WEAK VECTOR AND SCALAR POTENTIALS. APPLICATIONS TO POINCARÉ'S THEOREM AND KORN'S INEQUALITY IN SOBOLEV SPACES WITH NEGATIVE EXPONENTS

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In this paper, we present several results concerning vector potentials and scalar potentials with data in Sobolev spaces with negative exponents, in a not necessarily simply-connected, three-dimensional domain. We then apply these results to Poincaré's theorem and to Korn's inequality.

1. Weak versions of a classical theorem of Poincaré

In this work, (the results of which were announced in [2]), Ω is a bounded open connected subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary Γ. The notation ${}_{X'}\langle , \rangle_X$ denotes the duality pairing between a topological space X and its dual X'. The letter C denotes a constant that is not necessarily the same at its various occurrences.

We begin with a weak version of a well-known theorem of Poincaré. Here as elsewhere in this paper, "weak" means that the result to which it is attached holds as well in Sobolev spaces with negative exponents.

Theorem 1.1. *Let* $\boldsymbol{f} \in H^{-m}(\Omega)^3$ *for some integer* $m \geq 0$. *Then the following properties are equivalent:*

(i) ${}_{H^{-m}(\Omega)^3}\langle \boldsymbol{f},\ \boldsymbol{\varphi}\rangle_{H_0^m(\Omega)^3} = 0$ *for all* $\boldsymbol{\varphi} \in V_m = \{\boldsymbol{\varphi} \in H_0^m(\Omega)^3;\ div\ \boldsymbol{\varphi} = 0\}$,
(ii) ${}_{H^{-m}(\Omega)^3}\langle \boldsymbol{f},\ \boldsymbol{\varphi}\rangle_{H_0^m(\Omega)^3} = 0$ *for all* $\boldsymbol{\varphi} \in \mathcal{V} = \{\boldsymbol{\varphi} \in \mathscr{D}(\Omega)^3;\ div\ \boldsymbol{\varphi} = 0\}$,
(iii) *There exists a distribution* $\chi \in H^{-m+1}(\Omega)$*, unique up to an additive constant, such that* $\boldsymbol{f} = \boldsymbol{grad}\ \chi$ *in* Ω.

If in addition Ω *is simply-connected, the above properties are equivalent to:*
(iv) $\boldsymbol{curl}\ \boldsymbol{f} = \boldsymbol{0}$ *in* Ω.

Proof. For the equivalence between (i), (ii) and (iii), we refer to [4]. Since the implication (iii) $\Longrightarrow$ (iv) clearly holds, it remains to prove that (iv) $\Longrightarrow$ (iii).

To begin with, let $\boldsymbol{f} \in H^{-m}(\Omega)^3$ be such that $\mathbf{curl}\ \boldsymbol{f} = \mathbf{0}$ in Ω. We then use the same argument as in [8]: We know that there exist a unique $\boldsymbol{u} \in H_0^m(\Omega)^3$ and a unique $p \in H^{-m+1}(\Omega)/\mathbb{R}$ (see [5]) such that

$$\Delta^m \boldsymbol{u} + \mathbf{grad}\ p = \boldsymbol{f} \quad \text{and} \quad \text{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega. \tag{1.1}$$

Hence $\Delta^m \mathbf{curl}\ \boldsymbol{u} = \mathbf{0}$ in Ω so that the hypoellipticity (see [10]) of the polyharmonic operator Δ^m implies that $\mathbf{curl}\ \boldsymbol{u} \in \mathcal{C}^\infty(\Omega)^3$. Since div $\boldsymbol{u} = 0$, we deduce that $\Delta \boldsymbol{u} = \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{u} \in \mathcal{C}^\infty(\Omega)^3$. This also implies that $\Delta^m \boldsymbol{u}$ belongs to $\mathcal{C}^\infty(\Omega)^3$ and is an irrotational vector field. By the classical Poincaré theorem, there exists $q \in \mathcal{C}^\infty(\Omega)^3$ such that $\Delta^m \boldsymbol{u} = \mathbf{grad}\ q$. Thus, $\boldsymbol{f} = \mathbf{grad}\ (p+q)$ and, thanks to [4] proposition 2.10, the function $p+q$ belongs to the space $H^{-m+1}(\Omega)$. □

We can give another proof of the implication (iv) $\Longrightarrow$ (iii) by using the following theorem:

Theorem 1.2. *Assume that both* Ω *and* $\mathbb{R}^3 \setminus \Omega$ *are simply-connected. Let* $\boldsymbol{u} \in H_0^m(\Omega)^3$*,* $m \geq 0$*, be a vector field that satisfies div* $\boldsymbol{u} = 0$ *in* Ω*. Then there exists a vector potential* $\boldsymbol{\psi}$ *in* $H_0^{m+1}(\Omega)^3$ *such that*

$$\boldsymbol{u} = \boldsymbol{curl}\ \boldsymbol{\psi}, \quad div\ \Delta^{m+1}\boldsymbol{\psi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C\|\boldsymbol{u}\|_{H^m(\Omega)^3}. \tag{1.2}$$

Proof. Let $\boldsymbol{u} \in H_0^m(\Omega)^3$ be such that div $\boldsymbol{u} = 0$ in Ω and let $\widetilde{\boldsymbol{u}}$ denote the extension of $\boldsymbol{u}$ by $\mathbf{0}$ in $\mathbb{R}^3 \setminus \Omega$. Thus $\widetilde{\boldsymbol{u}} \in H_0^m(\mathbb{R}^3)^3$, div $\widetilde{\boldsymbol{u}} = 0$ in $\mathbb{R}^3$, and there exist an open ball B containing $\overline{\Omega}$ and a vector field $\boldsymbol{w} \in H_0^{m+1}(B)^3$ such that $\widetilde{\boldsymbol{u}} = \mathbf{curl}\ \boldsymbol{w}$, div $\Delta^{m+1}\boldsymbol{w} = 0$ in B, and

$$\|\boldsymbol{w}\|_{H^{m+1}(B)^3} \leq C\|\boldsymbol{u}\|_{H^m(B)^3}.$$

The open set $\Omega' := B \setminus \overline{\Omega}$ is bounded, has a Lipschitz-continuous boundary and is simply-connected. Furthermore, the vector field $\boldsymbol{w}' := \boldsymbol{w}|_{\Omega'}$ belongs to $H^{m+1}(\Omega')^3$ and satisfies $\mathbf{curl}\ \boldsymbol{w}' = \mathbf{0}$ in Ω'. Therefore, there exists a function $\chi' \in H^1(\Omega')$ such that $\boldsymbol{w}' = \mathbf{grad}\ \chi'$ in Ω'. Hence in fact $\chi' \in H^{m+2}(\Omega')$ and the estimate

$$\|\chi'\|_{H^{m+2}(\Omega')} \leq C\|\boldsymbol{w}'\|_{H^{m+1}(\Omega')^3}$$

holds. Since the function $\chi' \in H^{m+2}(\Omega')$ can be extended to a function $\widetilde{\chi}$ in $H^{m+2}(\mathbb{R}^3)$, with

$$\|\widetilde{\chi}\|_{H^{m+2}(\mathbb{R}^3)} \leq C\|\chi'\|_{H^{m+2}(\Omega')} \leq C\|\boldsymbol{w}'\|_{H^{m+1}(\Omega')^3},$$

the vector field $\widetilde{\boldsymbol{\varphi}} := \boldsymbol{w} - \mathbf{grad}\ \widetilde{\chi}$ belongs to the space $H^{m+1}(B)^3$ and satisfies $\widetilde{\boldsymbol{\varphi}}|_{\Omega'} = \mathbf{0}$. Then the restriction $\boldsymbol{\varphi} := \widetilde{\boldsymbol{\varphi}}|_{\Omega}$ belongs to the space $H_0^{m+1}(\Omega)^3$, satisfies the estimate (1.2), and $\mathbf{curl}\ \widetilde{\boldsymbol{\varphi}} = \mathbf{curl}\ \boldsymbol{w} = \widetilde{\boldsymbol{u}}$ in B. Thus $\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\varphi}$ in Ω. Let now p denote the unique solution in the space $H_0^{m+2}(\Omega)$ of $\Delta^{m+2}p = \mathrm{div}\ \Delta^{m+1}\boldsymbol{\varphi}$, so that the estimate

$$\|p\|_{H^{m+2}(\Omega)} \leq C\|\boldsymbol{\varphi}\|_{H^{m+1}(\Omega)^3}$$

holds. Then the function $\boldsymbol{\psi} = \boldsymbol{\varphi} - \mathbf{grad}\ p$ satisfies (1.2). □

We can give yet another proof of the above implication (iv) $\Longrightarrow$ (iii): Consider again the solution $\boldsymbol{u} \in H_0^m(\Omega)^3$ to (1.1) and let $\boldsymbol{v} \in H_0^{m+1}(\Omega)^3$ denote the vector potential of $\boldsymbol{u}$ as given by theorem 1.2. We then have $\Delta^m \mathbf{curl}\ \boldsymbol{u} = \mathbf{0}$. If $m = 2k$, for some integer $k \geq 1$, then

$$\begin{aligned} {}_{H^{-m-1}(\Omega)^3}\langle \Delta^m \mathbf{curl}\ \boldsymbol{u},\ \boldsymbol{v}\rangle_{H_0^{m+1}(\Omega)^3} &= {}_{H^{-1}(\Omega)^3}\langle \Delta^k \mathbf{curl}\ \boldsymbol{u},\ \Delta^k \boldsymbol{v}\rangle_{H_0^1(\Omega)^3} \\ &= \int_\Omega \Delta^k \boldsymbol{u} \cdot \Delta^k \mathbf{curl}\ \boldsymbol{v}\ d\boldsymbol{x} \\ &= \|\Delta^k \boldsymbol{u}\|^2_{L^2(\Omega)^3}. \end{aligned}$$

This implies that $\Delta^k \boldsymbol{u} = \mathbf{0}$ in Ω and thus $\boldsymbol{u} = \mathbf{0}$ since $\boldsymbol{u} \in H_0^m(\Omega)^3$. The case $m = 2k+1$ follows by a similar argument. □

2. Scalar Potentials

Let Γ_i, $0 \leq i \leq I$, denote the connected components of the boundary Γ of the domain Ω, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$. We do not assume that Ω is simply-connected, however we assume that there exist J connected and oriented surfaces Σ_j, $1 \leq j \leq J$ contained in Ω, with the following properties: each surface Σ_j is an open subset of a smooth manifold, the boundary of Σ_j is contained in Γ for $1 \leq j \leq J$, the intersection $\overline{\Sigma_i} \cap \overline{\Sigma_j}$ is empty for $i \neq j$, and finally the open set $\Omega^\circ = \Omega \setminus \bigcup_{j=1}^J \Sigma_j$ is simply-connected and pseudo-Lipschitz in the sense of [1]. Each such surface Σ_j is called a cut. Finally, let $[\cdot]_j$ denote the jump of a function over each cut Σ_j, $1 \leq j \leq J$.

We then define the spaces

$$H(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathbf{curl}\ \boldsymbol{v} \in L^2(\Omega)^3\},$$
$$H(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathrm{div}\ \boldsymbol{v} \in L^2(\Omega)\},$$

each one being equipped with the graph norm, and their subspaces

$$H_0(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in H(\mathbf{curl}, \Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0}\ \text{ on } \Gamma\},$$
$$H_0(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in H(\mathrm{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0\ \text{ on } \Gamma\}.$$

For any function q in $H^1(\Omega^\circ)$, $\mathbf{grad}\ q$ denotes the gradient of q in the sense of distributions in $\mathscr{D}'(\Omega^\circ)$. It belongs to $L^2(\Omega^\circ)^3$ and therefore can be extended to

$L^2(\Omega)^3$. In order to distinguish this extension from the gradient of q in $\mathscr{D}'(\Omega)$, we denote it by $\widetilde{\mathbf{grad}}\; q$. Finally, we remark that the space

$$K_T(\Omega) := \{\boldsymbol{w} \in H(\mathbf{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega);\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ and div } \boldsymbol{w} = 0 \text{ in } \Omega\}$$

is of dimension equal to J: As shown in [1] Prop. 3.14, it is spanned by the vector fields $\widetilde{\mathbf{grad}}\; q_j^T$, $1 \le j \le J$, where each function $q_j^T \in H^1(\Omega^\circ)$, which is unique up to an additive constant, satisfies

$$\begin{cases} \Delta q_j^T = 0 & \text{in } \Omega^\circ, \\ \partial_n q_j^T = 0, & \text{on } \Gamma, \\ [q_j^T]_k = \text{constant},\ [\partial_n q_j^T]_k = 0,\ \langle \partial_n q_j^T\,,\, 1\rangle_{\Sigma_k} = \delta_{jk} & \text{for } 1 \le k \le J. \end{cases} \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_{\Sigma_k}$ denotes the duality pairing between the spaces $H^{-1/2}(\Sigma_k)$ and $H^{1/2}(\Sigma_k)$.

Theorem 2.1. *Given any function $\boldsymbol{f} \in L^2(\Omega)^3$ that satisfies*

$$\boldsymbol{curl}\, \boldsymbol{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\ d\boldsymbol{x} = 0 \qquad \text{for all } \boldsymbol{v} \in K_T(\Omega), \tag{2.2}$$

there exists a scalar potential χ in $H^1(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{grad}\ \chi \quad \text{in } \Omega \quad \text{and} \qquad \|\chi\|_{H^1(\Omega)} \le C\|\boldsymbol{f}\|_{L^2(\Omega)^3}. \tag{2.3}$$

Proof. It suffices to show that, given any vector field $\boldsymbol{v} \in H_0(\mathrm{div}, \Omega)$ such that div $\boldsymbol{v} = 0$ in Ω, there holds $\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\ d\boldsymbol{x} = 0$. Let

$$\boldsymbol{z} = \sum_{j=1}^{J} \langle \boldsymbol{v}\cdot\boldsymbol{n},\ 1\rangle_{\Sigma_j}\ \widetilde{\mathbf{grad}}\ q_j^T$$

and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{z}$. According to [1], theorem 3.17, there exists a vector potential $\boldsymbol{\psi} \in L^2(\Omega)^3$ that satisfies $\boldsymbol{w} = \mathbf{curl}\ \boldsymbol{\psi}$, div $\boldsymbol{\psi} = 0$ in Ω and $\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0}$ on Γ. Hence

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\ d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\mathbf{curl}\ \boldsymbol{\psi}\ d\boldsymbol{x} = 0.$$

The result is then a consequence of theorem 1.1: there exists a function $\chi \in H^1(\Omega)$ satisfying (2.3). ◻

Remark 2.1. **(1)** Any function $\boldsymbol{f} \in L^2(\Omega)^3$ that satisfies $\mathbf{curl}\ \boldsymbol{f} = 0$ in Ω can be decomposed as:

$$\boldsymbol{f} = \mathbf{grad}\ \chi + \widetilde{\mathbf{grad}}\ p, \quad \text{with} \quad \chi \in H^1(\Omega) \quad \text{and} \quad \widetilde{\mathbf{grad}}\ p \in K_T(\Omega).$$

Such a result was alluded to in [11].
(2) The second condition in (2.2) is trivially satisfied when Ω is simply-connected, since $K_T(\Omega) = \{\mathbf{0}\}$ in this case.

Theorem 2.2. *Given any distribution $\boldsymbol{f} \in H_0(\mathrm{div},\Omega)'$ that satisfies*

$$\boldsymbol{curl}\, \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega \quad \text{and} \quad {}_{H_0(\mathrm{div},\Omega)'}\langle \boldsymbol{f}\ ,\ \boldsymbol{v}\rangle_{H_0(\mathrm{div},\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in K_T(\Omega), \tag{2.4}$$

there exists a scalar potential χ in $L^2(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{grad}\ \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{2.5}$$

Proof. Let $\boldsymbol{f} \in H_0(\mathrm{div},\Omega)'$ be such that $\mathbf{curl}\ \boldsymbol{f} = \boldsymbol{0}$ in Ω. Hence (see proposition 1 of [6]) there exist $\boldsymbol{\psi} \in L^2(\Omega)^3$ and $\chi_0 \in L^2(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\ \chi_0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\chi_0\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{2.6}$$

Observe that, thanks to the density of $\mathscr{D}(\Omega)^3$ in $H_0(\mathrm{div},\Omega)$,

$${}_{H_0(\mathrm{div},\Omega)'}\langle \mathbf{grad}\ \chi_0,\ \boldsymbol{v}\rangle_{H_0(\mathrm{div},\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in K_T(\Omega).$$

Therefore, the function $\boldsymbol{\psi} \in L^2(\Omega)^3$ satisfies relations (2.2). By theorem 2.1, there exists a function $p \in H^1(\Omega)$ such that

$$\boldsymbol{\psi} = \mathbf{grad}\ p \quad \text{in } \Omega \quad \text{and} \quad \|p\|_{H^1(\Omega)} \leq C\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}.$$

Hence the function $\chi = p + \chi_0$ satisfies the announced properties. □

Remark 2.2. Note that this theorem is an extension of the equivalence (iii) $\Longleftrightarrow$ (iv) in theorem 1.1 with $m = 1$ to the case where Ω is not simply-connected.

More generally, let us introduce, for any integer $m \geq 0$, the space

$$H_0^m(\mathrm{div},\Omega) := \{\boldsymbol{v} \in H_0(\mathrm{div},\Omega);\ \mathrm{div}\,\boldsymbol{v} \in H_0^m(\Omega)\},$$

which coincides with $H_0(\mathrm{div},\Omega)$ for $m = 0$. Its dual space, denoted by $H^{-m}(\mathrm{div},\Omega)$, can then be characterized by

$$H^{-m}(\mathrm{div},\Omega) = \{\boldsymbol{\psi} + \mathbf{grad}\ \chi;\ \boldsymbol{\psi} \in H_0(\mathrm{div},\Omega)',\ \chi \in H^{-m}(\Omega)\}.$$

One can also show that $\mathscr{D}(\Omega)^3$ is dense in $H_0^m(\mathrm{div},\Omega)$ and that the following Green formula holds for any $\chi \in H^{-m}(\mathrm{div},\Omega)$ and $\boldsymbol{v} \in H_0^m(\mathrm{div},\Omega)$:

$${}_{H^{-m}(\mathrm{div},\Omega)}\langle \mathbf{grad}\ \chi,\ \boldsymbol{v}\rangle_{H_0^m(\mathrm{div},\Omega)} + {}_{H^{-m}(\Omega)}\langle \chi,\ \mathrm{div}\boldsymbol{v}\rangle_{H_0^m(\Omega)} = 0. \tag{2.7}$$

As a consequence of theorem 2.2, it is easy to prove the following theorem, which shows that property (iv) in theorem 1.1 also holds when Ω is not simply-connected.

Theorem 2.3. *For any distribution $\boldsymbol{f} \in H^{-m}(\mathrm{div},\Omega)$ that satisfies (2.4), there exists a scalar potential χ in $H^{-m}(\Omega)$ such that*

$$\boldsymbol{f} = \boldsymbol{grad}\ \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{H^{-m}(\Omega)} \leq C\|\boldsymbol{f}\|_{H^{-m}(\mathrm{div},\Omega)}. \tag{2.8}$$

Proof. We give the proof when $m = 1$; the general case is similar. Let $\boldsymbol{f} \in H^{-1}(\mathrm{div},\Omega)$ satisfy (2.4). Then, there exist $\boldsymbol{\psi} \in H_0(\mathrm{div},\Omega)'$ and $\chi_0 \in H^{-1}(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\ \chi_0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H_0(\mathrm{div},\Omega)'} + \|\chi_0\|_{H^{-1}(\Omega)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\mathrm{div},\Omega)}. \tag{2.9}$$

Observe that, thanks to (2.9), we have

$$ {}_{H^{-1}(\mathrm{div},\Omega)}\langle \mathbf{grad}\ \chi_0,\ \boldsymbol{v}\rangle_{H^1_0(\mathrm{div},\Omega)} = -\ {}_{H^{-1}(\Omega)}\langle \chi_0,\ \mathrm{div}\, \boldsymbol{v}\rangle_{H^1_0(\Omega)} \ = 0 $$

for all $\boldsymbol{v} \in K_T(\Omega)$. By theorem 2.2, there exists a function $p \in L^2(\Omega)$ such that $\boldsymbol{\psi} = \mathbf{grad}\ p$ and the estimate (2.5) holds. Then the function $\chi = \chi_0 + p$ satisfies the announced properties. □

3. Vector potentials in $H^m_0(\Omega)^3$

First, we recall some results concerning the existence of tangential vector potential (see [1] for proofs).

Below, $\langle \cdot, \cdot \rangle_{\Gamma_i}$ denotes the duality pairing between the spaces $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$. Given any function $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$ that satisfies

$$ \mathrm{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \boldsymbol{u} \cdot \boldsymbol{n}\ , 1\rangle_{\Gamma_i} = 0, \quad 0 \le i \le I, \tag{3.1} $$

there exists a vector potential $\boldsymbol{\psi}$ in $L^2(\Omega)^3$ such that

$$ \boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}, \qquad \mathrm{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{3.2} $$

satisfying the estimate

$$ \|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \le C \|\boldsymbol{u}\|_{L^2(\Omega)^3}. \tag{3.3} $$

Moreover, there exists a unique vector field $\boldsymbol{\psi} \in L^2(\Omega)^3$ satisfying (3.2) and such that

$$ \langle \boldsymbol{\psi} \cdot \boldsymbol{n}\ , 1\rangle_{\Sigma_j} = 0, \quad 1 \le j \le J, \tag{3.4} $$

and the estimate (3.3) holds. When Ω is of class $\mathscr{C}^{1,1}$, then $\boldsymbol{\psi}$ belongs to $H^1(\Omega)^3$ and the estimate

$$ \|\boldsymbol{\psi}\|_{H^1(\Omega)^3} \le C \|\boldsymbol{u}\|_{L^2(\Omega)^3} \tag{3.5} $$

holds. If moreover $\boldsymbol{u} \in H^m(\Omega)^3$ and Ω is of class $\mathscr{C}^{m+1,1}$, for some integer $m \ge 0$, then $\boldsymbol{\psi}$ belongs to $H^{m+1}(\Omega)^3$ and the estimate

$$ \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \le C \|\boldsymbol{u}\|_{H^m(\Omega)^3} \tag{3.6} $$

holds. We also recall the result concerning the existence of normal vector potentials (see again [1] for proofs). For any vector field $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$ that satisfies

$$ \mathrm{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma \quad \text{and } \langle \boldsymbol{u} \cdot \boldsymbol{n}\ , 1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J, \tag{3.7} $$

there exists a vector potential $\boldsymbol{\psi}$ in $L^2(\Omega)^3$ such that

$$ \boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}, \quad \mathrm{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma, \tag{3.8} $$

and the estimate

$$ \|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \le C \|\boldsymbol{u}\|_{L^2(\Omega)^3} \tag{3.9} $$

holds. Moreover, there exists a unique vector field $\boldsymbol{\psi} \in L^2(\Omega)^3$ satisfying (3.8) and such that

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{n} \,,\, 1 \rangle_{\Gamma_i} = 0, \quad 0 \le i \le I\,, \tag{3.10}$$

and the estimate (3.9) holds. When $\boldsymbol{u}$ is more regular, then (3.5) and (3.6) are also satisfied.

Remark 3.1. Let $\boldsymbol{u}$ be a vector field in $H(\mathrm{div}, \Omega)$ that satisfies:

$$\mathrm{div}\; \boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma.$$

Using the same arguments as those of theorem 2.1, it is easy to verify that

$$\langle \boldsymbol{u} \cdot \boldsymbol{n} \,,\, 1 \rangle_{\Sigma_j} = 0, \quad 1 \le j \le J,$$

if and only if

$$\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\; q_j^T \; d\boldsymbol{x} = 0 \quad \text{for all } 1 \le j \le J.$$

Another kind of less standard but useful vector potential is given by the following theorem.

Theorem 3.1. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{1,1}$. For any function $\boldsymbol{u}$ in $H(\mathrm{div}, \Omega)$ satisfying (3.7), there exists a vector potential $\boldsymbol{\psi}$ in $H_0^1(\Omega)^3$, such that*

$$\boldsymbol{u} = \boldsymbol{curl}\; \boldsymbol{\psi} \text{ and } \mathit{div}\; \Delta\boldsymbol{\psi} = 0 \text{ in } \Omega, \quad \|\boldsymbol{\psi}\|_{H^1(\Omega)^3} \le C\|\boldsymbol{u}\|_{L^2(\Omega)^3}. \tag{3.11}$$

Proof. Given any vector field $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$ satisfying (3.7), we associate the vector potential $\boldsymbol{\psi}_0 \in H^1(\Omega)^3$ satisfying (3.8) and the estimate

$$\|\boldsymbol{\psi}_0\|_{H^1(\Omega)^3} \le C\|\boldsymbol{u}\|_{L^2(\Omega)^3}.$$

That Γ is of class $\mathscr{C}^{1,1}$ implies that the normal trace $\boldsymbol{\psi}_0 \cdot \boldsymbol{n}$ belongs to $H^{1/2}(\Gamma)$. Hence, the fourth-order problem

$$\Delta^2 \chi = 0 \quad \text{in } \Omega, \quad \chi = 0 \quad \text{and} \quad \partial_n \chi = \boldsymbol{\psi}_0 \cdot \boldsymbol{n} \quad \text{on } \Gamma$$

has a unique solution χ in $H^2(\Omega)$ satisfying the estimate

$$\|\chi\|_{H^2(\Omega)} \le C\|\boldsymbol{\psi}_0 \cdot \boldsymbol{n}\|_{H^{1/2}(\Gamma)} \le C\|\boldsymbol{u}\|_{L^2(\Omega)^3}.$$

Then the vector field

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{grad}\; \chi$$

satisfies (3.11). □

The vector field $\boldsymbol{\psi}$ given by the previous theorem is unique up to vector fields belonging to the space

$$K_0^1(\Omega) := \{\boldsymbol{w} \in H_0^1(\Omega)^3;\; \mathbf{curl}\; \boldsymbol{w} = \mathbf{0} \text{ and } \mathrm{div}\; (\Delta\boldsymbol{w}) = 0 \text{ in } \Omega\}$$

(see proposition 3.1 below).

Corollary 3.1. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{m+1,1}$, for some integer $m \geq 0$. For any vector field $\boldsymbol{u} \in H^m(\Omega)^3$ that satisfies (3.7), there exists a vector potential $\boldsymbol{\psi}$ in $(H^{m+1}\Omega) \cap H_0^1(\Omega))^3$ satisfying*

$$\boldsymbol{u} = \boldsymbol{curl}\, \boldsymbol{\psi} \text{ and } div\, \Delta\boldsymbol{\psi} = 0 \text{ in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C\|\boldsymbol{u}\|_{H^m(\Omega)^3}.$$

Proof. Under the given assumptions, the vector potential $\boldsymbol{\psi}$ given by the previous theorem belongs to $H^{m+1}(\Omega)^3$ and its normal trace $\boldsymbol{\psi} \cdot \boldsymbol{n}$ belongs to $H^{m+1/2}(\Gamma)$, on the one hand. On the other hand, the solution χ to the fourth-order problem found in the previous belongs to $H^{m+2}(\Omega)^3$. □

We now characterize the space $K_0^1(\Omega)$.

Prop 3.1. Assume that the boundary of the domain Ω is of class $\mathscr{C}^{1,1}$. Then the space $K_0^1(\Omega)$ is spanned by the vector fields **grad** q_i^1, $1 \leq i \leq I$, where each q_i^1 is the unique solution in $H^2(\Omega)$ to the problem

$$\begin{array}{llll} \Delta^2 q_i^1 = 0 & \text{in } \Omega, & & \\ q_i^1\big|_{\Gamma_0} = 0 & \text{and} \quad q_i^1\big|_{\Gamma_k} = \delta_{ik}, & & 1 \leq k \leq I, \\ \partial_n q_i^1 = 0 & \text{on } \Gamma, & & \\ \langle \partial_n \Delta q_i^1\,,\, 1\rangle_{\Gamma_k} = \delta_{ik} & \text{and} \quad \langle \partial_n \Delta q_i^1\,,\, 1\rangle_{\Gamma_0} = -1,\ 1 \leq k \leq I. & & \end{array} \tag{3.13}$$

Proof. First, we prove that the space $K_0^1(\Omega)$ and the space

$$G^1{:=}\{\mathbf{grad}\ q \in H_0^1(\Omega)^3; \quad \Delta^2 q = 0 \quad \text{in } \Omega\}$$

coincide. First, it is clear that G^1 is included in $K_0^1(\Omega)$. Second, given $\boldsymbol{w} \in K_0^1(\Omega)$, let $\widetilde{\boldsymbol{w}}$ denote the extension by zero of $\boldsymbol{w}$ to an open ball B containing $\overline{\Omega}$. Since **curl** $\widetilde{\boldsymbol{w}} = \mathbf{0}$ in B, $\widetilde{\boldsymbol{w}}$ is the gradient of a function $q \in H^2(B)$. Moreover, $q = 0$ in $B \setminus \overline{\Omega}$, so that $q' := q|_\Omega$ belongs to $H_0^2(\Omega)$. Since $\boldsymbol{w} = \mathbf{grad}\ q'$, one finds that $\boldsymbol{w}$ belongs to G^1.
Moreover, it is clear that the set of vector fields **grad** q_i, $1 \leq i \leq I$, where $q_i \in H^2(\Omega)$ is the unique solution to

$$\begin{array}{ll} \Delta^2 q_i = 0 & \text{in } \Omega, \\ q_i\big|_{\Gamma_0} = 0 & \text{and} \quad q_i\big|_{\Gamma_k} = \delta_{ik},\ 1 \leq k \leq I, \\ \partial_n q_i = 0 & \text{on } \Gamma, \end{array} \tag{3.14}$$

spans G^1 $(= K_0^1(\Omega))$.

One still has to check the last line of (3.13). Introduce now

$$M_2{:=}\{r \in H^2(\Omega);\ r\big|_{\Gamma_0} = 0 \text{ and } r\big|_{\Gamma_k} = \ \delta_{ik},\ 1 \leq k \leq I,\ \ \partial_n r = 0 \text{ on } \Gamma\}.$$

For $1 \leq i \leq I$, the problem: find q_i^1 in M_2 such that

$$\forall r \in M_2, \qquad \int_\Omega \Delta q_i^1\, \Delta r\, d\boldsymbol{x} = -r\big|_{\Gamma_i}, \tag{3.15}$$

has a unique solution. Furthermore, the following Green's formula can be proven by a density argument, for any functions q and r in M_2 with $\Delta^2 q$ in $L^2(\Omega)$:

$$\int_\Omega (\Delta^2 q) r \, d\boldsymbol{x} = \int_\Omega \Delta q \, \Delta r \, d\boldsymbol{x} + \sum_{i=1}^{I} r\big|_{\Gamma_i} \langle \partial_n(\Delta q), 1\rangle_{\Gamma_i}.$$

This formula implies that the solution q_i^1 to (3.15) satisfies (3.13). The vector fields $\mathbf{grad}\ q_i^1$, $1 \le i \le I$, are clearly linearly independent and they belong to $K_0^1(\Omega)$. Consequently, they form a basis of $K_0^1(\Omega)$. ◻

Prop 3.2. Assume that the boundary of the domain Ω is of class $\mathscr{C}^{1,1}$. Given any function $\boldsymbol{u}$ in $H(\mathrm{div}, \Omega)$ satisfying (3.7), there exists a unique vector potential $\boldsymbol{\psi}$ in $H_0^1(\Omega)^3$ satisfying

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}, \quad \mathrm{div}\ \Delta\boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and } \langle \partial_n(\mathrm{div}\ \Delta\boldsymbol{\psi})\,,\, 1\rangle_{\Gamma_i} = 0, \quad 0 \le i \le I. \tag{3.16}$$

Moreover, the estimate (3.5) holds.

Proof. Let $(\boldsymbol{\psi}_0 - \mathbf{grad}\ \chi)$ be the potential vector of $\boldsymbol{u}$ given in the proof of theorem 3.1. Then the vector field

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{grad}\ \chi + \sum_{i=1}^{I} \langle \partial_n(\Delta\chi)\,,\, 1\rangle_{\Gamma_i} \mathbf{grad}\ q_i^1$$

satisfies (3.16) (note that the quantities $\langle \partial_n(\Delta\chi), 1\rangle_{\Gamma_i}$ are well defined since $\Delta^2\chi = 0$). ◻

Corollary 3.2. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{m+1,1}$ for some integer $m \ge 0$. Given any function $\boldsymbol{u}$ in $H^m(\Omega)^3$ that satisfies (3.7), there exists a unique vector potential $\boldsymbol{\psi}$ in $(H^{m+1}\Omega) \cap H_0^1(\Omega))^3$ satisfying*

$$\boldsymbol{u} = \boldsymbol{curl}\ \boldsymbol{\psi}, \ \ \mathit{div}\ \Delta\boldsymbol{\psi} = 0 \text{ in } \Omega \quad \text{and} \quad \langle \partial_n(\mathit{div}\ \Delta\boldsymbol{\psi})\,,\, 1\rangle_{\Gamma_i} = 0, \quad 0 \le i \le I$$

and the estimate (3.6).

Theorem 3.2. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{2,1}$. Given any function $\boldsymbol{u}$ in $H_0^1(\Omega)^3$ that satisfies*

$$\mathit{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega \qquad \text{and} \quad \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0, \quad 1 \le j \le J, \tag{3.18}$$

there exists a vector potential $\boldsymbol{\psi}$ in $H_0^2(\Omega)^3$ such that

$$\boldsymbol{u} = \boldsymbol{curl}\ \boldsymbol{\psi} \quad \text{and} \quad \mathit{div}\ \Delta^2\boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^2(\Omega)^3} \le C\|\boldsymbol{u}\|_{H^1(\Omega)^3}. \tag{3.19}$$

Proof. Given $\boldsymbol{u}$ in $H_0^1(\Omega)^3$ that satisfies (3.18), let $\boldsymbol{\varphi} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ denote the vector potential given by corollary 3.2. The sixth-order problem

$$\Delta^3\chi = 0 \quad \text{in } \Omega, \quad \chi = \frac{\partial\chi}{\partial\boldsymbol{n}} = 0 \quad \text{and} \quad \frac{\partial^2\chi}{\partial\boldsymbol{n}^2} = \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} \quad \text{on } \Gamma, \tag{3.20}$$

has a unique solution $\chi \in H^3(\Omega)$ that satisfies the estimate

$$\|\chi\|_{H^3(\Omega)} \le C\|\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}\|_{H^{1/2}(\Gamma)^3} \le C\|\boldsymbol{\varphi}\|_{H^2(\Omega)^3} \le C\|\boldsymbol{u}\|_{H^1(\Omega)^3}.$$

Note that the last boundary condition in (3.20) can be written as

$$\left(\frac{\partial}{\partial \boldsymbol{n}}\mathbf{grad}\ \chi\right)\cdot \boldsymbol{n} = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}\cdot \boldsymbol{n}.$$

For any unit tangent vector $\boldsymbol{\tau}$ on Γ, we have:

$$\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}\cdot \boldsymbol{\tau} = \frac{\partial \boldsymbol{\varphi}_i}{\partial x_j} n_j \tau_i = \frac{\partial \boldsymbol{\varphi}_j}{\partial x_i}\tau_i n_j = \frac{\partial \boldsymbol{\varphi}_j}{\partial \boldsymbol{\tau}} n_j = 0.$$

Also, one can show that $(\partial_n \mathbf{grad}\ \chi)\cdot \boldsymbol{\tau} = 0$, which implies that the relation $\partial_n \mathbf{grad}\ \chi = \partial_n \boldsymbol{\varphi}$ holds. So, the vector field $\boldsymbol{\psi} = \boldsymbol{\varphi} - \mathbf{grad}\ \chi$ belongs to $H^2(\Omega)^3$ and satisfies (3.19). □

The vector field $\boldsymbol{\psi}$ given by Theoerm 3.2 is unique up to vector fields in the space

$$K_0^2(\Omega) := \{\boldsymbol{w} \in H_0^2(\Omega)^3;\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \quad \text{and} \quad \text{div}\ \Delta^2 \boldsymbol{w} = 0 \text{ in } \Omega\},$$

which we now characterize.

Prop 3.3. Assume that the boundary of the domain Ω is of class $\mathscr{C}^{2,1}$. Then the space $K_0^2(\Omega)$ is spanned by the vector fields $\mathbf{grad}\ q_i^2$, $1 \le i \le I$, where each function q_i^2 is the unique solution in $H^3(\Omega)$ to the problem

$$\begin{aligned}
&\Delta^3 q_i^2 = 0 && \text{in } \Omega, && \\
&q_i^2\big|_{\Gamma_0} = 0 && \text{and} \quad q_i^2\big|_{\Gamma_k} = \delta_{ik}, && 1 \le k \le I,\\
&\partial_n q_i^2 = \partial_n^2 q_i^2 = 0 && \text{on } \Gamma, && \\
&\langle \partial_n(\Delta^2 q_i^2)\,,\,1\rangle_{\Gamma_k} = \delta_{ik} && \text{and} \quad \langle \partial_n(\Delta^2 q_i^2)\,,\,1\rangle_{\Gamma_0} = -1,\ 1 \le k \le I. &&
\end{aligned} \tag{3.21}$$

Proof. First, we prove that the space $K_0^2(\Omega)$ coincides with the space

$$G^2 := \{\mathbf{grad}\ q \in H_0^2(\Omega)^3; \quad \Delta^3 q = 0 \quad \text{in } \Omega\},$$

using the same argument as in proposition 3.1. We next note that the set of vector fields $\mathbf{grad}\ q_i,\ 1 \le i \le I$, where $q_i \in H^3(\Omega)$ is the unique solution to the problem

$$\begin{aligned}
&\Delta^3 q_i = 0 && \text{in } \Omega,\\
&q_i\big|_{\Gamma_0} = 0 && \text{and} \quad q_i\big|_{\Gamma_k} = \delta_{ik},\ 1 \le k \le I,\\
&\partial_n q_i = \partial_n^2 q_i = 0 && \text{on } \Gamma,
\end{aligned} \tag{3.22}$$

spans $K_0^2(\Omega)$.

Let now

$$M_3 := \{r \in H^3(\Omega);\ r\big|_{\Gamma_0} = 0,\ r\big|_{\Gamma_k} = \delta_{ik},\ 1 \le k \le I\ ,\ \partial_n r = \partial_n^2 r = 0 \text{ on } \Gamma\}.$$

For $1 \le i \le I$, the problem: find q_i^2 in M_3 such that

$$\forall r \in M_3, \quad \int_\Omega \textbf{grad } \Delta q_i^2 \cdot \textbf{grad } \Delta r \, d\boldsymbol{x} = r\big|_{\Gamma_i}, \tag{3.23}$$

has a unique solution. Furthermore, the following Green's formula can be proved by a density argument, for any functions q and r in M_3 with $\Delta^3 q$ in $L^2(\Omega)$:

$$\int_\Omega (\Delta^3 q) r \, d\boldsymbol{x} = -\int_\Omega \textbf{grad } \Delta q \cdot \textbf{grad } \Delta r \, d\boldsymbol{x} + \sum_{i=1}^{I} r\big|_{\Gamma_i} \langle \partial_n (\Delta^2 q), \rangle_{\Gamma_i}.$$

This formula shows that the solution q_i^2 of (3.23) satisfies (3.21). The vector fields $\textbf{grad } q_i^2$, $1 \le i \le I$, are clearly linearly independent and they belong to $K_0^2(\Omega)$. Consequently, they form a basis of $K_0^2(\Omega)$. □

Corollary 3.3. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{2,1}$. Given any function $\boldsymbol{u}$ in $H_0^1(\Omega)^3$ such that (3.18) holds, there exists a unique vector potential $\boldsymbol{\psi}$ in $H_0^2(\Omega)^3$ satisfying*

$$\boldsymbol{u} = \textbf{\textit{curl}}\ \boldsymbol{\psi}, \ \ div\ \Delta^2 \boldsymbol{\psi} = 0 \text{ in } \Omega \quad \text{and } \langle \partial_n (div\ \Delta \boldsymbol{\psi}) \,, 1\rangle_{\Gamma_i} = 0, \quad 0 \le i \le I,$$

with the corresponding estimate.

More generally, we can prove using the same arguments:

Theorem 3.3. *Assume that boundary of the domain Ω is of class $\mathscr{C}^{m+1,1}$ for some integer $m \ge 1$. Given any vector field $\boldsymbol{u}$ in $H_0^m(\Omega)^3$ that satisfies (3.18), there exists a vector potential $\boldsymbol{\psi}$ in $H_0^{m+1}(\Omega)^3$ such that*

$$\boldsymbol{u} = \textbf{\textit{curl}}\ \boldsymbol{\psi} \text{ and } div\ \Delta^{m+1} \boldsymbol{\psi} = 0 \text{ in } \Omega \text{ and } \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \le C \|\boldsymbol{u}\|_{H^m(\Omega)^3}. \tag{3.25}$$

Moreover, there exists a unique vector potential $\boldsymbol{\psi}$ in $H_0^{m+1}(\Omega)^3$, satisfying (3.25) and

$$\langle \partial_n div\ \Delta \boldsymbol{\psi}^{m+1} \,, 1\rangle_{\Gamma_i} = 0, \quad 0 \le i \le I. \tag{3.26}$$

Remark 3.2. Similar results are found in Borchers & Sohr [7], but with different proof.

Let Ω be a domain with a boundary of class $\mathscr{C}^{m+1,1}$ for some integer $m \ge 1$ and let $\boldsymbol{u}$ in $H_0^m(\Omega)^3$ be such that div $\boldsymbol{u} = 0$. If Ω is simply-connected ($J = 0$), and Γ is connected ($I = 0$), then there exists a unique vector potential $\boldsymbol{\psi}$ in $H_0^{m+1}(\Omega)^3$ satisfying (3.25).

4. Weak vector potentials

First, we note that the continuous embeddings $H_0(\textbf{curl}, \Omega)' \hookrightarrow H^{-1}(\Omega)^3$ and $H_0(\text{div}, \Omega)' \hookrightarrow H^{-1}(\Omega)^3$ hold. Besides, given any $\boldsymbol{f} \in H^{-1}(\Omega)^3$, we know that there exist a unique $\boldsymbol{u} \in H_0^1(\Omega)^3$ and $\chi \in L^2(\Omega)$ such that

$$\boldsymbol{f} = -\Delta \boldsymbol{u} + \textbf{grad } \chi \quad \text{and} \quad \text{div } \boldsymbol{u} = 0 \ \text{ in } \Omega, \tag{4.1}$$

and satisfying the estimate

$$\|\boldsymbol{u}\|_{H^1(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)^3}.$$

Letting $\boldsymbol{\xi} = \mathbf{curl}\ \boldsymbol{u}$, we obtain the decomposition $\boldsymbol{f} = \mathbf{curl}\ \boldsymbol{\xi} + \mathbf{grad}\ \chi$ with div $\boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi} \cdot \boldsymbol{n} = 0$ on Γ. Since $\boldsymbol{\xi} \in L^2(\Omega)^3$ and $\chi \in L^2(\Omega)$, it follows that $\mathbf{curl}\ \boldsymbol{\xi} \in H_0(\mathbf{curl}, \Omega)'$ and $\mathbf{grad}\ \chi \in H_0(\mathrm{div}, \Omega)'$, so that

$$H^{-1}(\Omega)^3 = H_0(\mathbf{curl}, \Omega)' + H_0(\mathrm{div}, \Omega)'. \tag{4.2}$$

Prop 4.1. Assume that the boundary of the domain Ω is of class $\mathscr{C}^{1,1}$. Then, for any $\boldsymbol{f}$ in the dual space $H_0(\mathrm{div}, \Omega)'$, there exist a unique $\boldsymbol{u} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ and $\chi \in L^2(\Omega)$ solution to (4.1) and satisfying the estimate

$$\|\boldsymbol{u}\|_{H^2(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}.$$

Proof. Let $\boldsymbol{f}$ be in the dual space of $H_0(\mathrm{div}, \Omega)$. We know (see proposition 1 of [6]) that there exist $\boldsymbol{\psi} \in L^2(\Omega)^3$ and $\chi_0 \in L^2(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\ \chi_0 \quad \text{and } \|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\chi_0\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{4.3}$$

Thanks to the regularity of Γ, there exist $\boldsymbol{u} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ and $p \in H^1(\Omega)$ satisfying

$$\boldsymbol{\psi} = -\Delta\boldsymbol{u} + \mathbf{grad}\ p \quad \text{and} \quad \mathrm{div}\ \boldsymbol{u} = 0 \ \text{ in } \Omega, \tag{4.4}$$

with

$$\|\boldsymbol{u}\|_{H^2(\Omega)^3} + \|p\|_{H^1(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{\psi}\|_{L^2(\Omega)^3}.$$

Then $\boldsymbol{u}$ and $\chi = p + \chi_0$ satisfy the announced properties. □

We next consider the space

$$K_N(\Omega) := \{\boldsymbol{w} \in H_0(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega);\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ and div } \boldsymbol{w} = 0 \text{ in } \Omega\}$$

which is of dimension I. As shown in proposition 3.18 of [1], this space is spanned by the vector fields $\mathbf{grad}\ q_i^N$, $1 \leq i \leq N$, where each function $q_i^N \in H^1(\Omega)$ is the unique solution to the problem

$$\begin{array}{lll} \Delta q_i^N = 0 & \text{in } \Omega, & \\ q_i^N = 0 & \text{on } \Gamma_0, & \langle \partial_n q_i^N\ ,\ 1\rangle_{\Gamma_0} = -1, \\ q_i^N = \text{constant} & \text{on } \Gamma_k, & \langle \partial_n q_i^N\ ,\ 1\rangle_{\Gamma_k} = \delta_{ik},\ \text{for } 1 \leq k \leq I. \end{array} \tag{4.5}$$

Theorem 4.1. *Given any distribution* $\boldsymbol{f}$ *in the dual space* $H_0(\mathbf{curl}, \Omega)'$ *that satisfies*

$$\mathit{div}\ \boldsymbol{f} = 0 \ \text{ in } \Omega \text{ and } {}_{H_0(\mathbf{curl},\Omega)'}\langle \boldsymbol{f}\ ,\ \boldsymbol{v}\rangle_{H_0(\mathbf{curl},\Omega)} = 0 \quad \text{for all} \quad \boldsymbol{v} \in K_N(\Omega), \tag{4.6}$$

there exists a vector potential $\boldsymbol{\xi}$ *in* $L^2(\Omega)^3$ *such that*

$$\boldsymbol{f} = \boldsymbol{curl}\ \boldsymbol{\xi}, \quad \text{with} \quad \mathit{div}\ \boldsymbol{\xi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\xi} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{4.7}$$

and such that the following estimate holds:

$$\|\boldsymbol{\xi}\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H_0(\mathbf{curl},\Omega)'}. \tag{4.8}$$

Proof. Let $\boldsymbol{f}$ be in the dual space $H_0(\mathbf{curl},\Omega)'$. According to corollary 5 of [6], there exist $\boldsymbol{\psi} \in L^2(\Omega)^3$ and $\boldsymbol{\xi}_0 \in L^2(\Omega)^3$ with div $\boldsymbol{\xi}_0 = 0$ in Ω and $\boldsymbol{\xi}_0 \cdot \boldsymbol{n} = 0$ on Γ, such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\ \boldsymbol{\xi}_0$ and such that the estimate

$$\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\boldsymbol{\xi}_0\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H_0(\mathbf{curl},\Omega)'}$$

holds. Thanks to the density of $\mathscr{D}(\Omega)^3$ in $H_0(\mathbf{curl},\Omega)$, we deduce that for all $\boldsymbol{v} \in K_N(\Omega)$, we have

$$_{H_0(\mathbf{curl},\Omega)'}\langle \mathbf{curl}\ \boldsymbol{\xi}_0\ ,\ \boldsymbol{v}\rangle_{H_0(\mathbf{curl},\Omega)} = 0.$$

Since div $\boldsymbol{f} = 0$, it follows that div $\boldsymbol{\psi} = 0$. Then, thanks to the orthogonality relations

$$_{H_0(\mathbf{curl},\Omega)'}\langle \boldsymbol{f}\ ,\ \mathbf{grad}\ q_i^N\rangle_{H_0(\mathbf{curl},\Omega)} = 0 \quad \text{for all } i = 1,\dots,I,$$

the relations $\langle \boldsymbol{\psi}\cdot\boldsymbol{n}\ ,\ 1\rangle_{\Gamma_i} = 0$ are satisfied for all $i = 1,\dots,I$. There thus exists a vector potential $\boldsymbol{\varphi} \in L^2(\Omega)^3$ (see theorem 3.12 of [1]) such that $\boldsymbol{\psi} = \mathbf{curl}\ \boldsymbol{\varphi}$, with div $\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ, and such that

$$\|\boldsymbol{\varphi}\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{\psi}\|_{L^2(\Omega)^3}.$$

Hence, the vector field $\boldsymbol{\xi} = \boldsymbol{\xi}_0 + \boldsymbol{\varphi}$ possesses the announced properties. □

Remark 4.1. The previous theorem has been established in [6] when Γ is connected, in which case $K_N = \{\mathbf{0}\}$.

For any integer $m \geq 0$, let us introduce the space

$$H_0^m(\mathbf{curl},\Omega) := \{\boldsymbol{v} \in H_0(\mathbf{curl},\Omega);\ \mathbf{curl}\ \boldsymbol{v} \in H_0^m(\Omega)^3\}.$$

We can easily characterize its dual space, as:

$$H^{-m}(\mathbf{curl},\Omega) = \{\boldsymbol{\psi} + \mathbf{curl}\ \boldsymbol{\xi};\ \boldsymbol{\psi} \in H_0(\mathbf{curl},\Omega)',\ \boldsymbol{\xi} \in H^{-m}(\Omega)^3\}.$$

We can prove that $\mathscr{D}(\Omega)^3$ is dense in $H_0^m(\mathbf{curl},\Omega)$ and that the following Green formula holds: for any $\boldsymbol{\xi} \in H^{-m}(\mathbf{curl},\Omega)$ and $\boldsymbol{v} \in H_0^m(\mathbf{curl},\Omega)$

$$_{H^{-m}(\mathbf{curl},\Omega)}\langle \mathbf{curl}\boldsymbol{\xi},\ \boldsymbol{v}\rangle_{H_0^m(\mathbf{curl},\Omega)} +\ _{H^{-m}(\Omega)^3}\langle \boldsymbol{\xi},\ \mathbf{curl}\boldsymbol{v}\rangle_{H_0^m(\Omega)^3} = 0. \tag{4.9}$$

By using the decomposition (1.1) with $(m+1)$ instead of m, it is easy to prove (as in Section 2) that

$$H^{-m-1}(\Omega)^3 = H^{-m}(\mathbf{curl},\Omega) + H^{-m}(\mathrm{div}\ ,\Omega), \quad \text{for } m \geq 1.$$

Theorem 4.2. *For any distribution $\boldsymbol{f}$ in the dual space $H^{-1}(\mathbf{curl}, \Omega)$ that satisfies*

$$\textit{div}\, \boldsymbol{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \boldsymbol{f}\ ,\ \boldsymbol{v} \rangle\ = 0, \quad \text{for all } \boldsymbol{v} \in K_N(\Omega) \tag{4.10}$$

there exists a vector potential $\boldsymbol{\xi}$ in $H^{-1}(\Omega)^3$ such that

$$\boldsymbol{f} = \boldsymbol{curl}\ \boldsymbol{\xi}, \quad \textit{div}\ \boldsymbol{\xi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \|\boldsymbol{\xi}\|_{H^{-1}(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H^{-1}(\mathbf{curl},\Omega)}. \tag{4.11}$$

Proof. Given $\boldsymbol{f}$ in the dual space $H^{-1}(\mathbf{curl}, \Omega)$, there exist $\boldsymbol{f}_0 \in H_0(\mathbf{curl}, \Omega)'$ and $\boldsymbol{\xi}_0 \in H^{-1}(\Omega)^3$ such that $\boldsymbol{f} = \boldsymbol{f}_0 + \mathbf{curl}\ \boldsymbol{\xi}_0$, and satisfying the estimate

$$\|\boldsymbol{f}_0\|_{H_0(\mathbf{curl},\Omega)'} + \|\boldsymbol{\xi}_0\|_{H^{-1}(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H^{-1}(\mathbf{curl},\Omega)}.$$

Since $\boldsymbol{\xi}_0 \in H^{-1}(\Omega)^3$, there exists $\boldsymbol{\theta}_0 \in L^2(\Omega)^3$ satisfying div $\boldsymbol{\theta}_0 = 0$ in Ω, $\boldsymbol{\theta}_0 \cdot \boldsymbol{n} = 0$ on Γ, and there exists $\chi \in L^2(\Omega)$ such that $\boldsymbol{\xi}_0 = \mathbf{curl}\ \boldsymbol{\theta}_0 + \mathbf{grad}\ \chi$ and

$$\|\boldsymbol{\theta}_0\|_{L^2(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{\xi}_0\|_{H^{-1}(\Omega)^3}.$$

Since $\boldsymbol{f}_0 \in H_0(\mathbf{curl}, \Omega)'$, then $\boldsymbol{f}_0 = \boldsymbol{\psi}_0 + \mathbf{curl}\ \boldsymbol{\varphi}_0$, with $\boldsymbol{\psi}_0 \in L^2(\Omega)^3$, $\boldsymbol{\varphi}_0 \in L^2(\Omega)^3$, div $\boldsymbol{\varphi}_0 = 0$ in Ω, $\boldsymbol{\varphi}_0 \cdot \boldsymbol{n} = 0$ on Γ and

$$\|\boldsymbol{\psi}_0\|_{L^2(\Omega)^3} + \|\boldsymbol{\varphi}_0\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{f}_0\|_{H_0(\mathbf{curl},\Omega)'}.$$

Then $\boldsymbol{f} = \boldsymbol{\psi}_0 + \mathbf{curl}\ \boldsymbol{\varphi}_0 + \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{\theta}_0 = \boldsymbol{\psi}_0 + \mathbf{curl}\ \boldsymbol{\mu}$, with $\boldsymbol{\mu} = \boldsymbol{\varphi}_0 + \mathbf{curl}\ \boldsymbol{\theta}_0$, div $\boldsymbol{\mu} = 0$ in Ω, and the estimate

$$\|\boldsymbol{\psi}_0\|_{L^2(\Omega)^3} + \|\boldsymbol{\mu}\|_{H^{-1}(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H^{-1}(\mathbf{curl},\Omega)}$$

holds.

Thanks to the density of $\mathscr{D}(\Omega)^3$ in $H_0^1(\mathbf{curl}, \Omega)$, we infer that

$${}_{H^{-1}(\mathbf{curl},\Omega)}\langle \mathbf{curl}\ \boldsymbol{\mu}\ ,\ \boldsymbol{v} \rangle_{H_0^1(\mathbf{curl},\Omega)}\ = 0, \quad \text{for all } \boldsymbol{v} \in K_N(\Omega).$$

Since div $\boldsymbol{f} = 0$, div $\boldsymbol{\psi}_0 = 0$ and therefore the condition $\langle \boldsymbol{\psi}_0 \cdot \boldsymbol{n}\ , 1 \rangle_{\Gamma_i} = 0$ is automatically satisfied for any $i = 0, \dots, I$. Then by (3.1), there exists a vector potential $\boldsymbol{\varphi} \in L^2(\Omega)^3$ such that

$$\boldsymbol{\psi}_0 = \mathbf{curl}\ \boldsymbol{\varphi}, \quad \text{div}\ \boldsymbol{\varphi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma,$$

and

$$\|\boldsymbol{\varphi}\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{\psi}_0\|_{L^2(\Omega)^3}.$$

Hence, the vector field $\boldsymbol{\xi} = \boldsymbol{\mu} + \boldsymbol{\varphi}$ satisfies the announced properties. □

More generally, we can prove:

Theorem 4.3. *Given any integer $m \geq 0$ and any distribution $\boldsymbol{f}$ in the dual space $H^{-m}(\mathbf{curl}, \Omega)$ that satisfies (4.10), there exists a vector potential $\boldsymbol{\xi}$ in $H^{-m}(\Omega)^3$ such that*

$$\boldsymbol{f} = \boldsymbol{curl}\ \boldsymbol{\xi}, \quad \text{with} \qquad \textit{div}\ \boldsymbol{\xi} = 0 \quad \text{in } \Omega, \quad \text{and } \|\boldsymbol{\xi}\|_{H^{-m}(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H^{-m}(\mathbf{curl},\Omega)}.$$

5. Weak versions of Korn's inequality

Finally, we consider tensor fields. The next theorem generalizes theorem 3.2 of [8] and theorem 7 of [3] to Sobolev spaces with negative exponents.

In what follows, the subscript $_s$ denotes a space of symmetrix matrix fields.

Theorem 5.1. *Assume that Ω is simply-connected. Given an integer $m \geq 0$, let $\boldsymbol{e} = (e_{ij}) \in H_s^{-m}(\Omega)^{3\times 3}$ be a symmetric matrix field that satisfies the following compatibility conditions for all $i, j, k, l \in \{1, 2, 3\}$:*

$$\mathscr{R}_{ijkl} := \frac{\partial^2 e_{ik}}{\partial x_l \partial x_j} + \frac{\partial^2 e_{jl}}{\partial x_k \partial x_i} - \frac{\partial^2 e_{jk}}{\partial x_l \partial x_i} - \frac{\partial^2 e_{il}}{\partial x_k \partial x_j} = 0 \quad \text{in} \quad H^{-m-2}(\Omega). \tag{5.1}$$

Then there exists a vector field $\boldsymbol{v} \in H^{-m+1}(\Omega)^3$ such that $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ and $\boldsymbol{v}$ is unique up to vector fields in the space $R(\Omega) = \{\boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega;\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}$.

Proof. Given $\boldsymbol{e} = (e_{ij}) \in H_s^{-m}(\Omega)^{3\times 3}$, let $f_{ijk} := \partial_j e_{ik} - \partial_i e_{jk}$. Then $f_{ijk} \in H^{-m-1}(\Omega)$ and, thanks to the compatibility conditions (5.1), we have

$$\frac{\partial}{\partial x_l} f_{ijk} = \frac{\partial}{\partial x_k} f_{ijl}.$$

Hence the implication (iii) $\Longrightarrow$ (iv) in theorem 1.1 shows that there exist distributions $p_{ij} \in H^{-m}(\Omega)$, unique up to additive constants, such that $\partial_k p_{ij} = f_{ijk}$.

Besides, since $\partial_k p_{ij} = -\partial_k p_{ji}$, we can choose the distributions p_{ij} in such a way that $p_{ij} + p_{ji} = 0$. Noting that the distributions $q_{ij} := e_{ij} + p_{ij}$ belong to $H^{-m}(\Omega)$ and satisfy $\partial_k q_{ij} = \partial_j q_{ik}$, we again resort to theorem 1.1 to assert the existence of distributions $v_i \in H^{-m+1}(\Omega)$, unique up to additive constants, such that $\partial_j v_i = q_{ij}$. □

For any integer $m \geq 0$, let

$$E(\Omega) := \{\boldsymbol{e} \in H_s^{-m}(\Omega)^{3\times 3},\ \mathscr{R}_{ijkl}(\boldsymbol{e}) = 0\}$$

and

$$\dot{H}^{-m+1}(\Omega)^3 := H^{-m+1}(\Omega)^3 / R(\Omega).$$

By the previous theorem, given any $\boldsymbol{e} = (e_{ij}) \in E(\Omega)$, there exists a unique $\dot{\boldsymbol{v}} = (\dot{v}_i) \in \dot{H}^{-m+1}(\Omega)^3$ such that $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$. We may thus define a linear mapping $\mathcal{F} : E(\Omega) \to \dot{H}^{-m+1}(\Omega)^3$ by $\mathcal{F}(\boldsymbol{e}) = \dot{\boldsymbol{v}}$. Using the same method as in [8], we can then prove the following Korn's inequality in Sobolev spaces with negative exponents:

Theorem 5.2. *The linear mapping $\mathcal{F} : E(\Omega) \to \dot{H}^{-m+1}(\Omega)^3$ is an isomorphism. Besides, there exists a constant $C \geq 0$ such that*

$$\inf_{\boldsymbol{r} \in R(\Omega)} \|\boldsymbol{v} + \boldsymbol{r}\|_{H^{-m+1}(\Omega)^3} \leq C \sum_{i,j} \|\varepsilon_{ij}(\boldsymbol{v})\|_{H^{-m}(\Omega)} \quad \text{for all } \boldsymbol{v} \in H^{-m+1}(\Omega)^3,$$

and

$$\|\boldsymbol{v}\|_{H^{-m+1}(\Omega)^3} \leq C(\|\boldsymbol{v}\|_{H^{-m}(\Omega)^3} + \sum_{i,j} \|\varepsilon_{ij}(\boldsymbol{v})\|_{H^{-m}(\Omega)}) \quad \text{for all } \boldsymbol{v} \in H^{-m+1}(\Omega)^3$$

where $\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$.

Remark 5.1. Analogous techniques would likewise extend to Sobolev spaces with negative exponents the results obtained for non-simply connected domains in [9], [12] and [13].

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