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SOLVING ELECTROMAGNETIC EIGENVALUE PROBLEMS IN POLYHEDRAL DOMAINS WITH NODAL FINITE ELEMENTS

Annalisa Buffa · Patrick Ciarlet, Jr. · Erell Jamelot

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Abstract A few years ago, Costabel and Dauge proposed a variational setting, which allows one to solve numerically the time-harmonic Maxwell equations in $3D$ polyhedral geometries, with the help of a continuous approximation of the electromagnetic field. In order to remove spurious eigenmodes, their method required a parameterization of the variational formulation. In order to avoid this difficulty, we use a mixed variational setting instead of the parameterization, which allows us to handle the divergence-free constraint on the field in a straightforward manner. The numerical analysis of the method is carried out, and numerical examples are provided to show the efficiency of our approach.

Introduction

In a landmark paper [19], Costabel and Dauge proposed a method, which allowed one to discretize the electromagnetic field with a continuous approximation, in $3D$, convex or non-convex, polyhedra. In this way, they provided a generalization of the method earlier developed by Heintzé *et al* [26,6], which relied also on a continuous approximation of the field, but worked only in $3D$, convex polyhedra.
As a matter of fact, in order to be able to solve Maxwell's equations in a non-convex polyhedron with a continuous and conforming discretization, one has to overcome a

Annalisa Buffa
IMATI-CNR,
Via Ferrata 1, 27100 Pavia, Italy
E-mail: annalisa@imati.cnr.it
Patrick Ciarlet, Jr.
Laboratoire POEMS, UMR 7231 CNRS/ENSTA/INRIA,
École Nationale Supérieure de Techniques Avancées,
32, boulevard Victor, 75739 Paris Cedex 15, France
E-mail: patrick.ciarlet@ensta.fr
Erell Jamelot
Laboratoire POEMS, UMR 7231 CNRS/ENSTA/INRIA,
École Nationale Supérieure de Techniques Avancées,
32, boulevard Victor, 75739 Paris Cedex 15, France
E-mail: erell.jamelot@ensta.org

very difficult mathematical problem. It turns out that the discretized spaces are always included in a closed, strict subspace – sometimes called the subspace of *regular fields* – of the space of all possible fields. In other words, one cannot hope to approximate the part of the field, if it exists, which belongs to the orthogonal of the subspace of regular fields.
Over the past decade, several methods have been devised to address this problem. Apart from the method considered by Costabel and Dauge [19,20], at least two other remedies can be used. First, the *singular complement method*, in which one approximates explicitly the remaining part of the field. It works well in $2D$ geometries (see [5, 4,25,23,27]), and $2\frac{1}{2}D$ geometries (see [14,29,3]). Second, one can relax the boundary condition, and compute the field in a larger space: this allows one to recover the desired density property, since the sum of all discretized spaces is again dense in the space of possible fields. This approach has been studied recently [13,28]. Then, one can use a weaker norm to measure the divergence of the field: one replaces the L^2-norm by a weighted L^2-norm. This *weighted approach*, introduced by Costabel and Dauge [19,20], leads again to a larger space for the fields, where once again the density property is recovered. We focus hereafter on the weighted approach. Note however that all results remain valid, should one prefer to use the singular complement method in $2D$ and $2\frac{1}{2}D$ geometries (see the concluding remarks).

In order to solve the time-harmonic Maxwell equations, Costabel and Dauge then proceeded by adding a *regularization term*, with a parameter s: this resulted in the *weighted regularization method*, refered too as WRM later on. The drawback of this technique is that the regularized operator has two sequences of eigenvalues: one is correct and the other is spurious. This is a feature of the problem reformulation and has nothing to do with pollution of numerical schemes. The spurious eigenvalues vary with the parameter s and can be recognized performing the eigenvalue computation with several values of the parameter. The difficulty is that one has to deal with a $3D$ cloud of points, i. e. eigenvalues sorted by increasing magnitude, for given values of s and a given meshsize. Then, one has to keep all the right modes and, correlatively, reject all the spurious ones.
To get around this difficulty, we propose to use a constrained formulation, namely we add a constraint on the divergence of the field. This approach has been presented for the continuous problem in the Annex of [13]. Our aim here is to carry out the numerical analysis of the constrained formulation. For that, we rely mainly on the general approximation theory for this kind of problem, which can be found in [10].

The outline of the paper is as follows. In the next Section, we recall the time-harmonic Maxwell equations, which we express as a set of second-order PDEs. In Section 2, we introduce the functional framework. Then, we build the continuous variational formulations to be solved in Section 3. In Section 4, we prove the convergence of the discretized eigenmodes towards the exact eigenmodes. And, in Section 5, we propose some numerical examples to illustrate the behavior of our method. Concluding remarks follow.

1 Setting of the problem

Hereafter, we consider the case of a resonator cavity Ω, bounded by a perfect conductor. The domain $\Omega \subset \mathbb{R}^3$ is a bounded, simply connected, open polyhedron with a Lipschitz, connected, boundary $\partial\Omega$. Let $\mathbf{n}$ be the unit outward normal to $\partial\Omega$. The goal is to find eigenmodes of electromagnetic oscillations. Let c be the light velocity. The electromagnetic eigenmodes are non-zero solutions to the time-harmonic Maxwell equations

$$\imath\omega\mathcal{E} - c^2\mathbf{curl}\,\mathcal{B} = 0 \text{ in } \Omega, \tag{1}$$
$$\imath\omega\mathcal{B} + \mathbf{curl}\,\mathcal{E} = 0 \text{ in } \Omega, \tag{2}$$
$$\operatorname{div}\mathcal{E} = 0 \text{ in } \Omega, \tag{3}$$
$$\operatorname{div}\mathcal{B} = 0 \text{ in } \Omega, \tag{4}$$
$$\mathcal{E} \times \mathbf{n} = 0 \text{ on } \partial\Omega, \tag{5}$$
$$\mathcal{B} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{6}$$

Above, $\mathcal{E}$ and $\mathcal{B}$ are respectively the electric field and magnetic induction, and ω is the time-frequency. When $\omega \neq 0$, equations (3)-(4) are clearly redundant, being straightforward consequences of equations (1)-(2). However, since we want to investigate methods which include explicitly (3)-(4) as *constraints*, we have to mention them.

It is well known that one of the fields can be eliminated. Multiplying (1) by $\imath\omega$ and adding $\mathbf{curl}$ of (2) we get a vector wavelike equation for $\mathcal{E}$, which reads $-\omega^2\mathcal{E} + c^2\mathbf{curl}\,\mathbf{curl}\,\mathcal{E} = 0$ in Ω. Similarly, there holds $-\omega^2\mathcal{B} + c^2\mathbf{curl}\,\mathbf{curl}\,\mathcal{B} = 0$ in Ω. Then, each system of equations below is equivalent to (1)-(6).
The electric eigenvalue problem (PE):
Find $\mathcal{E}$ and ω such that

$$c^2\mathbf{curl}\,\mathbf{curl}\,\mathcal{E} = \omega^2\mathcal{E} \text{ in } \Omega, \tag{7}$$
$$\operatorname{div}\mathcal{E} = 0 \text{ in } \Omega, \tag{8}$$
$$\mathcal{E} \times \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{9}$$

The magnetic eigenvalue problem (PB):
Find $\mathcal{B}$ and ω such that

$$c^2\mathbf{curl}\,\mathbf{curl}\,\mathcal{B} = \omega^2\mathcal{B} \text{ in } \Omega, \tag{10}$$
$$\operatorname{div}\mathcal{B} = 0 \text{ in } \Omega, \tag{11}$$
$$\mathcal{B} \cdot \mathbf{n} = 0 \text{ and } \mathbf{curl}\,\mathcal{B} \times \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{12}$$

From now on, we consider the eigenvalue problem in $\mathcal{E}$ (PE).

2 Functional spaces

Recall that our aim is to discretize the field with a continuous, conforming approximation. As we mentioned already, the correct functional spaces will be different, depending on the convexity of the domain [19,20].

Let us begin with Lebesgue and Sobolev spaces, which are needed both in the convex and in the non-convex cases. Let $L^2(\Omega)$ be the usual Lebesgue space of measurable and square integrable functions over Ω. Its canonical norm and scalar product are respectively denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)_0$. Then, $H^1(\Omega)$ will denote the space of $L^2(\Omega)$ functions with gradients in $L^2(\Omega)^3$. We then introduce

$$\mathcal{H}(\mathbf{curl}\,,\Omega) := \{\mathcal{F} \in L^2(\Omega)^3 \,|\, \mathbf{curl}\,\mathcal{F} \in L^2(\Omega)^3\}\,,$$
$$\mathcal{H}_0(\mathbf{curl}\,,\Omega) := \{\mathcal{F} \in \mathcal{H}(\mathbf{curl}\,,\Omega) \,|\, \mathcal{F} \times \mathbf{n}_{|\partial\Omega} = 0\}\,.$$

When Ω is convex, we introduce also

$$\mathcal{X} := \{\mathcal{F} \in \mathcal{H}_0(\mathbf{curl}\,,\Omega) \,|\, \operatorname{div}\mathcal{F} \in L^2(\Omega)\}.$$

According to the set of equations governing the behavior of $\mathcal{E}$, there holds $\mathcal{E} \in \mathcal{X}$. According to Costabel [18], the graph norm and the semi-norm: $||\mathcal{F}||^2_{\mathcal{X}} := ||\mathbf{curl}\,\mathcal{F}||^2_0 + ||\operatorname{div}\mathcal{F}||^2_0$ are equivalent norms on $\mathcal{X}$. Also (see [1]), $\mathcal{X}$ is a subset of $H^1(\Omega)^3$: $\mathcal{X} \cap H^1(\Omega)^3$ and $\mathcal{X}$ coincide.

On the other hand, when Ω is non-convex, $\mathcal{X} \cap H^1(\Omega)^3$ is a strict, closed subspace [1] of $\mathcal{X}$: $\mathcal{X} \cap H^1(\Omega)^3$ is not dense in $\mathcal{X}$ anymore. Therefore the need of another choice of functional space.

Stating that the domain Ω is non-convex amounts to considering that $\partial\Omega$ includes a set of reentrant edges E, with dihedral angles $(\pi/\alpha_e)_{e\in E}$ such that $1/2 < \alpha_e < 1$. Let d denote the distance to the set of reentrant edges E: $d(\mathbf{x}) = dist(\mathbf{x}, \cup_{e\in E}\bar{e})$. We define the notation $d_{\mathcal{O}} := \inf\{d(\mathbf{x}),\ \mathbf{x} \in \mathcal{O}\}$, for any subset $\mathcal{O}$ of Ω.
Introduce the weight w_γ, a smooth non-negative function of $\mathbf{x}$. It behaves locally as d^γ in the neighborhood of reentrant edges and corners (we shall write $w_\gamma \simeq d^\gamma$), and is bounded above and below by a strictly positive constant outside a neighborhood of E (this corresponds to the simplified weights of [19], Subsection 4.5).
Let $L^2_\gamma(\Omega)$ be the following weighted space, with $||.||_{0,\gamma}$ norm:

$$L^2_\gamma(\Omega) := \{v \in L^2_{\mathrm{loc}}(\Omega) \,|\, w_\gamma\, v \in L^2(\Omega)\}\,,\ ||v||_{0,\gamma} := ||w_\gamma\, v||_0.$$

We then define

$$\mathcal{X}_\gamma := \{\mathcal{F} \in \mathcal{H}_0(\mathbf{curl}\,,\Omega) \,|\, \operatorname{div}\mathcal{F} \in L^2_\gamma(\Omega)\}.$$

In the non-convex case, it is easy to see that $\mathcal{X} \subset \mathcal{X}_\gamma$. The advantage of using a weaker topology has been pointed out by Costabel and Dauge in [19]. Their results read as follows[1]: there exists $\gamma_{min} \in]0, 1/2[$ such that for all $\gamma \in]\gamma_{min}, 1[$:

- on $\mathcal{X}_\gamma$, the graph norm and the semi-norm:

$$||\mathcal{F}||^2_{\mathcal{X}_\gamma} := ||\mathbf{curl}\,\mathcal{F}||^2_0 + ||\operatorname{div}\mathcal{F}||^2_{0,\gamma}$$

  are equivalent norms ;

[1] More precisely, one has $\gamma_{\min} := 2 - \sigma^D_\Delta$, where $\sigma^D_\Delta \in]\frac{3}{2}, 2[$ is the *minimum singularity exponent* for the Laplace problem with homogeneous Dirichlet boundary condition:

$$\{\phi \in H^1(\Omega) \,|\, \Delta\phi \in L^2(\Omega),\ \phi_{|\partial\Omega} = 0\} \subset \textstyle\bigcap_{s<\sigma^D_\Delta} H^s(\Omega),$$
$$\{\phi \in H^1(\Omega) \,|\, \Delta\phi \in L^2(\Omega),\ \phi_{|\partial\Omega} = 0\} \not\subset H^{\sigma^D_\Delta}(\Omega).$$

– $\mathcal{X}_\gamma \cap H^1(\Omega)^3$ is dense in $\mathcal{X}_\gamma$.

Finally, we introduce two functional spaces, which will be useful to characterize Lagrange multipliers. So, let

$$\mathbb{V}^1_\gamma := \{v \in L^2_{\text{loc}}(\Omega) \,|\, w_{\gamma-1} v \in L^2(\Omega),\ w_\gamma \nabla v \in L^2(\Omega)^3\},$$
$$\overset{\circ}{\mathbb{V}}{}^1_\gamma := \text{closure of } \mathcal{D}(\Omega) \text{ in } \mathbb{V}^1_\gamma\,.$$

It can be proved [19] that there holds $\overset{\circ}{\mathbb{V}}{}^1_\gamma = \{v \in \mathbb{V}^1_\gamma \,|\, v_{|\partial\Omega} = 0\}$. Note that given $v \in \mathbb{V}^1_\gamma$, one has $\nabla(w_\gamma v) \in L^2(\Omega)^3$. Moreover, on $\overset{\circ}{\mathbb{V}}{}^1_\gamma$, the graph norm and the semi-norm $||v||_{\mathbb{V}} = ||\nabla(w_\gamma v)||_0$ are equivalent norms. This implies[2] in particular the orthogonal decomposition

$$L^2(\Omega)^3 = \mathcal{H}(\operatorname{div}{}^0, \Omega) \overset{\perp}{\oplus} \nabla(w_\gamma \overset{\circ}{\mathbb{V}}{}^1_\gamma), \tag{13}$$

where $\mathcal{H}(\operatorname{div}{}^0, \Omega) := \{v \in L^2(\Omega)^3 \,|\, \operatorname{div} v = 0\}$. Finally, in the absence of weights ($w_0 \equiv 1$), one recovers that $\overset{\circ}{\mathbb{V}}{}^1_0 = H^1_0(\Omega)$.

In the following, we shall keep the index $_\gamma$ everywhere, to emphasize the use of the weighted norms in the non-convex case. However, it is clear that all results are valid in the convex case with the stronger norms (see Section 6).

3 Variational formulations

Set $\lambda = \omega^2/c^2$. Since the electric field belongs to

$$\mathcal{K}_\gamma := \{\mathcal{F} \in \mathcal{X}_\gamma \,|\, \operatorname{div} \mathcal{F} = 0\},$$

one can rewrite (PE) as
Find $(\mathcal{E}, \lambda) \in \mathcal{K}_\gamma \times \mathbb{R}^+$ *such that*

$$\mathbf{curl}\,\mathbf{curl}\,\mathcal{E} = \lambda\mathcal{E} \text{ in } \Omega. \tag{14}$$

It is common knowledge that the equivalent variational formulation is
Find $(\mathcal{E}, \lambda) \in \mathcal{K}_\gamma \times \mathbb{R}^+$ *such that*

$$(\mathbf{curl}\,\mathcal{E}, \mathbf{curl}\,\mathcal{F})_0 = \lambda(\mathcal{E}, \mathcal{F})_0,\ \forall \mathcal{F} \in \mathcal{K}_\gamma. \tag{15}$$

However, since it is difficult to build a conforming discretization in $\mathcal{K}_\gamma$, the divergence-free condition on the field is usually prescribed as a natural condition. This is the reason why one solves the eigenproblem in $\mathcal{X}_\gamma$ (or in $\mathcal{H}_0(\mathbf{curl}\,, \Omega)$, but this is another story...). We report below two possible approaches (see [19,20,13]).

[2] Given $f \in L^2(\Omega)^3$, solve
Find $u \in \overset{\circ}{\mathbb{V}}{}^1_\gamma$ *such that*

$$(\nabla(w_\gamma u), \nabla(w_\gamma u'))_0 = (f, \nabla(w_\gamma u'))_0,\ \forall u' \in \overset{\circ}{\mathbb{V}}{}^1_\gamma.$$

Thanks to the equivalence of norms, this problem is well-posed and admits a unique solution. Taking $u' \in \mathcal{D}(\Omega)$ yields $w_\gamma(\Delta(w_\gamma u) - \operatorname{div} f) = 0$ in the sense of distributions, so that $\Delta(w_\gamma u) = \operatorname{div} f$. Now, $\nabla(w_\gamma u)$ is in $L^2(\Omega)^3$, and $\zeta = f - \nabla(w_\gamma u)$ moreover belongs to $\mathcal{H}(\operatorname{div}{}^0, \Omega)$. The orthogonality of $\mathcal{H}(\operatorname{div}{}^0, \Omega)$ and $\nabla(w_\gamma \overset{\circ}{\mathbb{V}}{}^1_\gamma)$ is straightforward.

First approach. One remarks that if $(\mathcal{E}, \lambda)$ solves (14), then one can take the L^2 scalar product between $\mathbf{curl}\,\mathbf{curl}\,\mathcal{E}$ and a test field $\mathcal{F}$ of $\mathcal{X}_\gamma$ and integrate by parts, and then add the weighted L^2_γ scalar product between $\operatorname{div}\mathcal{E}$ and the divergence of the test field $\operatorname{div}\mathcal{F}$, to reach

$$(\mathbf{curl}\,\mathcal{E}, \mathbf{curl}\,\mathcal{F})_0 + (\operatorname{div}\mathcal{E}, \operatorname{div}\mathcal{F})_{0,\gamma} = \lambda(\mathcal{E}, \mathcal{F})_0.$$

Therefore, the eigenpair $(\mathcal{E}, \lambda)$ also solves
Find $(\mathcal{E}, \lambda) \in \mathcal{X}_\gamma \times \mathbb{R}^+$ *such that*

$$(\mathcal{E}, \mathcal{F})_{\mathcal{X}_\gamma} = \lambda(\mathcal{E}, \mathcal{F})_0 \ \forall \mathcal{F} \in \mathcal{X}_\gamma. \tag{16}$$

Unfortunately, the reciprocal assertion is not true: in other words, the fact that $(\mathcal{E}, \lambda)$ is a solution to (16) does not guarantee that it is an eigenpair of the original problem (14). The reason is that there exist solutions to (16) which are not divergence-free: in this sense we call these eigenpairs *spurious*.

To address this problem, Costabel and Dauge [19, 20] chose to consider instead the *parameterized eigenproblem*
Find $(\mathcal{E}_s, \lambda_s) \in \mathcal{X}_\gamma \times \mathbb{R}^+$ *such that*

$$(\mathbf{curl}\,\mathcal{E}_s, \mathbf{curl}\,\mathcal{F})_0 + s\,(\operatorname{div}\mathcal{E}_s, \operatorname{div}\mathcal{F})_{0,\gamma} = \lambda_s(\mathcal{E}_s, \mathcal{F})_0 \ \forall \mathcal{F} \in \mathcal{X}_\gamma, \tag{17}$$

where $s > 0$ is a parameter. In this way, the equivalence with the original problem can be restored. As a matter of fact, if one lets s vary, the true eigenpairs will be independent of s (they are divergence-free), whereas the spurious solutions will vary with s (they have a non-vanishing divergence).

Second approach. As it is advocated in [13], one can take into account the constraint on the divergence of the field, via the introduction of a Lagrange multiplier. In this way, one recovers the equivalence with the original problem (14) (see the Annex of [13]). The *mixed eigenproblem* to be solved reads
Find $(\mathcal{E}, p, \lambda) \in \mathcal{X}_\gamma \times L^2_\gamma(\Omega) \times \mathbb{R}^+$ *such that*

$$\begin{cases} (\mathcal{E}, \mathcal{F})_{\mathcal{X}_\gamma} + (p, \operatorname{div}\mathcal{F})_{0,\gamma} = \lambda(\mathcal{E}, \mathcal{F})_0 \ \forall \mathcal{F} \in \mathcal{X}_\gamma \\ (q, \operatorname{div}\mathcal{E})_{0,\gamma} = 0, \ \forall q \in L^2_\gamma(\Omega). \end{cases} \tag{18}$$

To prove the equivalence with the original problem, one follows the construction below. Notice that if $(\mathcal{E}, p, \lambda)$ is a solution to (18), then $\operatorname{div}\mathcal{E} = 0$ according to (18b). Moreover, the field[3] $\mathcal{F}_p = \nabla({\Delta_D}^{-1}p)$ belongs to $\mathcal{X}_\gamma$, since by construction $\mathcal{F}_p \in \mathcal{H}_0(\mathbf{curl}\,, \Omega)$, $\mathbf{curl}\,\mathcal{F}_p = 0$ and $\operatorname{div}\mathcal{F}_p = p$. As a consequence it can be put in (18a) to reach

$$||p||^2_{0,\gamma} = \lambda(\mathcal{E}, \nabla({\Delta_D}^{-1}p))_0 = 0 \text{ by integration by parts.}$$

Therefore, any solution to (18) is such that $p = 0$ and $\mathcal{E} \in \mathcal{K}_\gamma$. This proves the equivalence with the original problem, since $(\mathcal{E}, \lambda)$ is an eigenpair of (14).

In what follows, we focus on the discretization and on the numerical analysis of the second approach.

[3] Here, ${\Delta_D}^{-1} : L^2_\gamma(\Omega) \to H^1_0(\Omega)$ is such that, for any $q \in L^2_\gamma(\Omega)$, $\phi = {\Delta_D}^{-1}q$ is the solution to the Laplace problem with homogeneous Dirichlet boundary condition:
Find $\phi \in H^1_0(\Omega)$ *such that*

$$\Delta\phi = q \text{ in } \Omega.$$

This problem is well-posed since $L^2_\gamma(\Omega) \subset H^{-1}(\Omega)$.

4 Discretization and convergence results

4.1 Abstract theory

We recall here the general theory (see [10]), which allows one to prove convergence results, for eigenvalue problems set in a *mixed* variational framework. So, consider

- V and Q two Hilbert spaces, with norms $\|\cdot\|_V$ and $\|\cdot\|_Q$;
- a a bilinear, continuous, symmetric, positive, semidefinite form on $V \times V$;
- b a bilinear, continuous form on $V \times Q$;
- (f, g) an element of $V' \times Q'$.

Introduce the abstract mixed variational problem:
Find $(u, p) \in V \times Q$ *such that*

$$\begin{cases} a(u, v) + b(v, p) = \langle f, v \rangle, \ \forall v \in V \\ b(u, q) = \langle g, q \rangle, \ \forall q \in Q. \end{cases} \tag{19}$$

For this problem, we make two assumptions:

(A1) *inf-sup condition*: $\exists \beta > 0$ such that

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \, \|q\|_Q} \geq \beta.$$

(A2) *coercivity on the kernel* : $\exists \alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|_V^2, \ \forall v \in \mathbb{K},$$

where the kernel $\mathbb{K}$ is equal to $\{v \in V \mid b(v, q) = 0, \ \forall q \in Q\}$.

Theorem 4.1 (Babuska-Brezzi)[24, 12] *Assume that (A1) and (A2) are fulfilled. There exists one, and only one, solution* (u, p) *to the mixed variational problem (19), which depends continuously on the data* (f, g).

Note that if (A1) and (A2) hold, we can introduce the continuous and linear mapping T from V' to V, such that $\mathsf{T}f$ is the first element (called u above) of the unique solution to problem (19), with data $(f, 0)$. Since a is symmetric, T is selfadjoint.

Then, one can consider the conforming discretization of (19). To that aim, let V_h and Q_h be two finite dimensional subspaces of V and Q respectively, and introduce the discrete mixed variational problem:
Find $(u_h, p_h) \in V_h \times Q_h$ *such that*

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = \langle f, v_h \rangle, \ \forall v_h \in V_h \\ b(u_h, q_h) = \langle g, q_h \rangle, \ \forall q_h \in Q_h. \end{cases} \tag{20}$$

For this discrete problem, we make two assumptions:

(DA1) *discrete inf-sup condition*: $\forall h$, $\exists \beta(h) > 0$ such that

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \, \|q_h\|_Q} \geq \beta(h).$$

(DA2) *coercivity on the discrete kernel*: $\exists\alpha' > 0$ such that

$$\forall h,\ a(v_h, v_h) \geq \alpha' \|v_h\|_V^2,\ \forall v_h \in \mathbb{K}_h,$$

with the discrete kernels $\mathbb{K}_h := \{v_h \in V_h \,|\, b(v_h, q_h) = 0,\ \forall q_h \in Q_h\}$.

Theorem 4.2 (Babuska-Brezzi)[24,12] *Assume that (A1), (A2), (DA1) and (DA2) are fulfilled. There exists one, and only one, solution (u_h, p_h) to the discrete mixed variational problem (20), with the error estimate below: $\exists C > 0$ such that*

$$\|u - u_h\|_V \leq C \left\{ \left(1 + \frac{1}{\beta(h)}\right) \inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q \right\}.$$

Assume that (A1), (A2), (DA1) and (DA2) hold. For a given h, we can also introduce the discrete, continuous and linear mapping T_h from V' to V, such that $\mathsf{T}_h f$ is the first element of the unique solution to problem (20), with data $(f, 0)$.

We then consider the eigenproblem, set in a *mixed* variational framework. We need a third Hilbert space, called L, such that $V \subset L$ algebraically and topologically, and moreover V is a dense subset of L. The scalar product of L and its norm are respectively denoted by $(\cdot,\cdot,)_L$ and $\|\cdot\|_L$. Hereafter, we identify L' with L. The eigenproblem to be solved reads:
Find $(u, p, \lambda) \in V \times Q \times \mathbb{R}$ such that

$$\begin{cases} a(u, v) + b(v, p) = \lambda(u, v)_L,\ \forall v \in V \\ b(u, q) = 0,\ \forall q \in Q. \end{cases} \tag{21}$$

Note that, according to the definition of the operator T, (21) can be rewritten equivalently $\lambda\mathsf{T}u = u$, where T is considered from L to V. For this eigenproblem, we make one additional assumption:

(E) The operator $\mathsf{T} : L \to V$ is *compact* (and nonnegative).

The discrete approximation of (21) is:
Find $(u_h, p_h, \lambda_h) \in V_h \times Q_h \times \mathbb{R}$ such that

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = \lambda_h(u_h, v_h)_L,\ \forall v_h \in V_h \\ b(u_h, q_h) = 0,\ \forall q_h \in Q_h. \end{cases} \tag{22}$$

The discrete eigenproblem can be rewritten equivalently $\lambda_h \mathsf{T}_h u_h = u_h$. So, to establish the convergence of the solution of the discrete eigenproblem towards the solution of the exact eigenproblem, one has to prove that

$$\lim_{h\to 0^+} \sup_{f \in L} \frac{\|(\mathsf{T} - \mathsf{T}_h) f\|_V}{\|f\|_L} = 0. \tag{23}$$

Indeed, this uniform convergence in $\mathcal{L}(L, V)$ implies convergence of eigenvectors and eigenvalues, for T selfadjoint and compact [7]. Moreover, the (uniform) rate of convergence on the eigenvectors and eigenvalues is bounded by the rate of convergence in (23). From [10], we introduce *sufficient conditions* that ensure (23). A few additional definitions are needed, before these conditions are stated. Let S_0 denote the subspace of $V \times Q$, made of solutions to problem (19) with right-hand sides $(f, 0)$, where $f \in L$. Then, consider

$$V_0 = \{v \in V \ : \ \exists q \in Q \text{ s.t. } (v, q) \in S_0\},$$
$$Q_0 = \{q \in Q \ : \ \exists v \in V \text{ s.t. } (v, q) \in S_0\}.$$

These two spaces are endowed with their *natural* norms, that is $\|v\|_{V_0}$ (resp. $\|q\|_{Q_0}$) is equal to $\|f\|_L$ (resp. $\|f\|_L$), with f such that $(f, 0)$ is the right-hand side of (19) with solution (v, q_f) (resp. solution (v_f, q)).

The first condition is the *weak approximability* of Q_0.

Definition 4.1 The weak approximability of Q_0 is verified provided there exists $r_1 : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\lim_{h\to 0^+} r_1(h) = 0$ and

$$\sup_{v_h \in \mathbb{K}_h} \frac{b(v_h, q_0)}{\|v_h\|_V} \leq r_1(h)\|q_0\|_{Q_0}, \ \forall q_0 \in Q_0. \tag{24}$$

Inequality (24) actually corresponds to an approximability property, since one has $b(v_h, q^I) = 0$, $\forall q^I \in Q_h$ and $\forall v_h \in \mathbb{K}_h$.

The second condition is the *strong approximability* of V_0.

Definition 4.2 The strong approximability of V_0 is verified provided there exists $r_2 : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\lim_{h\to 0^+} r_2(h) = 0$ and

$$\forall v_0 \in V_0, \ \exists v^I \in \mathbb{K}_h \text{ s.t. } \|v_0 - v^I\|_V \leq r_2(h)\|v_0\|_{V_0}. \tag{25}$$

We note that, according to Theorem 4.2, provided the weak approximability is fulfilled, it is sufficient that

$$\lim_{h\to 0^+} \left\{ \frac{1}{\beta(h)} \sup_{v_0 \in V_0} \inf_{v_h \in V_h} \left(\frac{\|v_0 - v_h\|_V}{\|v_0\|_{V_0}} \right) \right\} = 0, \tag{26}$$

to ensure the strong approximability. Indeed, the solution to the discrete mixed problem with $g = 0$ automatically belongs to the discrete kernel $\mathbb{K}_h$ and, as such, it can play the role of v^I in (25).

Then, the following convergence result holds [10].

Theorem 4.3 *Assume that* T *is a (selfadjoint) compact operator from* L *to* V *(E). Assume moreover that the weak approximability of* Q_0 *(24), the strong approximability of* V_0 *(25) and the coercivity on the discrete kernel (DA2) are verified. Then, there exists* $r_3 : \mathbb{R}^+ \to \mathbb{R}^+$*, such that* $\lim_{h\to 0^+} r_3(h) = 0$ *and*

$$\sup_{f \in L} \frac{\|(\mathsf{T} - \mathsf{T}_h)f\|_V}{\|f\|_L} \leq r_3(h). \tag{27}$$

Proof According to [10], the result is true, with

$$r_3(h) = \left(1 + \frac{\|a\|}{\alpha'}\right) r_2(h) + \frac{1}{\alpha'} r_1(h). \square$$

Once the convergence in operator norm is established, one can obtain error bounds on the eigenvalues and eigenspaces, thanks to techniques *à la* Babuska-Osborn (see [7, 9] for details). Let λ be an eigenvalue, and E_λ be the eigenspace associated to it. Let λ_h be the average of the discrete eigenvalues converging to λ, and E_{λ_h} the sum of the discrete eigenspaces associated to λ_h. One wants to measure the error $|\lambda - \lambda_h|$. As far as the eigenspaces are concerned, one wants to bound the *gap* between them. It can be defined mathematically as follows: given V_1 and V_2 two subspaces of V, introduce

$$\hat\delta(V_1, V_2) = \max(\delta(V_1, V_2), \delta(V_2, V_1)), \text{ where } \delta(V_1, V_2) = \sup_{\substack{v_1 \in V_1 \\ \|v_1\|_V = 1}} \inf_{v_2 \in V_2} \|v_1 - v_2\|_V.$$

One can prove the

Theorem 4.4 *There exists $C > 0$ such that*

$$|\lambda - \lambda_h| < C\,\varepsilon_\lambda(h)^2 \textit{ and } \hat\delta(E_\lambda, E_{\lambda_h}) < C\,\varepsilon_\lambda(h), \tag{28}$$

with the approximation error

$$\varepsilon_\lambda(h) = \sup_{\substack{v \in E_\lambda \\ \|v\|_V = 1}} \inf_{v_h \in V_h} \|v - v_h\|_V. \tag{29}$$

4.2 Continuous and conforming discretization

Here, we describe briefly the discretization method that we use, together with some accompanying error estimates.

We consider a series of regular triangulations $(\mathcal{T}_h)_h$ of Ω, indexed by the meshsizes h, and made of tetrahedra. We use the conforming, continuous[4] P_k Lagrange finite element ($k \geq 1$) to discretize elements of $\mathcal{X}_\gamma$. Then, we choose the Zero Near Singularity $P_{k+1} - P_k$ Finite Element of Ciarlet, Jr. and Hechme [16] to discretize the mixed variational formulation in $\mathcal{X}_\gamma \times L^2_\gamma(\Omega)$. The choice of this Finite Element is justified later on. So, consider

$$\begin{cases} \mathcal{X}_h := \{\mathcal{F}_h \in \mathcal{C}^0(\bar\Omega)^3 \,|\, \mathcal{F}_h \times \mathbf{n}_{|\partial\Omega} = 0,\ \mathcal{F}_{h|K} \in P_{k+1}(K)^3,\ \forall K \in \mathcal{T}_h\} \\ M_h := \{q_h \in \mathcal{C}^0(\bar\Omega) \,|\, w_{2\gamma} q_{h|K} \in P_k(K),\ \forall K \in \mathcal{T}_h,\ q_{h|E_h} = 0\} \end{cases}. \tag{30}$$

Here, we set $E_h = \cup_{T \in \mathcal{T}_h, \partial T \cap E \neq \emptyset} T$. In other words, imposing $q_{h|E_h} = 0$ means that the discrete multipliers vanish on tetrahedra that neighbor the reentrant edges and corners. Moreover, given $(\mathcal{F}_h, q_h) \in \mathcal{X}_h \times M_h$, if we let $\underline{q}_h = w_{2\gamma} q_h$, we note the identity $(q_h, \operatorname{div} \mathcal{F}_h)_{0,\gamma} = (\underline{q}_h, \operatorname{div} \mathcal{F}_h)_0$.

[4] By construction, the discretized field and the test-fields are continuous, and piecewise smooth. So, they belong naturally to $H^1(\Omega)^3$. Since the discretization method is conforming, they also belong to $\mathcal{X}_\gamma$. Then, in order to be able to apply the classical Galerkin theory, a necessary condition is that $\mathcal{X}_\gamma \cap H^1(\Omega)^3$ be *dense* in $\mathcal{X}_\gamma$, which is precisely the result proven by Costabel and Dauge [19], when $\gamma \in]\gamma_{min}, 1[$.

Remark 4.1 Costabel and Dauge [19,20] noted a number of restrictions on the use of – lower degree – nodal finite elements to approximate the electric field. Foremost, there must be "enough" gradients (i.e. $\mathcal{F}_h = \nabla\varphi_h$) in $\mathcal{X}_h$. It turns out that this problem can be addressed thanks to the construction of the so-called $\mathcal{C}^1$ interpolants in Computer Aided Design. We refer for instance to [31], and to [16] and Refs therein for examples. In particular, the use of the lower degree P_1 or P_2 finite elements is possible, in 2D and in 3D.

The discrete approximation (22) reads:
Find $(\mathcal{E}_h, p_h, \lambda_h) \in \mathcal{X}_h \times M_h \times \mathbb{R}^+$ *such that*

$$\begin{cases} (\mathcal{E}_h, \mathcal{F}_h)_{\mathcal{X}_\gamma} + (p_h, \operatorname{div}\mathcal{F}_h)_{0,\gamma} = \lambda_h(\mathcal{E}_h, \mathcal{F}_h)_0 \ \forall \mathcal{F}_h \in \mathcal{X}_h \\ (q_h, \operatorname{div}\mathcal{E}_h)_{0,\gamma} = 0, \ \forall q_h \in M_h. \end{cases} \tag{31}$$

We recall below error estimates from [19,16], when one solves the static version of problem (7)-(9). It will be useful to establish the convergence result for the eigenproblem. The static problem amounts to solving

$$\mathbf{curl}\,\mathbf{curl}\,\mathcal{E} = \mathcal{J} \text{ in } \Omega, \tag{32}$$

$$\operatorname{div}\mathcal{E} = 0 \text{ in } \Omega, \tag{33}$$

$$\mathcal{E} \times \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{34}$$

Above, one considers $\mathcal{J} \in L^2(\Omega)^3$ such that $\operatorname{div}\mathcal{J} = 0$.

First, consider the plain variational formulation
Find $\mathcal{E} \in \mathcal{X}_\gamma$ *such that*

$$(\mathcal{E}, \mathcal{F})_{\mathcal{X}_\gamma} = (\mathcal{J}, \mathcal{F})_0 \ \forall \mathcal{F} \in \mathcal{X}_\gamma. \tag{35}$$

To approximate (35), one uses the P_k Lagrange finite element ($k \geq 1$) and builds a discretized variational formulation, with solution $\mathcal{E}_h^0$. Costabel and Dauge [19] proved the worst case estimate

$$||\mathcal{E} - \mathcal{E}_h^0||_{\mathcal{X}_\gamma} \leq C_\varepsilon h^{\tau-\varepsilon}||\mathcal{J}||_0, \ \forall \varepsilon > 0.$$

where $\tau = \min(\gamma - \gamma_{min}, \sigma_\Delta^N - 1)$, and $\sigma_\Delta^N \in]\frac{3}{2}, 2[$ is the *minimum singularity exponent* for the Laplace problem with homogeneous Neumann boundary condition[5]. It is common knowledge that this estimate improves when one uses graded meshes (cf. [2]), or provided $\mathcal{E}$ is smoother (when k increases).
From now on, we replace the exponent $(\tau - \varepsilon) \in]0, \tau[$ by μ. The previous estimate is reformulated as

$$||\mathcal{E} - \mathcal{E}_h^0||_{\mathcal{X}_\gamma} \leq C_\mu h^{\mu}||\mathcal{J}||_0, \ \forall \mu \in]0, \tau[. \tag{36}$$

However, one wants to approximate $\mathcal{E}$ by an element of the discrete kernel, i. e. some $\mathcal{E}_h^1 \in \mathcal{X}_h$ such that $(q_h, \operatorname{div}\mathcal{E}_h^1)_{0,\gamma} = 0$, for all $q_h \in M_h$. To that aim, consider the mixed variational formulation
Find $(\mathcal{E}, p) \in \mathcal{X}_\gamma \times L^2_\gamma(\Omega)$ *such that*

$$\begin{cases} (\mathcal{E}, \mathcal{F})_{\mathcal{X}_\gamma} + (p, \operatorname{div}\mathcal{F})_{0,\gamma} = (\mathcal{J}, \mathcal{F})_0 \ \forall \mathcal{F} \in \mathcal{X}_\gamma \\ (q, \operatorname{div}\mathcal{E})_{0,\gamma} = 0, \ \forall q \in L^2_\gamma(\Omega). \end{cases} \tag{37}$$

[5] When $\Omega \subset \mathbb{R}^2$, $\sigma_\Delta^N = \sigma_\Delta^D$, so that $\tau = \gamma - \gamma_{min}$. When $\Omega \subset \mathbb{R}^3$, it can happen that $\sigma_\Delta^N < \sigma_\Delta^D$.

Using the $P_{k+1} - P_k$ Finite Element of [16] on a series of triangulations $(\mathcal{T}_h)_h$, one can build a discretized variational formulation with solution $(\mathcal{E}_h^1, p_h)$. Then, one can apply Theorem 4.2, to reach the error estimate

$$||\mathcal{E} - \mathcal{E}_h^1||_{\mathcal{X}_\gamma} \leq C_\mu \left(1 + \frac{1}{\beta(h)}\right) h^\mu ||\mathcal{J}||_0, \tag{38}$$

where $\beta(h)$ is the 'constant', which appears in the discrete inf-sup condition (DA1). To bound $\beta(h)$ from below, we refer to [16] for an extended discussion. In particular, the "obvious" choice of the Taylor-Hood Finite Element is ill-advised in our case, and a condition like $q_{h|E_h} = 0$ must be imposed to the discrete multipliers. One finds

Proposition 4.1 *There exists $C_{is} > 0$, such that $\beta(h) \geq C_{is}$ for all h.*

Next, we establish an approximability property for the Lagrange multiplier. To that aim, we introduce

$$\mathcal{Q}_\gamma := \{q_0 \in L^2_\gamma(\Omega) \,|\, w_\gamma\, q_0 \in \overset{\circ}{\mathbb{V}}{}^1_\gamma\}.$$

Proposition 4.2 *There exists $C_{app} > 0$, such that*

$$\inf_{q_h \in M_h} ||q_0 - q_h||_{0,\gamma} \leq C_{app}\, h^{1-\gamma} \|q_0\|_{\mathcal{Q}_\gamma},\ \forall q_0 \in \mathcal{Q}_\gamma. \tag{39}$$

Proof We follow the guidelines of the proof of Proposition 8.3 of [19], but we have to use Clément type interpolation operators since functions $q \in \mathcal{Q}_\gamma$ are not, in general, continuous. Note that $q \in \mathcal{Q}_\gamma$, means that $w_{2\gamma-1} q \in L^2(\Omega)$ and $w_{2\gamma} \nabla q \in L^2(\Omega)^3$. For all $T \in \mathcal{T}_h$, let $\widetilde{T}$ be the set of tetrahedra T' such that $T \cap T' \neq \emptyset$ (We recall here that the tetrahedra are closed subsets of $\mathbb{R}^3$). Let $V_h := \{\underline{q}_h \in \mathcal{C}^0(\bar{\Omega}) \,|\, \underline{q}_{h|K} \in P_1(K),\ \forall K \in \mathcal{T}_h\}$. It is known [30] that there exists a projection $\Pi_h : L^2_{\mathrm{loc}}(\Omega) \to V_h$ such that:

$$\|q - \Pi_h q\|_{L^2(T)} \leq Ch|q|_{H^1(\widetilde{T})}. \tag{40}$$

We construct a cut-off function χ_h such that:

$$0 \leq \chi_h \leq 1 \quad \chi_h(\mathbf{x}) = 1 \text{ if } d(\mathbf{x}) \leq 4h, \quad \chi_h(\mathbf{x}) = 0 \text{ if } d(\mathbf{x}) \geq 8h,$$

where one recalls that d denotes the distance from the set of reentrant edges.

For any $T \in \mathcal{T}_h$, without loss of generality, we suppose that if $\chi_h(\mathbf{x}) = 0,\ \forall \mathbf{x} \in T$, then $d_{\widetilde{T}} \geq h$. Given $q_0 \in \mathcal{Q}_\gamma$, we set $q_h = w_{-2\gamma} \Pi_h (1 - \chi_h) w_{2\gamma} q_0$, and we prove (39), with $\|q_0\|^2_{\mathcal{Q}_\gamma} \simeq \|q_0\|^2_{0,2\gamma-1} + \|\nabla q_0\|^2_{0,2\gamma}$. We have that $q_{h|E_h} = 0$ and moreover $q_h \in M_h$. We have also:

$$\|q_0 - q_h\|^2_{0,\gamma} \leq 2\|\chi_h q_0\|^2_{0,\gamma} + 2\|(1 - \chi_h) q_0 - q_h\|^2_{0,\gamma} = 2(I + II).$$

and we estimate the two terms separately. We compute:

$$\begin{aligned} I = \sum_{T \in \mathcal{T}_h} \int_T w_{2\gamma} |\chi_h q_0|^2 &= \sum_{T \in \mathcal{T}_h \,:\, d_T < 8h} \int_T w_{2\gamma} |\chi_h q_0|^2 \leq \sum_{T \in \mathcal{T}_h \,:\, d_T < 8h} \int_T w_{2\gamma} |q_0|^2 \\ &\leq C \sum_{T \in \mathcal{T}_h \,:\, d_T < 8h} h^{2-2\gamma} \int_T w_{4\gamma-2} |q_0|^2 \leq C h^{2(1-\gamma)} \|q_0\|^2_{0,2\gamma-1}. \end{aligned} \tag{41}$$

Before we estimate the second term II, let us make some preliminary (elementary) computations. Set $q = (1-\chi_h)w_{2\gamma}q_0$:

$$\begin{aligned} II &= \|(1-\chi_h)q_0 - q_h\|_{0,\gamma}^2 = \|w_\gamma(1-\chi_h)q_0 - w_{-\gamma}\Pi_h(1-\chi_h)w_{2\gamma}q_0\|_0^2 \\ &= \|w_{-\gamma}\left(w_{2\gamma}(1-\chi_h)q_0 - \Pi_h(1-\chi_h)w_{2\gamma}q_0\right)\|_0^2 = \|w_{-\gamma}\left(q-\Pi_h q\right)\|_0^2 \\ &= \sum_{T\in\mathcal{T}_h} \|w_{-\gamma}\left(q-\Pi_h q\right)\|_{L^2(T)}^2. \end{aligned}$$

Thanks to the definition of the cut-off function, we know that $(1-\chi_h)(\mathbf{x}) = 0$ if $d(\mathbf{x}) \le 4h$. As a consequence, in the above summation, one can exclude tetrahedra T such that $d_T \le h$ or, equivalently, tetrahedra of E_h. Now, on a tetrahedron T of $\mathcal{T}_h\backslash E_h$, we have $\|w_{-\gamma}\|_{L^\infty(T)} \le C\,h^{-\gamma}$. So, we can write, using the rule $\|f\,g\|_{L^2} \le \|f\|_{L^\infty}\|g\|_{L^2}$ and with the help of (40),

$$\begin{aligned} II &= \sum_{T\in\mathcal{T}_h\backslash E_h} \|w_{-\gamma}\left(q-\Pi_h q\right)\|_{L^2(T)}^2 \le C\,h^{-2\gamma}\sum_{T\in\mathcal{T}_h\backslash E_h}\|q-\Pi_h q\|_{L^2(T)}^2 \\ &\le C\,h^{2-2\gamma}\sum_{T\in\mathcal{T}_h\backslash E_h}|q|_{H^1(\widetilde{T})}^2. \end{aligned} \tag{42}$$

Next, we develop the expression of ∇q, which writes

$$\nabla q = (1-\chi_h)\nabla(w_{2\gamma}q_0) + w_{2\gamma}q_0\nabla(1-\chi_h).$$

We then evaluate the norm of each term.
For the first term, since $\|1-\chi_h\|_\infty \le 1$, one has

$$\|(1-\chi_h)\nabla(w_{2\gamma}q_0)\|_{L^2(\widetilde{T})} \le \|\nabla(w_{2\gamma}q_0)\|_{L^2(\widetilde{T})}.$$

For the second term, we note that the support of $\nabla(1-\chi_h)$ is included in $\{\mathbf{x}\,|\,4h \le d(\mathbf{x}) \le 8h\}$, and that $\|\nabla(1-\chi_h)\|_{L^\infty} \le C\,h^{-1}$. Furthermore, in this region, one has $\|w_1\|_{L^\infty} \le C\,h$. One gets

$$\begin{aligned} \|w_{2\gamma}q_0\nabla(1-\chi_h)\|_{L^2(\widetilde{T})} &= \|w_{2\gamma-1}q_0 w_1\nabla(1-\chi_h)\|_{L^2(\widetilde{T})} \\ &\le C\,\|w_{2\gamma-1}q_0\|_{L^2(\widetilde{T})} = C\,\|q_0\|_{L^2_{2\gamma-1}(\widetilde{T})}. \end{aligned}$$

Using (42), we then reach an estimate of II:

$$\begin{aligned} II &\le C\;h^{2-2\gamma}\sum_{T\in\mathcal{T}_h}\left(\|\nabla(w_{2\gamma}q_0)\|_{L^2(\widetilde{T})}^2 + \|q_0\|_{L^2_{2\gamma-1}(\widetilde{T})}^2\right) \\ &\le C h^{2(1-\gamma)}\left(\|\nabla(w_{2\gamma}q_0)\|_0^2 + \|q_0\|_{0,2\gamma-1}^2\right). \end{aligned} \tag{43}$$

The last inequality is obtained thanks to the following observation. Since the series of triangulations is regular, there exists an upper bound (independent of h) to the number of times any given tetrahedron T' occurs in the sets $\widetilde{T}$, when T spans $\mathcal{T}_h$.
Adding up (41) and (43) ends the proof. $\square$

4.3 Convergence results

In this last Subsection, we verify that we can apply the abstract theory [10] (reported in Subsection 4.1) to our situation. We add *s to highlight the *ad hoc* functional spaces, the forms, etc. that we consider specifically. We have:

- $V^\star := \mathcal{X}_\gamma$ with norm $||\cdot||_{V^\star} := ||\cdot||_{\mathcal{X}_\gamma}$.
- $Q^\star := L^2_\gamma(\Omega)$ with norm $||\cdot||_{Q^\star} := ||\cdot||_{0,\gamma}$.
- $a^\star(u,v) := (u,v)_{\mathcal{X}_\gamma}$; $b^\star(v,q) := (\operatorname{div} v, q)_{0,\gamma}$.
- The kernel $\mathbb{K}^\star := \mathcal{K}_\gamma$.
- The (selfadjoint) operator $\mathsf{T}^\star : (V^\star)' \to V^\star$, which associates to any f the first element of the unique solution to problem (19)*, with data $(f,0)$.
- $L^\star := L^2(\Omega)^3$ with norm $||\cdot||_{L^\star} := ||\cdot||_0$; $V^\star \subset L^\star$ and $V^\star$ is dense in $L^\star$.

The *inf-sup condition* (A1)* holds with $\beta^\star = 1$. As a matter of fact, given $q \in Q^\star$, consider $v_q = \nabla({\Delta_D}^{-1} q) \in V^\star$, such that $b^\star(v_q, q) = ||q||_{Q^\star} ||v_q||_{V^\star}$. The *coercivity on the kernel* (A2)* is clear, since $a^\star$ is the scalar product of $X^\star$.

The proof that the (restriction of) the operator $\mathsf{T}^\star$ is *compact* from $L^\star$ to $V^\star$, that is (E)*, is postponed until Corollary 4.2. Before that, we characterize the subspaces $V_0^\star$ and $Q_0^\star$. Given $f \in L^\star$, we thus study the solution to the problem
Find $(u,p) \in \mathcal{X}_\gamma \times L^2_\gamma(\Omega)$ *such that*

$$\begin{cases} (u,v)_{\mathcal{X}_\gamma} + (p, \operatorname{div} v)_{0,\gamma} = (f,v)_0 \ \forall v \in \mathcal{X}_\gamma \\ (q, \operatorname{div} u)_{0,\gamma} = 0, \ \forall q \in L^2_\gamma(\Omega). \end{cases} \tag{44}$$

There holds

Theorem 4.5 *Let (u,p) be the solution to (44).*
On the one hand, $\operatorname{div} u = 0$ *and* $\mathbf{curl}\,\mathbf{curl}\, u \in L^2(\Omega)^3$.
On the other hand, $w_\gamma p \in \overset{\circ}{\mathbb{V}}{}^1_\gamma$.

Proof According to (44b) $\operatorname{div} u = 0$, so that $u \in \mathcal{K}_\gamma$ and (44a) reduces to

$$(\mathbf{curl}\, u, \mathbf{curl}\, v)_0 + (p, \operatorname{div} v)_{0,\gamma} = (f,v)_0 \ \forall v \in \mathcal{X}_\gamma.$$

To proceed, let us apply to f the not so standard orthogonal decomposition (13) of $L^2(\Omega)^3$: $\exists!(\zeta, q) \in \mathcal{H}(\operatorname{div}^0, \Omega) \times \overset{\circ}{\mathbb{V}}{}^1_\gamma$ such that $f = \zeta + \nabla(w_\gamma q)$. Then, $w_\gamma p + q$ belongs to $L^2(\Omega)$, and one can solve the auxiliary problem
Find $z \in \overset{\circ}{\mathbb{V}}{}^1_\gamma$ *such that*

$$(\nabla(w_\gamma z), \nabla(w_\gamma z'))_0 = -(w_\gamma p + q, z')_0, \ \forall z' \in \overset{\circ}{\mathbb{V}}{}^1_\gamma.$$

By construction, $w_\gamma \Delta(w_\gamma z) = w_\gamma p + q$. Thus, if we let $v = \nabla(w_\gamma z)$, we have $v \in \mathcal{X}_\gamma$, with $\mathbf{curl}\, v = 0$ and $w_\gamma \operatorname{div} v = w_\gamma p + q$. Using this test-field in the reduced version of (44a) yields

$$(w_\gamma p, w_\gamma p + q)_0 = (f, \nabla(w_\gamma z))_0 \overset{\perp}{=} (\nabla(w_\gamma q), \nabla(w_\gamma z))_0 \overset{z'=q}{=} -(w_\gamma p + q, q)_0.$$

In other words, $||w_\gamma p + q||_0^2 = 0$, i.e. $w_\gamma p = -q$ (in $L^2(\Omega)$): this proves the improved regularity of the Lagrange multiplier p.

Now, let us take a test-field v of $\mathcal{D}(\Omega)^3$ in the reduced version of (44a), with f replaced again by $\zeta + \nabla(w_\gamma q)$:

$$\begin{aligned}\langle \mathbf{curl}\,\mathbf{curl}\, u - \zeta, v\rangle &= (\nabla(w_\gamma q), v)_0 - (w_\gamma p, w_\gamma \mathrm{div}\, v)_0 \\ &= -(w_\gamma q, \mathrm{div}\, v)_0 - (w_\gamma p, w_\gamma \mathrm{div}\, v)_0 \text{ by integration by parts} \\ &= -(q + w_\gamma p, w_\gamma \mathrm{div}\, v)_0 = 0.\end{aligned}$$

In other words, $\mathbf{curl}\,\mathbf{curl}\, u = \zeta$, so the conclusion follows. □

Remark 4.2 One can characterize the spaces $V_0^\star$ and $Q_0^\star$ completely. After some elementary computations, one reaches

$$V_0^\star = \{v \in \mathcal{K}_\gamma \,|\, \mathbf{curl}\,\mathbf{curl}\, v \in L^2(\Omega)^3\},\ Q_0^\star = \{q \in L^2_\gamma(\Omega) \,|\, w_\gamma q \in \overset{\circ}{\mathbb{V}}{}^1_\gamma\}.$$

Now, it is possible to measure elements of $V_0^\star$ and $Q_0^\star$. Using the definition of the norms, one gets immediately the

Corollary 4.1 *On $V_0^\star$ and $Q_0^\star$, the norms are respectively*

$$||v||_{V_0^\star} = \Big(||\mathbf{curl}\, v||^2 + ||\mathbf{curl}\,\mathbf{curl}\, v||^2\Big)^{1/2}, \tag{45}$$

$$||q||_{Q_0^\star} = ||\nabla(w_{2\gamma} q)||_0. \tag{46}$$

This last result allows us to prove the compactness result $(\mathrm{E})^\star$.

Corollary 4.2 *The restriction of the operator $\mathsf{T}^\star$ from $L^\star$ to $V^\star$ is* compact.

Proof Consider a bounded sequence $(f_n)_n$ in $L^2(\Omega)^3$. The corresponding solutions (u_n, p_n) are bounded in $V_0^\star \times Q_0^\star$. In particular, thanks to the identification (45), the sequence $(y_n)_n$, with $y_n = \mathbf{curl}\, u_n$, is bounded in the space

$$\mathcal{Y} := \{\mathcal{F} \in \mathcal{H}(\mathbf{curl}\,, \Omega) \,|\, \mathrm{div}\, \mathcal{F} \in L^2(\Omega),\ \mathcal{F} \cdot \mathbf{n}_{|\partial\Omega} = 0\}.$$

According to the fundamental result of Weber [32] on vector fields with vanishing normal trace, $\mathcal{Y}$ is compactly imbedded in $L^2(\Omega)^3$. In other words, there exists $y \in L^2(\Omega)^3$ such that $\lim_{n\to+\infty} y_n = y$ in $L^2(\Omega)^3$ (we identify the subsequence of $(y_n)_n$ with $(y_n)_n$ itself).
Obviously, the sequence $(u_n)_n$ is bounded in $\mathcal{X}$: according to the second fundamental result of Weber [32] on vector fields with vanishing tangential trace, $\mathcal{X}$ is also compactly imbedded in $L^2(\Omega)^3$. Therefore, there exists $u \in L^2(\Omega)^3$ such that $\lim_{n\to+\infty} u_n = u$ in $L^2(\Omega)^3$. Since $\mathrm{div}\, u_n = 0$, for all n, one has $\mathrm{div}\, u = 0$ thanks to the uniqueness of the limit. Moreover, $y_n = \mathbf{curl}\, u_n$ converges to y in $L^2(\Omega)^3$, and, by identification, $\mathbf{curl}\, u = y$. Then, the subsequence $(u_n)_n$ converges to u in $\mathcal{H}(\mathbf{curl}\,, \Omega)$, so that $u \in \mathcal{H}_0(\mathbf{curl}\,, \Omega)$. Finally one concludes first that u belongs to $\mathcal{X}_\gamma$, and second that the subsequence $(u_n)_n$ converges to u in $\mathcal{X}_\gamma$, which is the desired result. □

To be able to conclude favorably, there remains to study the weak approximability $(24)^\star$ of $Q_0^\star$, and the strong approximability $(25)^\star$ of $V_0^\star$. Here, we use the discretization described in Subsection 4.2.

Let us start with the weak approximability of $Q_0^\star$. Consider v_h, a non-zero element of the discrete kernel $\mathbb{K}_h^\star$, and q_0 an element of $Q_0^\star$. We estimate the quotient $|(\operatorname{div} v_h, q_0)_{0,\gamma}|/||v_h||_{\mathcal{X}_\gamma}$ in the following way:

$$
\begin{aligned}
&(\operatorname{div} v_h, q_0)_{0,\gamma} = (\operatorname{div} v_h, q_0 - q^I)_{0,\gamma},\ \forall q^I \in Q_h^\star\,;\\
&|(\operatorname{div} v_h, q_0)_{0,\gamma}| \le ||\operatorname{div} v_h||_{0,\gamma}||q_0 - q^I||_{0,\gamma} \le ||v_h||_{\mathcal{X}_\gamma}||q_0 - q^I||_{0,\gamma},\ \forall q^I \in Q_h^\star\,;\\
&\frac{|(\operatorname{div} v_h, q_0)_{0,\gamma}|}{||v_h||_{\mathcal{X}_\gamma}} \le \inf_{q^I \in Q_h^\star} ||q_0 - q^I||_{0,\gamma}.
\end{aligned}
$$

Recall that q_0 belongs to $Q_0^\star$, which is a subset of $\mathcal{Q}_\gamma$. The weak approximability $(24)^\star$ of $Q_0^\star$ is therefore a consequence of Proposition 4.2 and of the above, with $r_1^\star(h) = C_{app} h^{1-\gamma}$.

Let us turn now to the strong approximability of $V_0^\star$. Comparing (26) to (38) and noting that $\|\mathcal{J}\|_0 = \|\mathcal{E}\|_{V_0^\star}$, one finds that $r_2^\star(h) \le C_\mu(1 + \beta(h)^{-1})h^\mu$, for $\mu \in]0, \tau[$. According to Proposition 4.1, $\beta(h)$ is bounded from below by C_{is}, so one can actually consider $r_2^\star(h) = C'_\mu h^\mu$.

We conclude with the error estimates. From (36), we know that the bound in the approximation error (29) behaves like $\varepsilon^\star(h) = C_\mu h^\mu$, for $\mu \in]0, \tau[$ (it can be better – smaller – provided the eigenfield is smooth). Therefore, we have the following error bounds for the discretization of the electric eigenproblem (PE):

- Approximation of the eigenvalue: $|\lambda - \lambda_h| < C_\mu^2 h^{2\mu}$,
- Gap: $\hat\delta(E_\lambda, E_{\lambda_h}) < C_\mu h^\mu$.

5 Numerical tests

Here, we highlight our convergence results on a practical example, taken from the benchmark of Monique Dauge (see [22]). The domain Ω_2 we consider is L-shaped, with straight sides and corners in (0,0), (1,0), (1,1), (-1,1), (-1,-1), (0,-1). It possesses a single reentrant corner, located at the origin. Below, we reproduce the values of the first five eigenvalues (with repetition), up to seven digits (which is well below the 11-digit claimed accuracy of the benchmark). The values are

$\lambda_1 = 1.475622$, $\lambda_2 = 3.534031$, $\lambda_3 = 9.869604$, $\lambda_4 = 9.869604$, and $\lambda_5 = 11.38948$.

In addition, it is reported that the first eigenvector has the *strong unbounded singularity*, which means that it does not belong to $H^1(\Omega_2)^2$.

According to the theory, the weight w can be chosen as $w = r^\gamma$, with r the distance to the origin, and $\gamma \in]1/3, 1[$. We compute the solution using a value of the parameter γ set to $\gamma = .95$, on a series of *quasi-uniform meshes*. To discretize the problem, we consider the $P_2 - P_1$ Zero Near Singularity finite element pair, cf. (30). The meshes are respectively made of 738, 2952 and 11808 triangles, with 410, 1557 and 6065 vertices. The corresponding meshsizes are $h = 7.84\,10^{-2}$, $h = 3.92\,10^{-2}$ and $h = 1.96\,10^{-2}$. The computation of the first five eigenvalues $(\lambda_{k,h})_{1\le k\le 5}$ is carried out with the help of Matlab. The relative errors $r_{k,h} = |\lambda_{k,h} - \lambda_k|/\lambda_k$, $1 \le k \le 5$, are reported on Table 1. We note that on the coarsest mesh, all eigenvalues are already very well-approximated, with the exception of the first one, for which convergence is in addition slower.

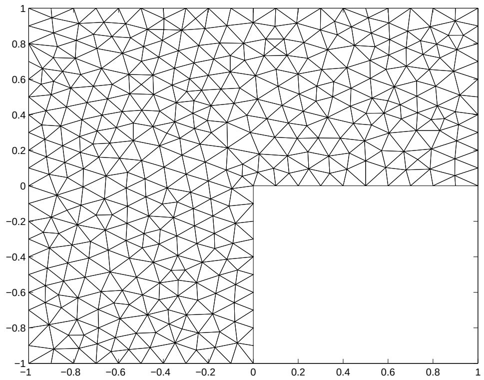

Fig. 1 Coarsest quasi-uniform mesh.

$mesh$	$r_{1,h}$	$r_{2,h}$	$r_{3,h}$	$r_{4,h}$	$r_{5,h}$
$uniform1$	$1.3e-2$	$3.3e-4$	$9.4e-5$	$1.1e-4$	$9.9e-3$
$uniform2$	$8.0e-3$	$6.2e-5$	$2.3e-5$	$2.5e-5$	$1.3e-5$
$uniform3$	$4.4e-3$	$1.2e-5$	$5.5e-6$	$6.2e-6$	$5.3e-6$

Table 1 Relative errors on quasi-uniform meshes with $\gamma = 0.95$.

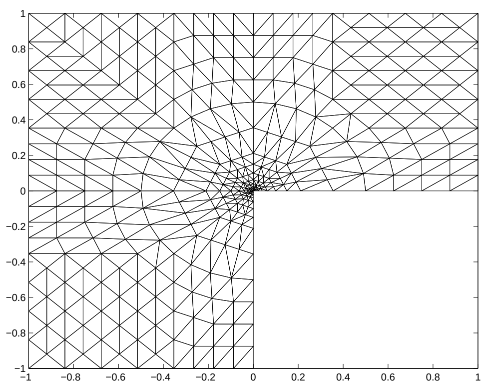

Fig. 2 Coarsest graded mesh.

To improve the convergence towards the first eigenvalue, we then make some computations, on a series of suitably *graded meshes*[6], with the same value of the parameter γ, that is $\gamma = 0.95$. See [11] for a rigorous study of the benefit of graded meshes for the computation of Maxwell eigenvalues. It is important to note that the number of triangles and vertices are similar, since the graded meshes are respectively made of 648, 2664 and 10728 triangles, with 362, 1410 and 5522 vertices. The results on these graded meshes are reported on Table 2.

To complete this series of experiments, let us emphasize the crucial importance of the weight w_γ. Without the weight, it is expected that the eigenvalue(s) corresponding

[6] Generated by Beate Jung, from the University of Chemnitz, Germany, with a grading parameter $\mu = 1/3$ (see [2]). According to the same Ref., the grading is isotropic, and as a consequence the series of triangulations is regular.

mesh	$r_{1,h}$	$r_{2,h}$	$r_{3,h}$	$r_{4,h}$	$r_{5,h}$
*graded*1	$2.4e-3$	$7.9e-5$	$2.1e-4$	$2.5e-4$	$1.9e-4$
*graded*2	$5.4e-4$	$1.8e-5$	*$4.7e-1$*	$5.1e-5$	*$8.5e-2$*
*graded*3	$1.6e-4$	$4.5e-6$	*$2.0e-2$*	$1.3e-5$	*$4.6e-2$*

Table 2 Relative errors on graded meshes with $\gamma = 0.95$.

mesh	$\lambda'_{1,h}$	$\lambda'_{2,h}$	$\lambda'_{3,h}$	$\lambda'_{4,h}$	$\lambda'_{5,h}$
*graded*1	3.553	6.073	9.872	9.872	11.40
*graded*2	3.535	6.068	9.870	9.870	11.39
*graded*3	3.534	6.071	9.870	9.870	11.39

Table 3 Computed eigenvalues on graded meshes with $\gamma = 0$.

to singular eigenvector(s) are not captured numerically (in our case, the first one). So let us choose $\gamma = 0$: the values for the smallest five computed eigenvalues $(\lambda'_{k,h})_{1\le k\le 5}$ are reported on Table 3. Note that we use in this case the well-known Taylor-Hood finite element, since it satisfies a *uniform* discrete inf-sup condition in the absence of weights [8,15]. We obtain the expected result, if one notices that the values of $\lambda'_{1,h}$ of Table 3 actually converge to λ_2. What is more (confusing?), we see are that all five computed eigenvalues converge numerically to a limit value. The reason is simple. As a matter of fact, if one applies the convergence analysis carried out earlier on, it turns out that, when $\gamma = 0$, we approximate *precisely* the mixed eigenproblem below
Find $(\mathcal{E}', p', \lambda') \in (\mathcal{X} \cap H^1(\Omega)^3) \times L^2(\Omega) \times \mathbb{R}^+$ *such that*

$$\begin{cases} (\mathcal{E}', \mathcal{F})_{\mathcal{X}} + (p', \operatorname{div} \mathcal{F})_0 = \lambda'(\mathcal{E}', \mathcal{F})_0 \ \forall \mathcal{F} \in (\mathcal{X} \cap H^1(\Omega)^3) \\ (q, \operatorname{div} \mathcal{E}')_0 = 0, \ \forall q \in L^2(\Omega). \end{cases}$$

(Compare to the exact eigenproblem (18).)
In particular, one cannot prove that $p' = 0$ in the above. As a matter of fact, this can be false, since the field $\mathcal{F}_{p'}$ (see the Second approach, as described in Section 3) does not necessarily belong to $H^1(\Omega)^3$.

6 Concluding remarks

To solve Maxwell eigenvalue problems with a continuous approximation of the field, we introduced an *equivalent* mixed eigenproblem. We presented the numerical analysis of this mixed formulation together with some numerical experiments, assuming one uses the weighted method of Costabel and Dauge to discretize the electric field, completed with the Zero Near Singularity finite element pair of Ciarlet, Jr. and Hechme to discretize the Lagrange multiplier. For additional numerical results and comparisons, we refer to [17]. In particular, the use of higher degree finite element pairs improves the convergence rate, as expected. This accounts for the so-called *p*-version of finite element methods (from a numerical point of view). Following [21], a theoretical study of the *p*- and *hp*- versions of finite element methods could be investigated.

We note that a similar analysis can be carried out, when one uses the singular complement method to discretize the electromagnetic field in a $2D$ non-convex polygon Ω, here with the $P_{k+1} - P_k$ Taylor-Hood finite element. In which case, one uses the

stronger norms $\|\cdot\|_0$ and $\|\cdot\|_{\mathcal{X}}$ (without the weights).
The estimate corresponding to the weak approximability stems from (39), with $\gamma = 0$: $r_1^*(h) = C\, h$.
To obtain the strong approximability bound, the needed error estimates for the static case can be found in [27]. Let the angles at the reentrant corners of $\partial\Omega$ be denoted π/α_j, and set $\alpha = \min_j \alpha_j$ ($\sigma_\Delta^D = 1 + \alpha$). Estimate (36) is replaced by

$$||\mathcal{E} - \mathcal{E}_h^0||_{\mathcal{X}} \le C_\varepsilon h^{2\alpha-1-\varepsilon} ||\mathcal{J}||_0, \ \forall\varepsilon > 0.$$

Now, consider the static problem written in mixed form (similarly to (37)). When it is discretized, it satisfies a *uniform* [8,15] discrete inf-sup condition: $\beta(h) \ge C_{is}^* > 0$, $\forall h$, in (DA1). If $(\mathcal{E}_h^1, p_h)$ denotes the solution to the discretized problem, there holds

$$||\mathcal{E} - \mathcal{E}_h^1||_{\mathcal{X}} \le C_\varepsilon h^{2\alpha-1-\varepsilon} ||\mathcal{J}||_0, \ \forall\varepsilon > 0.$$

One gets $r_2^*(h) = C_\varepsilon h^{2\alpha-1-\varepsilon}$. Error bounds on the eigenvalues and on the gap follow, with $\varepsilon^*(h) = C_\varepsilon h^{2\alpha-1-\varepsilon}$. Again, the use of graded meshes shall improve the overall quality of the computations.

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