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WEIGHTED REGULARIZATION FOR COMPOSITE MATERIALS IN ELECTROMAGNETISM

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Abstract. In this paper, a weighted regularization method for the time-harmonic Maxwell equations with perfect conducting or impedance boundary condition in composite materials is presented. The computational domain Ω is the union of polygonal or polyhedral subdomains made of different materials. As a result, the electromagnetic field presents singularities near geometric singularities, which are the interior and exterior edges and corners. The variational formulation of the weighted regularized problem is given on the subspace of $\mathcal{H}(\mathbf{curl};\Omega)$ whose fields $\boldsymbol{u}$ satisfy $w^\alpha \operatorname{div}(\varepsilon \boldsymbol{u}) \in L^2(\Omega)$ and have vanishing tangential trace or tangential trace in $L^2(\partial\Omega)$. The weight function $w(\boldsymbol{x})$ is equivalent to the distance of $\boldsymbol{x}$ to the geometric singularities and the minimal weight parameter α is given in terms of the singular exponents of a scalar transmission problem. A density result is proven that guarantees the approximability of the solution field by piecewise regular fields. Numerical results for the discretization of the source problem by means of Lagrange Finite Elements of type P_1 and P_2 are given on uniform and appropriately refined two-dimensional meshes. The performance of the method in the case of eigenvalue problems is adressed.

Résumé. Dans cet article, nous présentons une approche de type régularisation à poids pour résoudre les équations de Maxwell harmoniques en temps, avec condition aux limites de conducteur parfait ou d'impédance. Le domaine de calcul Ω est la réunion de sous-domaines polygonaux ou polyédriques contenant des milieux matériels différents. Le champ électromagnétique produit est par voie de conséquence singulier au voisinage des singularités géométriques, formées des arêtes et coins extérieurs et intérieurs. La formulation variationnelle de type régularisation à poids est construite dans un sous-espace de $\mathcal{H}(\mathbf{curl};\Omega)$, dont les éléments $\boldsymbol{u}$ sont tels que $w^\alpha \operatorname{div}(\varepsilon \boldsymbol{E})$ appartienne à $L^2(\Omega)$, avec une trace tangentielle dans $L^2(\partial\Omega)$ éventuellement nulle. Le poids $w(\boldsymbol{x})$ considéré se comporte comme la distance de $\boldsymbol{x}$ aux singularités géométriques, alors que l'exposant minimal α est déterminé en fonction des exposants singuliers d'un problème de transmission scalaire. Nous démontrons un résultat de densité garantissant l'approximabilité du champ cherché par des champs réguliers par morceaux. Nous présentons ensuite des résultats numériques sur le problème source, obtenus à l'aide d'Eléments Finis de Lagrange P_1 et P_2 sur des maillages bidimensionnels uniformes ou raffinés. Nous évaluons enfin la performance de la méthode sur le problème aux valeurs propres, avec le même type de discrétisation.

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1. Introduction

The question of approximability of the solution of Maxwell's equations by means of nodal finite elements has been widely studied in the last ten years (see e.g. [2, 4, 5, 10, 18, 32] for perfect conducting boundary conditions and homogeneous materials). In a regular domain of class C^1 as well as in a convex polyhedron, the discretization of the time-harmonic Maxwell equations can be performed *via* standard Lagrange Finite Elements by solving an equivalent regularized variational formulation similar to the vector Helmholtz equation (see [25]). In a non-convex polyhedron, however, this approximation fails since the electromagnetic field does in general present singularities near the reentrant edges and corners (see e.g. [7, 8, 17]) and the discretization space is no longer dense in the vector space of the variational formulation. The same situation does occur in composite materials where the electric permittivity and the magnetic permeability are piecewise constant functions. The electromagnetic field then presents singularities near the exterior and interior edges and corners of the different subdomains [19].

In order to overcome the lack of density, several possibilities have been studied. The singular complement method [3] and singular field method [26] add explicitly the singularities to the discretization space according to the splitting of the electromagnetic field into a regular part and a singular part deriving from a scalar potential. Another possibility is the penalization of the perfect conducting boundary condition by an impedance-like condition. From a theoretical point of view, the density result of the FE-space in the variational space holds true for any homogeneous material (see [13, 16]) and some composite materials (see [29]). The numerical performances of this method, however, are rather poor. The idea of weighted regularization has been developed in [18] for homogeneous materials. It consists in looking for the solution in the subspace of $\mathcal{H}(\mathbf{curl};\Omega)$ of fields with divergence in a weighted L^2-space, whereas the classical regularized formulation corresponds to the L^2-space without weight.

In this paper, we study the method of weighted regularization for composite materials and prove the density of the space of piecewise regular vector fields in the space of the weighted regularization method, for an appropriate choice of the weight parameter. The idea of the proof is similar to the proof in [29] where the case of classical regularization with impedance boundary condition has been addressed. It consists in proving that the orthogonal of the closure of the space of piecewise regular vector fields is reduced to $\{0\}$. However, if the density result for classical regularization with impedance boundary condition always holds true in the case of homogeneous materials, it may fail for some composite materials. On the contrary, the method of weighted regularization allows one to choose the weight parameter depending on the singularities of a scalar second-order transmission problem and hence, the density result may be recovered for any composite material.

The paper is organized as follows: the theoretical aspects of the problem are dealt with in section 2. More precisely, in §2.1, we give the geometric setting and the functional framework including a perfect conducting boundary as well as an impedance boundary condition. We also address equivalence between the weighted regularized formulation and the original Maxwell equations. In §2.2, we show that the density problem for vector fields can be reduced to a similar density problem for the associated scalar potentials. The weight function in two dimensions will be defined in §2.3. The proof of the density result in a two-dimensional domain is developed for a more general family of two-dimensional scalar problems depending on a real parameter. This turns out to be useful in order to deal with the three-dimensional case where the real parameter represents the (local) edge variable. Subsection 2.4 is devoted to the proof of the density result in a polyhedron. In section 3, we state precisely the discretization by means of Lagrange Finite Elements of type P_k and give a basic convergence proof. Finally, section 4 is devoted to a series of numerical tests performed in two dimensions. In §4.1, we present the resolution of the static problem with source term for the electric field in an L-shaped domain with three subdomains. Depending on the value of the electric permittivity, the main singularity of the electric field can be arbitrarily strong and thus it is challenging for any numerical method. The numerical results show clearly that the weighted regularization method does converge to the exact singular solution whereas the classical regularization method does not. Further, we provide numerical convergence rates for Finite Elements of types P_1 and P_2 on uniform and refined meshes. Next, we study in §4.2 the performance of the weighted regularization method for the eigenvalue problem and we compare our results to a benchmark in the case of an interior singularity in a "checkerboard-like" domain decomposed into four subdomains.

2. Weighted regularization in the case of mixed boundary conditions

2.1. Setting of the problem

In this section we will define precisely the geometric setting, which is the same as the one in [29]. Further, we introduce the variational formulation of the weighted regularization problem as well as the associated functional spaces. Whenever possible, we adopt the notations of [29].

We are concerned with an open bounded set $\Omega \subset \mathbb{R}^d$ where $d = 2$ or 3. We assume that Ω is a Lipschitz polygon ($d = 2$) or a Lipschitz polyhedron ($d = 3$) which means that Ω is a Lipschitz domain with piecewise linear ($d = 2$) or plane ($d = 3$) boundary $\partial\Omega$. We denote by $\boldsymbol{n}$ the unit outward normal vector to $\partial\Omega$. We further assume that Ω is connected and simply connected and that its boundary $\partial\Omega$ is connected[1].

It follows from the Maxwell equations that the electric field $\boldsymbol{E}$ is a solution to

$$\mathbf{curl}\left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}\right) - \omega^2\varepsilon\boldsymbol{E} = i\omega\boldsymbol{J}, \tag{1}$$

where the time variation is assumed to be in $e^{-i\omega t}$, with $\omega \in \mathbb{R}$. In the sequel, we set

$$\boldsymbol{f} = i\omega\boldsymbol{J}.$$

The coefficients ε and μ are, respectively, the permittivity and the permeability of the medium in Ω, and $\boldsymbol{J} \in L^2(\Omega)^d$ is a datum which represents the impressed current density. We assume that $\boldsymbol{J}$ (and thus $\boldsymbol{f}$) is divergence-free which amounts to saying that the electric charge density vanishes in the whole domain Ω.

In the case of composite materials, the electromagnetic coefficients ε and μ are given by piecewise constant functions. This defines a partition $\mathcal{P}$ of Ω into a finite number of subdomains $\Omega_1, \dots, \Omega_J$ such that on each Ω_j we have $\varepsilon(x) = \varepsilon_j > 0$ and $\mu(x) = \mu_j > 0$.

We assume that each subdomain is itself a polygon ($d = 2$) or a polyhedron ($d = 3$) with Lipschitz boundary, and we denote by F_{jk} the edges or faces of $\partial\Omega_j \cap \partial\Omega_k$. We distinguish between the sets $\mathcal{F}_{int}$ and $\mathcal{F}_{ext}$ of interior faces (contained in Ω) and exterior faces (contained in $\partial\Omega$). Without loss of generality, we may assume that the subdomains are connected and simply connected and have a connected boundary (see a similar remark in [29]).

In order to deal both with the boundary condition of a perfect conductor and an impedance boundary condition, let $\{\Gamma_D, \Gamma_I\}$ denote a partition of $\partial\Omega$ such that

$$\begin{aligned} &\overline{\Gamma_D \cup \Gamma_I} = \partial\Omega, \\ &\overset{\circ}{\Gamma}_D \cap \overset{\circ}{\Gamma}_I = \emptyset. \end{aligned} \tag{2}$$

This induces a partition of $\mathcal{F}_{ext}$ into $\mathcal{F}_D = \{F \in \mathcal{F}_{ext} \mid F \subset \Gamma_D\}$ and $\mathcal{F}_I = \{F \in \mathcal{F}_{ext} \mid F \subset \Gamma_I\}$.

The electric field then satisfies the following mixed boundary condition:

$$\begin{aligned} &\boldsymbol{E} \times \boldsymbol{n} = 0 && \text{on } \Gamma_D, \\ &\boldsymbol{n} \times (\boldsymbol{E} \times \boldsymbol{n}) + \lambda(\boldsymbol{n} \times \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}) = 0 && \text{on } \Gamma_I. \end{aligned} \tag{3}$$

[1]On the one hand, the case where $\partial\Omega$ consists of a finite number of connected components could be easily included, but would result in more complicated notations. On the other hand, the case of a multiply connected domain is more involved, since one has to deal with cuts, and we refer to [1,31] for a more detailed discussion. However, our results should carry over to this more general setting, since they depend only on *local* geometry considerations.

Above, λ is a complex number proportional to the characteristic impedance of the surrounding conductor and satisfying

$$\Re e\, \lambda \leq 0.$$

The variational formulation of problem (1)–(3) is given on the functional space

$$\left\{ \boldsymbol{u} \in \mathcal{H}(\mathbf{curl};\Omega) \,\middle|\, \operatorname{div}(\varepsilon \boldsymbol{u}) = 0;\ (\boldsymbol{u}\times\boldsymbol{n})_{|\Gamma_D} = 0;\ (\boldsymbol{u}\times\boldsymbol{n})_{|\Gamma_I} \in L^2(\Gamma_I)^n \right\},$$

where $n = 1$ if $d = 2$ and $n = 3$ if $d = 3$, and enters within the framework of the Fredholm alternative. Hence, (1)–(3) admits a unique solution $\boldsymbol{u}$ satisfying $\operatorname{div}(\varepsilon\boldsymbol{u}) = 0$ if, and only if, $\omega^2 \notin \sigma(\mathbf{curl}, \operatorname{div}\varepsilon^0)$ where $\sigma(\mathbf{curl}, \operatorname{div}\varepsilon^0)$ is the discrete spectrum of the involved partial differential operator.

As in [18] for the homogeneous case, we consider a weighted regularized formulation of problem (1)-(3). To this end, we denote by Y a (separable) Hilbert space with scalar product $< \cdot,\cdot >_Y$ such that

$$L^2(\Omega) \hookrightarrow Y \hookrightarrow H^{-1}(\Omega). \tag{4}$$

The variational space $\boldsymbol{W}[Y]$ is then given by

$$\boldsymbol{W}[Y] = \left\{ \boldsymbol{u} \in \mathcal{H}(\mathbf{curl};\Omega) \,\middle|\, \operatorname{div}(\varepsilon \boldsymbol{u}) \in Y;\ (\boldsymbol{u}\times\boldsymbol{n})_{|\Gamma_D} = 0;\ (\boldsymbol{u}\times\boldsymbol{n})_{|\Gamma_I} \in L^2(\Gamma_I)^n \right\}. \tag{5}$$

The space $\boldsymbol{W}[Y]$ is equipped with its canonical norm

$$||\boldsymbol{u}||_{\boldsymbol{W}[Y]} = \left(||\boldsymbol{u}||^2_{0,\Omega} + ||\mathbf{curl}\,\boldsymbol{u}||^2_{0,\Omega} + ||\operatorname{div}(\varepsilon\boldsymbol{u})||^2_Y + \|\boldsymbol{u}\times\boldsymbol{n}\|^2_{0,\Gamma_I} \right)^{1/2}.$$

The variational formulation corresponding to the space Y now reads as follows

$$(\mathcal{P}[Y]) \qquad \left\{ \begin{array}{l} \text{Find } \boldsymbol{u} \in \boldsymbol{W}[Y] \text{ such that} \\ a(\boldsymbol{u},\boldsymbol{v}) - \omega^2(\varepsilon\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})\ \forall \boldsymbol{v} \in \boldsymbol{W}[Y], \end{array} \right.$$

where the sesquilinear form $a(\cdot,\cdot)$ is given by

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) \quad = \quad & \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\overline{\mathbf{curl}\,\boldsymbol{v}}\,dx + s < \operatorname{div}(\varepsilon\boldsymbol{u}), \operatorname{div}(\varepsilon\boldsymbol{v}) >_Y \\ & -\lambda^{-1}\int_{\Gamma_I} (\boldsymbol{u}\times\boldsymbol{n})\cdot\overline{(\boldsymbol{v}\times\boldsymbol{n})}\,ds. \end{aligned} \tag{6}$$

Here, $s > 0$ is a real parameter, but it could be defined as a positive piecewise constant function.

Equivalence between problems $(\mathcal{P}[Y])$ and (1)–(3) involves a scalar transmission operator $\Delta^{\mathrm{Dir}}_\varepsilon[Y] = \operatorname{div}\varepsilon\operatorname{grad}$ with range in Y and Dirichlet boundary condition. The domain of $\Delta^{\mathrm{Dir}}_\varepsilon[Y]$ is given by

$$\mathcal{D}(\Delta^{\mathrm{Dir}}_\varepsilon[Y]) := \left\{ \varphi \in H^1_0(\Omega) \mid \operatorname{div}(\varepsilon\operatorname{grad}\varphi) \in Y \right\}. \tag{7}$$

In the sequel, we note $\Delta_\varepsilon\varphi = \operatorname{div}\varepsilon\operatorname{grad}\varphi$ for any φ in $H^1_0(\Omega)$. Since Y is a subspace of $H^{-1}(\Omega)$, the definition of $\mathcal{D}(\Delta^{\mathrm{Dir}}_\varepsilon[Y])$ is natural. Indeed, for $\varphi \in H^1_0(\Omega)$, we have $q = \Delta_\varepsilon\varphi \in H^{-1}(\Omega)$ in the sense of

$$\int_\Omega \varepsilon\operatorname{grad}\varphi\cdot\operatorname{grad}\psi\,dx = -\langle q,\psi\rangle_{H^{-1}(\Omega)-H^1_0(\Omega)}\ \forall\psi \in H^1_0(\Omega)$$

where $\langle\cdot,\cdot\rangle_{H^{-1}(\Omega)-H^1_0(\Omega)}$ denotes the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

Further, the Riesz representation Theorem yields the existence of a bounded operator K_ε such that

$$\begin{array}{rccl} K_\varepsilon & : & \mathcal{D}(\Delta^{\mathrm{Dir}}_\varepsilon[Y]) & \longrightarrow & Y \\ & & \varphi & \longmapsto & K_\varepsilon\varphi \end{array} \tag{8}$$

where $K_\varepsilon \varphi$ is the unique element in Y such that

$$< p, K_\varepsilon \varphi >_Y = \langle p, \varphi \rangle_{H^{-1}(\Omega)-H^1_0(\Omega)} \ \forall p \in Y.$$

We are now able to state the following equivalence result:

Theorem 2.1. *Let $\boldsymbol{f} \in L^2(\Omega)^d$ be divergence-free, $\operatorname{div} \boldsymbol{f} = 0$ in Ω and assume that $\omega \neq 0$. Let $\boldsymbol{u}$ be a solution to $(\mathcal{P}[Y])$. If the range of the operator $\Delta_\varepsilon^{\mathrm{Dir}}[Y] + \frac{\omega^2}{s} K_\varepsilon$ is dense in Y, then $\operatorname{div}(\varepsilon \boldsymbol{u}) = 0$ in Ω and $\boldsymbol{u}$ is a solution to (1–3).*

The idea of the proof is the same as in [18] and is omitted here. It is obvious that any solution of (1)–(3) satisfies $(\mathcal{P}[Y])$. Under the condition of Theorem 2.1, problem $(\mathcal{P}[Y])$ has thus a unique solution whenever $\omega^2 \notin \sigma(\mathbf{curl}, \operatorname{div} \varepsilon^0)$.

Remark 2.2.
1. The result of Theorem 2.1 carries over to the case $\omega = 0$, since the range of $\Delta_\varepsilon^{\mathrm{Dir}}[Y]$ is the whole space Y provided that $Y \subset H^{-1}(\Omega)$.
2. If the imbedding of Y in $H^{-1}(\Omega)$ is compact, we can prove in a similar way as in [18], that $\boldsymbol{W}[Y]$ is compactly imbedded in $L^2(\Omega)^d$ (see also [21] for the case $Y = L^2(\Omega)$). The sesquilinear form $a(\cdot, \cdot)$ is thus coercive on the space $\boldsymbol{W}[Y]$.
3. As in [18], the space Y will be defined later on as a weighted L^2-space. Therefore, the range of $\Delta_\varepsilon^{\mathrm{Dir}}[Y] + \frac{\omega^2}{s} K_\varepsilon$ will be dense in Y if and only if $\frac{\omega^2}{s}$ does not belong to the spectrum of a scalar positive self-adjoint operator with compact inverse (see [18] for details). Hence, taking s such that $\frac{\omega^2}{s}$ is smaller than the smallest eigenvalue of this operator guarantees the equivalence between $(\mathcal{P}[Y])$ and the original problem.

Let us finally introduce the following spaces of piecewise regular functions

$$PH^s(\Omega; \mathcal{P}) = \left\{ \varphi \in L^2(\Omega) \mid \varphi_j \in H^s(\Omega_j),\ j = 1, \dots, J \right\}, \tag{9}$$

where φ_j denotes the restriction of φ to Ω_j. We denote by $P\boldsymbol{H}^s(\Omega; \mathcal{P})$ the corresponding spaces of vector fields.

The remainder of this first part is to show that the space $\boldsymbol{W}[Y] \cap P\boldsymbol{H}^1(\Omega; \mathcal{P})$ is dense in $\boldsymbol{W}[Y]$ for an appropriated choice of the space Y. As mentioned before, the main application is the possibility to approximate the problem $(\mathcal{P}[Y])$ by means of nodal finite elements.

2.2. Scalar potentials

With regard to the density results that we address here, we prove in this subsection that it is sufficient to deal with the question in terms of scalar potentials only. We introduce the following functional space

$$H[Y] = \left\{ \varphi \in H^1(\Omega) \,\middle|\, \Delta_\varepsilon \varphi \in Y;\ \varphi_{|\Gamma_D} = 0;\ \varphi_{|\Gamma_I} \in H^1(\Gamma_I);\ l(\varphi) = 0 \right\} \tag{10}$$

where l is a continuous linear form on $H^1(\Gamma_I)$ such that $l(1) \neq 0$. The space $H[Y]$ is equipped with its canonical norm

$$||\varphi||_{H[Y]} = \left(||\varphi||^2_{1,\Omega} + ||\Delta_\varepsilon \varphi||^2_Y + \sum_{F \in \mathcal{F}_I} ||\varphi||^2_{1,F} \right)^{1/2}. \tag{11}$$

It is a space of scalar potentials associated with the space of vector fields $\boldsymbol{W}[Y]$ in the sense that

$$\mathbf{grad}\, H[Y] \subset \boldsymbol{W}[Y].$$

Notice that in general, scalar potentials are uniquely determined up to an additive constant. Here, the linear form l is introduced in the space $H[Y]$ in order to fix this constant. In the case where $\partial\Omega$ is not connected, one linear form for each connected component including a part from Γ_I should be included in $H[Y]$.

The first step will be the decomposition of the elements of $\boldsymbol{W}[Y]$ into a (piecewise) regular part and a singular part, the singular part deriving from a scalar potential.

Theorem 2.3. *Let $\boldsymbol{u} \in \boldsymbol{W}[Y]$. There is a scalar function $\varphi \in H[Y]$ and a piecewise regular vector field $\boldsymbol{u}_R \in \boldsymbol{W}[Y] \cap P\boldsymbol{H}^1(\Omega;\mathcal{P})$ such that*

$$\boldsymbol{u} = \boldsymbol{u}_R + \operatorname{grad}\varphi. \tag{12}$$

Further, there is a constant $c > 0$ independent from $\boldsymbol{u}$ such that

$$||\boldsymbol{u}_R||_{PH^1(\Omega;\mathcal{P})} + ||\varphi||_{H[Y]} \le c\,||\boldsymbol{u}||_{\boldsymbol{W}[Y]}. \tag{13}$$

Proof. Let $\boldsymbol{u} \in \boldsymbol{W}[Y]$. Since $\operatorname{div}(\varepsilon\boldsymbol{u}) \in Y \subset H^{-1}(\Omega)$, there is a unique function $\chi \in H^1_0(\Omega)$ such that $\Delta_\varepsilon\chi = \operatorname{div}(\varepsilon\boldsymbol{u})$. Thus, the vector field $\boldsymbol{v} = \boldsymbol{u} - \operatorname{grad}\chi$ satisfies

$$\left|\begin{array}{l} \mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{u} \text{ in } \Omega \\ \operatorname{div}(\varepsilon\boldsymbol{v}) = 0 \text{ in } \Omega \\ \boldsymbol{v}\times\boldsymbol{n} = 0 \text{ on } \Gamma_D \\ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{u}\times\boldsymbol{n} \text{ on } \Gamma_I. \end{array}\right.$$

Hence, $\boldsymbol{v}$ belongs to the standard regularization space $\boldsymbol{W}[L^2(\Omega)]$. From [29] (Theorem 3.2), we deduce the existence of a regular vector potential $\boldsymbol{u}_R \in \boldsymbol{W}[L^2(\Omega)] \cap P\boldsymbol{H}^1(\Omega;\mathcal{P})$ satisfying

$$\left|\begin{array}{l} \mathbf{curl}\,\boldsymbol{u}_R = \mathbf{curl}\,\boldsymbol{v} \text{ in } \Omega \\ \operatorname{div}(\varepsilon\boldsymbol{u}_R) \in L^2(\Omega) \\ \boldsymbol{u}_R\times\boldsymbol{n} = 0 \text{ on } \partial\Omega \end{array}\right.$$

as well as the estimate

$$||\boldsymbol{u}_R||_{PH^1(\Omega;\mathcal{P})} + ||\operatorname{div}(\varepsilon\boldsymbol{u}_R)||_{0,\Omega} \le c\,||\mathbf{curl}\,\boldsymbol{v}||_{0,\Omega}. \tag{14}$$

Since $\mathbf{curl}(\boldsymbol{u}-\boldsymbol{u}_R) = \mathbf{curl}(\boldsymbol{v}-\boldsymbol{u}_R) = 0$ in Ω, there is a unique scalar potential $\varphi \in H^1(\Omega)$ such that

$$\begin{array}{l} \operatorname{grad}\varphi = \boldsymbol{u}-\boldsymbol{u}_R \text{ in } \Omega \text{ and} \\ l(\varphi) = 0. \end{array}$$

We obviously have $\Delta_\varepsilon\varphi \in Y$. Moreover, $\varphi_{|\Gamma_I}$ belongs to $H^1(\Gamma_I)$ since

$$\operatorname{grad}_T\varphi_{|F} = \operatorname{grad}\varphi_{|F}\times\boldsymbol{n} = \boldsymbol{u}_{|F}\times\boldsymbol{n} \in L^2(F)^n \;\; \forall F \subset \Gamma_I.$$

This shows that φ belongs to $H[Y]$.

We prove in Lemma 2.4 below that

$$||\varphi||_{H[Y]} \le c\left(||\Delta_\varepsilon\varphi||_Y^2 + \sum_{F\in\mathcal{F}_I}||\operatorname{grad}_T\varphi||_{0,F}^2\right)^{1/2} \quad \forall\varphi \in H[Y].$$

The estimate of $||\Delta_\varepsilon\varphi||_Y$ follows from the continuous imbedding of $L^2(\Omega)$ in the vector space Y and (14), taking into account that $\mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{u}$:

$$\begin{array}{rcl} ||\Delta_\varepsilon\varphi||_Y &\le& ||\operatorname{div}(\varepsilon\boldsymbol{u})||_Y + c\,||\operatorname{div}(\varepsilon\boldsymbol{u}_R)||_{0,\Omega} \\ &\le& c(||\operatorname{div}(\varepsilon\boldsymbol{u})||_Y + ||\mathbf{curl}\,\boldsymbol{u}||_{0,\Omega}) \\ &\le& c\,||\boldsymbol{u}||_{\boldsymbol{W}[Y]} \end{array}$$

whereas the second term is equal to

$$\sum_{F\in\mathcal{F}_I} ||\boldsymbol{u}\times\boldsymbol{n}||^2_{0,F}$$

according to the definition of φ. This proves (13). $\square$

In the proof of Theorem 2.3, we made use of the following equivalence result between norms:

Lemma 2.4. *Let Y be such that (4) holds. The application*

$$\begin{array}{rcccl} |\cdot|_{H[Y]} & : & H[Y] & \longrightarrow & \mathbb{R}^+ \\ & & \varphi & \longmapsto & \left(||\Delta_\varepsilon\varphi||^2_Y + \sum_{F\in\mathcal{F}_I} ||\mathrm{grad}_T\,\varphi||^2_{0,F}\right)^{1/2} \end{array}$$

defines a norm on $H[Y]$ equivalent to the canonical norm $||\cdot||_{H[Y]}$.

Proof. It is obvious that $|\cdot|_{H[Y]}$ defines a semi-norm on $H[Y]$. Now, let $\varphi\in H[Y]$ be such that $|\varphi|_{H[Y]}=0$. Hence, $\Delta_\varepsilon\varphi=0$ on Ω which yields $\varphi=0$ on Ω if $\Gamma_D\neq\emptyset$. If $\Gamma_D=\emptyset$, we have $\mathrm{grad}_T\,\varphi=0$ on all exterior faces. Hence, $\varphi_{|\partial\Omega}$ is a constant and this constant must be 0 since $l(\varphi)=0$ and $l(1)\neq 0$.

We next prove equivalence between $|\cdot|_{H[Y]}$ and the canonical norm. Let $\varphi\in H[Y]$. There is a unique function $r\in H^1(\Omega)$ such that

$$\begin{array}{ll} \Delta_\varepsilon r = 0 & \text{in } \Omega \\ r=\varphi & \text{on } \partial\Omega. \end{array}$$

It follows from classical results in variational theory and the continuous imbedding $H^1(\Gamma_I)\hookrightarrow H^{1/2}(\Gamma_I)$ that

$$||r||_{1,\Omega} \le c\|\varphi\|_{1/2,\partial\Omega} \le c\left(\sum_{F\in\mathcal{F}_I} ||\varphi||^2_{1,F}\right)^{1/2}. \tag{15}$$

Next, let $\tilde\varphi=\varphi-r$. The function $\tilde\varphi$ is the variational solution in $H^1_0(\Omega)$ to the following Dirichlet problem with data in Y:

$$\begin{array}{ll} \Delta_\varepsilon\tilde\varphi=\Delta_\varepsilon\varphi & \text{in } \Omega \\ \varphi=0 & \text{on } \partial\Omega \end{array}.$$

We deduce from Poincaré's inequality and the definition of the parameter ε that

$$\begin{array}{rcl} ||\tilde\varphi||^2_{1,\Omega} & \le & c\int_\Omega \varepsilon|\,\mathrm{grad}\,\tilde\varphi|^2\,dx \\ & = & -c\langle\Delta_\varepsilon\varphi,\tilde\varphi\rangle_{H^{-1}(\Omega)-H^1_0(\Omega)} \\ & \le & c\|\Delta_\varepsilon\varphi\|_{-1,\Omega}\,||\tilde\varphi||_{1,\Omega}\,, \end{array}$$

and thus

$$||\tilde\varphi||_{1,\Omega}\le c\,||\Delta_\varepsilon\varphi||_Y \tag{16}$$

since the imbedding $Y\hookrightarrow H^{-1}(\Omega)$ is continuous. Finally, we deduce from (15) and (16) that

$$||\varphi||_{1,\Omega}\le c\left(||\Delta_\varepsilon\varphi||^2_Y+\sum_{F\in\mathcal{F}_I}||\varphi||^2_{1,F}\right)^{1/2}$$

and the result follows from the equivalence between the H^1-norm and the seminorm $\sum_{F\in\mathcal{F}_I}|\cdot|_{1,F}$ in the space $\{w\in H^1(\Gamma_I)\mid l(w)=0\}$. $\square$

Note that the above decomposition (12) is not unique. For instance take $\psi \in \mathcal{D}(\Omega_j)$ for a fixed $j \in \{1, \cdots, J\}$ and let $\boldsymbol{u}'_R = \boldsymbol{u}_R + ||\boldsymbol{u}||_{\boldsymbol{W}[Y]} \operatorname{grad} \psi$ and $\varphi' = \varphi - ||\boldsymbol{u}||_{\boldsymbol{W}[Y]} \psi$. Obviously,

$$\boldsymbol{u} = \boldsymbol{u}'_R + \operatorname{grad} \varphi'$$

and $\boldsymbol{u}'_R$ and φ' satisfy (13).

Nevertheless, due to the decomposition (12) and estimate (13), we are able to define a linear continuous application $\Phi : \boldsymbol{W}[Y] \longrightarrow H[Y]$ which maps any vector field $\boldsymbol{u} \in \boldsymbol{W}[Y]$ on the corresponding scalar potential $\varphi \in H[Y]$ in the sense that

$$\boldsymbol{u} - \operatorname{grad}(\Phi(\boldsymbol{u})) \in \boldsymbol{W}[Y] \cap P\boldsymbol{H}^1(\Omega; \mathcal{P}) \text{ and} \tag{17}$$

$$\Phi(\operatorname{grad} \varphi) = \varphi \ \forall \varphi \in H[Y]. \tag{18}$$

Since $\operatorname{grad} H[Y] \subset \boldsymbol{W}[Y]$, Φ is well defined and onto. Moreover, Φ maps regular vector fields on regular scalar potentials, i.e.

$$\Phi\left(\boldsymbol{W}[Y] \cap P\boldsymbol{H}^1(\Omega; \mathcal{P})\right) \subset H[Y] \cap PH^2(\Omega; \mathcal{P}). \tag{19}$$

Indeed, let $\boldsymbol{u} \in \boldsymbol{W}[Y] \cap P\boldsymbol{H}^1(\Omega; \mathcal{P})$. Due to (17), we have $\operatorname{grad}(\Phi(\boldsymbol{u})) \in P\boldsymbol{H}^1(\Omega; \mathcal{P})$ which implies that

$$\Phi(\boldsymbol{u}) = \varphi \in PH^2(\Omega; \mathcal{P}).$$

We are now able to state the main result of this subsection:

Theorem 2.5. *The space of vector fields $\boldsymbol{W}[Y] \cap P\boldsymbol{H}^1(\Omega; \mathcal{P})$ is dense in $\boldsymbol{W}[Y]$ if, and only if, the corresponding space of scalar potentials $H[Y] \cap PH^2(\Omega; \mathcal{P})$ is dense in $H[Y]$.*

Proof. The proof of Theorem 2.5 follows directly from the definition and the properties of the application Φ. We refer to [29] (Proof of Theorem 3.1) for details. □

2.3. Two-dimensional results

In this subsection, we prove the density result in the case of a polygon for an appropriate choice of the space Y. We further state some preliminary results which will be helpful for the edge singularities in three dimensions. In this subsection, Ω is a fixed polygon of the plane with the assumptions of §2.1.

Let us start with the definition of the space Y. For $\alpha \in]-1, 1[$, we denote

$$Y = \left\{ g \in H^{-1}(\Omega) \,\middle|\, w^\alpha g \in L^2(\Omega) \right\}, \tag{20}$$

where the weight function w is assumed to be positive on Ω. There are several possibilities to define the function w (see [18]). Roughly speaking, w will be chosen to be equivalent to the distance function to the set of vertices of the subdomains. The space Y is a Hilbert space equipped with the scalar product

$$< f, g >_Y = \int_\Omega w^{2\alpha}(\boldsymbol{x}) f(\boldsymbol{x}) g(\boldsymbol{x})\, dx$$

In order to provide a rigorous definition of the weight function w, we introduce the following notations. Let $\mathcal{S}$ be the set of vertices of at least one Ω_j. The set of exterior vertices will be denoted by $\mathcal{S}_{ext}$,

$$\mathcal{S}_{ext} = \{ S \in \mathcal{S} \mid S \in \partial\Omega \}.$$

This set is split into two subsets, namely,

$$\mathcal{S}_D = \mathcal{S}_{ext} \cap \overset{\circ}{\Gamma}_D$$
$$\mathcal{S}_I = \mathcal{S}_{ext} \setminus \mathcal{S}_D.$$

The set of interior vertices is given by $\mathcal{S}_{int} = \mathcal{S} \setminus \mathcal{S}_{ext}$.

Definition 2.6 (Weight function in two dimensions)**.** Let $\Omega \subset \mathbb{R}^2$ be a polygon. For any vertex $S \in \mathcal{S}$, let (r_S, θ_S) denote the local polar coordinates with respect to S. The weight function w is defined by

$$w(\boldsymbol{x}) = \prod_{S \in \mathcal{S}_0} r_S \tag{21}$$

where $\mathcal{S}_0$ is a subset of $\mathcal{S}$.

This definition is similar to the one of *simplified weights* in [18]. Notice that $w(\boldsymbol{x})$ is equivalent to the distance function $d(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \mathcal{S})$. Moreover, in a sufficiently small neigbourhood $\mathcal{V}_S$ of the vertex S containing no other vertex of Ω, the weight function is equivalent to r_S if the weight is "active", whereas $w(\boldsymbol{x}) \approx 1$ far away from the vertices. Let us now introduce

$$L^2_\alpha(\Omega) = \left\{ g \in H^{-1}(\Omega) \,\middle|\, \left(\prod_{S \in \mathcal{S}_0} r_S \right)^\alpha g \in L^2(\Omega) \right\}.$$

The following result shows that $L^2_\alpha(\Omega)$ is an admissible choice for the space Y:

Proposition 2.7. *Let $Y = L^2_\alpha(\Omega)$. Then (4) does hold for any $\alpha \in [0, 1[$.*

Proof. Since $\alpha \geq 0$ and w is continuous on $\bar{\Omega}$, the imbedding $L^2(\Omega) \hookrightarrow L^2_\alpha(\Omega)$ is obvious.

On the other hand, we deduce from a classic Hardy inequality (see for instance [33], Lemma 4.1 p. 38) that

$$H^1(\Omega) \hookrightarrow L^2_{-\alpha}(\Omega)$$

for all $\alpha \in [0, 1[$, if one recalls that the weight function w is equivalent to the distance function near the vertices and equivalent to 1 anywhere else.

The result of the Proposition follows by duality since $(L^2_{-\alpha}(\Omega))' = L^2_\alpha(\Omega)$. $\square$

From now on, let Y be as in Proposition 2.7. For $\xi \in \mathbb{R}$, we introduce the space of dual singularities $\mathcal{N}_{\varepsilon,\xi}[Y]$ defined as the orthogonal in Y of $(\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})(H[Y] \cap PH^2(\Omega; \mathcal{P}))$ with respect to the scalar product of Y. In other words, an element $g \in Y$ belongs to $\mathcal{N}_{\varepsilon,\xi}[Y]$ if, and only if,

$$< g, (\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})\varphi >_Y = 0 \;\forall \varphi \in H[Y] \cap PH^2(\Omega; \mathcal{P}). \tag{22}$$

We next recall the space of standard dual singularities $\mathcal{N}_{\varepsilon,Dir,\xi}$ defined as follows: $g \in \mathcal{N}_{\varepsilon,Dir,\xi}$ if, and only if, $g \in L^2(\Omega)$ and

$$\int_\Omega g(\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})\varphi \, dx = 0 \;\forall \varphi \in \mathcal{D}(\Delta_\varepsilon^{\mathrm{Dir}}[L^2(\Omega)]) \cap PH^2(\Omega; \mathcal{P}) \tag{23}$$

where $\mathcal{D}(\Delta_\varepsilon^{\mathrm{Dir}}[L^2(\Omega)])$ is defined in analogy with (7). Taking into account the definition of the scalar product $< \cdot, \cdot >_Y$, we are now able to state the following link between $\mathcal{N}_{\varepsilon,\xi}[Y]$ and $\mathcal{N}_{\varepsilon,Dir,\xi}$:

Proposition 2.8. *Let $\xi \in \mathbb{R}$. For any $g \in \mathcal{N}_{\varepsilon,\xi}[Y]$, the function g_α defined by*

$$g_\alpha = w^{2\alpha} g$$

belongs to the space of standard dual singularities, $\mathcal{N}_{\varepsilon,Dir,\xi}$.

Proof. (1) Let $g \in \mathcal{N}_{\varepsilon,\xi}[Y]$. Since g belongs to $Y = L^2_\alpha(\Omega)$, the function $w^\alpha g$ belongs to $L^2(\Omega)$. The definition of the weight function w then guarantees that $g_\alpha = w^{2\alpha} g \in L^2(\Omega)$.

(2) In order to prove that g_α satisfies the orthogonality relation (23), let $\varphi \in \mathcal{D}(\Delta_\varepsilon^{\rm Dir}[L^2(\Omega)]) \cap PH^2(\Omega;\mathcal{P})$. Then, φ also belongs to $H[Y] \cap PH^2(\Omega;\mathcal{P})$, since $L^2(\Omega) \subset L^2_\alpha(\Omega)$ and $\varphi_{|\partial\Omega} = 0$. Hence

$$\int_\Omega g_\alpha(\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})\varphi\,dx = < g, (\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})\varphi >_Y = 0$$

which proves (23).

□

In view of the forthcoming Theorem 2.10, we need to recall the singularities of the transmission problem involving the operator Δ_ε with domain $\mathcal{D}(\Delta_\varepsilon^{\rm Dir}[L^2(\Omega)])$. (see [28, 34, 35] for details).

For $S \in \mathcal{S}_{ext}$, let $\Lambda_{\varepsilon,S}$ be the set of positive singular exponents of the operator $\Delta_\varepsilon^{\rm Dir}[L^2(\Omega)]$ that we now describe shortly. Without loss of generality we may assume that the set of subdomains Ω_j having S as vertex is $\{\Omega_j\}_{j=1}^{J_S}$, for some positive integer J_S. For $j \in \{1, \cdots, J_S\}$ let ω_j be the interior opening of Ω_j at S and set $\sigma_0 = 0$ and $\sigma_j = \sigma_{j-1} + \omega_j$. Then a real number λ belongs to $\Lambda_{\varepsilon,S}$ if, and only if, there exists a non trivial solution $\phi_\lambda \in H^1(]0, \sigma_{J_S}[)$, $\phi_\lambda = (\phi_{\lambda,j})_{j=1}^{J_S}$, to

$$\phi''_{\lambda,j} + \lambda^2\phi_{\lambda,j} = 0 \text{ in }]\sigma_{j-1}, \sigma_j[,\ j = 1, \cdots, J_S, \tag{24}$$
$$\phi_{\lambda,1}(0) = \phi_{\lambda,J_S}(\sigma_{J_S}) = 0, \tag{25}$$
$$[\phi_\lambda]_{\sigma_{j-1}} = [\varepsilon\phi'_\lambda]_{\sigma_{j-1}} = 0,\ j = 1, \cdots, J_S - 1, \tag{26}$$

where (r_S, θ_S) are the local polar coordinates with respect to S, the half-line $\theta_S = \sigma_j$ containing an edge of Ω_j, for $j = 1, \cdots, J_S$ while the half-line $\theta_S = 0$ contains an edge of Ω_1 (see Fig. 1).

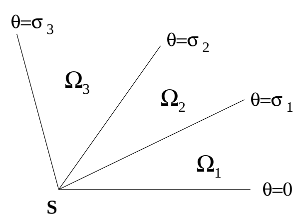

Figure 1. Subdomains having a common vertex ($J_S = 3$).

Note that in the homogeneous case, i.e., $\varepsilon_j = \varepsilon$, for all $j = 1, \cdots, J_S$, the set $\Lambda_{\varepsilon,S}$ is equal to $\{\frac{k\pi}{\sigma_{J_S}} : k \in \mathbb{N}, k \neq 0\}$ and is independent of ε. In the inhomogeneous case this set is not explicitly known but may be approximated numerically (see e.g. [28, 34, 35]).

We proceed similarly for $S \in \mathcal{S}_{int}$, replacing the Dirichlet boundary condition (25) by the transmission conditions

$$\phi_{\lambda,1}(0) = \phi_{\lambda,J_S}(2\pi), \quad \varepsilon_1\phi'_{\lambda,1}(0) = \varepsilon_{J_S}\phi'_{\lambda,J_S}(2\pi).$$

Let us notice that if $S \in \mathcal{S}_{ext}$ then $\lambda \in \Lambda_{\varepsilon,S}$ is simple (see [35]). In other words, the solution ϕ_λ to (24)-(25) is unique up to a multiplicative factor. On the other hand, if $S \in \mathcal{S}_{int}$, then $\lambda \in \Lambda_{\varepsilon,S}$ has a finite multiplicity and in that case λ is repeated in $\Lambda_{\varepsilon,S}$ according to its multiplicity.

The standard singularities of the operator $\Delta_\varepsilon^{\rm Dir}[L^2(\Omega)]$ at the vertex $S \in \mathcal{S}$ are

$$S_{S,\lambda} = \eta_S r^\lambda \phi_\lambda,\ \text{for } \lambda \in \Lambda_{\varepsilon,S}, \tag{27}$$

where $\eta_S = \eta_S(r)$ is an appropriate cut-off function ($\eta_S \equiv 1$ in a neighbourhood of S and $\eta_S \equiv 0$ in a neighbourhood of the other vertices).

Next, we need to characterize the elements of the space $\mathcal{N}_{\varepsilon,Dir,\xi}$. To this end, we recall Proposition 2.8. of [29] (for any details, see [34] for the case $\xi = 0$ and [24] for $\xi \neq 0$). Let us begin with some classical Grisvard notations. For $\ell = 1, 2$, let $H^{\ell-1/2}(\partial\Omega)$ be the range of the trace mapping, starting from $H^{\ell}(\Omega)$; for all faces $F \in \mathcal{F}_{ext}$, the restrictions of those sets to F is denoted by $H^{\ell-1/2}(F)$. Then, define $\widetilde{H}^{\ell-1/2}(F)$ as the set of elements of $H^{\ell-1/2}(F)$ whose continuation by zero to $\partial\Omega$ belongs to $H^{\ell-1/2}(\partial\Omega)$. Finally, let $\widetilde{H}^{1/2-\ell}(F)$ denote the dual space of $\widetilde{H}^{\ell-1/2}(F)$. In the same manner, one can introduce similar spaces for the interior faces, starting from interior domains.

Proposition 2.9. *$g \in \mathcal{N}_{\varepsilon,Dir,\xi}$ if, and only if, $g \in L^2(\Omega)$ is solution to*

$$\begin{aligned}
&(\Delta - \xi^2\mathbb{I})g = 0 \ \text{ in } \Omega_j \, \forall j,\\
&g = 0 \ \text{in} \ \ \widetilde{H}^{-1/2}(F) \, \forall F \in \mathcal{F}_{ext}.\\
&[g] = 0 \ \text{in} \ \ \widetilde{H}^{-1/2}(F) \, \forall F \in \mathcal{F}_{int},\\
&[\varepsilon\partial_n g] = 0 \ \text{ in } \widetilde{H}^{-3/2}(F) \, \forall F \in \mathcal{F}_{int}.
\end{aligned}$$

In order to give an appropriate basis of $\mathcal{N}_{\varepsilon,Dir,\xi}$, we set for any vertex $S \in \mathcal{S}$ and all $\lambda \in \Lambda_{\varepsilon,S}$,

$$g_{S,\lambda,\xi} = \eta_S e^{-|\xi| r} r^{-\lambda} \phi_\lambda - v_{S,\lambda,\xi}, \tag{28}$$

where $v_{S,\lambda,\xi} \in H^1_0(\Omega)$ is the unique variational solution to

$$(\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})v_{S,\lambda,\xi} = (\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})(\eta_S e^{-|\xi| r} r^{-\lambda}\phi_\lambda), \tag{29}$$

i.e., is the unique solution to

$$\begin{aligned}
&\int_\Omega \varepsilon(\operatorname{grad} v_{S,\lambda,\xi} \cdot \operatorname{grad} w + \xi^2 v_{S,\lambda,\xi} w)\, dx\\
&\qquad = -\int_\Omega (\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})(\eta_S e^{-|\xi| r} r^{-\lambda}\phi_\lambda) w \, dx, \forall w \in H^1_0(\Omega).
\end{aligned}$$

Notice that this problem is well defined since the right hand side of (29) belongs to $L^q(\Omega)$ with $q < \frac{2}{1+\lambda}$ (see Lemma 4.4 and 4.5 of [27]).

The function $g_{S,\lambda,\xi}$ belongs to $\mathcal{N}_{\varepsilon,Dir,\xi}$ and satisfies (thanks to Green's formula, see Proposition 2.5.5 of [24]))

$$\int_\Omega (\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})(\eta_T r^\mu \phi_\mu) g_{S,\lambda,\xi}\, dx = 2\lambda\delta_{S,T}\delta_{\lambda,\mu}. \tag{30}$$

Furthermore, under the assumption

$$1 \notin \Lambda_{\varepsilon,S}, \forall S \in \mathcal{S}, \tag{31}$$

the set $\{g_{S,\lambda,\xi}\}_{\lambda\in\Lambda_{\varepsilon,S}\cap]0,1[,S\in\mathcal{S}}$ is a basis of $\mathcal{N}_{\varepsilon,Dir,\xi}$.

The following Theorem provides an appropriate condition on the weight exponent α such that the density result for the scalar potentials (and thus also for the corresponding space of vector fields) holds true:

Theorem 2.10. *Let Y be as in Proposition 2.7. Let the subset $\mathcal{S}_0 \subset \mathcal{S}$ satisfy the following inclusion:*

$$\{S \in \mathcal{S}_D \cup \mathcal{S}_{int} \mid \Lambda_{\varepsilon,S}\cap]0,1[\neq \emptyset\} \cup \{S \in \mathcal{S}_I \mid \Lambda_{\varepsilon,S}\cap]0,1/2[\neq \emptyset\} \subset \mathcal{S}_0. \tag{32}$$

Further, let α be such that

$$\alpha > 1 - \min\left\{ \bigcup_{S\in\mathcal{S}_D\cup\mathcal{S}_{int}} (\Lambda_{\varepsilon,S}\cap]0,1[) \cup \bigcup_{S\in\mathcal{S}_I} (\Lambda_{\varepsilon,S}\cap]0,1/2[) \right\} \tag{33}$$

Assume that (31) holds. Then $H[Y] \cap PH^2(\Omega;\mathcal{P})$ is dense in $H[Y]$.

Proof. In order to prove the density result, we will characterize the elements of some complementary space $\mathcal{O}[Y]$ that wedefine here. Let

$$H_0[Y] = \left\{ \varphi \in H^1(\Omega) \,\middle|\, \Delta_\varepsilon \varphi \in Y;\ \varphi_{|\Gamma_D} = 0;\ \varphi_{|F} \in H_0^1(F), \forall F \in \mathcal{F}_I \right\}.$$

As in [29], we prove that $H_0[Y]$ is continuously imbedded in $H[Y]$ if we choose the linear form l in the definition of $H[Y]$ as follows:

$$l(\varphi) = \sum_{S \in \mathcal{S}_I} \varphi(S).$$

Now, let $\xi \in \mathbb{R}$ and let $\mathcal{O}[Y]$ be the orthogonal complement of $\overline{H_0[Y] \cap PH^2(\Omega;\mathcal{P})}$ for the inner product

$$\begin{aligned}(\varphi,\psi)_{\xi,Y} &= \;<(\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})\varphi, (\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I})\psi>_Y \\ &\quad + \sum_{F \in \mathcal{F}_I} \left\{ (\operatorname{grad}_T \varphi, \operatorname{grad}_T \psi)_{0,F} + \xi^2 (\varphi,\psi)_{0,F} \right\}.\end{aligned}$$

Then

$$H[Y] = \overline{H[Y] \cap PH^2(\Omega;\mathcal{P})} \oplus \mathcal{O}[Y]$$

since the complementary space of $H_0[Y]$ in $H[Y]$ is spanned by a finite number of functions that belong to $H[Y] \cap PH^2(\Omega;\mathcal{P})$. Notice that arguments, similar to those of Lemma 2.4, allow one to show that the norm $\|\varphi\|_{\xi,Y} = (\varphi,\varphi)_{\xi,Y}^{1/2}$ is equivalent to $||\cdot||_{H[Y]}$ with equivalence constants that depend on ξ.

As in [29] (Proposition 4.3), we are able to prove that for any $f \in \mathcal{O}[Y]$, there is a unique $g \in \mathcal{N}_{\varepsilon,\xi}[Y]$ with

$$(\Delta_\varepsilon - \varepsilon\xi^2\mathbb{I}) f = g \text{ in } Y, \tag{34}$$

$$(\Delta_T - \xi^2\mathbb{I}) f = -\varepsilon \partial_\nu g_\alpha \text{ in } H^{-1}(F), \forall F \in \mathcal{F}_I, \tag{35}$$

$$\|f\|_{\xi,Y} \le c \left(||g||_Y + \sum_{F \in \mathcal{F}_I} \|\varepsilon \partial_\nu g_\alpha\|_{-1,F} \right) \tag{36}$$

where $g_\alpha = w^{2\alpha} g$ is the standard dual singularity in $\mathcal{N}_{\varepsilon,Dir,\xi}$ corresponding to g, according to Proposition 2.8. The function g_α is thus uniquely represented as

$$g_\alpha = \sum_{S \in \mathcal{S}} \sum_{\lambda \in \Lambda_{\varepsilon,S} \cap]0,1[} c_{\lambda,S}\, g_{S,\lambda,\xi}.$$

As in the proof of Theorem 4.4 in [29], condition (35) implies that

$$c_{\lambda,S} = 0 \ \forall \lambda \in \Lambda_{\varepsilon,S} \cap [1/2, 1[,\ \forall S \in \mathcal{S}_I$$

since $\partial_\nu g_{S,\lambda,\xi} \approx r_S^{\lambda-1}$ near S.

Now, let $\lambda \in \Lambda_{\varepsilon,S} \cap]0,1[$ for a vertex $S \in \mathcal{S}_D \cup \mathcal{S}_{int}$ or $\lambda \in \Lambda_{\varepsilon,S} \cap]0,1/2[$ for $S \in \mathcal{S}_I$. Taking into account that $w^\alpha g$ belongs to $L^2(\Omega)$, we deduce that $w^{-\alpha} g_{S,\lambda,\xi} \in L^2(\Omega)$ whenever $c_{\lambda,S} \neq 0$. But

$$w^{-\alpha} g_{S,\lambda,\xi} \approx r_S^{-(\alpha+\lambda)} \Phi_{\lambda,S}(\theta_S)$$

near S, and $r_S^{-(\alpha+\lambda)} \Phi_{\lambda,S}$ belongs to $L^2(\Omega)$ if, and only if, $\alpha+\lambda < 1$ which is in contradiction with the assumption on α. Therefore $c_{\lambda,S} = 0$ for any λ which yields $g = 0$ in Ω.

Finally, we deduce from (36) that $f = 0$ in Ω which completes the proof. $\square$

Remark 2.11. One could also consider *general weights* with an exponent that depends on the vertex S of $\mathcal{S}$. Namely, one could replace w^α by

$$\prod_{S\in\mathcal{S}} r_S^{\alpha_S}, \tag{37}$$

with $(\alpha_S)_{S\in\mathcal{S}}$ in $]0,1]^{|\mathcal{S}|}$ such that

$$\begin{cases} \alpha_S > 1 - \min\left(\Lambda_{\varepsilon,S}\cap]0,1[\right) & \text{if } S\in\mathcal{S}_D\cup\mathcal{S}_{int}\ ; \\ \alpha_S > 1 - \min\left(\Lambda_{\varepsilon,S}\cap]0,1/2[\right) & \text{if } S\in\mathcal{S}_I. \end{cases} \tag{38}$$

2.4. Density results in three-dimensional domains

In this subsection we investigate a suitable condition on the weight exponent α in order to obtain the density result in the case of a three dimensional Lipschitz-polyhedron.

In order to define the weight function w, we introduce the following notations which describe the domain Ω near the geometric singularities.

Let $\mathcal{S}$ (resp. $\mathcal{E}$) be the set of vertices (resp. edges) of at least one Ω_j. The subscripts *ext* and *int* will denote exterior and interior vertices or edges as before, and the set $\mathcal{S}_{ext}$ (resp. $\mathcal{E}_{ext}$) admits the following splitting, according to the different boundary conditions:

$$\mathcal{S}_D = \mathcal{S}_{ext}\cap \overset{\circ}{\Gamma}_D,\ \mathcal{S}_I = \mathcal{S}_{ext}\setminus\mathcal{S}_D,$$
$$\mathcal{E}_D = \mathcal{E}_{ext}\cap \overset{\circ}{\Gamma}_D,\ \mathcal{E}_I = \mathcal{E}_{ext}\setminus\mathcal{E}_D.$$

For a vertex $S\in\mathcal{S}$, let Γ_S be the polyhedral cone which coincides with Ω near S and let G_S be the intersection of Γ_S with the unit sphere. We shall use local spherical coordinates (r_S,σ_S) centered at S. To each edge e adjacent to the vertex S, corresponds a corner of G_S denoted by S_e. A neighbourhood of the point S_e may thus be mapped on an infinite plane sector which can be written in polar coordinates as

$$C_{S,e} = \{(\vartheta_{S,e},\varphi_{S,e}) \mid \vartheta_{S,e}>0,\ 0<\varphi_{S,e}<\omega_{S,e}\}.$$

Next, let $e\in\mathcal{E}$ be an (exterior or interior) edge with opening angle $\omega_e\in]0,2\pi]$ ($\omega_e=2\pi$ if, and only if, $e\in\mathcal{E}_{int}$). Without loss of generality, we may assume that e is supported by the z-axis and we denote (r_e,θ_e,z) the corresponding cylindrical coordinates. In particular, we have

$$r_e(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x},\bar{e})\ \forall \boldsymbol{x}\in\Omega.$$

Let us fix $R_e>0$ and $h_e>0$ and introduce the two-dimensional domain

$$\Omega_e := \{(r_e\cos\theta_e, r_e\sin\theta_e) \mid 0<r_e<R_e, 0<\theta_e<\omega_e\}$$

such that the dihedral cone

$$D_e = \Omega_e\times\mathbb{R} \tag{39}$$

coincides with Ω for any $z\in]-h_e,h_e[$ and does contain no other edge nor any vertex of Ω. To each Ω_j containing e, there corresponds a unique set $\Omega_{e,j}\subset\Omega_e$. Therefore the partition $\mathcal{P}$ induces a natural partition $\mathcal{P}_e$ of Ω_e (and thus D_e) for which ε and μ are piecewise constant and depend only on θ. Namely, we take

$$\varepsilon_{e,j} = \varepsilon_j \text{ on } \Omega_{e,j}\times\mathbb{R},$$
$$\mu_{e,j} = \mu_j \text{ on } \Omega_{e,j}\times\mathbb{R}.$$

We finally denote $\Gamma_{e,0}$ (resp. $\Gamma_{e,\omega}$) the edges of Ω_e and $F_{e,0}=\Gamma_{e,0}\times\mathbb{R}$ (resp. $F_{e,\omega}$) the corresponding exterior faces of D_e containing e.

If we denote by $d_{\mathcal{S}}(\boldsymbol{x})$ (resp. $d_{\mathcal{E}}$) the distance function to the set $\mathcal{S}$ (resp. $\mathcal{E}$), i. e.

$$d_{\mathcal{S}}(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \mathcal{S}) \text{ and } d_{\mathcal{E}}(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \mathcal{E}),$$

we clearly have

$$d_{\mathcal{S}} \approx r_S$$

in any sufficiently small neighbourhood $\mathcal{V}_S$ of the vertex S, and

$$d_{\mathcal{E}} \approx r_e$$

in $\Omega_e \times]-h_e, h_e[$ for sufficiently small numbers R_e and h_e.

In order to define the weight function, we need to introduce another distance function ρ_e taking into account the edge/vertex interaction. Let $e \in \mathcal{E}$ be the segment between the two vertices S and S'. Then we define ρ_e by

$$r_e = \rho_e r_S r_{S'}. \tag{40}$$

In a sufficiently small neighbourhood of the vertex S, the function ρ_e is equivalent to the angular distance $\vartheta_{S,e}$ near the edge e, while

$$\rho_e \approx d_{\mathcal{E}} \text{ far from } \mathcal{S}.$$

The definition of the weight function then reads as follows (see the definition of global weights in [18]):

Definition 2.12 (Weight function in three dimensions)**.** Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz-polyhedron. The weight function w is defined by

$$w(\boldsymbol{x}) = \left(\prod_{S \in \mathcal{S}_0} r_S\right)\left(\prod_{e \in \mathcal{E}_0} \rho_e\right) \tag{41}$$

where $\mathcal{S}_0 \subset \mathcal{S}$ and $\mathcal{E}_0 \subset \mathcal{E}$ satisfy the following compatibility condition: if $e \in \mathcal{E}_0$ is an edge with end points S and S', then $S \in \mathcal{S}_0$ and $S' \in \mathcal{S}_0$.

It has been proven in [18] that an equivalent definition is

$$w(\boldsymbol{x}) = \operatorname{dist}\left(\boldsymbol{x}, \mathcal{S}_0 \cup \mathcal{E}_0\right).$$

This corresponds to the *simple weights* of Costabel-Dauge, where the set $\mathcal{S}_0 \cup \mathcal{E}_0$ is a so-called *wire basket*, in the spirit of [36].

As in two dimensions, we have the

Proposition 2.13. *Let $Y = L^2_\alpha(\Omega)$ with a weight function as in Definition 2.12. Then (4) does hold for any $\alpha \in [0, 1[$.*

Proof. As in two dimensions, the first imbedding $L^2(\Omega) \hookrightarrow L^2_\alpha(\Omega)$ is obvious. For the second imbedding, we proceed by duality, proving that

$$H^1_0(\Omega) \hookrightarrow L^2_{-\alpha}(\Omega).$$

Near an edge, this follows as in two dimensions from the classical Hardy inequality. Near the vertices, we may use Proposition 5.1. in [30] since the definition of weights therein is equivalent to Definition 2.12. □

We next describe the vertex and edge singularities of the operator Δ_ε with Dirichlet boundary condition, i. e. with domain $\mathcal{D}(\Delta_\varepsilon^{\mathrm{Dir}}[L^2(\Omega)])$. The set $\Lambda_{\varepsilon,S}$ of positive vertex singular exponents is related to the spectrum of the nonnegative Laplace-Beltrami operator $L_{\varepsilon,S}$. More precisely, it is the Friedrichs extension of the triple $(H_{\varepsilon,S}, V_S, a_{\varepsilon,S})$, where $H_{\varepsilon,S} = L^2(G_S)$ with the inner product

$$(\psi, \phi)_\varepsilon = \int_{G_S} \varepsilon \psi \phi \, d\sigma,$$

the space V_S being equal to $H_0^1(G_S)$ if $S \in \mathcal{S}_{ext}$, and $V_S = H^1(G_S)$ if $S \in \mathcal{S}_{int}$, and finally

$$a_{\varepsilon,S} : V_S \times V_S \to \mathbb{R} : (\psi, \phi) \to \int_{G_S} \varepsilon \operatorname{grad}_T \psi \cdot \operatorname{grad}_T \phi \, d\sigma.$$

The operator $L_{\varepsilon,S}$ is a nonnegative selfadjoint operator on $H_{\varepsilon,S}$ with a compact inverse. Let $0 \le \nu_1 \le \nu_2 \cdots$ be its eigenvalues repeated according to their multiplicity. We further denote by $\phi_j \in V_S$ the eigenfunction associated with ν_j. According to [19], we have

$$\Lambda_{\varepsilon,S} \setminus \mathbb{N} = \left\{ -\frac{1}{2} + \sqrt{\nu_j + \frac{1}{4}} \mid j \ge 1 \right\} \setminus \mathbb{N}$$

and $0 \notin \Lambda_{\varepsilon,S}$. For $\lambda \in \Lambda_{\varepsilon,S}$, we will denote by ϕ_λ the eigenfunction ϕ_j for which $\lambda = -\frac{1}{2} + \sqrt{\nu_j + \frac{1}{4}}$ (with the above convention ϕ_λ is uniquely defined). As in two dimensions, for $S \in \mathcal{S}$ the standard singularities of the operator Δ_ε at the vertex $S \in \mathcal{S}$ are given by (27).

As the edge e of Ω corresponds to a vertex S_e of Ω_e, the set $\Lambda_{\varepsilon,e}$ of edge singular exponents is given by

$$\Lambda_{\varepsilon,e} = \Lambda_{\varepsilon,S_e},$$

where $\Lambda_{\varepsilon,S_e}$ is the set of corner singularities defined in §2.3 (here at S_e in Ω_e). In other words, the edge singularities are induced by the corner singularities at S_e in Ω_e.

The goal of this subsection is to show the following density result:

Theorem 2.14. *Let $\mathcal{E}_0 \subset \mathcal{E}$ such that*

$$\{e \in \mathcal{E}_D \cup \mathcal{E}_{int} \mid \Lambda_{\varepsilon,e} \cap]0,1[\ne \emptyset\} \cup \{e \in \mathcal{E}_I \mid \Lambda_{\varepsilon,e} \cap]0,1/2[\ne \emptyset\} \subset \mathcal{E}_0. \tag{42}$$

Let $\mathcal{S}_0 \subset \mathcal{S}$ such that

$$\{S \in \mathcal{S}_D \cup \mathcal{S}_{int} \mid \Lambda_{\varepsilon,S} \cap]0,1/2[\ne \emptyset\} \subset \mathcal{S}_0 \tag{43}$$

and assume that $\mathcal{E}_0$ and $\mathcal{S}_0$ satisfy the compatibility condition of Definition 2.12. Assume further that

$$1/2 \notin \Lambda_{\varepsilon,S}, \forall S \in \mathcal{S} \text{ and } 1 \notin \Lambda_{\varepsilon,e}, \forall e \in \mathcal{E}. \tag{44}$$

Let $Y = L^2_\alpha(\Omega)$ where $\alpha \in [0,1[$ satisfies

$$\alpha > 1 - \min(\Lambda_{\varepsilon,e} \cap]0,1[) \ \forall e \in \mathcal{E}_0 \cap (\mathcal{E}_D \cup \mathcal{E}_{int}) \tag{45}$$
$$\alpha > 1 - \min(\Lambda_{\varepsilon,e} \cap]0,1/2[) \ \forall e \in \mathcal{E}_0 \cap \mathcal{E}_I \tag{46}$$
$$\alpha > 1/2 - \min(\Lambda_{\varepsilon,S} \cap]0,1/2[) \ \forall S \in \mathcal{S}_0 \cap (\mathcal{S}_D \cup \mathcal{S}_{int}) \tag{47}$$

Then the space $H[Y] \cap PH^2(\Omega; \mathcal{P})$ is dense in $H[Y]$.

The arguments of the proof of Theorem 2.14 are similar to those in [29] (Theorem 5.1). In a first step, we reduce the density problem from $H[Y]$ to that of the closed subspace

$$H_0[Y] = \left\{ \varphi \in H \;\middle|\; \varphi_{|F} \in H_0^1(F) \ \forall F \in \mathcal{F}_I \right\}$$

which makes sense if we define the linear form l involved in the definition of $H[Y]$ by

$$l(\varphi) = \sum_{e \in \mathcal{E}_I} \int_e \varphi(s) \, ds$$

(we recall that this definition is meaningful since the trace on a face $F \in \mathcal{F}_I$ of any element of $H[Y]$ belongs to $H^1(F) \hookrightarrow L^1(e)$). As in [29] (Proposition 5.3) we have the

Proposition 2.15. *If $H_0[Y] \cap PH^2(\Omega;\mathcal{P})$ is dense in $H_0[Y]$, then $H[Y] \cap PH^2(\Omega;\mathcal{P})$ is dense in $H[Y]$.*

As in two dimensions, the proof of Theorem 2.14 relies on a careful analysis of the dual singularities associated with the weighted space Y, that are defined by:

$$\mathcal{N}_\varepsilon[Y] = \left\{ g \in Y \,\middle|\, < g, \Delta_\varepsilon \varphi >_Y = 0 \ \forall \varphi \in H_0[Y] \cap PH^2(\Omega;\mathcal{P}) \right\}. \tag{48}$$

The standard dual singularities are given by

$$\mathcal{N}_\varepsilon = \left\{ g \in L^2(\Omega) \,\middle|\, (g, \Delta_\varepsilon \varphi) = 0 \ \forall \varphi \in \mathcal{D}(\Delta_\varepsilon^{\mathrm{Dir}}[L^2(\Omega)]) \cap PH^2(\Omega;\mathcal{P}) \right\}. \tag{49}$$

From [29] (Proposition 5.5 and Lemma 5.6) we recall the following characterization of the elements of $\mathcal{N}_\varepsilon$:

Proposition 2.16. *Let $g \in \mathcal{N}_\varepsilon$. Then g is solution to*

$$\Delta g = 0 \ \text{ in } \Omega_j \, \forall j, \tag{50}$$
$$g = 0 \ \text{in } \ \widetilde{H}^{-1/2}(F) \, \forall F \in \mathcal{F}_{ext}. \tag{51}$$
$$[g] = 0 \ \text{in } \ \widetilde{H}^{-1/2}(F) \, \forall F \in \mathcal{F}_{int}, \tag{52}$$
$$[\varepsilon \partial_n g] = 0 \ \text{ in } \widetilde{H}^{-3/2}(F) \, \forall F \in \mathcal{F}_{int}. \tag{53}$$

Moreover, g belongs to $\bigcup_j \mathcal{C}^\infty(\overline{\Omega_j} \setminus \mathcal{V})$ where $\mathcal{V}$ is any neighbourhood of the geometric singularities of $\bar{\Omega}$ (edges and corners of at least one $\bar{\Omega}_j$).

Proof of Theorem 2.14. Let $\mathcal{O}[Y] \subset H_0[Y]$ be the orthogonal space of $\overline{H_0[Y] \cap PH^2(\Omega;\mathcal{P})}$ and take $f \in \mathcal{O}[Y]$, i. e.

$$< \Delta_\varepsilon f, \Delta_\varepsilon \varphi >_Y + \sum_{F \in \mathcal{F}_I} (\mathrm{grad}_T\, f, \mathrm{grad}_T\, \varphi)_{0,F} = 0 \ \forall \varphi \in H_0[Y] \cap PH^2(\Omega;\mathcal{P}).$$

Now, let $g = \Delta_\varepsilon f$. Since $L^2(\Omega) \hookrightarrow Y$, we get

$$< g, \Delta_\varepsilon \varphi >_Y = 0 \ \forall \varphi \in \mathcal{D}(\Delta_\varepsilon^{\mathrm{Dir}}[L^2(\Omega)]) \cap PH^2(\Omega;\mathcal{P}).$$

As in Proposition 2.8, the function $g_\alpha = w^{2\alpha} g$ thus belongs to $\mathcal{N}_\varepsilon$ and satisfies

$$w^{-\alpha} g_\alpha \in L^2(\Omega).$$

Moreover, applying the arguments of the proof of Proposition 5.7 in [29], we show that

$$-\varepsilon \partial_n g_\alpha = \Delta_T f \in H^{-1}(F) \ \forall F \in \mathcal{F}_I.$$

According to Proposition 2.20 below, these supplementary regularity results guarantee that g_α belongs to $H^1(\Omega)$. On the other hand, g_α is a solution to the homogeneous problem $\Delta_\varepsilon g_\alpha = 0$ in Ω and $g_\alpha = 0$ on $\partial\Omega$ (see Proposition 2.16). This implies that g_α vanishes in Ω and so does $g = w^{-2\alpha} g_\alpha$.

Finally, $f \in H_0[Y]$ is the variational solution to the homogenous problem

$$\begin{array}{ll} \Delta_\varepsilon f = 0 & \text{in } \Omega, \\ f = 0 & \text{on } \Gamma_D, \\ \Delta_T f = 0 & \text{on } \Gamma_I. \end{array}$$

Hence, $f = 0$ in Ω which proves the density result. $\square$

According to the proof of Theorem 2.14, we shall consider in the sequel a function $g_\alpha \in \mathcal{N}_\varepsilon$ satisfying

$$w^{-\alpha} g_\alpha \in L^2(\Omega) \tag{54}$$

and

$$\varepsilon \partial_n g_\alpha \in H^{-1}(F)\ \forall F \in \mathcal{F}_I. \tag{55}$$

In order to describe its behaviour near an edge e, we introduce a cut-off function φ_e with respect to e which is given by

$$\varphi_e(r_e, \theta_e, z) = \psi(r_e)\chi(z) \tag{56}$$

with $\psi \in \mathrm{C}^\infty([0,\infty[)$, $\psi \equiv 1$ if $0 \le r \le r_0/3$, $\psi \equiv 0$ if $r \ge 2r_0/3$ and $\chi \in \mathcal{D}(]-h,h[)$, $\chi \equiv 1$ on $[-h/2, h/2]$. Thus $\varphi_e \equiv 1$ in the neigbourhood of an interior part of e, and φ_e vanishes near any other geometric singularity of Ω.

In Lemma 2.17 below we prove that the elements of $\mathcal{N}_\varepsilon$ coincide with those of the corresponding space on the infinite cone D_e modulo a function of class H^1. To this end, let us introduce the space

$$\mathcal{N}_\varepsilon(D_e) = \left\{ \varphi \in L^2(D_e) \,\middle|\, (\varphi, \Delta_\varepsilon \psi)_{D_e} = 0\, \forall \psi \in PH^2(D_e; \mathcal{P}_e) \cap (H_0[Y] \cap H^1_0(D_e)) \right\}.$$

For any function φ of $L^2(\Omega_e \times]-h,h[)$, $\tilde\varphi$ denotes its extension by zero on D_e. We now prove the

Lemma 2.17. *Let $g_\alpha \in \mathcal{N}_\varepsilon$ with $0 \le \alpha < 1$. Let φ_e be as in (56). There is a unique function $g^* \in H^1_0(D_e)$ such that*

$$g_{0,\alpha} := g^* - \widetilde{\varphi_e g} \in \mathcal{N}_\varepsilon(D_e). \tag{57}$$

Moreover, if g_α satisfies (54) and (55), then

$$d_e^{-\alpha} g_{0,\alpha} \in L^2(D_e) \text{ and} \tag{58}$$

$$\varepsilon \partial g_{0,\alpha} \in H^{-1}(F_{e,0}) \text{ if } e \in \mathcal{E}_I \tag{59}$$

where $d_e(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, e)$ denotes the distance function with respect to the edge e.

Proof. (57) and (59) have been proved in Lemma 5.8 in [29].

Now, suppose that $g_\alpha \in \mathcal{N}_\varepsilon$ satisfies in addition (54). We deduce from Hardy's inequality that g^*/d_e belongs to $L^2(D_e)$ since $g^* \in H^1_0(\Omega)$ and $d_e = r_e$ in D_e. Hence, $d_e^{-\alpha} g^* \in L^2(D_e)$ for all $\alpha \in [0,1[$ since $d_e^{-\alpha} < d_e^{-1}$ near e.

In order to prove that $d_e^{-\alpha} \widetilde{\varphi_e g} \in L^2(D_e)$, we notice that

$$d_e^{-\alpha} \widetilde{\varphi_e g_\alpha} \approx \varphi_e w^{-\alpha} g_\alpha \text{ on } \Omega_e \times]-h,h[$$

whereas $d_e^{-\alpha} \widetilde{\varphi_e g_\alpha} = 0$ anywhere else. We thus conclude with the help of condition (54). □

The main tool to investigate edge singularities is the partial Fourier transform in the edge variable z: for a given function $v \in L^2(D_e)$, we denote

$$\mathcal{F}v(x', \xi) = \hat v(x', \xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(x', z) e^{-i\xi z} dz.$$

Then we have the following

Lemma 2.18. *Let $g \in \mathcal{N}_\varepsilon(D_e)$. Then $\hat g(\cdot, \xi) \in \mathcal{N}_{\varepsilon, Dir, \xi}$ for almost every $\xi \in \mathbb{R}$ (here $\mathcal{N}_{\varepsilon, Dir, \xi}$, defined in §2.3, is based on Ω_e instead of Ω). If in addition, g satisfies (58) and (59), then*

$$r_e^{-\alpha} \hat g(\cdot, \xi) \in L^2(\Omega_e) \text{ and} \tag{60}$$

$$\varepsilon \partial_n \hat g \in H^{-1}(\Gamma_{e,0}) \text{ if } e \in \mathcal{E}_I \tag{61}$$

for almost every $\xi \in \mathbb{R}$.

Proof. The first part has been proved in Lemma 5.9 in [29]. We deduce from Lemma 2.17 that $d_e^{-\alpha} g \in L^2(D_e)$. (60) then follows since $d_e(\boldsymbol{x}) = r_e$ does not depend on z and thus

$$\widehat{d_e^{-\alpha} g}(\cdot, \xi) = r^{-\alpha} \hat{g}.$$

In the same way, we get (61) since the normal vector $\boldsymbol{n}$ is invariant in z. □

The following Proposition yields a condition on α in order to get H^1-regularity of g_α near the edges:

Proposition 2.19. *Let $0 \leq \alpha < 1$. Consider an edge $e \in \mathcal{E}$ and assume that $1 \neq \Lambda_{\varepsilon,e}$. Let $g_\alpha \in \mathcal{N}_\varepsilon$ satisfy (54) and (55). Assume that $\alpha > 1 - \min(\Lambda_{\varepsilon,e} \cap]0,1[)$ if $e \in (\mathcal{E}_D \cup \mathcal{E}_{int}) \cap \mathcal{E}_0$, and $\alpha > 1 - \min(\Lambda_{\varepsilon,e} \cap]0,1/2[)$ if $e \in \mathcal{E}_I \cap \mathcal{E}_0$. Then $\varphi_e g_\alpha \in H^1(\Omega)$ where φ_e is a cut-off function defined as in (56).*

Proof. Lemma 2.17 implies that there exists $g_{0,\alpha} \in \mathcal{N}_\varepsilon(D_e)$ satisfying (58) and (59) such that $g_{0,\alpha} + \widetilde{\varphi_e g_\alpha} \in H_0^1(D_e)$. Under the given assumptions on α, we then deduce from Lemma 2.18 and Section 2.3 that

$$\widehat{g_{0,\alpha}}(\cdot, \xi) = 0$$

for almost every $\xi \in \mathbb{R}$ which yields

$$g_{0,\alpha} = 0 \text{ on } D_e.$$

In other words,

$$\widetilde{\varphi_e g_\alpha} \in H_0^1(D_e)$$

which completes the proof. □

We are now able to prove the following global regularity result:

Proposition 2.20. *Let $0 \leq \alpha < 1$. Let $g_\alpha \in \mathcal{N}_\varepsilon$ satisfy (54) and (55). Assume further conditions (44), (45), (46) and (47) to be true. Then $g_\alpha \in H^1(\Omega)$.*

Proof. Under the given assumptions, we already know that g_α exhibits the H^1-regularity away from the corners. As the function g_α belongs to $\mathcal{N}_\varepsilon$ (and thus to $L^2(\Omega)$), one infers the following decomposition near a vertex $S \in \mathcal{S}$

$$g_\alpha(r_S, \sigma_S) = \sum_{l \in \mathbb{N}} g_l(r_S) \phi_l(\sigma_S),$$

where ϕ_l denotes the orthonormalized eigenfunction corresponding to the (nonnegative) eigenvalues ν_l of the Laplace-Beltrami operator $L_{e,S}$ on G_S (for the inner product $(\cdot,\cdot)_\varepsilon$). For $l \in \mathbb{N}$, the coefficient g_l is given by

$$g_l(r_S) = a_l r_S^{\lambda_l} + b_l r_S^{\mu_l}$$

where

$$\lambda_l = -\frac{1}{2} + \sqrt{\nu_l + \frac{1}{4}} \text{ and } \mu_l = -\frac{1}{2} - \sqrt{\nu_l + \frac{1}{4}}$$

(see [23] for details). Notice that $\lambda_l \geq 0$ and $\mu_l \leq -1$ since $\nu_l \geq 0$ for all $l \in \mathbb{N}$.

As the function g_α belongs to $L^2(\Omega)$, we notice that $b_l = 0$ for any $l \in \mathbb{N}$ such that $\mu_l \leq -3/2$.

Now, let $S \in \mathcal{S}_I$. Since $g_\alpha \in \mathcal{N}_\varepsilon$, is satisfies a Dirichlet boundary condition on $\partial\Omega$ (see Proposition 2.16). Hence, all eigenvalues ν_l are positive which implies $\mu_l \neq -1$. Moreover, there is at least one face $F \in \mathcal{F}_I$ such that S is a vertex of F. Taking into account that $\varepsilon \partial_n g_\alpha \in H^{-1}(F)$ thanks to (55), we deduce as in [29] that $b_l = 0$ for all $\mu_l \in]-3/2, -1[$. Therefore

$$g_\alpha(r_S, \sigma_S) = \sum_{l \in \mathbb{N}} a_l r_S^{\lambda_l} \phi_l(\sigma_S)$$

and g_α belongs to H^1 in the cone $\Gamma_S(R')$ with basis G_S and height R' for any $R' < R_S$.

Next, take a vertex $S \in \mathcal{S}_D$. Again, $\mu_l < -1$ thanks to the Dirichlet boundary condition. If $\Lambda_{\varepsilon,S} \cap]0,1/2[= \emptyset$, we have $\mu_l < -3/2$ and hence

$$g_\alpha(r_S,\sigma_S) = \sum_{l\in\mathbb{N}} a_l r_S^{\lambda_l}\phi_l(\sigma_S).$$

We thus conclude as before.

Now, let $S \in \mathcal{S}_0$ such that $\Lambda_{\varepsilon,S} \cap]0,1/2[\neq \emptyset$. Taking into account that g_α satisfies (54), we must have

$$a_l r_S^{-\alpha+\lambda_l}\phi_l(\sigma_S) \in L^2(\Omega) \tag{62}$$

and

$$b_l r_S^{-(\alpha+\lambda_l+1)}\phi_l(\sigma_S) \in L^2(\Omega) \tag{63}$$

for any λ_l where we used that $\lambda_l + \mu_l = -1$. (62) is always satisfied since $\lambda_l \geq 0$ and $0 \leq \alpha < 1$. Thanks to (47), $\alpha > \frac{1}{2} - \lambda_l$ for any $\lambda_l \in \Lambda_{\varepsilon,S} \cap]0,1/2[$. Hence, property (63) is satisfied if and only if $b_l = 0$ for all $l \in \mathbb{N}$ because $\alpha + \lambda_l + 1 > \frac{3}{2}$.

Again, we conclude that $g_\alpha \in H^1(\Gamma_S(R'))$ for any $R' < R$.

Finally, let $S \in \mathcal{S}_{int}$. Now, $\nu_1 = 0$ is an eigenvalue of the operator $L_{e,S}$ and $\phi_1 = c_\varepsilon$ denotes the associated (constant) eigenfunction. g_α thus splits as follows,

$$g_\alpha = g_1(r_S)\phi_1(\sigma_S) + \sum_{l\geq 2} g_l(r_S)\phi_l(\sigma_S). \tag{64}$$

But g_α belongs to $\mathcal{N}_\varepsilon$ and thus

$$\int_\Omega g_\alpha \Delta_\varepsilon \eta_S \, dx = 0,$$

where $\eta_S = \eta_S(r_S)$ is any regular cut-off function such that $\eta_S \equiv 1$ near the vertex S and $\eta_S \equiv 0$ near the other vertices. Indeed, such a function belongs to $\mathcal{D}(\Delta_\varepsilon^{\mathrm{Dir}}[L^2(\Omega)]) \cap PH^2(\Omega;\mathcal{P})$ and is admissible in the orthogonality relation that defines $\mathcal{N}_\varepsilon$ (see (49)). As in [24], we prove that

$$\int_\Omega g_\alpha \Delta_\varepsilon \eta_S \, dx = c_\varepsilon \int_0^\infty g_1(r_S)(\eta_S''(r_S) + \frac{2}{r_S}\eta_S'(r_S))r_S^2 \, dr$$

since

$$\int_{G_S} \varepsilon \phi_l(\sigma_S)\, d\sigma = 0$$

for all $l \geq 2$. It follows that the integral of the second term in (64) vanishes. But $g_1(r_S) = a_1 + b_1 r_S^{-1}$ and an elementary calculation shows that

$$\int_0^\infty g_1(r_S)(\eta_S''(r_S) + \frac{2}{r_S}\eta_S'(r_s))r_S^2 \, dr = -b_1$$

which yields $b_1 = 0$. We then conclude as in the case $S \in \mathcal{S}_D$ that $g_\alpha \in H^1(\Gamma_S(R'))$ for any $R' < R$.

□

3. Discretization and convergence

In this section, we describe the discretization of problem $(\mathcal{P}[Y])$ by means of *conforming* nodal finite elements of order k, and we prove convergence of the numerical method.

3.1. Discretization

Consider a family of simplicial meshes $(\mathcal{T}_h)_h$ of Ω, with $h = \max_{T_l \in \mathcal{T}_h} h_l$, which is compatible with the partition $\mathcal{P}$ (in the sense that all simplices lie in exactly one Ω_j, $j = 1, \ldots, J$). With $\mathcal{T}_h$, we associate the space of vector finite elements

$$\boldsymbol{X}_h = \left\{ \boldsymbol{v}_h \in P\boldsymbol{H}^1(\Omega; \mathcal{P}) \;\middle|\; \boldsymbol{v}_{h|T_l} \in \mathbb{P}_k(T_l)^d, \; \forall T_l \in \mathcal{T}_h \right\} \tag{65}$$

where $d = 2$ or $d = 3$. Let $\{M_I\}_{1,\ldots,\mathsf{nbn}}$ be the set of nodes of the mesh $\mathcal{T}_h$. The discretization space $\boldsymbol{V}_h$ is the subspace of $\boldsymbol{X}_h$ defined by $\boldsymbol{V}_h = \boldsymbol{X}_h \cap \boldsymbol{W}[Y] \subset P\boldsymbol{H}^1(\Omega; \mathcal{P}) \cap \boldsymbol{W}[Y]$, that is

$$\boldsymbol{V}_h = \{ \boldsymbol{v}_h \in \boldsymbol{X}_h \mid \text{"}[\boldsymbol{v}_h \times \boldsymbol{n}](M_I) = 0\text{"};\ \text{"}[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}](M_I) = 0\text{"}\ \forall M_I \in \mathcal{F}_{int}\ \text{"}(\boldsymbol{v}_h \times \boldsymbol{n})(M_I) = 0\text{"}\ \forall M_I \in \mathcal{F}_D \}\,. \tag{66}$$

This discretization is conforming in the sense that $\boldsymbol{V}_h$ is a subspace of the vector space involved in the variational formulation of the continuous problem $\mathcal{P}[Y]$. The elements of $\boldsymbol{V}_h$ are continuous on each subdomain Ω_j and satisfy the transmission (resp. boundary conditions) pointwise on the interfaces $F \subset \mathcal{F}_{int}$ (resp. boundary faces $F \subset \mathcal{F}_D$) since the restriction of Lagrange Finite Elements to the element faces is unisolvent.

Note that the discrete transmission (resp. boundary conditions) "$[\boldsymbol{v}_h \times \boldsymbol{n}](M_I) = 0$", "$[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}](M_I) = 0$" (resp. "$(\boldsymbol{v}_h \times \boldsymbol{n})(M_I) = 0$") are ambiguous on the set of vertices $\mathcal{S}$ of the domain Ω (and also on the set of edges $\mathcal{E}$ if $d = 3$) and will be specified hereafter for a two-dimensional problem. In three dimensions, the ideas are the same, but the implementation is more technical (see for instance [6]). For simplicity, we also assume that $\Gamma_I = \emptyset$, i. e. $\mathcal{F}_D = \mathcal{F}_{ext}$.

We start our investigation with boundary nodes belonging to a single subdomain. For each boundary node situated at the interior of a boundary face, we apply a rotation in $\mathbb{R}^2$ which maps the canonical basis $(\vec{e}_x, \vec{e}_y)$ on a local basis of the normal and tangential vectors. In the latter basis the vector boundary condition becomes decoupled and standard elimination techniques apply. Next, let the boundary node be the vertex of a single subdomain Ω_j. The two boundary faces that form the vertex have linearly independent normal vectors and it follows from the continuity of the fields of $\boldsymbol{X}_h$ in $\bar{\Omega}_j$ that two linearly independent vanishing boundary conditions have to be imposed at the vertex. The zero value of any field $\boldsymbol{u}_h \in \boldsymbol{V}_h$ at this vertex is thus completely determined by the boundary conditions.

Next, we describe how the transmission conditions are taken into account at the interfaces. The first step is a replication of the degrees of freedom according to the number of subdomains the associated node does belong to. To fix ideas, let $M_I \subset \overset{\circ}{F}_{e,e'}$ be an interior node of the interface $F_{e,e'} = \bar{\Omega}_e \cap \bar{\Omega}_{e'} \in \mathcal{F}_{int}$. M_I belongs to subdomains $\bar{\Omega}_e$ and $\bar{\Omega}_{e'}$ and the associated degrees of freedom will thus be doubled. Let $\vec{U}_I^e = U_{I,x}^e \vec{e}_x + U_{I,y}^e \vec{e}_y$ (resp. $\vec{U}_I^{e'} = U_{I,x}^{e'} \vec{e}_x + U_{I,y}^{e'} \vec{e}_y$) be the (vectorial) unknown associated to M_I on Ω_e (resp. $\Omega_{e'}$). The transmission conditions at M_I read

$$\varepsilon_e \vec{U}_I^e \cdot \vec{n}_{ee'} = \varepsilon_{e'} \vec{U}_I^{e'} \cdot \vec{n}_{ee'} \quad \vec{n}_{ee'} \times \vec{U}_I^e = \vec{n}_{ee'} \times \vec{U}_I^{e'} \tag{67}$$

where $\vec{n}_{ee'} = n_{ee',x} \vec{e}_x + n_{ee',y} \vec{e}_y$ is the unit normal vector on $F_{e,e'}$. (67) can be written in matrix form

$$\mathbb{D}_e \mathbb{R}_{e,e'}^t \vec{U}_I^e = \mathbb{D}_{e'} \mathbb{R}_{e,e'}^t \vec{U}_I^{e'} \tag{68}$$

with

$$\mathbb{D}_e = \begin{pmatrix} \varepsilon_e & 0 \\ 0 & 1 \end{pmatrix},\ \mathbb{D}_{e'} = \begin{pmatrix} \varepsilon_{e'} & 0 \\ 0 & 1 \end{pmatrix},\ \text{and } \mathbb{R}_{e,e'} = \begin{pmatrix} n_{ee',x} & -n_{ee',y} \\ n_{ee',y} & n_{ee',x} \end{pmatrix},$$

three elements of $\mathcal{M}_2(\mathbb{R})$. Hence, $\vec{U}_I^{e'}$ can be eliminated in terms of $\vec{U}_I^e$. The matrix $\mathbb{R}_{ee'}$ performs the transformation of the canonical basis into the local basis of the normal and tangential vectors.

The situation is more involved if M_I coincides with a vertex $S \in \mathcal{S}$. Let $m \in \mathbb{N}$ denote the number of subdomains containing M_I as a vertex. If $M_I \in \mathcal{S}_{ext}$, then there are $m - 1$ interfaces $F_{e,e+1}$ having M_I as an endpoint. Thus, $\boldsymbol{u}_h$ has to satisfy $2(m-1)$ transmission conditions at M_I. For $e = 1, \cdots, m$, let $\vec{U}_I^e = U_{I,x}^e \vec{e}_x + U_{I,y}^e \vec{e}_y$ denote the degrees of freedom associated with M_I on the subdomain $\bar{\Omega}_e$. Applying successively the formula (68), $\vec{U}_I^e$ can be eliminated in terms of $\vec{U}_I^1$ for all $e = 2, \cdots, m$ and we have

$$\vec{U}_I^m = \mathbb{T}\vec{U}_I^1, \text{ where } \mathbb{T} = \prod_{e=1}^{m-1} \mathbb{R}_{e,e+1}\mathbb{D}_{e+1}^{-1}\mathbb{D}_e\mathbb{R}_{e,e+1}^t. \tag{69}$$

But both Ω_1 and Ω_m have a boundary face with M_I as its endpoint. Let Γ_1 (resp. Γ_m) denote this boundary face and $\vec{n}_1$ (resp. $\vec{n}_m$) its outer normal unit vector. Then the boundary conditions read

$$\vec{n}_1 \times \vec{U}_I^1 = 0 \text{ and } \vec{n}_m \times \vec{U}_I^m = 0 \tag{70}$$

or, taking into account (69),

$$\vec{n}_1 \times \vec{U}_I^1 = 0 \text{ and } \vec{n}_m \times \mathbb{T}\vec{U}_I^1 = 0. \tag{71}$$

Above, (71) is a linear system in the unknowns $\vec{U}_I^1 = U_{I,x}^1\vec{e}_x + U_{I,y}^1\vec{e}_y$ which admits a non trivial solution if and only if its matrix is singular. In this case[2], the two boundary conditions in (70) are in fact the same, and we can apply the same techniques as for boundary nodes belonging to a single subdomain. Otherwise, the values of $\boldsymbol{u}_h$ at M_I are entirely determined by the boundary condition, i. e. $\vec{U}_I^e = 0$ for all $e = 1, \cdots, m$, and no degree of freedom is associated with the node M_I.

A similar situation occurs if M_I coincides with an interior vertex. Assume again that M_I does belong to m subdomains. Since $M_I \in \mathcal{S}_{int}$, there are now m interfaces $F_{e,e+1}$ having M_I as an endpoint with the convention that $F_{m,m+1} = F_{m,1}$. The unknowns $(\vec{U}_I^e)_{e=1}^m$ satisfy the block linear system

$$\begin{pmatrix} \mathbb{M}_{1,1} & -\mathbb{M}_{1,2} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbb{M}_{2,2} & -\mathbb{M}_{2,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbb{M}_{m-1,m-1} & -\mathbb{M}_{m-1,m} \\ -\mathbb{M}_{m,1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbb{M}_{m,m} \end{pmatrix} \begin{pmatrix} \vec{U}_I^1 \\ \vec{U}_I^2 \\ \vdots \\ \vec{U}_I^{m-1} \\ \vec{U}_I^m \end{pmatrix} = 0 \tag{72}$$

where $\mathbb{M}_{e,e} = \mathbb{D}_e\mathbb{R}_{e,e+1}^t$ and $\mathbb{M}_{e,e+1} = \mathbb{D}_{e+1}\mathbb{R}_{e,e+1}^t$. Let $\mathbb{M}_{\text{int}} \in \mathcal{M}_{2m}(\mathbb{R})$ be the matrix in (72). Again, this system admits a non trivial solution if and only if its matrix $\mathbb{M}_{\text{int}}$ is singular. In this case, it may easily be seen that $\mathbb{M}_{\text{int}}$ is of rank $2(m-1)$ and there are thus 2 degrees of freedom associated with the node M_I. Otherwise, the values of $\boldsymbol{u}_h$ at M_I are entirely determined by the transmission conditions and we have necessarily $\vec{U}_I^e = 0$ for any $e \in \{1, \ldots, m\}$.

The following three examples illustrate the different situations that may occur. In the first example (see Figure 2, left), M_I is a boundary node belonging to two subdomains $\bar{\Omega}_1$ and $\bar{\Omega}_2$. The normal vector on the interface $F_{1,2}$ is given by $\vec{n}_{12} = -\vec{e}_x$ and the matrix $\mathbb{R}_{1,2}$ thus reads

$$\mathbb{R}_{1,2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have $\vec{U}_I^2 = \mathbb{T}\vec{U}_I^1$ with

$$\mathbb{T} = \mathbb{R}_{1,2}\mathbb{D}_2^{-1}\mathbb{D}_1\mathbb{R}_{1,2}^t = \begin{pmatrix} \frac{\varepsilon_1}{\varepsilon_2} & 0 \\ 0 & 1 \end{pmatrix}.$$

The outer normal vectors on Γ_1 and Γ_2 are given by $\vec{n}_1 = \vec{n}_2 = \vec{e}_y$ and the linear system (70) thus reads as follows in this first case

$$\begin{pmatrix} -1 & 0 \\ -\frac{\varepsilon_1}{\varepsilon_2} & 0 \end{pmatrix} \vec{U}_I^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

[2]As can be expected by looking directly at the boundary conditions when $\vec{n}_1 = \vec{n}_m$.

The degree of freedom associated with the node M_I is thus $U^1_{I,y}$, while the others, prescribed by the transmission and boundary conditions, vanish.

In the second example (Figure 2, middle), M_I is a boundary node that belongs to three subdomains $\bar\Omega_1$, $\bar\Omega_2$ and $\bar\Omega_3$. Eliminating $\vec{U}^3_I$ in terms of $\vec{U}^1_I$ yields

$$\vec{U}^3_I = \begin{pmatrix} \frac{\varepsilon_1}{\varepsilon_2} & 0 \\ 0 & \frac{\varepsilon_2}{\varepsilon_3} \end{pmatrix} \vec{U}^1_I .$$

The outer normal vectors on Γ_1 and Γ_3 are respectively given by $\vec{n}_1 = -\vec{e}_y$ and $\vec{n}_3 = \vec{e}_x$. Hence, the linear system (70) reads

$$-U^1_{I,x} = 0 \text{ and } \frac{\varepsilon_2}{\varepsilon_3} U^2_{I,y} = 0$$

which implies in turn $\vec{U}^e_I = 0$ for all $e \in \{1, \dots, 3\}$. No degree of freedom is associated with M_I.

Finally, the last example deals with an interior vertex (Figure 2, right). M_I belongs to four subdomains and the matrix $\mathbb{M}_{\text{int}} \in \mathcal{M}_8(\mathbb{R})$ of the linear system (72) is now given by

$$\mathbb{M}_{\text{int}} = \left(\begin{array}{cc|cc|cc|cc} -\varepsilon_1 & 0 & \varepsilon_2 & 0 & & & & \\ 0 & 1 & 0 & -1 & \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline & & 0 & \varepsilon_2 & 0 & -\varepsilon_3 & & \\ \multicolumn{2}{c|}{\mathbf{0}} & -1 & 0 & 1 & 0 & \multicolumn{2}{c}{\mathbf{0}} \\ \hline & & & & \varepsilon_3 & 0 & -\varepsilon_4 & 0 \\ \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c|}{\mathbf{0}} & 0 & 1 & 0 & -1 \\ \hline 0 & \varepsilon_1 & & & & & 0 & -\varepsilon_4 \\ -1 & 0 & \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c|}{\mathbf{0}} & 1 & 0 \end{array} \right).$$

An elementary calculus yields $\det(\mathbb{M}_{\text{int}}) = (\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4)^2$. Hence $\mathbb{M}_{\text{int}}$ is singular if and only if $\varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_4$. In all the other cases, no degree of freedom is associated with M_I and $\vec{U}^e_I = 0$ for all $e \in \{1, \dots, 4\}$.

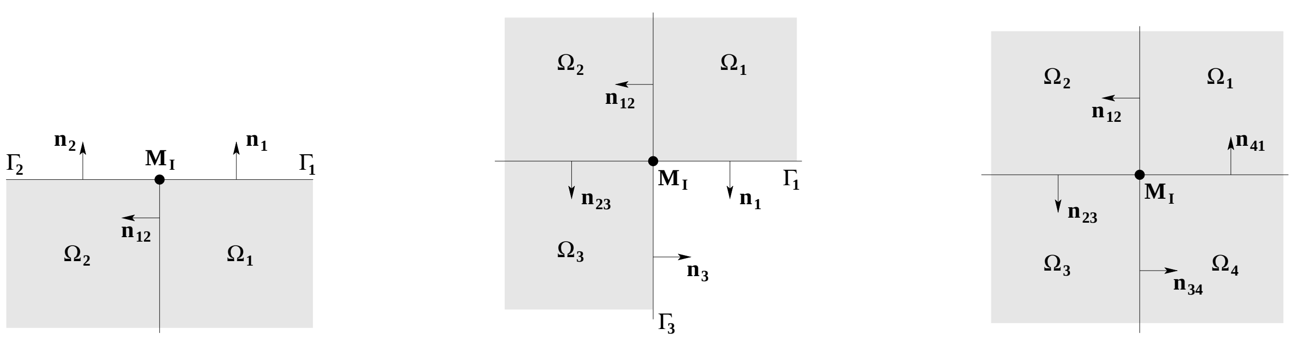

FIGURE 2. Boundary and transmission conditions at vertices.

At first glance, it may seem surprising to constrain the fields of $\boldsymbol{V}_h$ to vanish at a vertex $S \in \mathcal{S}$ at which the exact solution field presents an unbounded singularity. The density result however shows that in presence of an appropriate weight function, the fields in $\boldsymbol{V}_h$ are able to recover the singular behavior in the energy norm. Notice however that no pointwise convergence can be obtained.

3.2. **Convergence**

To fix ideas, asssume that V_h is given by Lagrange finite elements of order k. The discrete problem is given on the space $\boldsymbol{V}_h$ by

$$(\mathcal{P}_h[Y]) \qquad \begin{cases} \text{Find } \boldsymbol{u}_h \in \boldsymbol{V}_h \text{ such that} \\ a(\boldsymbol{u}_h, \boldsymbol{v}_h) - \omega^2(\varepsilon \boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{cases}$$

One can prove by a classical contradiction argument (the proof is omitted here), that there exists $h_\omega > 0$ such that, for all $h < h_\omega$, the discrete problem $(\mathcal{P}_h[Y])$ has one, and only one, solution $\boldsymbol{u}_h$.

The following theorem yields the convergence of the nodal finite element method.

Theorem 3.1. *Let $Y = L^2_\alpha(\Omega)$ where $\alpha \in [0,1[$ satisfies conditions (45), (46) and (47). Assume the condition of Theorem 2.1 to be true and let $\omega^2 \in \mathbb{R}^+ \setminus \sigma(\mathbf{curl}, \operatorname{div} \varepsilon^0)$. Let $\boldsymbol{u}$ be the solution to $(\mathcal{P}[Y])$ with $\boldsymbol{f} \in L^2(\Omega)$. Consider a family of meshes $(\mathcal{T}_h)_h$. Let $\boldsymbol{u}_h$ be the solution to the discrete problem $(\mathcal{P}_h[Y])$ where the discretization space is defined by (66). Then, there exists $h_0 > 0$ and $C(\omega) > 0$ such that*

$$||\boldsymbol{u} - \boldsymbol{u}_h||_{\boldsymbol{W}[Y]} \le C(\omega) \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} ||\boldsymbol{u} - \boldsymbol{v}_h||_{\boldsymbol{W}[Y]}, \ \forall h < h_0. \tag{73}$$

It follows that

$$\lim_{h \to 0} ||\boldsymbol{u} - \boldsymbol{u}_h||_{\boldsymbol{W}[Y]} = 0. \tag{74}$$

Finally, if $\boldsymbol{u} \in \boldsymbol{W}[Y] \cap P\boldsymbol{H}^s(\Omega; \mathcal{P})$ with $s > 1$, one has

$$||\boldsymbol{u} - \boldsymbol{u}_h||_{\boldsymbol{W}[Y]} \le C h^{\min(k, s-1)}, \ \forall h \le h_0. \tag{75}$$

Proof. Let us prove (73) first, with the help of a variant of Céa's Lemma. Indeed, the orthogonality relation between problems $(\mathcal{P}[Y])$ and $(\mathcal{P}_h[Y])$ reads

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) - \omega^2(\boldsymbol{u} - \boldsymbol{u}_h, \varepsilon(\boldsymbol{u}_h - \boldsymbol{v}_h)) = 0 \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \tag{76}$$

and thanks to the coercivity of $a(\cdot,\cdot)$ on $\boldsymbol{W}[Y]$, there are constants $c > 0$ and $C(\omega) > 0$ such that

$$c\,||\boldsymbol{u} - \boldsymbol{u}_h||^2_{\boldsymbol{W}[Y]} - \omega^2 \|\varepsilon^{1/2}(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_{L^2(\Omega)^d} \le C(\omega)\,||\boldsymbol{u} - \boldsymbol{u}_h||_{\boldsymbol{W}[Y]}\,||\boldsymbol{u} - \boldsymbol{v}_h||_{\boldsymbol{W}[Y]} \tag{77}$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$. Now, consider the sequence

$$\boldsymbol{w}_h = \frac{\boldsymbol{u} - \boldsymbol{u}_h}{||\boldsymbol{u} - \boldsymbol{u}_h||_{\boldsymbol{W}[Y]}}.$$

Since $||\boldsymbol{w}_h||_{\boldsymbol{W}[Y]}$ is bounded, there is a sub-sequence $(w_{h'})$ which converges weakly in $\boldsymbol{W}[Y]$ to an element $\boldsymbol{w} \in \boldsymbol{W}[Y]$. Now, let $\boldsymbol{v} \in \boldsymbol{W}[Y]$. Thanks to the density result of Theorem 2.5, there is a sequence $(\boldsymbol{v}_h)$, with $\boldsymbol{v}_h \in \boldsymbol{V}_h$, that converges strongly in $\boldsymbol{W}[Y]$ to $\boldsymbol{v}$. From (76), we get $a(\boldsymbol{w}_{h'}, \boldsymbol{v}_{h'}) - \omega^2(\varepsilon \boldsymbol{w}_{h'}, \boldsymbol{v}_{h'}) = 0$. Thus,

$$a(\boldsymbol{w}, \boldsymbol{v}) - \omega^2(\varepsilon \boldsymbol{w}, \boldsymbol{v}) = a(\boldsymbol{w} - \boldsymbol{w}_{h'}, \boldsymbol{v}) - \omega^2(\varepsilon(\boldsymbol{w} - \boldsymbol{w}_{h'}), \boldsymbol{v}) + a(\boldsymbol{w}_{h'}, \boldsymbol{v} - \boldsymbol{v}_{h'}) - \omega^2(\varepsilon \boldsymbol{w}_{h'}, \boldsymbol{v} - \boldsymbol{v}_{h'}).$$

The right hand side tends to 0 if $h' \to 0$ due to the weak convergence of $(\boldsymbol{w}_{h'})$ to $\boldsymbol{w}$ and the strong convergence of $\boldsymbol{v}_{h'}$ to $\boldsymbol{v}$. Hence,

$$a(\boldsymbol{w}, \boldsymbol{v}) - \omega^2(\varepsilon \boldsymbol{w}, \boldsymbol{v}) = 0 \ \forall \boldsymbol{v} \in \boldsymbol{W}[Y].$$

Since $\omega \notin \sigma(\mathbf{curl}, \operatorname{div} \varepsilon^0)$, problem $(\mathcal{P}[Y])$ has a unique solution. Thus, $\boldsymbol{w} = 0$, and the whole sequence $(\boldsymbol{w}_h)$ converges weakly to 0 in $\boldsymbol{W}[Y]$. We conclude from the compact imbedding of $\boldsymbol{W}[Y]$ into $L^2(\Omega)^d$ that $(\boldsymbol{w}_h)$ converges strongly to 0 in $L^2(\Omega)^d$.

Thus, there is $h_0 > 0$ such that

$$\|\varepsilon^{1/2}(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_{L^2(\Omega)^d} \le \frac{c}{2\omega^2}\,||\boldsymbol{u} - \boldsymbol{u}_h||^2_{\boldsymbol{W}[Y]} \ \forall h < h_0.$$

Consequently, we deduce from (77) that

$$||\boldsymbol{u}-\boldsymbol{u}_h||_{\boldsymbol{W}[Y]} \leq \frac{2C(\omega)}{c}\,||\boldsymbol{u}-\boldsymbol{v}_h||_{\boldsymbol{W}[Y]}\ \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\ \forall h < h_0,$$

which yields (73).

In [19] (Theorem 2.1), the density of $\boldsymbol{W}[Y] \cap P\boldsymbol{H}^s(\Omega;\mathcal{P})$, $s > 1$, in $\boldsymbol{W}[Y] \cap P\boldsymbol{H}^1(\Omega;\mathcal{P})$ has been proven in the case $\Gamma_I = \emptyset$. The generalization of this result to the case $\Gamma_I \neq \emptyset$ is straightforward. Under the given assumptions on α, $\boldsymbol{W}[Y] \cap P\boldsymbol{H}^s(\Omega;\mathcal{P})$ is thus dense in $\boldsymbol{W}[Y]$ for any $s > 1$.

Now, let $s > 1$ be given, and let $\eta > 0$. There is $\boldsymbol{u}_R \in \boldsymbol{W}[Y] \cap P\boldsymbol{H}^s(\Omega;\mathcal{P})$ such that

$$||\boldsymbol{u}-\boldsymbol{u}_R||_{\boldsymbol{W}[Y]} \leq \eta.$$

On the other hand, if Π_h denotes the standard *piecewise* (with respect to the partition $\mathcal{P}$) interpolation operator for Lagrange finite elements, then $\Pi_h \boldsymbol{u}_R \in \boldsymbol{V}_h$. Indeed, $\boldsymbol{u}_R$ satisfies the transmission (resp. boundary) conditions on each node located on an interface (resp. boundary face), and so does $\Pi_h \boldsymbol{u}_R$. Since the restriction of standard Lagrange finite elements to the element faces is unisolvent, $\Pi_h \boldsymbol{u}_R$ satisfies the transmission (resp. boundary) conditions on any interface (resp. boundary face). Standard error analysis for Lagrange finite elements of type P_k yields the following estimation in the $PH^1(\Omega;\mathcal{P})$-norm:

$$||\boldsymbol{u}_R - \Pi_h \boldsymbol{u}_R||_{PH^1(\Omega;\mathcal{P})} \leq C(\boldsymbol{u}_R) h^{\min(k,s-1)} \tag{78}$$

where the constant $C(\boldsymbol{u}_R)$ does depend on $\boldsymbol{u}_R$, but is independent from the mesh size h.

We finally deduce from (73) and (78) that there is $h_0 > 0$ (depending on η) such that

$$||\boldsymbol{u}-\boldsymbol{u}_h||_{\boldsymbol{W}[Y]} \leq C\eta\ \forall h \leq h_0.$$

This proves (74.
The last estimate (75) follows by standard error analysis (replace $\boldsymbol{u}_R$ by $\boldsymbol{u}$ above). □

4. Numerical results

In this section, we provide numerical illustrations for the application of the weighted regularization method in two-dimensional polygons. We further restrict ourselves to the case $\Gamma_I = \emptyset$. According to Theorem 2.10, the space Y is realized as a weighted L^2-space. Thus, the variational space $\boldsymbol{W}_\alpha = \boldsymbol{W}[L^2_\alpha(\Omega)]$ is defined as

$$\boldsymbol{W}_\alpha = \left\{ \boldsymbol{u} \in \mathcal{H}(\mathbf{curl};\Omega) \,\middle|\, \operatorname{div}(\varepsilon \boldsymbol{u}) \in L^2_\alpha(\Omega);\ (\boldsymbol{u}\times\boldsymbol{n})_{|\partial\Omega} = 0 \right\}, \tag{79}$$

with *ad hoc* values of α (see Theorems 2.10 and 2.14). It is equipped with the semi-norm

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_\alpha} = \left(||\mathbf{curl}\,\boldsymbol{u}||^2_{0,\Omega} + \|\operatorname{div}(\varepsilon\boldsymbol{u})\|^2_{L^2_\alpha(\Omega)} \right)^{1/2},$$

which is equivalent to the full norm, thanks to the compact imbedding of $\boldsymbol{W}_\alpha$ into $L^2(\Omega)^2$.

Finally, we slightly modify the definition of the sesquilinear form $a(\cdot,\cdot)$ in order to get a better conditioning of the linear system. Actually, we take

$$a_\beta(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\overline{\mathbf{curl}\,\boldsymbol{v}}\,dx + \beta \sum_{j=1}^{J} \varepsilon_j^{-2} \int_{\Omega_j} w^{2\alpha} \operatorname{div}\varepsilon\boldsymbol{u}\,\overline{\operatorname{div}\varepsilon\boldsymbol{v}}\,dx \tag{80}$$

with $\beta > 0$.

4.1. **Source problem**

In this subsection, we provide numerical tests for the computation of the solution to problem ($\mathcal{P}[Y]$) on a two-dimensional L-shaped domain,

$$\Omega =]-1,1[^2 \setminus ([0,1] \times [-1,0]) .$$

We consider the static case where $\omega = 0$. The computational domain is split into three sub-domains according to Figure 3. Notice that the only singular vertex is located at $(0,0)$. Indeed, no singular behavior does occur near the other vertices of $\partial\Omega$ since they correspond to a convex opening angle in a homogeneous medium and the solution to ($\mathcal{P}[Y]$) is thus of class H^1 in a neighborhood of these vertices. The situation is similar near $(-1,0)$ and $(0,1)$. Indeed, the interfaces are orthogonal to the boundary, and classical extension techniques allow us to prove that any scalar potential has piecewise H^2-regularity near these vertices. We thus deduce from Theorem 2.3 that the solution to problem ($\mathcal{P}[Y]$) is piecewise H^1 near $(-1,0)$ and $(0,1)$.

We define the weight function w by

$$w(\boldsymbol{x}) = \min(r,1)$$

where (r,θ) are the polar coordinates with respect to the origin.

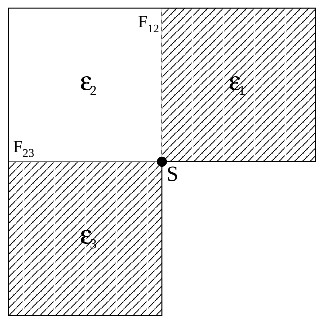

FIGURE 3. L-shaped domain with 3 subdomains.

The electromagnetic coefficients are

$$\mu_j = 1 \forall j = 1:3;\ \varepsilon_2 = 1,\ \varepsilon_1 = \varepsilon_3 = \varepsilon > 0.$$

We are then able to construct a family of vector fields that belong to the space

$$\left\{ \boldsymbol{u} \in \mathcal{H}(\mathbf{curl};\Omega) \,\middle|\, \operatorname{div}(\varepsilon \boldsymbol{u}) \in L^2_\alpha(\Omega) \right\}.$$

To this end, we define the scalar potential

$$S_\lambda(r,\theta) = r^\lambda \phi(\theta)$$

where $\lambda > 0$ is solution to the non-linear equation

$$\tan\frac{\lambda\pi}{4}\tan\frac{\lambda\pi}{2} = \varepsilon \tag{81}$$

and $\phi = (\phi_j)_{j=1:3}$ is given by

$$\begin{aligned}
&\phi_1(\theta) = \sin(\lambda\theta) && \text{if } 0 \le \theta < \tfrac{\pi}{2} \\
&\phi_2(\theta) = \eta\cos(\lambda(\theta - \tfrac{3\pi}{4})) && \text{if } \tfrac{\pi}{2} \le \theta < \pi, \qquad \eta = \frac{\sin\frac{\lambda\pi}{2}}{\cos\frac{\lambda\pi}{4}} \\
&\phi_3(\theta) = \sin(\lambda(\tfrac{3\pi}{2} - \theta)) && \text{if } \pi \le \theta \le \tfrac{3\pi}{2}.
\end{aligned}$$

Notice that ϕ satisfies equations (24)–(26) and thus λ is a singular exponent with respect to the vertex $S = (0, 0)$.

Now, let

$$\boldsymbol{E}_\lambda = \operatorname{grad} S_\lambda.$$

We have

$$\mathbf{curl}\, \boldsymbol{E}_\lambda = 0, \text{ and } \operatorname{div}(\varepsilon \boldsymbol{E}_\lambda) = 0 \text{ in } \Omega$$

Further, $\boldsymbol{E}_\lambda$ has a vanishing tangential trace on those segments that form the reentrant corner at $S = (0, 0)$, i.e.

$$\boldsymbol{E}_\lambda \times \boldsymbol{n} = 0 \text{ for } \theta = 0 \text{ and } \theta = 3\pi/2.$$

Notice however, that $\boldsymbol{E}_\lambda$ does not satisfy the perfect conductor condition on the whole boundary $\partial\Omega$. We thus have to deal with a non-homogeneous boundary condition. Numerically, this is achieved by a transformation into local coordinates and a technique of pseudo-elimination involving a discrete lifting of $\boldsymbol{E}_\lambda \times \boldsymbol{n}$ on each edge of the boundary which vanishes on the interior nodes of the mesh. Notice that such a lifting determines completely the solution field on the vertices of $\partial\Omega$ since two linearly independent components have to be fixed. We get the following regularity result for $\boldsymbol{E}_\lambda$,

$$\boldsymbol{E}_\lambda \in P\boldsymbol{H}^s(\Omega; \mathcal{P}) \; \forall s < \lambda.$$

However, if $\lambda_0 > 0$ is solution to (81), so is $\lambda_k = \lambda_0 + 4k$ for $k \in \mathbb{N}$. We thus get a family of vector fields that become more and more regular as k increases. It is clear that the smallest positive value λ_0, solution to (81) depends on the choice of the parameter ε. More precisely, if ε tends to zero, so does λ_0. Thus, the smaller is ε, the stronger is the singularity at $S = (0, 0)$ of the corresponding vector field $\boldsymbol{E}_{\lambda_0}$.

Now, we choose the right hand side $\boldsymbol{f}$ in such a way that $\boldsymbol{E}_\lambda$ is the exact solution to the problem. Since $\mathbf{curl}\, \boldsymbol{E}_\lambda = 0$, $\operatorname{div}(\varepsilon \boldsymbol{E}_\lambda) = 0$ and $\omega = 0$, this actually means that $\boldsymbol{f} = 0$.

We present the error $\boldsymbol{E}_\lambda - \boldsymbol{E}^h$ in the semi-norm

$$e_a = a_1(\boldsymbol{E}_\lambda - \boldsymbol{E}^h, \boldsymbol{E}_\lambda - \boldsymbol{E}^h) = \left(\left\| \mathbf{curl}(\boldsymbol{E}_\lambda - \boldsymbol{E}^h) \right\|_{0,\Omega}^2 + \sum_{j=1}^{3} \varepsilon_j^{-2} \| \operatorname{div}(\boldsymbol{E}_\lambda - \boldsymbol{E}^h) \|_{L^2_\alpha(\Omega_j)}^2 \right)^{1/2}$$

as well as in the L^2-norm

$$e_2 = \|\boldsymbol{E}_\lambda - \boldsymbol{E}^h\|_{L^2(\Omega)^2}.$$

For both norms, we give the numerical convergence rate

$$\tau_\ell = \frac{\log(e(h_{\ell-1})/e(h_\ell))}{\log(h_{\ell-1}/h_\ell)}$$

of two successive simulations corresponding to mesh parameters $h_{\ell-1}$ and h_ℓ respectively. Notice, that e_a may be computed exactly since $\mathbf{curl}\, \boldsymbol{E}_\lambda = 0$ and $\operatorname{div}(\varepsilon \boldsymbol{E}_\lambda) = 0$. Hence,

$$e_a^2 = (\boldsymbol{E}^h)^t \mathbb{A} \boldsymbol{E}^h$$

where $\mathbb{A}$ is the stiffness matrix corresponding to the sesquilinear form $a_1(\cdot, \cdot)$. The computation of the L^2-norm is a little bit more involved since $\boldsymbol{E}_\lambda$ does not belong to C^0 and thus, its standard interpolate does not exist. Instead, we write

$$e_2^2 = \|\boldsymbol{E}_\lambda\|_{L^2(\Omega)^2}^2 - 2(\boldsymbol{E}_\lambda, \boldsymbol{E}^h) + (\boldsymbol{E}^h)^t \mathbb{M} \boldsymbol{E}^h$$

where $\mathbb{M}$ denotes the mass matrix. The first term can be written as a one-dimensional integral which is computed using Simpson's rule. The second term is computed using Gauss quadrature of order 2. Higher order quadrature rules have been tested, but do not improve significantly the results.

In tables 1 to 3 below, N denotes the number of nodes and the number of degrees of freedom is thus given by $2N$.

P_1-FEM, uniform meshes

$\lambda = 4.535$		$\alpha = 0$				$\alpha = 0.95$			
$h/\sqrt{2}$	N	e_a	τ	e_2	τ	e_a	τ	e_2	τ
1/2	21	1.698e+01	–	2.096e+00	–	1.600e+01	–	3.421e+00	–
1/4	65	9.071e+00	0.9041	7.095e-01	1.5625	8.613e+00	0.8936	1.113e+00	1.6202
1/8	225	4.615e+00	0.9751	1.505e-01	2.2371	4.435e+00	0.9575	2.631e-01	2.0807
1/16	833	2.318e+00	0.9936	4.122e-02	1.8681	2.242e+00	0.9845	7.604e-02	1.7906
1/32	3201	1.160e+00	0.9984	9.108e-03	2.1782	1.128e+00	0.9902	2.033e-02	1.9032

P_2-FEM, uniform meshes

$\lambda = 4.535$		$\alpha = 0$				$\alpha = 0.95$			
$h/\sqrt{2}$	N	e_a	τ	e_2	τ	e_a	τ	e_2	τ
1/2	65	2.501e+00	–	5.695e-01	–	2.225e+00	–	5.724e-01	–
1/4	225	6.392e-01	1.9686	1.894e-01	1.5878	5.790e-01	1.9425	1.900e-01	1.5910
1/8	833	1.614e-01	1.9849	3.795e-02	2.3199	1.485e-01	1.9626	3.796e-02	2.3236
1/16	3201	4.057e-02	1.9926	1.053e-02	1.8494	3.742e-02	1.9892	1.053e-02	1.8497

TABLE 1. Regular solution on uniform meshes with different values of α, FEM of type P_1 and P_2.

First we show numerical results for a regular field with parameters $\varepsilon = 0.5$ and $k = 1$ and uniform meshes as shown in Figure 5. This is a validation situation for our code. The corresponding singular exponent is given by $\lambda \approx 4.535$ and thus $\boldsymbol{E}_\lambda$ belongs to $P\boldsymbol{H}^4(\Omega; \mathcal{P})$. We get optimal convergence rates in the semi-norm as well for the standard regularization ($\alpha = 0$) as for the weighted regularization ($\alpha > 0$) for both P_1 and P_2 experiments (Table 1).

Next, we show numerical results for the computation of a singular field. Indeed, for $k = 0$ and $\varepsilon = 0.5$ we get $\lambda \approx 0.535$ and thus $\boldsymbol{E}_\lambda \notin P\boldsymbol{H}^1(\Omega; \mathcal{P})$. In Table 2, we see that the numerical convergence rate tends to zero if $\alpha = 0$, whereas it is positive for $\alpha = 0.48$ or $\alpha = 0.95$. On one hand, this illustrates that standard regularization does not allow one to approximate the singular solution, but yields a spurious solution (see also Figure 6). On the other hand, according to Theorem 3.1, the weighted regularization method converges to the exact solution if the weight parameter α satisfies

$$1 - \min \Lambda_{\varepsilon,S} < \alpha < 1.$$

In the present case, $1 - \min \Lambda_{\varepsilon,S} \approx 0.465$ and $\alpha = 0.48$ or $\alpha = 0.95$ are suitable. However, following Table 2 and Figure 6, we see that the numerical convergence rate increases with α.

As shown in Figure 4, switching from uniform to geometric refined meshes (see Figure 5) improves significantly the numerical rate of convergence (from $\tau \approx 0.32$ to $\tau \approx 1.21$ in the semi-norm e_a for finite elements of type P_1). Here, the numerical convergence rate is obtained using least square calculations.

Table 3 contains results for $\alpha = 0.95$ and refined meshes. It clearly shows the advantage of using P_2 or higher degree FE-solutions (instead of P_1) for improving both the errors and the numerical rate of convergence. This is particularly striking for the visualization of the singular field (see Figure 6 where the radial component of the electric field is represented).

4.2. **Eigenvalue problem**

In this subsection, we carry out some numerical experiments on the computation of electromagnetic eigenmodes in a bounded cavity, encased in a perfect conducting material. In other words, we solve the eigenproblem related to (1) and (3) (with $\Gamma_I = \emptyset$), that is

P_1-FEM, uniform meshes

$\lambda = 0.535$		$\alpha = 0$			
$h/\sqrt{2}$	N	e_a	τ	e_2	τ
1/2	21	9.679e-01	–	7.186e-01	–
1/4	65	9.368e-01	0.0470	7.028e-01	0.0321
1/8	225	9.224e-01	0.0224	6.837e-01	0.0398
1/16	833	9.154e-01	0.0110	6.754e-01	0.0176
1/32	3201	9.119e-01	0.0055	6.705e-01	0.0104

P_1-FEM, uniform meshes

$\lambda = 0.535$		$\alpha = 0.48$			
$h/\sqrt{2}$	N	e_a	τ	e_2	τ
1/2	21	8.933e-01	–	6.528e-01	–
1/4	65	7.966e-01	0.1652	5.852e-01	0.1575
1/8	225	7.250e-01	0.1360	5.118e-01	0.1935
1/16	833	6.706e-01	0.1124	4.591e-01	0.1567
1/32	3201	6.274e-01	0.0960	4.139e-01	0.1496

P_1-FEM, uniform meshes

$\lambda = 0.535$		$\alpha = 0.95$			
$h/\sqrt{2}$	N	e_a	τ	e_2	τ
1/2	21	8.251e-01	–	6.119e-01	–
1/4	65	6.423e-01	0.3613	5.291e-01	0.2098
1/8	225	5.189e-01	0.3079	4.559e-01	0.2147
1/16	833	4.138e-01	0.3262	3.969e-01	0.2000
1/32	3201	3.395e-01	0.2857	3.435e-01	0.2088

Table 2. Singular solution on uniform meshes with different value of α, FEM of type P_1.

P_1-FEM, refined meshes

$\lambda = 0.535$		$\alpha = 0.95$			
h	N	e_a	τ	e_2	τ
0.471	58	5.865e-01	–	4.852e-01	–
0.347	135	4.454e-01	0.8966	4.091e-01	0.5556
0.287	314	3.186e-01	1.7693	2.993e-01	1.6504
0.236	672	2.598e-01	1.0357	2.201e-01	1.5621
0.158	2528	1.610e-01	1.1989	1.461e-01	1.0254

P_2-FEM, refined meshes

$\lambda = 0.535$		$\alpha = 0.95$			
h	N	e_a	τ	e_2	τ
0.471	203	3.819e-01	–	3.959e-01	–
0.347	497	2.166e-01	1.8466	2.449e-01	1.5648
0.287	1191	8.281e-02	5.0808	9.560e-02	4.9690
0.236	2585	2.259e-02	6.5997	1.609e-02	9.0536

Table 3. Singular solution on refined meshes, FEM of type P_1 and P_2.

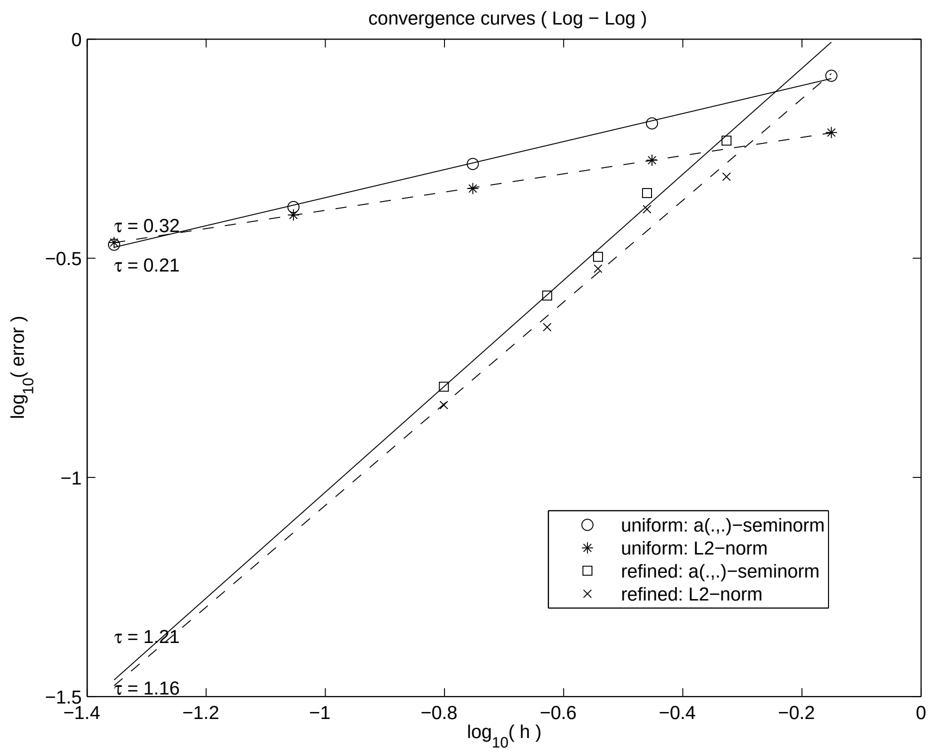

Figure 4. Singular solution: numerical rates of convergence for the P_1-FEM with uniform and refined meshes.

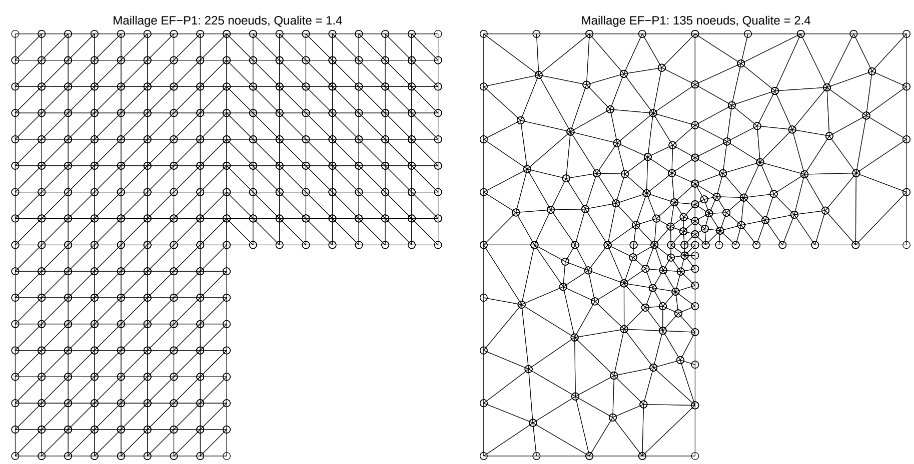

Figure 5. Exemple of meshes used for calculations. Left: uniform, 384 triangles (225 vertices in P_1). Right: geometric refinement, 228 triangles (135 vertices in P_1).

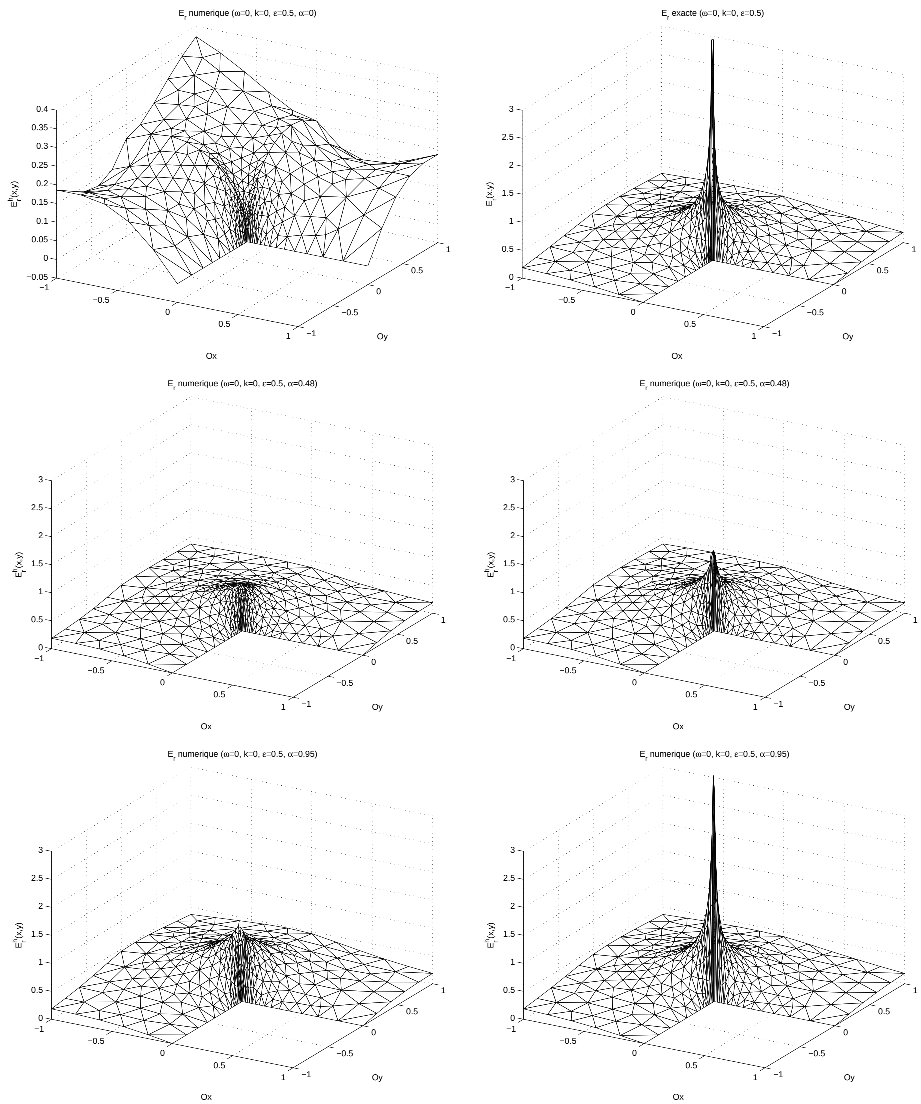

FIGURE 6. Radial component $\boldsymbol{E}_r$ of the singular solution: $\varepsilon_1 = 0.5, \varepsilon_2 = 1, \varepsilon_3 = 0.5$ with the same refined mesh of 564 triangles (314 vertices in P_1, 1191 vertices in P_2). Top left: P_1 solution with standard regularization ($\alpha = 0$). Top right: exact solution. Middle: P_1 (left) and P_2 (right) solution with weighted regularization ($\alpha = 0.48$). Bottom: P_1 (left) and P_2 (right) solution with weighted regularization ($\alpha = 0.95$).

Find $(\boldsymbol{E}, \omega)$ such that

$$\left\{ \begin{array}{ll} \mathbf{curl}\left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}\right) = \omega^2 \varepsilon \boldsymbol{E} & \text{in } \Omega, \\ \operatorname{div}(\varepsilon \boldsymbol{E}) = 0 & \text{in } \Omega, \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{82}$$

Note that we write down the constraint on the divergence of the field, which was implicit in the original formulation (1). As a matter of fact, it will be used explicitly to approximate the eigenmodes, via a mixed, augmented variational formulation (see (84) below). Let us describe briefly how it is constructed (we follow the Annex of [12]).

Let us introduce

$$\boldsymbol{K}_\alpha = \{\boldsymbol{u} \in \boldsymbol{W}_\alpha \mid \operatorname{div}(\varepsilon \boldsymbol{u}) = 0\}.$$

It is common knowledge that an equivalent variational formulation of the eigenproblem (82) is
Find $(\boldsymbol{E}, \omega) \in \boldsymbol{K}_\alpha \times \mathbb{R}^+$ such that

$$(\mu^{-1} \operatorname{\mathbf{curl}} \boldsymbol{E}, \operatorname{\mathbf{curl}} \boldsymbol{v})_{0,\Omega} = \omega^2 (\varepsilon \boldsymbol{E}, \boldsymbol{v})_{0,\Omega}, \quad \forall \boldsymbol{v} \in \boldsymbol{K}_\alpha. \tag{83}$$

However, it is difficult to build a conforming discretization in $\boldsymbol{K}_\alpha$, so the divergence-free condition on $\boldsymbol{E}$ is preferably taken into account as a natural condition. In other words, one solves the eigenproblem in $\boldsymbol{W}_\alpha$. There exist two approaches: the *parameterized* one is described in [18], and the *mixed* one in [11] (see also [9] for the abstract theory).
The first approach relies on the introduction in the left-hand side of a parameterized regularization term namely, with a parameter $s > 0$,

$$s\,(\operatorname{div} \varepsilon \boldsymbol{E}, \operatorname{div} \varepsilon \boldsymbol{v})_{L^2_\alpha(\Omega)}.$$

The idea is two-fold. One notices first that the left-hand side now defines a scalar product on $\boldsymbol{W}_\alpha$ for any $s > 0$. However, one captures both $\operatorname{div} \varepsilon\cdot$-free eigenfields and curl-free eigenfields. The first family corresponds to the actual electromagnetic eigenmodes, whereas the second family is made of *spurious* modes. So, one allows the parameter s to vary: for two different values of s, one recovers the same two families, but with different eigenvalues for the spurious modes. The second idea is thus to let s vary, to keep only the eigenmodes with the "numerically constant" eigenvalues, and to drop the others. For other alternatives based on this technique, we refer the interested reader to [15].

The second approach consists in keeping the constraint on the $\operatorname{div} \varepsilon\cdot$ of the eigenmodes explicitly in the variational formulation, thus resulting in a mixed approach. Also, one adds a stabilizing term like

$$(s \operatorname{div} \varepsilon \boldsymbol{E}, \operatorname{div} \varepsilon \boldsymbol{v})_{L^2_\alpha(\Omega)}$$

in the left-hand side, to deal again with a scalar product on $\boldsymbol{W}_\alpha$. Here, s is fixed, piecewise constant, with $s(\boldsymbol{x}) \geq s_0 > 0$ a.e.: following (80), we choose $s_j = \beta \varepsilon_j^{-2}$, $j = 1, \ldots, J$. Following [11, 12], one finds that the eigenproblem (82) is equivalent to the mixed, augmented variational formulation
Find $(\boldsymbol{E}, p, \omega) \in \boldsymbol{W}_\alpha \times L^2_{-\alpha}(\Omega) \times \mathbb{R}^+$ such that

$$\left\{ \begin{array}{l} a_\beta(\boldsymbol{E}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \omega^2 (\varepsilon \boldsymbol{E}, \boldsymbol{v})_{0,\Omega}, \quad \forall \boldsymbol{v} \in \boldsymbol{W}_\alpha \\ b(\boldsymbol{E}, q) = 0, \quad \forall q \in L^2_{-\alpha}(\Omega). \end{array} \right. \tag{84}$$

Above, the sesquilinear forms $a_\beta(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are respectively given by (80) and

$$b(\boldsymbol{u}, q) = \langle \operatorname{div} \varepsilon \boldsymbol{u}, q \rangle_{L^2_\alpha(\Omega) - L^2_{-\alpha}(\Omega)} = \int_\Omega \operatorname{div} \varepsilon \boldsymbol{u}\, q\, dx.$$

Indeed, one can prove (simply) that the Lagrange multiplier p that appears in (84) is always equal to zero, because the bilinear form b satisfies the inf-sup condition with respect to the spaces $\boldsymbol{W}_\alpha$ and $L^2_{-\alpha}(\Omega)$.

Then, one proceeds by discretizing the electric field as before (vector P_k Lagrange finite elements), whereas the multiplier is discretized with scalar P_{k-1} Lagrange finite elements. In particular, in order to guarantee the discrete inf-sup condition, this choice imposes that $k \geq 2$. There is one further restriction on the choice of the discretization of the Lagrange multiplier. If there exists vertices and/or edges located on the boundary that induce a singular behavior of the field (id est, either reentrant corners and/or edges in a homogeneous medium, or at the intersection of two or more media), one should use discretized multipliers that vanish in a neighborhood of these geometrical singularities. For details, we refer to [14], in which the case of a homogeneous medium with geometrical singularities is treated extensively.

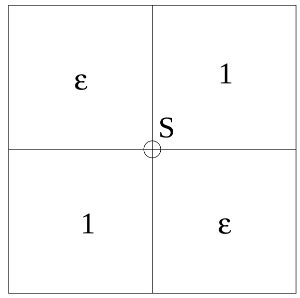

FIGURE 7. Checkerboard with 4 subdomains.

For illustrations purposes, let us consider the two-dimensional sample configuration of Figure 7: the checkerboard domain of interest is the square $\Omega =]-1, 1[^2$, and it is divided into four squares with different values of ε. We first remark that there is no singular behavior induced by the intersection of the interfaces with the boundary. Again, this stems from the fact that the interfaces are orthogonal to the boundary. Therefore, the set $\mathcal{S}_0$ reduces to the center $\{S\}$ of the square.

We carried out the numerical experiments with $\varepsilon = 0.5$, or $\varepsilon = 10^{-8}$, on a series of three successively refined graded meshes (labeled mesh #1, #2 and #3). The meshes contain 403 (resp. 1612, 6448) triangles and 231 (resp. 864, 3339) vertices. The discretizations using the Taylor-Hood $P_2 - P_1$ finite elements yield discrete problems with 1841 (resp. 7310, 29129) d.o.f. The weight is implemented with $\alpha = 0.95$ and the stabilization term with $\beta = 5$. Our results are compared to those obtained by M. Dauge (see [20]). The smallest 6 Maxwell eigenvalues up to eight digits are listed in Table 4.

ε	0.5	10^{-8}
λ_1	3.3175488	4.9348022
λ_2	3.3663242	7.2252112
λ_3	6.1863896	24.674005
λ_4	13.926323	24.674011
λ_5	15.082991	24.674011
λ_6	15.778866	27.868851

TABLE 4. Maxwell eigenvalues in the checkerboard domain (M. Dauge's computations).

The relative errors on the computed eigenvalues,

$$r_k = |\lambda_{h,k} - \lambda_k|/|\lambda_k|,$$

are reported in Tables 5 and 6.

When $\varepsilon = 10^{-8}$, we note that there is a triple eigenvalue at 24.674, which seems hard to capture numerically (see the residuals r_3, r_4 and r_5 of Table 6).

Finally, we conclude this series of experiments by the computation of eigenvalues (for $\varepsilon = 0.5$) using a formulation without weight, i.e. we set $\alpha = 0$. We report the first 6 computed eigenvalues in Table 7. As expected [11], since one solves a different (variational) problem, one fails to capture the singular eigenmodes (here λ_1 or λ_2), and new ones appear ($\lambda_{h,2}$).

mesh	#1	#2	#3
r_1	8.4e-4	1.6e-4	2.7e-5
r_2	9.6e-3	2.8e-3	1.1e-3
r_3	1.0e-3	2.2e-4	1.6e-5
r_4	1.9e-3	8.0e-4	1.7e-4
r_5	8.3e-4	7.5e-4	1.1e-3
r_6	3.5e-3	1.2e-3	2.7e-4

Table 5. Relative errors for $\varepsilon = 0.5$.

mesh	#1	#2	#3
r_1	1.4e-3	2.6e-4	6.2e-5
r_2	4.0e-3	4.4e-3	4.5e-3
r_3	2.5e-3	3.5e-2	1.9e-2
r_4	1.1e-2	3.3e-4	6.0e-5
r_5	5.8e-2	1.7e-2	2.8e-3
r_6	6.7e-2	1.1e-2	6.5e-4

Table 6. Relative errors for $\varepsilon = 10^{-8}$.

mesh	#1	#2	#3
$\lambda_{h,1}$	3.47142	3.43039	3.39922
$\lambda_{h,2}$	4.55538	4.77986	5.10793
$\lambda_{h,3}$	6.19368	6.18823	6.18672
$\lambda_{h,4}$	13.9537	13.9375	13.9286
$\lambda_{h,5}$	15.1186	15.0941	15.0848
$\lambda_{h,6}$	15.9944	15.9159	15.8680

Table 7. Computed eigenvalues in the absence of weights ($\varepsilon = 0.5$).

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