1(\Omega); \mathbb{R})$ defined by

$$L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v}\,\mathrm{d}\Gamma \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$$

vanishes for all $\boldsymbol{v} \in \mathbf{Ker}\,\nabla_s$, where ∇_s denotes the *symmetrized gradient operator*, i.e.,

$$\nabla_s \boldsymbol{v} := \frac{1}{2}\left(\nabla \boldsymbol{v}^T + \nabla \boldsymbol{v}\right) \quad \text{for any } \boldsymbol{v} \in \boldsymbol{D}'(\Omega).$$

Then the corresponding *pure traction problem of three-dimensional linearized elasticity* classically consists in finding $\dot{\boldsymbol{u}} \in \dot{\boldsymbol{H}}^1(\Omega) := \boldsymbol{H}^1(\Omega)/\mathbf{Ker}\,\nabla_s$ such that

$$J(\dot{\boldsymbol{u}}) = \inf_{\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)} J(\dot{\boldsymbol{v}}), \quad \text{where } J(\dot{\boldsymbol{v}}) := \frac{1}{2}\int_\Omega \mathbf{A}\nabla_s \dot{\boldsymbol{v}} : \nabla_s \dot{\boldsymbol{v}}\,\mathrm{d}x - L(\dot{\boldsymbol{v}}).$$

As is well known (cf. Duvaut and Lions [7]), this minimization problem has one and only one solution.

An *intrinsic approach* to the above pure traction problem consists in considering the linearized strain tensor field $\boldsymbol{\varepsilon} := \nabla_s \dot{\boldsymbol{u}} : \Omega \to \mathbb{S}^3$ as the primary unknown, instead of the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ itself. Accordingly, one first needs to characterize those 3×3 matrix fields $\boldsymbol{e} \in \mathbb{L}_s^2(\Omega)$ that can be written as $\boldsymbol{e} = \nabla_s \boldsymbol{v}$ for some vector fields $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. The *"Donati-like"* characterization given in the next theorem is not the only possible one; others, such as those found in [1] and [11], are equally possible, but they do not seem to be suitable for our purposes (an extensive discussion of Donati-like compatibility conditions is found in [1]).

Given a domain Ω in $\mathbb{R}^3$, define the Hilbert space

$$\mathbb{H}(\mathbf{div}; \Omega) := \left\{\boldsymbol{\mu} \in \mathbb{L}_s^2(\Omega);\ \mathbf{div}\,\boldsymbol{\mu} \in \boldsymbol{L}^2(\Omega)\right\},$$

and let $\boldsymbol{\nu} : \Gamma \to \mathbb{R}^3$ denote the unit outer normal vector field along the boundary of Ω (such a field is defined $\mathrm{d}\Gamma$-everywhere). Then, as shown in Geymonat and Krasucki [8,9] the density of the space $\mathbb{C}_s^\infty(\overline{\Omega})$ in the space $\mathbb{H}(\mathbf{div}; \Omega)$ then implies that the mapping $\boldsymbol{\mu} \in \mathbb{C}_s^\infty(\overline{\Omega}) \to \boldsymbol{\mu}\boldsymbol{\nu}|_\Gamma$ can be extended to a continuous linear mapping $\boldsymbol{\gamma} : \mathbb{H}(\mathbf{div}; \Omega) \to \boldsymbol{H}^{-1/2}(\Gamma)$.

The proof of the next theorem relies on various results from [1] and [8].

Theorem 2.1. *Let Ω be a domain in $\mathbb{R}^3$ and let there be given a matrix field $\boldsymbol{e} \in \mathbb{L}_s^2(\Omega)$. Then there exists a vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{e} = \nabla_s \boldsymbol{v}$ if and only if*

$$\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{M},$$

where the space $\mathbb{M}$ is defined as

$$\mathbb{M} := \left\{\boldsymbol{\mu} \in \mathbb{L}_s^2(\Omega);\ \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega),\ \boldsymbol{\gamma}\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1/2}(\Gamma)\right\}.$$

All other vector fields $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ that satisfy $\boldsymbol{e} = \nabla_s \widetilde{\boldsymbol{v}}$ are of the form $\widetilde{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega$ for some vectors $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$.

Thanks to Theorem 2.1, the pure traction problem of three-dimensional elasticity can then be recast as *a constrained quadratic minimization problem*, with $\boldsymbol{\varepsilon} := \nabla_s \dot{\boldsymbol{u}} \in \mathbb{L}_s^2(\Omega)$ as the primary unknown. The proof of the next theorem relies on Theorem 2.1, Banach open mapping theorem, and the Lax–Milgram lemma.

Theorem 2.2. *Let Ω be a domain in $\mathbb{R}^3$, and let the space $\mathbb{M}$ be defined as in Theorem* 2.1*. Define the Hilbert space*

$$\mathbb{E} := \left\{ \boldsymbol{e} \in \mathbb{L}^2_s(\Omega);\ \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \textit{for all } \boldsymbol{\mu} \in \mathbb{M} \right\},$$

and, for each $\boldsymbol{e} \in \mathbb{E}$, let $\boldsymbol{\mathcal{F}}(\boldsymbol{e})$ denote the unique element in the quotient space $\dot{\boldsymbol{H}}^1(\Omega)$ that satisfies $\nabla_s \boldsymbol{\mathcal{F}}(\boldsymbol{e}) = \boldsymbol{e}$ (Theorem 2.1*). Then the mapping $\boldsymbol{\mathcal{F}} : \mathbb{E} \to \dot{\boldsymbol{H}}^1(\Omega)$ defined in this fashion is an isomorphism between the Hilbert spaces $\mathbb{E}$ and $\dot{\boldsymbol{H}}^1(\Omega)$.*

The minimization problem: Find $\boldsymbol{\varepsilon} \in \mathbb{E}$ such that

$$j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e} \in \mathbb{E}} j(\boldsymbol{e}), \quad \textit{where } j(\boldsymbol{e}) := \frac{1}{2} \int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x - L \circ \boldsymbol{\mathcal{F}}(\boldsymbol{e}),$$

has one and only one solution $\boldsymbol{\varepsilon}$, and this solution satisfies $\boldsymbol{\varepsilon} = \nabla_s \dot{\boldsymbol{u}}$, where $\dot{\boldsymbol{u}}$ is the unique minimizer of the functional J in the space $\dot{\boldsymbol{H}}^1(\Omega)$.

3. Lagrangian approach to the pure traction problem

We now identify the *Lagrangian*, and consequently the *Lagrange multiplier* (as the second argument of the saddle-point of the Lagrangian), associated with the constrained quadratic minimization problem of Theorem 2.2. The spaces $\mathbb{M}$ and $\mathbb{E}$ used in the next theorem are those defined in Theorems 2.1 and 2.2.

Theorem 3.1. *Define the Lagrangian*

$$\mathcal{L}(\boldsymbol{e}, \boldsymbol{\mu}) := \frac{1}{2} \int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x - \ell(\boldsymbol{e}) + \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x \quad \textit{for all } (\boldsymbol{e}, \boldsymbol{\mu}) \in \mathbb{V} \times \mathbb{M},$$

where $\mathbb{V} := \mathbb{L}^2_s(\Omega)$ and $\ell : \mathbb{L}^2_s(\Omega) \to \mathbb{R}$ is any continuous linear extension of the continuous linear form $L \circ \mathcal{F} : \mathbb{E} \to \mathbb{R}$.

Then the Lagrangian $\mathcal{L}$ has a unique saddle-point $(\boldsymbol{\varepsilon}, \boldsymbol{\lambda}) \in \mathbb{V} \times \mathbb{M}$ over the space $\mathbb{V} \times \mathbb{M}$. Besides, the first argument $\boldsymbol{\varepsilon}$ of the saddle-point is the unique solution of the minimization problem of Theorem 2.2*, i.e.,*

$$\boldsymbol{\varepsilon} \in \mathbb{E}(\Omega) \quad \textit{and} \quad j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e} \in \mathbb{E}} j(\boldsymbol{e}),$$

and the second argument $\boldsymbol{\lambda} \in \mathbb{M}$ of the saddle-point is a Lagrange multiplier associated with this minimization problem.

Sketch of proof. Let the spaces $\mathbb{V}$ and $\mathbb{M} \subset \mathbb{V}$ be both equipped with the norm of the space $\mathbb{L}^2_s(\Omega)$. Combining the relations $\mathbb{M} \subset \mathbb{H}(\mathbf{div}; \Omega)$ and $\mathbf{div}\,\boldsymbol{\mu} = \mathbf{0}$ if $\boldsymbol{\mu} \in \mathbb{M}$ with the continuity of the mapping $\boldsymbol{\gamma} : \mathbb{H}(\mathbf{div}; \Omega) \to \boldsymbol{H}^{-1/2}(\Gamma)$, we first conclude that the space $\mathbb{M}$ is closed in $\mathbb{V}$. Hence both $\mathbb{V}$ and $\mathbb{M}$ are Hilbert spaces.

Define two bilinear forms $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and $b : \mathbb{V} \times \mathbb{M} \to \mathbb{R}$ by

$$a(\boldsymbol{\varepsilon}, \boldsymbol{e}) := \int_\Omega \mathbf{A}\boldsymbol{\varepsilon} : \boldsymbol{e}\,\mathrm{d}x \quad \text{for all } (\boldsymbol{\varepsilon}, \boldsymbol{e}) \in \mathbb{V} \times \mathbb{V},$$

$$b(\boldsymbol{e}, \boldsymbol{\mu}) := \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x \quad \text{for all } (\boldsymbol{e}, \boldsymbol{\mu}) \in \mathbb{V} \times \mathbb{M}.$$

Clearly, both bilinear forms are continuous. Besides, the bilinear form $a(\cdot,\cdot)$ is symmetric on $\mathbb{V}$, and $\mathbb{V}$-coercive.

Finally, the Babuška–Brezzi inf–sup condition (cf. [2] and [3]) follows from the inclusion $\mathbb{M} \subset \mathbb{V}$, which implies that, for each $\boldsymbol{\mu} \in \mathbb{M}$,

$$\sup_{\left\{ \substack{\boldsymbol{e} \in \mathbb{V} \\ \boldsymbol{e} \neq \mathbf{0}} \right.} \frac{\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x}{\|\boldsymbol{e}\|_{\mathbb{L}^2(\Omega)} \|\boldsymbol{\mu}\|_{\mathbb{L}^2(\Omega)}} \geqslant \frac{\int_\Omega \boldsymbol{\mu} : \boldsymbol{\mu}\,\mathrm{d}x}{\|\boldsymbol{\mu}\|^2_{\mathbb{L}^2(\Omega)}} = 1.$$

It then follows from classical results (see Brezzi and Fortin [4] or Girault and Raviart [10]) that the variational problem: Find $(\boldsymbol{\varepsilon}, \boldsymbol{\lambda}) \in \mathbb{V} \times \mathbb{M}$ such that

$$a(\boldsymbol{\varepsilon}, \boldsymbol{e}) + b(\boldsymbol{e}, \boldsymbol{\lambda}) = \ell(\boldsymbol{e}) \quad \text{for all } \boldsymbol{e} \in \mathbb{V},$$

$$b(\boldsymbol{\varepsilon}, \boldsymbol{\mu}) = 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{M},$$

has one and only one solution.

Besides $(\boldsymbol{\varepsilon}, \boldsymbol{\lambda}) \in \mathbb{V} \times \mathbb{M}$ is the unique saddle-point of the Lagrangian $\mathcal{L} : \mathbb{V} \times \mathbb{M} \to \mathbb{R}$ defined by

$$\mathcal{L}(\boldsymbol{e},\boldsymbol{\mu}) := \frac{1}{2}a(\boldsymbol{e},\boldsymbol{e}) - \ell(\boldsymbol{e}) + b(\boldsymbol{e},\boldsymbol{\mu}) \quad \text{for all } (\boldsymbol{e},\boldsymbol{\mu}) \in \mathbb{V}\times\mathbb{M},$$

i.e.,

$$\inf_{\boldsymbol{e}\in\mathbb{V}}\sup_{\boldsymbol{\mu}\in\mathbb{M}}\mathcal{L}(\boldsymbol{e},\boldsymbol{\mu}) = \sup_{\boldsymbol{\mu}\in\mathbb{M}}\mathcal{L}(\boldsymbol{\varepsilon},\boldsymbol{\mu}) = \mathcal{L}(\boldsymbol{\varepsilon},\boldsymbol{\lambda}) = \inf_{\boldsymbol{e}\in\mathbb{V}}\mathcal{L}(\boldsymbol{e},\boldsymbol{\lambda}) = \sup_{\boldsymbol{\mu}\in\mathbb{M}}\inf_{\boldsymbol{e}\in\mathbb{V}}\mathcal{L}(\boldsymbol{e},\boldsymbol{\mu}),$$

and $\boldsymbol{\varepsilon}$ is the unique solution to the constrained quadratic minimization problem

$$j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e}\in\mathbb{E}} J(\boldsymbol{e}), \quad \text{where } j(\boldsymbol{e}) := \frac{1}{2}a(\boldsymbol{e},\boldsymbol{e}) - \ell(\boldsymbol{e}) \text{ for all } \boldsymbol{e}\in\mathbb{V}.$$

In the language of optimization theory, $\boldsymbol{\lambda}\in\mathbb{M}$ is thus the *Lagrange multiplier* associated with the above constrained quadratic minimization problem. □

4. An intrinsic approach to the pure displacement problem

Consider now the *pure displacement problem of three-dimensional linearized elasticity*, which classically consists in finding $\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega)$ such that

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)} J(\boldsymbol{v}), \quad \text{where } J(\boldsymbol{v}) := \frac{1}{2}\int_\Omega \mathbf{A}\nabla_s\boldsymbol{v} : \nabla_s\boldsymbol{v}\,\mathrm{d}x - L(\boldsymbol{v}),$$

where

$$L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \text{for all } \boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$$

for some given body force density $\boldsymbol{f}\in\boldsymbol{L}^{6/5}(\Omega)$.

An *intrinsic approach* to the above pure displacement problem consists again in considering the linearized strain tensor $\boldsymbol{\varepsilon} := \nabla_s\boldsymbol{u} : \Omega\to\mathbb{S}^3$ as the primary unknown, instead of the displacement vector field $\boldsymbol{u}:\Omega\to\mathbb{R}^3$. Accordingly, we need to characterize those 3×3 matrix fields $\boldsymbol{e}\in\mathbb{L}_s^2(\Omega)$ that can be written as $\boldsymbol{e}=\nabla_s\boldsymbol{v}$ for some vector field $\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$. The following result, established in Theorem 4.2 in [1], constitutes such a characterization, again *of Donati type*.

Theorem 4.1. *Let Ω be a domain in $\mathbb{R}^3$ and let there be given a matrix field $\boldsymbol{e}\in\mathbb{L}_s^2(\Omega)$. Then there exists a vector field $\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$ such that $\boldsymbol{e}=\nabla_s\boldsymbol{v}$ if and only if*

$$\int_\Omega \boldsymbol{e}:\boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \textit{for all } \boldsymbol{\mu}\in\mathbb{M}_0,$$

where the space $\mathbb{M}_0$ is defined as

$$\mathbb{M}_0 := \{\boldsymbol{\mu}\in\mathbb{L}_s^2(\Omega);\ \mathbf{div}\,\boldsymbol{\mu}=\mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega)\}.$$

If this is the case, the vector field $\boldsymbol{v}$ is unique.

Thanks to Theorem 4.1, this problem can be again recast as *a constrained quadratic minimization problem* with $\boldsymbol{\varepsilon} := \nabla_s\boldsymbol{u}\in\mathbb{L}_s^2(\Omega)$ as the primary unknown (the proof is similar to that of Theorem 2.2).

Theorem 4.2. *Let Ω be a domain in $\mathbb{R}^3$, and let the space $\mathbb{M}_0$ be defined as in Theorem* 4.1. *Define the Hilbert space*

$$\mathbb{E}_0 := \left\{\boldsymbol{e}\in\mathbb{L}_s^2(\Omega);\ \int_\Omega \boldsymbol{e}:\boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \textit{for all } \boldsymbol{\mu}\in\mathbb{M}_0\right\},$$

and, for each $\boldsymbol{e}\in\mathbb{E}_0$, let $\boldsymbol{\mathcal{F}}_0(\boldsymbol{e})$ denote the unique element in the space $\boldsymbol{H}_0^1(\Omega)$ that satisfies $\nabla_s\boldsymbol{\mathcal{F}}_0(\boldsymbol{e})=\boldsymbol{e}$ (Theorem 4.1*). Then the mapping $\boldsymbol{\mathcal{F}}_0:\mathbb{E}_0\to\boldsymbol{H}_0^1(\Omega)$ defined in this fashion is an isomorphism between the Hilbert spaces $\mathbb{E}_0$ and $\boldsymbol{H}_0^1(\Omega)$.*

The minimization problem: Find $\boldsymbol{\varepsilon}\in\mathbb{E}_0$ such that

$$j_0(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e}\in\mathbb{E}_0} j_0(\boldsymbol{e}), \quad \textit{where } j_0(\boldsymbol{e}) := \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e}:\boldsymbol{e}\,\mathrm{d}x - L\circ\boldsymbol{\mathcal{F}}_0(\boldsymbol{e}),$$

has one and only one solution $\boldsymbol{\varepsilon}$, and this solution satisfies $\boldsymbol{\varepsilon}=\nabla_s\boldsymbol{u}$, where $\boldsymbol{u}$ is the unique minimizer of the functional J in the space $\boldsymbol{H}_0^1(\Omega)$.

5. A Lagrangian approach to the pure displacement problem

We now identify the *Lagrangian* and *Lagrange multiplier* associated with the constrained quadratic minimization problem of Theorem 4.2. The spaces $\mathbb{M}_0$ and $\mathbb{E}_0$ used in the next theorem are those defined in Theorems 4.1 and 4.2. The proof is similar to that of Theorem 3.1.

Theorem 5.1. *Define the Lagrangian*

$$\mathcal{L}_0(\boldsymbol{e},\boldsymbol{\mu}) := \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x - \ell_0(\boldsymbol{e}) + \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x \quad \text{for all } (\boldsymbol{e},\boldsymbol{\mu}) \in \mathbb{V}\times\mathbb{M}_0,$$

where $\mathbb{V} := \mathbb{L}^2_s(\Omega)$ *and* $\ell_0 : \mathbb{L}^2_s(\Omega) \to \mathbb{R}$ *is any continuous linear extension of the continuous linear form* $L \circ \mathcal{F}_0 : \mathbb{E}_0 \to \mathbb{R}$.

Then the Lagrangian $\mathcal{L}_0$ *has a unique saddle-point* $(\boldsymbol{\varepsilon},\boldsymbol{\lambda}) \in \mathbb{V}\times\mathbb{M}_0$ *over the space* $\mathbb{V}\times\mathbb{M}_0$*. Besides, the first argument* $\boldsymbol{\varepsilon}$ *of the saddle-point is the unique solution of the minimization problem of Theorem* 4.1*, i.e.,*

$$\boldsymbol{\varepsilon} \in \mathbb{E}_0(\Omega) \quad \text{and} \quad j_0(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e}\in\mathbb{E}_0} j_0(\boldsymbol{e}),$$

and the second argument $\boldsymbol{\lambda} \in \mathbb{M}_0$ *of the saddle-point is a Lagrange multiplier associated with this minimization problem.*

6. Concluding remarks

Another possible Lagrangian approach to the *pure traction problem* is based on the following observations. Recall that, for any matrix field $\boldsymbol{e} = (e_{ij}) \in \mathbb{D}'(\Omega)$, the matrix field $\mathbf{CURLCURL}\,\boldsymbol{e} \in \mathbb{D}'(\Omega)$ is defined by

$$(\mathbf{CURLCURL}\,\boldsymbol{e})_{ij} := \varepsilon_{ik\ell}\varepsilon_{jmn}\partial_{\ell n}e_{km},$$

where (ε_{ijk}) denotes the orientation tensor.

The classical *Saint Venant compatibility conditions* have been recently shown to remain sufficient under weak regularity assumptions. More specifically, Ciarlet and Ciarlet Jr. [5] have established the following *Saint Venant theorem in* $\mathbb{L}^2_s(\Omega)$: *Let* Ω *be a simply-connected domain in* $\mathbb{R}^3$ *and let* $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$ *be a matrix field that satisfies the Saint Venant compatibility conditions* $\mathbf{CURLCURL}\,\boldsymbol{e} = \mathbf{0}$ *in* $\mathbb{H}^{-2}_s(\Omega)$*. Then there exists a vector field* $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ *such that* $\boldsymbol{e} = \nabla_s\boldsymbol{v}$ in $\mathbb{L}^2_s(\Omega)$.

A natural question then arises as to whether a Lagrange multiplier can be associated with the constraint $\mathbf{CURLCURL}\,\boldsymbol{e} = \mathbf{0}$ in $\mathbb{H}^{-2}_s(\Omega)$. With the same space $\mathbb{V}$ and bilinear form $a : \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ as in Theorem 4.1, natural candidates for the corresponding space $\mathbb{M}$ and bilinear form $b : \mathbb{V}\times\mathbb{M} \to \mathbb{R}$ are:

$$\mathbb{M} = \mathbb{H}^2_{0,s}(\Omega) \quad \text{and} \quad b(\boldsymbol{e},\boldsymbol{\mu}) = \int_\Omega \boldsymbol{e} : \mathbf{CURLCURL}\,\boldsymbol{\mu}\,\mathrm{d}x \quad \text{for all } (\boldsymbol{e},\boldsymbol{\mu}) \in \mathbb{L}^2_s(\Omega)\times\mathbb{H}^2_{0,s}(\Omega),$$

since, given $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$, the constraint $\mathbf{CURLCURL}\,\boldsymbol{e} = \mathbf{0}$ in $\mathbb{H}^{-2}_s(\Omega)$ is equivalent to

$${}_{\mathbb{H}^{-2}_s(\Omega)}\langle \mathbf{CURLCURL}\,\boldsymbol{e},\boldsymbol{\mu}\rangle_{\mathbb{H}^2_{0,s}(\Omega)} = \int_\Omega \boldsymbol{e} : \mathbf{CURLCURL}\,\boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{H}^2_{0,s}(\Omega).$$

Using results from Geymonat and Krasucki [8], one can then show that the Babuška–Brezzi inf–sup condition is not satisfied, however (see [6]).

Another possible Lagrangian approach to the *pure traction problem* is based on the following observations. In Theorem 4.3 in [1] it was shown that, given $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$, there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{e} = \nabla_s\boldsymbol{v}$ in $\mathbb{L}^2_s(\Omega)$ if and only if

$$\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{H}^1_{0,s}(\Omega) \text{ such that } \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{L}^2(\Omega).$$

This extension does not seem to be amenable to a Lagrangian approach, however, again because the Babuška–Brezzi inf–sup condition is not satisfied by the corresponding space $\mathbb{M}$ and bilinear form b.

The present results could pave the way for a new class of numerical schemes for approximating linear elasticity problems, where the unknown to be approximated is the saddle-point found in either Theorem 3.1 or Theorem 5.1.

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