Mathematical Models and Methods in Applied Sciences

A NEW COMPACTNESS RESULT FOR ELECTROMAGNETIC WAVES. APPLICATION TO THE TRANSMISSION PROBLEM BETWEEN DIELECTRICS AND METAMATERIALS

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We consider the time-harmonic Maxwell equations, involving wave transmission between media with opposite sign dielectric and/or magnetic coefficients. We prove that, in the case of sign-shifting dielectric coefficients, the space of electric fields is compactly embedded in L^2. We build a three-field variational formulation equivalent to Maxwell system for sign-shifting magnetic coefficients and show that, under some suitable conditions, the formulation fits into the coercive plus compact framework.

Keywords: electromagnetic wave transmission problem; sign-shifting dielectric coefficient; sign-shifting magnetic coefficient; left-hand materials.

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Introduction

In recent years, a growing interest for new materials has arisen. At particular frequencies, they behave like materials with negative electric permittivity ϵ or/and negative magnetic permeability μ. They include superconductors, left-handed materials [16], etc. As a consequence, most mathematical approaches fail to resolve the corresponding electromagnetic models. Accordingly, these ”negative” materials raise many challenging questions, from both mathematical and numerical points of view.
We consider here the particular case of an interface between ”positive” (dielectric) and ”negative” materials. Our main objective is to study those interface problems, and to provide variational settings, which can be easily discretized, for instance via finite element methods.
In 2d configurations, they reduce to scalar problems involving terms like $-\mathrm{div}\,(\epsilon\nabla\cdot)$. Those scalar problems have been thoroughly investigated: we refer the reader to Refs. 9, 15, 5. It is now well understood that well-posedness depends crucially on the ratio of the values of ϵ taken from both sides of the interface. On the one hand, when its value is precisely equal to -1, the interface problem is ill-posed [9]. On the other hand, well-posedness in the Fredholm sense has been obtained when its absolute value is small enough (or large enough). This

result has been achieved under very weak assumptions (Lipschitz interface, L^∞ coefficient ϵ) in Ref. 3 where a variational formulation with an additional vector unknown is used (see also Ref. 4 for an alternate proof).
The main objective of this paper is to extend these results to the case of the 3d Maxwell equations, especially when both ϵ and μ exhibit a sign-shift at the interface. With no loss of generality, we shall focus on the electric field formulation. Here, the sign-shift of μ in the term $\boldsymbol{curl}\left(\frac{1}{\mu}\boldsymbol{curl}\,\cdot\right)$ raises similar difficulties as in the scalar case. In addition, a new difficulty appears in the constraint on the field $\operatorname{div}(\epsilon\boldsymbol{e}) = 0$, coming now from the sign-shift of ϵ.

The outline of the paper will be as follows. In section 1, we introduce the *ad hoc* mathematical framework of our study. In section 2, we specifically target the constraint involving a sign-shifting electric permittivity ϵ: we prove a new, Weber-like, compactness embedding result for the space of electric fields. Then, in the next section, we focus instead on Maxwell's equations with a sign-shifting magnetic permeability μ. We build a three-field variational formulation, thus generalizing the approach advocated in Ref. 3 for the scalar problem. Assembling the previous results allows us to prove the well-posedness of this variational formulation, in the general case of sign-shifting electric permittivity and magnetic permeability. Finally we give some concluding remarks, before recalling some elementary results in the Annex.

1. Derivation of the model and mathematical framework

Let Ω be an open, bounded and connected set of $\mathbb{R}^3$ with a Lipschitz polyhedral[a] $\partial\Omega$; let $\boldsymbol{n}$ be the unit outward normal to $\partial\Omega$.
It is assumed that the domain Ω can be partitioned into two simply connected sub-domains Ω_1 and Ω_2 with Lipschitz polyhedral boundaries: $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$; let $\boldsymbol{n}_i$ be the unit outward normal to $\partial\Omega_i$, $i = 1, 2$. Then, define the interface $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$. Finally, we introduce $\Gamma_i = \partial\Omega_i \setminus \Sigma$; it is assumed that Γ_1 and Γ_2 are connected.
Both assumptions on the geometry (Ω_1 and Ω_2 simply connected, Γ_1 and Γ_2 connected) can be removed. We introduce them for the ease of exposition.
In the sequel, we shall introduce functional spaces with elements defined on $\mathcal{O}$, or on (a part of) its boundary $\partial\mathcal{O}$, where $\mathcal{O}$ stands for an open, bounded and connected set with a Lipschitz polyhedral boundary. Typically, $\mathcal{O} \in \{\Omega, \Omega_1, \Omega_2\}$.
Hereafter we adopt the same notations as in Ref. 3: for all quantities v defined on Ω, $v_i := v|_{\Omega_i}$ (for $i = 1, 2$) and

$$\begin{cases} \text{If } v_i > 0 \text{ a. e. in } \Omega_i\text{: } v_i^{max} = \sup_{x\in\Omega_i} v_i(x),\ v_i^{min} = \inf_{x\in\Omega_i} v_i(x). \\ \text{If } v_i < 0 \text{ a. e. in } \Omega_i\text{: } v_i^{+} = \sup_{x\in\Omega_i} |v_i(x)|,\ v_i^{-} = \inf_{x\in\Omega_i} |v_i(x)|. \end{cases}$$

[a]Results can be generalized to the case of an open, bounded and connected set of $\mathbb{R}^3$ with a Lipschitz *curvilinear* polyhedral boundary. For short, we simply write that the boundaries are Lipschitz polyhedral boundaries.

Let ϵ and μ be respectively the dielectric permittivity and the magnetic permeability: we assume that ϵ, ϵ^{-1}, μ and μ^{-1} all belong to $L^\infty(\Omega)$. The time-harmonic Maxwell equations ($\omega \neq 0$), with perfect conductor boundary condition on $\partial\Omega$, read:

$$\left\{\begin{array}{l} \imath\omega\epsilon\mathcal{E} - \boldsymbol{curl}\mathcal{H} = -\mathcal{J} \\ \imath\omega\mu\mathcal{H} + \boldsymbol{curl}\mathcal{E} = 0 \\ \operatorname{div}(\epsilon\mathcal{E}) = \rho \\ \operatorname{div}(\mu\mathcal{H}) = 0 \\ \mathcal{E} \times \boldsymbol{n}|_{\partial\Omega} = 0 \end{array}\right. , \tag{1.1}$$

where $(\mathcal{E}, \mathcal{H})$ is the electromagnetic field. Quantities ρ and $\mathcal{J}$, respectively the charge and current densities, satisfy the relation $i\omega\rho + \operatorname{div}\mathcal{J} = 0$. We assume that $\rho \in L^2(\Omega)$. Provided $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{J}$ belong to $L^2(\Omega)$ component by component (we shall write $\mathcal{E} \in \boldsymbol{L}^2(\Omega)$ hereafter), one finds that $\mathcal{E}$ and $\mathcal{H}$ are both in $\boldsymbol{H}(\boldsymbol{curl};\Omega)$. Recall that

$$\left\{\begin{array}{l} \boldsymbol{H}(\boldsymbol{curl};\mathcal{O}) := \{\boldsymbol{p} \in \boldsymbol{L}^2(\mathcal{O}) \mid \boldsymbol{curl}\boldsymbol{p} \in \boldsymbol{L}^2(\mathcal{O})\}, \\ \boldsymbol{H}(\operatorname{div};\mathcal{O}) := \{\boldsymbol{p} \in \boldsymbol{L}^2(\mathcal{O}) \mid \operatorname{div}\boldsymbol{p} \in L^2(\mathcal{O})\}. \end{array}\right.$$

The norm on $\boldsymbol{H}(op;\mathcal{O})$ (for $op \in \{\boldsymbol{curl}, \operatorname{div}\}$) is equal to the graph norm.
We also introduce

$$\boldsymbol{H}_0(\boldsymbol{curl};\mathcal{O}) := \{\boldsymbol{p} \in \boldsymbol{H}(\boldsymbol{curl};\mathcal{O}) \mid \boldsymbol{p} \times \boldsymbol{n}|_{\partial\mathcal{O}} = 0\}.$$

Indeed, thanks to the boundary condition on $\mathcal{E}$, the electric field belongs to $\boldsymbol{H}_0(\boldsymbol{curl};\Omega)$. One can eliminate one of the two fields (below, $\mathcal{H}$), to find an *equivalent*, second order system of equations:

$$\left\{\begin{array}{l} \omega^2\epsilon\mathcal{E} - \boldsymbol{curl}\left(\dfrac{1}{\mu}\boldsymbol{curl}\mathcal{E}\right) = \imath\omega\mathcal{J} \text{ in } \Omega \\ \operatorname{div}(\epsilon\mathcal{E}) = \rho \text{ in } \Omega \\ \mathcal{E} \times \boldsymbol{n}|_{\partial\Omega} = 0 \end{array}\right. . \tag{1.2}$$

Let us consider the "electrostatic-like problem":
Find $\phi_e \in H_0^1(\Omega)$ *such that*

$$\operatorname{div}(\epsilon\nabla\phi_e) = \rho\,. \tag{1.3}$$

In the case when ϵ is a constant-sign element of $L^\infty(\Omega)$, solving the problem (1.3) is classical. When ϵ exhibits a sign-shift over Ω, (1.3) may be solved using the three-field variational formulation proposed in Refs. 3 (see also Refs. 4, 18 for another solution). More precisely, suppose for instance that $\epsilon_1 > 0$. Then, it is proven that the electrostatic problem is well-posed under the assumption that one of the two contrasts $R_1^\epsilon := \epsilon_2^-/\epsilon_1^{max}$ or $R_2^\epsilon := \epsilon_1^{min}/\epsilon_2^+$ is large enough. Note that this type of condition on contrasts (for ϵ and/or μ) will systematically appear throughout the paper as a sufficient condition.

By solving problem (1.3) and setting $\boldsymbol{e} = \mathcal{E} - \nabla\phi_e$, $\boldsymbol{j} = i\omega\mathcal{J} - \omega^2\epsilon\nabla\phi_e$, it is straightforward to prove that the system of equations (1.2) may be rewritten as

$$\left\{\begin{array}{l} \omega^2\epsilon\boldsymbol{e} - \boldsymbol{curl}\left(\frac{1}{\mu}\boldsymbol{curl}\boldsymbol{e}\right) = \boldsymbol{j} \text{ in } \Omega \\ \operatorname{div}(\epsilon\boldsymbol{e}) = 0 \text{ in } \Omega \\ \boldsymbol{e}\times\boldsymbol{n}|_{\partial\Omega} = 0 \end{array}\right. . \tag{1.4}$$

We note that, by contruction, $\boldsymbol{j} \in \boldsymbol{L}^2(\Omega)$ and $\operatorname{div}\boldsymbol{j} = 0$, and the field $\boldsymbol{e}$ belongs to the functional space $\boldsymbol{X}$ defined by

$$\boldsymbol{X} := \{\boldsymbol{p} \in \boldsymbol{H}_0(\boldsymbol{curl};\Omega) \mid \operatorname{div}(\epsilon\boldsymbol{p}) = 0 \text{ in } \Omega\}. \tag{1.5}$$

On the one hand, the "natural" variational formulation of (1.4) is:
Find $\boldsymbol{e} \in \boldsymbol{X}$ *such that*

$$\forall \boldsymbol{v} \in \boldsymbol{X},\ \left(\frac{1}{\mu}\boldsymbol{curl}\boldsymbol{e}, \boldsymbol{curl}\boldsymbol{v}\right)_{0,\Omega} - \omega^2(\epsilon\boldsymbol{e},\boldsymbol{v})_{0,\Omega} = -(\boldsymbol{j},\boldsymbol{v})_{0,\Omega}. \tag{1.6}$$

This "natural" formulation highlights the difficulties we have to cope with. The first one, if ϵ exhibits a sign-shift, since in this case, there exists no result ensuring that the embedding of the functional space $\boldsymbol{X}$ into $\boldsymbol{L}^2(\Omega)$ is compact. The second one, if μ exhibits a sign-shift, since $\left(\mu^{-1}\boldsymbol{curl}\boldsymbol{v}, \boldsymbol{curl}\boldsymbol{v}\right)_{0,\Omega}$ has no specific sign, so its coercivity does not hold. We note that the respective roles of ϵ and μ can be reversed, if one chooses instead to write the "natural" variational formulation in $\boldsymbol{h}$.

On the other hand, it is easy to prove that the system of equations (1.4) with a solution in $\boldsymbol{X}$ is equivalent to:
Find $(\boldsymbol{e}_1,\boldsymbol{e}_2) \in \boldsymbol{H}(\boldsymbol{curl};\Omega_1)\times\boldsymbol{H}(\boldsymbol{curl};\Omega_2)$ *such that*

$$\left\{\begin{array}{l} \omega^2\epsilon_1\boldsymbol{e}_1 - \boldsymbol{curl}\left(\frac{1}{\mu_1}\boldsymbol{curl}\boldsymbol{e}_1\right) = \boldsymbol{j}_1 \text{ in } \Omega_1 \\ \omega^2\epsilon_2\boldsymbol{e}_2 - \boldsymbol{curl}\left(\frac{1}{\mu_2}\boldsymbol{curl}\boldsymbol{e}_2\right) = \boldsymbol{j}_2 \text{ in } \Omega_2 \\ \operatorname{div}(\epsilon_i\boldsymbol{e}_i) = 0 \text{ in } \Omega_i \\ \boldsymbol{e}_i\times\boldsymbol{n}_{i|\Gamma_i} = 0 \quad i=1,2 \\ \boldsymbol{e}_1\times\boldsymbol{n}_{1|\Sigma} = \boldsymbol{e}_2\times\boldsymbol{n}_{1|\Sigma} \\ \epsilon_1\boldsymbol{e}_1\cdot\boldsymbol{n}_{1|\Sigma} = \epsilon_2\boldsymbol{e}_2\cdot\boldsymbol{n}_{1|\Sigma} \\ \frac{1}{\mu_1}\boldsymbol{curl}\boldsymbol{e}_1\times\boldsymbol{n}_{1|\Sigma} = \frac{1}{\mu_2}\boldsymbol{curl}\boldsymbol{e}_2\times\boldsymbol{n}_{1|\Sigma} \end{array}\right. . \tag{1.7}$$

In particular, $\boldsymbol{e}_i$ belongs to

$$\mathcal{X}_i := \{\boldsymbol{p} \in \boldsymbol{H}(\boldsymbol{curl};\Omega_i) \mid \operatorname{div}(\epsilon\boldsymbol{p}) = 0 \text{ in } \Omega_i,\ \boldsymbol{p}\times\boldsymbol{n}_{|\Gamma_i} = 0\}. \tag{1.8}$$

This setting shall be used hereafter, to build some well-posed variational formulations. In the sequel, we denote by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ respectively the canonical scalar product and norm of $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$, whereas we denote by $(\cdot,\cdot)_{0,i}$ and $\|\cdot\|_{0,i}$ resp. the canonical scalar product and norm of $L^2(\Omega_i)$ (and of $\boldsymbol{L}^2(\Omega_i)$), for $i = 1, 2$.

In order to build equivalent variational formulations, we shall need a few results on traces of vector fields of $\boldsymbol{H}(\boldsymbol{curl}; \mathcal{O})$. We follow Refs. 6, 7.

First (cf. Thms 3.9 and 3.10, and subsection 4.1 in Ref. 6), the following integration by parts formula holds

$$\forall \boldsymbol{f} \in \boldsymbol{H}(\boldsymbol{curl}; \mathcal{O}),\ \forall \boldsymbol{g} \in \boldsymbol{H}(\boldsymbol{curl}; \mathcal{O}),\\ (\boldsymbol{f}, \boldsymbol{curl} \boldsymbol{g})_{0,\mathcal{O}} - (\boldsymbol{curl} \boldsymbol{f}, \boldsymbol{g})_{0,\mathcal{O}} = \left\langle \boldsymbol{f} \times \boldsymbol{n}_{|\partial\mathcal{O}}, \boldsymbol{g}_{T|\partial\mathcal{O}} \right\rangle_{\partial\mathcal{O}},$$

with $\boldsymbol{g}_{T|\partial\mathcal{O}}$ the trace of the tangential components of $\boldsymbol{g}$. Above, $\langle\cdot,\cdot\rangle_{\partial\mathcal{O}}$ is a well-defined duality product between two different *ad hoc* Hilbert spaces of functions with support on the boundary $\partial\mathcal{O}$, and endowed with the "natural" – quotient – norm. Namely,

$$\boldsymbol{TL}(\partial\mathcal{O}) := \left\{ (\boldsymbol{p} \times \boldsymbol{n})_{|\partial\mathcal{O}} \mid \boldsymbol{p} \in \boldsymbol{H}(\boldsymbol{curl}; \mathcal{O}) \right\},\\ \boldsymbol{TR}(\partial\mathcal{O}) := \left\{ (\boldsymbol{p})_{T|\partial\mathcal{O}} \mid \boldsymbol{p} \in \boldsymbol{H}(\boldsymbol{curl}; \mathcal{O}) \right\}.$$

In addition, the trace mappings $\boldsymbol{v} \mapsto \boldsymbol{v} \times \boldsymbol{n}_{|\partial\mathcal{O}}$ and $\boldsymbol{v} \mapsto \boldsymbol{v}_{T|\partial\mathcal{O}}$ are onto, from $\boldsymbol{H}(\boldsymbol{curl}; \mathcal{O})$ to the same trace spaces, respectively $\boldsymbol{TL}(\partial\mathcal{O})$ and $\boldsymbol{TR}(\partial\mathcal{O})$ (cf. Thm 5.4 in Ref. 7).

Second (cf. Thms 3.15 and 3.16 and subsection 4.2 in Ref. 6), given $\gamma \subset \partial\mathcal{O}$ (also with a Lipschitz boundary $\partial\gamma$) and $\gamma' = \partial\mathcal{O} \setminus \bar{\gamma}$, consider $\boldsymbol{H}_{0,\gamma}(\boldsymbol{curl}; \mathcal{O}) := \{\boldsymbol{p} \in \boldsymbol{H}(\boldsymbol{curl}; \mathcal{O}) \mid \boldsymbol{p} \times \boldsymbol{n}_{|\gamma} = 0\}$; this space is endowed with the usual norm of $H(\boldsymbol{curl}; \mathcal{O})$. Then, one can prove the following integration by parts formula

$$\forall \boldsymbol{f} \in \boldsymbol{H}_{0,\gamma}(\boldsymbol{curl}; \mathcal{O}),\ \forall \boldsymbol{g} \in \boldsymbol{H}(\boldsymbol{curl}; \mathcal{O}),\\ (\boldsymbol{f}, \boldsymbol{curl} \boldsymbol{g})_{0,\mathcal{O}} - (\boldsymbol{curl} \boldsymbol{f}, \boldsymbol{g})_{0,\mathcal{O}} = \left\langle \boldsymbol{f} \times \boldsymbol{n}_{|\gamma'}, \boldsymbol{g}_{T|\gamma'} \right\rangle_{\gamma'},$$

The duality product $\langle\cdot,\cdot\rangle_{\gamma'}$ is again considered between appropriate Hilbert spaces:

$$\boldsymbol{TL}(\gamma') := \left\{ (\boldsymbol{p} \times \boldsymbol{n})_{|\gamma'} \mid \boldsymbol{p} \in \boldsymbol{H}_{0,\gamma}(\boldsymbol{curl}; \mathcal{O}) \right\},\\ \boldsymbol{TR}(\gamma') := \left\{ (\boldsymbol{p})_{T|\gamma'} \mid \boldsymbol{p} \in \boldsymbol{H}(\boldsymbol{curl}; \mathcal{O}) \right\}.$$

The trace mapping $\boldsymbol{v} \mapsto \boldsymbol{f} \times \boldsymbol{n}_{|\gamma'}$ is onto, from $\boldsymbol{H}_{0,\gamma}(\boldsymbol{curl}; \mathcal{O})$ to $\boldsymbol{TL}(\gamma')$ (cf. Thm 6.6 in Ref. 7).

Finally, for scalar fields that belong to $H^1(\mathcal{O})$, recall that

$$H^{1/2}_{00}(\gamma) := \{p \in H^{1/2}(\gamma) \mid \tilde{p} \in H^{1/2}(\partial\mathcal{O})\},$$

where $\tilde{p}$ is the continuation of p by zero to the whole boundary, is the "natural" space for traces on γ, whenever the trace vanishes on $\partial\mathcal{O} \setminus \gamma$. This Hilbert space is endowed with the "natural" norm $\|p\|_{H^{1/2}_{00}(\gamma)} := \|\tilde{p}\|_{H^{1/2}(\partial\mathcal{O})}$.

2. A compactness result for a sign-shifting ϵ: extension of the Weber embedding Theorem

When ϵ is sign-constant over the whole domain Ω, according to [17,13] the embedding of $\boldsymbol{X}$ into $\boldsymbol{L}^2(\Omega)$ is compact (we call this result the *Weber embedding Theorem*, as a tribute to the landmark paper of Weber [17]). However, when ϵ exhibits a sign-shift, there exists to our knowledge no result ensuring that the embedding of the functional space $\boldsymbol{X}$ into $\boldsymbol{L}^2(\Omega)$ is compact. We suppose in this section that $\epsilon_1 > 0$ and $\epsilon_2 < 0$. In order to extend further the range of our theory, we shall establish that, provided at least one of the two *global contrasts in* ϵ $R_1^\epsilon := \epsilon_2^- / \epsilon_1^{max}$ or $R_2^\epsilon := \epsilon_1^{min}/\epsilon_2^+$ is large enough, the embedding of the functional space

$$\boldsymbol{XY} := \{\boldsymbol{p} \in \boldsymbol{H}_0(\boldsymbol{curl};\Omega) \,|\, \mathrm{div}\,(\epsilon\boldsymbol{p}) \in L^2(\Omega)\}$$

into $\boldsymbol{L}^2(\Omega)$ is compact. In other words, we propose an extension of the *Weber embedding Theorem*. As a particular case, the embedding of $\boldsymbol{X}$ – which is a subset of $\boldsymbol{XY}$ – will be compact.

We follow the skeleton of the proof given by Hazard-Lenoir for the same result, when $\epsilon > 0$ a.e. (Appendix B of Ref. 13). For that, we study separately the case of curl-free elements (the space $\boldsymbol{Y}$ defined below) and the case of divergence-free elements (the space $\boldsymbol{X}$).

First, let us consider the embedding of

$$\boldsymbol{Y} := \{\boldsymbol{p} \in \boldsymbol{XY} \,|\, \boldsymbol{curl}\boldsymbol{p} = 0 \text{ in } \Omega\}$$

into $\boldsymbol{L}^2(\Omega)$.

Theorem 2.1. *The embedding of the functional space $\boldsymbol{Y}$ into $\boldsymbol{L}^2(\Omega)$ is compact if at least one of the global contrasts R_1^ϵ or R_2^ϵ is large enough.*

Proof: We carry out the proof in the case of a large contrast R_1^ϵ.
NB. The proof, in the case of a large contrast R_2^ϵ, proceeds symmetrically, with the roles of Ω_1 and Ω_2 reversed.
Let $\left(\boldsymbol{U}^k\right)_{k\in\mathbb{N}}$ be a bounded sequence of $\boldsymbol{Y}$. In particular, we deal with curl-free fields: as Ω is simply connected (see page 31 in Ref. 11) and as its boundary is connected, one can replace each $\boldsymbol{U}^k$ by $\nabla\varphi^k$, with $\varphi^k \in H_0^1(\Omega)$. Our aim is to prove that a subsequence of $(\nabla\varphi^k)_k$ converges in $\boldsymbol{L}^2(\Omega)$.
Note that, since finding this subsequence is an iterative process (one extracts a subsequence, then a subsubsequence, etc.), we keep the same notation for all subsequences of a given sequence.
By construction, φ^k solves
Find $\varphi^k \in H_0^1(\Omega)$ *such that*

$$\mathrm{div}\,(\epsilon\nabla\varphi^k) = \mathrm{div}\,(\epsilon\boldsymbol{U}^k) \text{ in } \Omega. \tag{2.1}$$

$\Big($According to Corollary 4.3 of Ref. 3, this problem is well-posed for a large contrast R_1^ϵ.$\Big)$
Let us consider p_i^k solution to

Find $p_i^k \in H^1_{0,\Gamma_i}(\Omega_i)$ *such that*

$$\mathrm{div}\,(\epsilon_i \nabla p_i^k) = \mathrm{div}\,(\epsilon_i \boldsymbol{U}_i^k) \text{ in } \Omega_i, \quad \epsilon_i \frac{\partial p_i^k}{\partial n}|_\Sigma = 0 \text{ in } \left(H^{1/2}_{00}(\Sigma)\right)'. \tag{2.2}$$

The sequence $(p_i^k)_k$ is bounded in $H^1(\Omega_i)$, thus by the Sobolev embedding Theorem we can extract a subsequence – still called $(p_i^k)_k$ – that converges in $L^2(\Omega_i)$. Moreover, since there holds

$$\left|\left(\epsilon_i \nabla(p_i^k - p_i^l), \nabla(p_i^k - p_i^l)\right)_{0,i}\right| \le \|\mathrm{div}\,(\epsilon_i(\boldsymbol{U}_i^k - \boldsymbol{U}_i^l))\|_{0,i}\|(p_i^k - p_i^l)\|_{0,i}\ ,$$

the subsequence $(p_i^k)_k$ actually converges in $H^1(\Omega_i)$. Let us introduce the auxiliary (sub)sequences of term $u_i^k := \varphi_i^k - p_i^k$; these fields belong to $H^1(\Omega_i)$. Moreover, (u_1^k, u_2^k) satisfies the system of equations

$$\begin{cases} \mathrm{div}\,(\epsilon_i \nabla u_i^k) = 0 \text{ in } \Omega_i \\ u_i^k|_{\Gamma_i} = 0 \\ u_1^k|_\Sigma - u_2^k|_\Sigma = h_\Sigma^k \\ \epsilon_1 \partial_{\boldsymbol{n}_1} u^k_{1|\Sigma} = -|\epsilon_2| \partial_{\boldsymbol{n}_1} u^k_{2|\Sigma} \end{cases}, \tag{2.3}$$

where the jump is equal to $h_\Sigma^k := -(p_1^k - p_2^k)|_\Sigma$. By construction, the sequence $(h_\Sigma^k)_k$ converges in $H^{1/2}_{00}(\Sigma)$.

Let us set $u^{kl} = u^k - u^l$ and $h_\Sigma^{kl} = h_\Sigma^k - h_\Sigma^l$. From the definition of u^k, we have, integrating by parts,

$$\begin{aligned}\left(\epsilon_2 \nabla u_2^{kl}, \nabla u_2^{kl}\right)_{0,2} &= \langle \epsilon_2 \partial_{\boldsymbol{n}_2} u_2^{kl}, u_2^{kl}\rangle_\Sigma \\ &= -_{H^{1/2}_{00}(\Sigma)'}\langle \epsilon_2 \frac{\partial u_2^{kl}}{\partial n}, h_\Sigma^{kl}\rangle_{H^{1/2}_{00}(\Sigma)} - \left(\epsilon_1 \nabla u_1^{kl}, \nabla u_1^{kl}\right)_{0,1}.\end{aligned} \tag{2.4}$$

This leads to the inequality

$$\epsilon_2^- \|\nabla u_2^{kl}\|^2_{0,2} \le \|\epsilon_2 \frac{\partial u_2^{kl}}{\partial n}\|_{H^{1/2}_{00}(\Sigma)'} \|h_\Sigma^{kl}\|_{H^{1/2}_{00}(\Sigma)} + \epsilon_1^{max}\|\nabla u_1^{kl}\|^2_{0,1}. \tag{2.5}$$

To bound the last term of (2.5), we use (implicitly) a Dirichlet-to-Neumann operator in the process: we go from Ω_1 to the interface Σ, and then from Σ to Ω_2. In other words, it is possible to consider (2.3) as a problem where the unknown is defined on Ω_1, i. e. u_1^k or u_1^l. From Proposition A.1, we can then verify that

$$\|\nabla u_1^{kl}\|_{0,1} \le \mathcal{C}^{int}_{\epsilon_1} \|u_1^{kl}\|_{H^{1/2}_{00}(\Sigma)} \le \mathcal{C}^{int}_{\epsilon_1}\left(\|u_2^{kl}\|_{H^{1/2}_{00}(\Sigma)} + \|h_\Sigma^{kl}\|_{H^{1/2}_{00}(\Sigma)}\right),$$

where the local contrast $\mathcal{C}^{int}_{\epsilon_1}$ is equal to the ratio $\epsilon_1^{max}/\epsilon_1^{min}$. Next, the trace operator, from $H^1_{0,\Gamma_2}(\Omega_2)$ to $H^{1/2}_{00}(\Sigma)$ is linear and continuous. Let $\mathcal{C}_{1/2}$ be its norm (see the Annex after Proposition A.1 for a discussion): one has $\|u_2^{kl}\|_{H^{1/2}_{00}(\Sigma)} \le \mathcal{C}_{1/2}\|\nabla u_2^{kl}\|_{0,2}$. Putting everything back together, we obtain

$$\begin{aligned}&\left(\epsilon_2^- - (\mathcal{C}^{int}_{\epsilon_1}\mathcal{C}_{1/2})^2 \epsilon_1^{max}\right)\|\nabla u_2^{kl}\|^2_{0,2} \\ &\quad \le \|h_\Sigma^{kl}\|_{H^{1/2}_{00}(\Sigma)}\Big\{\epsilon_1^{max}\left(\mathcal{C}^{int}_{\epsilon_1}\right)^2\left[\|h_\Sigma^{kl}\|_{H^{1/2}_{00}(\Sigma)} + 2\|u_2^{kl}\|_{H^{1/2}_{00}(\Sigma)}\right] \\ &\qquad + \|\epsilon_2 \frac{\partial u_2^{kl}}{\partial n}\|_{H^{1/2}_{00}(\Sigma)'}\Big\}.\end{aligned} \tag{2.6}$$

Recall that the subsequence $(u_2^k)_k$ is bounded in $H^1(\Omega_2)$, so $(u^k_{2|\Sigma})_k$ and $(\epsilon_2\partial_n u^k_{2|\Sigma})_k$ are bounded sequences of respectively $H^{1/2}_{00}(\Sigma)$ and $H^{1/2}_{00}(\Sigma)'$ (cf. Proposition A.1). As a consequence, the right-hand side of (2.6) goes to zero when $k, l \to \infty$. From the definition of R_1^ϵ, we deduce that, *provided*

$$R_1^\epsilon > (\mathcal{C}^{int}_{\epsilon_1}\mathcal{C}_{1/2})^2\,, \tag{2.7}$$

holds, the subsequence $(u_2^k)_k$ actually is a Cauchy sequence in $H^1(\Omega_2)$, so it converges. The same is true for $(p_2^k)_k$. Therefore, in the sub-domain Ω_2, we conclude that, since $\nabla\varphi_2^k = \nabla(u_2^k + p_2^k)$, the subsequence $(\nabla\varphi_2^k)_k$ converges in $\boldsymbol{L}^2(\Omega_2)$.

In order to end the proof, we must show that some subsequence of $(\nabla\varphi_1^k)$ also converges in $\boldsymbol{L}^2(\Omega_1)$. To this aim, let us recall the "natural" variational formulation of (2.1):
Find $\varphi^k \in H^1_0(\Omega)$ *such that*

$$\left(\epsilon\nabla\varphi^k, \nabla v\right)_0 = \left(\epsilon\boldsymbol{U}^k, \nabla v\right)_0, \quad \forall v \in H^1_0(\Omega). \tag{2.8}$$

Let us set $\boldsymbol{U}^{kl} = \boldsymbol{U}^k - \boldsymbol{U}^l$, $\varphi^{kl} = \varphi^k - \varphi^l$ and choose in (2.8) the test field $v = \varphi^{kl}$. We have, after integrating by parts,

$$(\epsilon_1\nabla\varphi_1^{kl}, \nabla\varphi_1^{kl})_{0,1} - (|\epsilon_2|\nabla\varphi_2^{kl}, \nabla\varphi_2^{kl})_{0,2} = -(\mathrm{div}\,(\epsilon\boldsymbol{U}^{kl}), \varphi^{kl})_0\,.$$

As $(\varphi^k)_k$ is bounded in $H^1(\Omega)$, we can extract a subsequence that converges in $L^2(\Omega)$. Since a (sub)sequence $(\varphi_2^k)_k$ converges in $H^1(\Omega_2)$ (*provided (2.7) holds*), the convergence of $(\nabla\varphi_1^k)_k$ in $\boldsymbol{L}^2(\Omega_1)$ follows.

Going back to $\boldsymbol{U}^k = \nabla\varphi^k$, we conclude that we can extract a subsequence of $(\boldsymbol{U}^k)_k$ that converges in $\boldsymbol{L}^2(\Omega)$. ■

Second, let us study the embedding of $\boldsymbol{X}$ into $\boldsymbol{L}^2(\Omega)$. In order to achieve a compactness result similar to Theorem 2.1, we add two geometry-related assumptions. For that, let

$$\boldsymbol{W}_T(\mathcal{O}) := \{\boldsymbol{w} \in \boldsymbol{H}(\boldsymbol{curl};\mathcal{O}) \mid \mathrm{div}\,\boldsymbol{w} \in L^2(\mathcal{O}),\ \boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\mathcal{O}} = 0\}\,.$$

We recall that, according to the Weber embedding Theorem, $\boldsymbol{W}_T(\mathcal{O})$ is compactly embedded in $\boldsymbol{L}^2(\mathcal{O})$.

The first assumption writes

$$\exists\chi \in \mathcal{C}^\infty(\bar\Omega) \text{ such that} \begin{cases} \chi = 1 \text{ in a neighborhood of } \Sigma \\ \boldsymbol{w} \mapsto \chi\boldsymbol{w} \text{ is continuous, from } \boldsymbol{W}_T(\Omega) \text{ to } \boldsymbol{H}^1(\Omega). \end{cases} \tag{2.9}$$

The second assumption writes

$$\exists\chi \in \mathcal{C}^\infty(\bar\Omega) \text{ s. t.} \begin{cases} \chi = 1 \text{ in a neighborhood of } \Sigma \\ \boldsymbol{w}_i \mapsto \chi\boldsymbol{w}_i \text{ is continuous, from } \boldsymbol{W}_T(\Omega_i) \text{ to } \boldsymbol{H}^1(\Omega_i),\ i = 1, 2. \end{cases} \tag{2.10}$$

These two assumptions are independent of the coefficients ϵ and μ. On the one hand, (2.9) is verified if, and only if, the domain Ω is locally convex *or* if the boundary $\partial\Omega$ is smooth in a neighborhood of the intersection of the interface Σ with $\partial\Omega$ (note that $\Sigma \cap \partial\Omega$ is equal to $\partial\Gamma_1 \cap \partial\Gamma_2$). One the other hand, (2.10) is verified if, and only if, both Ω_i are locally convex in a neighborhood $\partial\Gamma_1 \cap \partial\Gamma_2$ *and* if Σ is smooth. Finally, we consider that the two functions χ can be merged into a single one. Figure 1 pictures an example of an admissible

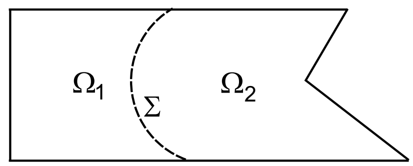

Fig. 1. An admissible configuration.

geometry: both assumptions (2.9) and (2.10) hold true. Figure 2 pictures two unadmissible geometries: on the left, assumption (2.9) is violated; on the right; assumption (2.10) is violated.

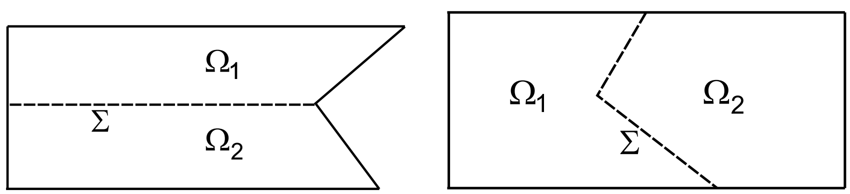

Fig. 2. Two unadmissible configurations.

Proposition 2.1. *Assume that (2.9) and (2.10) hold. Then, the embedding of the functional space $\boldsymbol{X}$ into $\boldsymbol{L}^2(\Omega)$ is compact if at least one of the global contrasts R_1^ϵ or R_2^ϵ is large enough.*

Remark 2.1. We shall explain how the first assumption (2.9) can be removed later on. In this way, a configuration like the one depicted on the left of Figure 2 is now admissible. We proceed in two steps, since its removal adds another layer of technicalities. The final result is established at Theorem 2.2.

Proof: We again carry out the proof in the case of a large contrast R_1^ϵ.
NB. Once again, the case of a large contrast R_2^ϵ is handled similarly.
Let $\left(\boldsymbol{W}^k\right)_{k\in\mathbb{N}}$ be a bounded sequence of $\boldsymbol{X}$. Let us introduce and focus on the problem below:
Find $\boldsymbol{\phi}^k \in \boldsymbol{L}^2(\Omega)$ *such that*

$$\left\{ \begin{array}{l} \boldsymbol{curl}\boldsymbol{\phi}^k = \epsilon \boldsymbol{W}^k \text{ in } \Omega \\ \operatorname{div} \boldsymbol{\phi}^k = 0 \text{ in } \Omega \\ \boldsymbol{\phi}^k \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \end{array} \right. . \qquad (2.11)$$

It is well-known that this "magnetic"-like problem is well-posed in the simply-connected domain Ω (see for instance Ref. 8). In particular, $\boldsymbol{\phi}^k$ belongs to $\boldsymbol{W}_T(\Omega)$, and $(\boldsymbol{\phi}^k)_k$ is bounded in $\boldsymbol{W}_T(\Omega)$ (and in $\boldsymbol{L}^2(\Omega)$).
Our aim is to prove that a (sub)sequence of $(\boldsymbol{curl}\boldsymbol{\phi}^k)_k$ converges in $\boldsymbol{L}^2(\Omega)$.
Note that since $\boldsymbol{W}^k$ belongs to $\boldsymbol{X}$, one has $\epsilon^{-1}\boldsymbol{curl}\boldsymbol{\phi}^k \times \boldsymbol{n}|_{\partial\Omega} = 0$, so $\boldsymbol{\phi}^k$ actually solves
Find $\boldsymbol{\phi}^k \in \boldsymbol{H}(\boldsymbol{curl};\Omega)$ *such that*

$$\begin{cases} \boldsymbol{curl}\left(\dfrac{1}{\epsilon}\boldsymbol{curl}\boldsymbol{\phi}^k\right) = \boldsymbol{curl}\boldsymbol{W}^k \text{ in } \Omega \\ \operatorname{div}\boldsymbol{\phi}^k = 0 \text{ in } \Omega \\ \boldsymbol{\phi}^k \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \\ \dfrac{1}{\epsilon}\boldsymbol{curl}\boldsymbol{\phi}^k \times \boldsymbol{n}|_{\partial\Omega} = 0 \end{cases}. \tag{2.12}$$

Following a procedure analogous to the proof of the previous Theorem, we are going to isolate the trace of $\boldsymbol{\phi}^k$ on the interface Σ, $\boldsymbol{\phi}^k_{|\Sigma}$, which belongs to $\boldsymbol{H}^{1/2}(\Sigma)$ (cf. assumption (2.9)).
In a first step, for $i = 1, 2$, let us consider $\boldsymbol{p}_i^k$ solution to the regularized problem
Find $\boldsymbol{p}_i^k \in \boldsymbol{H}(\boldsymbol{curl};\Omega_i)$ *such that*

$$\begin{cases} \boldsymbol{curl}\left(\dfrac{1}{\epsilon_i}\boldsymbol{curl}\boldsymbol{p}_i^k\right) - sg(\epsilon_i)\nabla(\operatorname{div}\boldsymbol{p}_i^k) = \boldsymbol{curl}\boldsymbol{W}_i^k \text{ in } \Omega_i \\ \boldsymbol{p}_i^k \cdot \boldsymbol{n}_i|_{\partial\Omega_i} = 0 \\ \dfrac{1}{\epsilon_i}\boldsymbol{curl}\boldsymbol{p}_i^k \times \boldsymbol{n}_i|_{\partial\Omega_i} = 0 \end{cases}. \tag{2.13}$$

Above, $sg(\epsilon_i)$ is equal to the sign of ϵ_i.
To begin with, let us prove a few results on the two sequences $(\boldsymbol{p}_i^k)_k$, with the help of the Annex. Set i to 1 or 2. According to Proposition A.2, the regularized problem defines a unique $\boldsymbol{p}_i^k$, the $\boldsymbol{W}_T(\Omega_i)$-norm of which is bounded by $\|\boldsymbol{curl}\boldsymbol{W}_i^k\|_{0,i}$. It follows from Corollary A.1 that there exists a subsequence, still denoted by $(\boldsymbol{p}_i^k)_k$, that converges in $\boldsymbol{W}_T(\Omega_i)$. Furthermore, from assumption (2.10), we deduce that $(\boldsymbol{p}_i^k|_\Sigma)_k$ converges in $\boldsymbol{H}^{1/2}(\Sigma)$. Finally (cf. Proposition A.2) $(\operatorname{div}\boldsymbol{p}_i^k)_k$ is bounded in $H^1(\Omega_i)$.
Now, let $\boldsymbol{g}_\Sigma^k := (\boldsymbol{p}_2^k - \boldsymbol{p}_1^k)_{|\Sigma}$: by construction, $(\boldsymbol{g}_\Sigma^k)_k$ converges in $\boldsymbol{H}^{1/2}(\Sigma)$ and one has $(\boldsymbol{g}_\Sigma^k)_T = \boldsymbol{g}_\Sigma^k$ (since $\boldsymbol{g}_\Sigma^k \cdot \boldsymbol{n}_1 = 0$). Next, we define the vector field $\boldsymbol{u}_i^k := \boldsymbol{\phi}_i^k - \boldsymbol{p}_i^k$. This field, which belongs to $\boldsymbol{H}(\boldsymbol{curl};\Omega_i)$, satisfies the system of equations

$$\begin{cases} \boldsymbol{curl}\left(\dfrac{1}{\epsilon_i}\boldsymbol{curl}\boldsymbol{u}_i^k\right) - sg(\epsilon_i)\nabla(\operatorname{div}\boldsymbol{u}_i^k) = 0 \text{ in } \Omega_i \\ \operatorname{div}\boldsymbol{u}_i^k = -\operatorname{div}\boldsymbol{p}_i^k \text{ in } \Omega_i \\ \boldsymbol{u}_i^k \cdot \boldsymbol{n}_{i|\Gamma_i} = 0 \\ \boldsymbol{u}_{1|\Sigma}^k - \boldsymbol{u}_{2|\Sigma}^k = \boldsymbol{g}_\Sigma^k \\ \dfrac{1}{\epsilon_i}\boldsymbol{curl}\boldsymbol{u}_i^k \times \boldsymbol{n}_{i|\Gamma_i} = 0 \\ \dfrac{1}{\epsilon_1}\boldsymbol{curl}\boldsymbol{u}_1^k \times \boldsymbol{n}_{1|\Sigma} = -\dfrac{1}{|\epsilon_2|}\boldsymbol{curl}\boldsymbol{u}_2^k \times \boldsymbol{n}_{1|\Sigma} \end{cases}, \tag{2.14}$$

Let us set $\boldsymbol{u}_i^{kl} := \boldsymbol{u}_i^k - \boldsymbol{u}_i^l$, $\boldsymbol{p}_i^{kl} := \boldsymbol{p}_i^k - \boldsymbol{p}_i^l$ and $\boldsymbol{g}_\Sigma^{kl} := \boldsymbol{g}_\Sigma^k - \boldsymbol{g}_\Sigma^l$. Our aim is to show that $(\boldsymbol{u}_i^{kl})_{kl}$ converges to zero in $\boldsymbol{H}(\boldsymbol{curl};\Omega_i)$ when $k,l \to \infty$.

Integrating by parts the first line of (2.14) for indices k and l with the test field $\boldsymbol{u}_i^{kl}$, we find

$$\left(\frac{1}{\epsilon_i}\boldsymbol{curl}\boldsymbol{u}_i^{kl}, \boldsymbol{curl}\boldsymbol{u}_i^{kl}\right)_{0,i} = \left\langle \frac{1}{\epsilon_i}\boldsymbol{curl}\boldsymbol{u}_i^{kl} \times \boldsymbol{n}_i, (\boldsymbol{u}_i^{kl})_T \right\rangle_\Sigma - sg(\epsilon_i)(\nabla(\operatorname{div}\boldsymbol{p}_i^{kl}), \boldsymbol{u}_i^{kl})_{0,i}. \qquad (2.15)$$

Then, we use two identities on Σ, namely ${\epsilon_1}^{-1}\boldsymbol{curl}\boldsymbol{u}_1^{kl} \times \boldsymbol{n}_1 = |\epsilon_2|^{-1}\boldsymbol{curl}\boldsymbol{u}_2^{kl} \times \boldsymbol{n}_2$, and $(\boldsymbol{u}_1^{kl})_T := \boldsymbol{g}_\Sigma^{kl} + (\boldsymbol{u}_2^{kl})_T$, to reach:

$$\frac{1}{\epsilon_1^{max}}\|\boldsymbol{curl}\boldsymbol{u}_1^{kl}\|_{0,1}^2 \le \left\langle \frac{1}{\epsilon_1}\boldsymbol{curl}\boldsymbol{u}_1^{kl} \times \boldsymbol{n}_1, \boldsymbol{g}_\Sigma^{kl} \right\rangle_\Sigma + \sum_{i=1,2}\|\nabla(\operatorname{div}\boldsymbol{p}_i^{kl})\|_{0,i}\|\boldsymbol{u}_i^{kl}\|_{0,i} + \frac{1}{\epsilon_2^-}\|\boldsymbol{curl}\boldsymbol{u}_2^{kl}\|_{0,2}^2. \qquad (2.16)$$

The next step consists in evaluating the terms in the right-hand-side.
As far as the first term on the right-hand side is concerned, note that $({\epsilon_1}^{-1}\boldsymbol{curl}\boldsymbol{u}_1^k \times \boldsymbol{n}_1|_\Sigma)_k$ is bounded in $\boldsymbol{TL}(\Sigma)$, since $({\epsilon_1}^{-1}\boldsymbol{curl}\boldsymbol{u}_1^k)_k$ is itself bounded in $\boldsymbol{H}(\boldsymbol{curl};\Omega_1)$. Moreover, as $(\boldsymbol{g}_\Sigma^k)_k$ converges in $\boldsymbol{TR}(\Sigma)$ this term goes to zero when $k,l \to \infty$.
About the second term, we recall that $(\nabla(\operatorname{div}\boldsymbol{p}_i^k))_k$ is a bounded sequence in $\boldsymbol{L}^2(\Omega_i)$. Then, one can extract a subsequence of $(\boldsymbol{u}_i^k)_k$ which converges in $\boldsymbol{L}^2(\Omega_i)$. Indeed, one has $\boldsymbol{u}_i^k = \boldsymbol{\phi}_i^k - \boldsymbol{p}_i^k$ by construction, and

- since $(\boldsymbol{p}_i^k)_k$ is bounded in $\boldsymbol{W}_T(\Omega_i)$, the Weber embedding Theorem tells us that there exists a subsequence that converges in $\boldsymbol{L}^2(\Omega_i)$;
- similarly, since $(\boldsymbol{\phi}^k)_k$ is bounded in $\boldsymbol{W}_T(\Omega)$, one can extract a subsequence that converges in $\boldsymbol{L}^2(\Omega)$. Its restriction to Ω_i converges in $\boldsymbol{L}^2(\Omega_i)$.

The third term in the right-hand side cannot be handled as straightforwardly. Let us proceed as follows. Thanks to the system of equations (2.14) governing $(\boldsymbol{u}_2^k)_k$, we infer first from Proposition A.3 that there exists $\mathcal{C}_{reg} > 0$ independent of $(\boldsymbol{u}_2^k)_k$ such that

$$\|\boldsymbol{curl}\boldsymbol{u}_2^{kl}\|_{0,2}^2 \le \mathcal{C}_{reg}\|\boldsymbol{u}_2^{kl}\|_{\boldsymbol{H}^{1/2}(\Sigma)} \le \mathcal{C}_{reg}\left(\|\boldsymbol{u}_1^{kl}\|_{\boldsymbol{H}^{1/2}(\Sigma)} + \|\boldsymbol{g}_\Sigma^{kl}\|_{\boldsymbol{H}^{1/2}(\Sigma)}\right).$$

Next, we infer from assumption (2.10) that there exists a constant $c > 0$ such that

$$\|\boldsymbol{curl}\boldsymbol{u}_2^{kl}\|_{0,2}^2 \le c\left(\|\boldsymbol{curl}\boldsymbol{u}_1^{kl}\|_{0,1}^2 + \|\operatorname{div}\boldsymbol{p}_1^{kl}\|_{0,1}^2 + \|\boldsymbol{g}_\Sigma^{kl}\|_{\boldsymbol{H}^{1/2}(\Sigma)}\right). \qquad (2.17)$$

From the above, we upgrade the estimate (2.16) to

$$\left(\frac{1}{\epsilon_1^{max}} - \frac{c}{\epsilon_2^-}\right)\|\boldsymbol{curl}\boldsymbol{u}_1^{kl}\|_{0,1}^2 \le \left\langle \frac{1}{\epsilon_1}\boldsymbol{curl}\boldsymbol{u}_1^{kl} \times \boldsymbol{n}_1, \boldsymbol{g}_\Sigma^{kl} \right\rangle_\Sigma + \sum_{i=1,2}\|\nabla(\operatorname{div}\boldsymbol{p}_i^{kl})\|_{0,i}\|\boldsymbol{u}_i^{kl}\|_{0,i} + \frac{c}{\epsilon_2^-}\left(\|\operatorname{div}\boldsymbol{p}_1^{kl}\|_{0,1}^2 + \|\boldsymbol{g}_\Sigma^{kl}\|_{\boldsymbol{H}^{1/2}(\Sigma)}\right).$$

Now, the last term in the right-hand side converges to zero when $k, l \to \infty$. Indeed,

- since $(\operatorname{div} \boldsymbol{p}_i^k)_k$ is bounded in $H^1(\Omega_i)$, we know from the Sobolev embedding Theorem that there exists a subsequence that converges in $L^2(\Omega_i)$;
- $\boldsymbol{g}_\Sigma^k$ converges in $\boldsymbol{H}^{1/2}(\Sigma)$.

Therefore, we see that

$$R_1^\epsilon > c \tag{2.18}$$

ensures that the subsequence $(\boldsymbol{curl} \boldsymbol{u}_1^k)_k$ actually is a Cauchy sequence in $\boldsymbol{L}^2(\Omega_1)$, so it converges. But we already proved that a subsequence $(\boldsymbol{curl} \boldsymbol{p}_1^k)_k$ converges in $\boldsymbol{L}^2(\Omega_1)$. As a consequence, $(\boldsymbol{curl} \boldsymbol{\phi}_1^k)_k$ does converge too. Therefore, in the sub-domain Ω_1, we conclude that, since $\boldsymbol{W}_1^k = {\epsilon_1}^{-1} \boldsymbol{curl} \boldsymbol{\phi}_1^k$, the subsequence $(\boldsymbol{W}_1^k)_k$ converges in $\boldsymbol{L}^2(\Omega_1)$.

To conclude the proof, we go back to (2.12) and we introduce the related "natural" variational formulation (cf. Ref. 8).
Find $\boldsymbol{\phi}^k \in \boldsymbol{W}_T(\Omega)$ *such that*

$$\left(\frac{1}{\epsilon} \boldsymbol{curl} \boldsymbol{\phi}^k, \boldsymbol{curl} \boldsymbol{w}\right)_0 = \left(\boldsymbol{curl} \boldsymbol{W}^k, \boldsymbol{w}\right)_0, \quad \forall \boldsymbol{w} \in \boldsymbol{W}_T(\Omega).$$

Set $\boldsymbol{\phi}^{kl} := \boldsymbol{\phi}^k - \boldsymbol{\phi}^l$ and $\boldsymbol{W}^{kl} := \boldsymbol{W}^k - \boldsymbol{W}^l$ and choose $\boldsymbol{w} = \boldsymbol{\phi}^{kl}$ in the above (for indices k and l). This yields

$$\left(\frac{1}{\epsilon_1} \boldsymbol{curl} \boldsymbol{\phi}_1^{kl}, \boldsymbol{curl} \boldsymbol{\phi}_1^{kl}\right)_{0,1} - \left(\frac{1}{|\epsilon_2|} \boldsymbol{curl} \boldsymbol{\phi}_2^{kl}, \boldsymbol{curl} \boldsymbol{\phi}_2^{kl}\right)_{0,2} = \left(\boldsymbol{curl} \boldsymbol{W}^{kl}, \boldsymbol{\phi}^{kl}\right)_0 .$$

We already noted that there exists a subsequence $(\boldsymbol{\phi}^k)_k$ that converges in $\boldsymbol{L}^2(\Omega)$. Since $(\boldsymbol{curl} \boldsymbol{\phi}_1^k)_k$ converges in $\boldsymbol{L}^2(\Omega_1)$, we infer that $(\boldsymbol{curl} \boldsymbol{\phi}_2^k)_k$ converges in $\boldsymbol{L}^2(\Omega_2)$. Thus $(\boldsymbol{W}_2^k)_k$ converges in $\boldsymbol{L}^2(\Omega_2)$, and so does $(\boldsymbol{W}^k)_k$ in $\boldsymbol{L}^2(\Omega)$, which ends the proof. ■

Theorem 2.2. *Assume that (2.10) holds. Then, the embedding of the functional space* $\boldsymbol{X}$ *into* $\boldsymbol{L}^2(\Omega)$ *is compact if at least one of the global contrasts* R_1^ϵ *or* R_2^ϵ *is large enough.*

Proof: We follow step by step the proof of Proposition 2.1, bearing in mind that, on the one hand, since assumption (2.10) still holds, all the $\boldsymbol{H}^1(\Omega_i)$-regularity results on $(\boldsymbol{p}_i^k)_k$ remain valid. On the other hand, without assumption (2.9), $\boldsymbol{\phi}_{|\Sigma}^k$ does not automatically belong to $\boldsymbol{H}^{1/2}(\Sigma)$. To address this difficulty, we rely on the continuous decomposition of elements of $\boldsymbol{W}_T(\Omega)$ into a regular part and a gradient part, first obtained by Birman and Solomyak[2]. Consider

$$\boldsymbol{W}_T^{reg}(\Omega) := \boldsymbol{W}_T(\Omega) \cap \boldsymbol{H}^1(\Omega),\ \Psi := \left\{\psi \in H^1(\Omega)/\mathbb{R} \mid \Delta\psi \in L^2(\Omega),\ \partial_n \psi_{|\partial\Omega} = 0\right\}.$$

The space of potentials Ψ can be endowed with the equivalent norm $\|\psi\|_\Psi := \|\Delta\psi\|_0$. According to Ref. 2, one can introduce the continuous splitting $\boldsymbol{W}_T(\Omega) = \boldsymbol{W}_T^{reg}(\Omega) + \nabla\Psi$, in the following sense:

$$\begin{cases} \exists \mathcal{C}_{BS} > 0,\ \forall \boldsymbol{w} \in \boldsymbol{W}_T(\Omega), \exists (\boldsymbol{w}_R, \psi) \in \boldsymbol{W}_T^{reg}(\Omega) \times \Psi, \\ \boldsymbol{w} = \boldsymbol{w}_R + \psi,\ \|\boldsymbol{w}_R\|_{\boldsymbol{W}_T(\Omega)} + \|\psi\|_\Psi \le \mathcal{C}_{BS} \|\boldsymbol{w}\|_{\boldsymbol{W}_T(\Omega)} \end{cases} .$$

In other words, $\nabla\Psi$ contains the singular parts of elements of $\boldsymbol{W}_T(\Omega)$ (if Ω is not convex). Thus, one can write the continuous splittings, for all k,

$$\boldsymbol{\phi}^k = \boldsymbol{\phi}_R^k + \nabla\psi^k,\ (\boldsymbol{\phi}_R^k, \psi^k) \in \boldsymbol{W}_T^{reg}(\Omega) \times \Psi.$$

The regular parts are handled as before, whereas the gradient parts have to be tackled separately. This plays a role only in the estimate of the third term in (2.16), since the other two terms can be estimated as before. To that aim, one has to consider a variant of Proposition A.3. More precisely, one considers a regularized problem with boundary data made up of two parts: a regular part, which belongs to the same trace space as in the original Proposition A.3, plus a singular part, equal to the trace on the interface of an element of $\nabla\Psi$. Since one has $\|\Delta\psi\|_0 = \|\nabla\psi\|_{\boldsymbol{W}_T(\Omega)}$, one reaches the same conclusion as before, *id est* (2.17), for an *ad hoc* constant c. Then, provided (2.18) holds, there exists a subsequence $(\boldsymbol{curl}\boldsymbol{u}_1^k)_k$ which converges in $\boldsymbol{L}^2(\Omega_1)$. The end of the proof is unchanged. ■

Remark 2.2. Probably, one should be able to handle the case of piecewise smooth interfaces, thus also removing assumption (2.10). The idea in this general case can be outlined as follows. Consider each edge and/or corner of the interface. One field, for instance $\boldsymbol{p}_1^k$, is locally regular: $\chi^\star\boldsymbol{p}_1^k \in \boldsymbol{H}^1(\Omega_1)$, for an *ad hoc* truncation function $\chi^\star$. Whereas the other field, $\boldsymbol{p}_2^k$ can be locally singular: $\chi^\star\boldsymbol{p}_2^k \notin \boldsymbol{H}^1(\Omega_2)$ is possible. Then, this behavior is inherited by $\boldsymbol{u}_1^k$ and $\boldsymbol{u}_2^k$. Therefore, in order to bound the third term as in (2.17), one should probably isolate the singular and regular parts of $\boldsymbol{u}_2^{kl}$, and proceed by controlling its singular part by $\boldsymbol{g}_\Sigma^{kl}$, resp. its regular part by $\boldsymbol{u}_1^{kl}$.

Theorem 2.3. *Assume that (2.10) holds. The embedding of the functional space $\boldsymbol{XY}$ into $\boldsymbol{L}^2(\Omega)$ is compact if at least one of the global contrasts R_1^ϵ or R_2^ϵ is large enough.*

Proof: It is based on the standard Helmholtz decomposition of vector fields, here on elements of $\boldsymbol{XY}$. Given $\boldsymbol{xy} \in \boldsymbol{XY}$, solve
Find $\phi \in H_0^1(\Omega)$ such that

$$\operatorname{div}(\epsilon\nabla\phi) = \operatorname{div}(\epsilon\boldsymbol{xy}) \text{ in } \Omega.$$

This problem is well-posed [3,4]: its solution ϕ exists (and it is unique) and moreover $\|\phi\|_{H^1(\Omega)} \le C(\epsilon)\,\|\operatorname{div}(\epsilon\boldsymbol{xy})\|_0$ (with $C(\epsilon) > 0$ independent of $\boldsymbol{xy}$).
Let $\boldsymbol{y} := \nabla\phi$ and $\boldsymbol{x} := \boldsymbol{xy} - \boldsymbol{y}$. By construction,

$$\begin{cases} \boldsymbol{y} \in \boldsymbol{Y},\ \text{with } \operatorname{div}(\epsilon\boldsymbol{y}) = \operatorname{div}(\epsilon\boldsymbol{xy}) \text{ in } \Omega \\ \boldsymbol{x} \in \boldsymbol{X},\ \text{with } \boldsymbol{curl}\boldsymbol{x} = \boldsymbol{curl}\boldsymbol{xy} \text{ in } \Omega \end{cases}.$$

We note that $\|\boldsymbol{y}\|_0 = \|\nabla\phi\|_0 \le C(\epsilon)\,\|\operatorname{div}(\epsilon\boldsymbol{xy})\|_0$, so it follows that $\|\boldsymbol{x}\|_0 \le \|\boldsymbol{xy}\|_0 + C(\epsilon)\,\|\operatorname{div}(\epsilon\boldsymbol{xy})\|_0$.
Combining the above, we deduce that, from any bounded sequence $(\boldsymbol{xy}^k)_k$ in $\boldsymbol{XY}$, one can build two *bounded* sequences $(\boldsymbol{y}^k)_k$ (in $\boldsymbol{Y}$) and $(\boldsymbol{x}^k)_k$ (in $\boldsymbol{X}$). From each sequence, we extract a subsequence that converges in $\boldsymbol{L}^2(\Omega)$, according to Theorems 2.1 and 2.2. Aggregating the two, we obtain a subsequence $(\boldsymbol{xy}^k)_k$ that converges in $\boldsymbol{L}^2(\Omega)$. ■

3. Recovering coerciveness in the case of a sign-shifting μ

In this section we assume that the sign of μ shifts from Ω_1 to Ω_2, with $\mu_1 > 0$ and $\mu_2 < 0$. We make no further assumption on the sign of ϵ. Recall that $\boldsymbol{j}$ belongs to $\boldsymbol{L}^2(\Omega)$ and that $\operatorname{div} \boldsymbol{j} = 0$. The aim is to build a three-field variational formulation, equivalent to (1.7), and then to show that this formulation is well-posed under suitable conditions.

3.1. *A three-field formulation*

To begin with, keeping both $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ leads to a reformulated – equivalent – definition of $\boldsymbol{X}$. According to the definition (1.8) of the spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, to which $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ belong respectively, and adding the compatibility conditions on the interface, we are led to introduce the functional space

$$\mathcal{X} := \{(\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{X}_1 \times \mathcal{X}_2 \mid \boldsymbol{v} \times \boldsymbol{n}_{1|\Sigma} = \boldsymbol{w} \times \boldsymbol{n}_1|_\Sigma,\ \epsilon_1 \boldsymbol{v} \cdot \boldsymbol{n}_{1|\Sigma} = \epsilon_2 \boldsymbol{w} \cdot \boldsymbol{n}_{1|\Sigma}\}$$

and the auxiliary unknown

$$\underline{\boldsymbol{e}_2} := \frac{1}{|\mu_2|} \boldsymbol{curl} \boldsymbol{e}_2 \, .$$

Now, let us consider the test functions $(\boldsymbol{v}_1,\ \boldsymbol{v}_2) \in \mathcal{X}$ and $\underline{\boldsymbol{v}_2} \in \boldsymbol{H}(\boldsymbol{curl}; \Omega_2)$ and

- take the $\boldsymbol{L}^2$−scalar product of the first Eq. of (1.7) with $\boldsymbol{v}_1$:

$$\omega^2 (\epsilon_1 \boldsymbol{e}_1, \boldsymbol{v}_1)_{0,1} - \left(\boldsymbol{curl} \left(\frac{1}{\mu_1} \boldsymbol{curl} \boldsymbol{e}_1 \right), \boldsymbol{v}_1 \right)_{0,1} = (\boldsymbol{j}_1, \boldsymbol{v}_1)_{0,1}$$

  and let us integrate by parts:

$$\omega^2 (\epsilon_1 \boldsymbol{e}_1, \boldsymbol{v}_1)_{0,1} - \left(\frac{1}{\mu_1} \boldsymbol{curl} \boldsymbol{e}_1, \boldsymbol{curl} \boldsymbol{v}_1 \right)_{0,1} - \left\langle \boldsymbol{v}_1 \times \boldsymbol{n}_1, \left(\frac{1}{\mu_1} \boldsymbol{curl} \boldsymbol{e}_1 \right)_T \right\rangle_\Sigma = (\boldsymbol{j}_1, \boldsymbol{v}_1)_{0,1}.$$

  Since, by definition, $\left(\frac{1}{\mu_1} \boldsymbol{curl} \boldsymbol{e}_1 \right)_T = -(\underline{\boldsymbol{e}_2})_T$ and $\boldsymbol{v}_1 \times \boldsymbol{n}_1 = \boldsymbol{v}_2 \times \boldsymbol{n}_1$ on Σ, we obtain

$$\left(\frac{1}{\mu_1} \boldsymbol{curl} \boldsymbol{e}_1, \boldsymbol{curl} \boldsymbol{v}_1 \right)_{0,1} - \omega^2 (\epsilon_1 \boldsymbol{e}_1, \boldsymbol{v}_1)_{0,1} - \langle \boldsymbol{v}_2 \times \boldsymbol{n}_1, (\underline{\boldsymbol{e}_2})_T \rangle_\Sigma = -(\boldsymbol{j}_1, \boldsymbol{v}_1)_{0,1} \, . \tag{3.1}$$

- take the $\boldsymbol{L}^2$−scalar product of the second Eq. of (1.7) with $\boldsymbol{curl} \underline{\boldsymbol{v}_2}$; multiply the resulting equality by a constant factor $\vartheta > 0$:

$$\vartheta \omega^2 (\epsilon_2 \boldsymbol{e}_2, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} + \vartheta (\boldsymbol{curl} \underline{\boldsymbol{e}_2}, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} = \vartheta (\boldsymbol{j}_2, \boldsymbol{curl} \underline{\boldsymbol{v}_2}) \, . \tag{3.2}$$

- consider the identity

$$\left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{e}_2, \boldsymbol{curl}\boldsymbol{v}_2\right)_{0,2} = \left(\boldsymbol{curl}\left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{e}_2\right), \boldsymbol{v}_2\right)_{0,2} - \left\langle \boldsymbol{v}_2 \times \boldsymbol{n}_2, \left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{e}_2\right)_T \right\rangle_\Sigma ;$$

according to the definition of $\underline{\boldsymbol{e}_2}$ this last equation leads to ($\boldsymbol{n}_2 = -\boldsymbol{n}_1$ on Σ)

$$\left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{e}_2, \boldsymbol{curl}\boldsymbol{v}_2\right)_{0,2} - (\boldsymbol{curl}\underline{\boldsymbol{e}_2}, \boldsymbol{v}_2)_{0,2} - \langle(\boldsymbol{v}_2 \times \boldsymbol{n}_1, \underline{\boldsymbol{e}_2})_T\rangle_\Sigma = 0\,. \quad (3.3)$$

- consider the identity

$$(\boldsymbol{e}_2, \boldsymbol{curl}\underline{\boldsymbol{v}_2})_{0,2} = (\boldsymbol{curl}\boldsymbol{e}_2, \underline{\boldsymbol{v}_2})_{0,2} + \langle \boldsymbol{e}_2 \times \boldsymbol{n}_2, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma ;$$

since $\boldsymbol{e}_2 \times \boldsymbol{n}_2|_\Sigma = -\boldsymbol{e}_1 \times \boldsymbol{n}_1|_\Sigma$, we get:

$$(|\mu_2|\underline{\boldsymbol{e_2}}, \underline{\boldsymbol{v}_2})_{0,2} - (\boldsymbol{e}_2, \boldsymbol{curl}\underline{\boldsymbol{v}_2}) - \langle \boldsymbol{e}_1 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma = 0\,. \quad (3.4)$$

Adding up the previous contributions (3.1)-(3.4), we introduce the variational formulation (3.5):

Find $U = ((\boldsymbol{e}_1, \boldsymbol{e}_2), \underline{\boldsymbol{e}_2}) \in \mathcal{X} \times \boldsymbol{H}(\boldsymbol{curl}; \Omega_2)$ *such that*

$$\forall V = ((\boldsymbol{v}_1, \boldsymbol{v}_2), \underline{\boldsymbol{v}_2}) \in \mathcal{X} \times \boldsymbol{H}(\boldsymbol{curl}; \Omega_2)\,,\ A^\vartheta(U, V) = L^\vartheta(V)\,. \quad (3.5)$$

We call (3.5) the *three-field formulation*.
The forms A^ϑ and L^ϑ are respectively defined by

$$\begin{aligned} A^\vartheta(U,V) := &\left(\frac{1}{\mu_1}\boldsymbol{curl}\boldsymbol{e}_1, \boldsymbol{curl}\boldsymbol{v}_1\right)_{0,1} - \omega^2(\epsilon_1 \boldsymbol{e}_1, \boldsymbol{v}_1)_{0,1} + \vartheta\omega^2(\epsilon_2 \boldsymbol{e}_2, \boldsymbol{curl}\underline{\boldsymbol{v}_2})_{0,2} \\ &+\vartheta(\boldsymbol{curl}\underline{\boldsymbol{e}_2}, \boldsymbol{curl}\underline{\boldsymbol{v}_2})_{0,2} + \left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{e}_2, \boldsymbol{curl}\boldsymbol{v}_2\right)_{0,2} \\ &-(\boldsymbol{curl}\underline{\boldsymbol{e}_2}, \boldsymbol{v}_2)_{0,2} + (|\mu_2|\underline{\boldsymbol{e}_2}, \underline{\boldsymbol{v}_2})_{0,2} - (\boldsymbol{e}_2, \boldsymbol{curl}\underline{\boldsymbol{v}_2})_{0,2} \\ &-2\langle \boldsymbol{v}_2 \times \boldsymbol{n}_1, (\underline{\boldsymbol{e}_2})_T\rangle_\Sigma - \langle \boldsymbol{e}_1 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma \end{aligned} \quad (3.6)$$

and

$$L^\vartheta(V) := -(\boldsymbol{j}_1, \boldsymbol{v}_1)_{0,1} + \vartheta(\boldsymbol{j}_2, \boldsymbol{curl}\underline{\boldsymbol{v}_2})_{0,2}\,. \quad (3.7)$$

It is important to note that, in the definition of the bilinear form A^ϑ, the two boundary terms $\langle \boldsymbol{v}_2 \times \boldsymbol{n}_1, (\underline{\boldsymbol{e}_2})_T\rangle_\Sigma$ and $\langle \boldsymbol{e}_1 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma$ are independent of the coefficients ϵ and μ. In addition, we remark that this is true for any choice of the strictly positive factor ϑ (which we will fit to some optimal value when we establish the coercivity of A^ϑ).

3.2. *Equivalence with the initial problem*

Proposition 3.1. *The three-field formulation (3.5) is equivalent to problem (1.7).*

Proof: It is already known that (3.5) follows from (1.7), so let us focus on the reciprocal assertion.

To begin with, we note that, since $\boldsymbol{e}_i \in \mathcal{X}_i$ ($i = 1, 2$), one has $\operatorname{div} \boldsymbol{e}_i = 0$ in Ω_i.
Next, one finds that $\boldsymbol{e}_1 \times \boldsymbol{n}_1|_\Sigma = \boldsymbol{e}_2 \times \boldsymbol{n}_1|_\Sigma$, $\epsilon_1 \boldsymbol{e}_1 \cdot \boldsymbol{n}_1|_\Sigma = \epsilon_2 \boldsymbol{e}_2 \cdot \boldsymbol{n}_1|_\Sigma$ and $\boldsymbol{e}_i \times \boldsymbol{n}_i|_{\Gamma_i} = 0$ ($i = 1, 2$), according to the definition of $\mathcal{X}$.

Let us choose in (3.5) test functions $\boldsymbol{v}_1$ which span $(\mathcal{D}(\Omega_1))^3$, $(\boldsymbol{v}_2, \underline{\boldsymbol{v}_2}) = (0, 0)$ and differentiate in the sense of distributions in $(\mathcal{D}'(\Omega_1))^3$:

$$\left\langle \omega^2 \epsilon_1 \boldsymbol{e}_1 - \boldsymbol{curl} \left(\frac{1}{\mu_1} \boldsymbol{curl} e_1 \right) - \boldsymbol{j}_1, \boldsymbol{v}_1 \right\rangle = 0.$$

Thus the first Eq. of (1.7) is recovered.

From there, we shall establish simultaneously that $|\mu_2| \underline{\boldsymbol{e}_2} = \boldsymbol{curl} \boldsymbol{e}_2$ and that the second Eq. of (1.7) is recovered.
We introduce two elements of $\boldsymbol{L}^2(\Omega)$: $\boldsymbol{\tau} := \boldsymbol{curl} \boldsymbol{e}_2 - |\mu_2| \underline{\boldsymbol{e}_2}$, and $\boldsymbol{\eta} := \omega^2 \epsilon_2 \boldsymbol{e}_2 + \boldsymbol{curl} \underline{\boldsymbol{e}_2} - \boldsymbol{j}_2$, and prove that both fields vanish over Ω_2. To start with, we know that $\operatorname{div} \boldsymbol{\eta} = 0$ in the whole of Ω, since $(\boldsymbol{e}_1, \boldsymbol{e}_2) \in \mathcal{X}$ and $\operatorname{div} \boldsymbol{j} = 0$.

Choose first in (3.5) $(\boldsymbol{v}_1, \boldsymbol{v}_2) = (0, 0)$ and $\underline{\boldsymbol{v}_2} \in (\mathcal{D}(\Omega_2))^3$:

$$\begin{aligned} \vartheta \omega^2 (\epsilon_2 \boldsymbol{e}_2, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} + \vartheta (\boldsymbol{curl} \underline{\boldsymbol{e}_2}, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} - \vartheta (\boldsymbol{j}_2, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} \\ + (|\mu_2| \underline{\boldsymbol{e}_2}, \underline{\boldsymbol{v}_2})_{0,2} - (\boldsymbol{e}_2, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} = 0\,. \end{aligned}$$

After differentiating in the sense of distributions, this last equation leads to

$$\vartheta \boldsymbol{curl} (\omega^2 \epsilon_2 \boldsymbol{e}_2 + \boldsymbol{curl} \underline{\boldsymbol{e}_2} - \boldsymbol{j}_2) + |\mu_2| \underline{\boldsymbol{e}_2} - \boldsymbol{curl} \boldsymbol{e}_2 = 0 \text{ in } (\mathcal{D}'(\Omega_2))^3,$$

which implies

$$\vartheta \boldsymbol{curl} \boldsymbol{\eta} = \boldsymbol{\tau} \text{ in } \boldsymbol{L}^2(\Omega_2). \tag{3.8}$$

Then, let us take in (3.5) $(\boldsymbol{v}_1, \underline{\boldsymbol{v}_2}) = (0, 0)$ and $\boldsymbol{v}_2 \in (\mathcal{D}(\Omega_2))^3$:

$$\left(\frac{1}{|\mu_2|} \boldsymbol{curl} \boldsymbol{e}_2, \boldsymbol{curl} \boldsymbol{v}_2 \right)_{0,2} - (\boldsymbol{curl} \underline{\boldsymbol{e}_2}, \boldsymbol{v}_2)_{0,2} = 0\,.$$

Let us differentiate again in the sense of distributions to obtain

$$\boldsymbol{curl} \left(\frac{1}{|\mu_2|} \boldsymbol{curl} \boldsymbol{e}_2 - \underline{\boldsymbol{e}_2} \right) = 0 \text{ in } (\mathcal{D}'(\Omega_2))^3 \tag{3.9}$$

that we may rewrite as

$$\boldsymbol{curl} \left(\frac{1}{|\mu_2|} \boldsymbol{\tau} \right) = 0\,. \tag{3.10}$$

Now, let us prove that the tangential trace of $\boldsymbol{\eta}$ vanishes over $\partial\Omega_2$. For that, choose in (3.5) $(\boldsymbol{v}_1, \boldsymbol{v}_2) = (0, 0)$ and $\underline{\boldsymbol{v}_2} \in \boldsymbol{H}(\boldsymbol{curl}, \Omega_2)$:

$$\begin{aligned} \vartheta \omega^2 (\epsilon_2 \boldsymbol{e}_2, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} + \vartheta (\boldsymbol{curl} \underline{\boldsymbol{e}_2}, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} + (|\mu_2| \underline{\boldsymbol{e}_2}, \underline{\boldsymbol{v}_2})_{0,2} \\ - (\boldsymbol{e}_2, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} - \langle \boldsymbol{e}_1 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T \rangle_\Sigma - \vartheta (\boldsymbol{j}_2, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} = 0\,. \end{aligned}$$

In terms of $\boldsymbol{\eta}$, this reads:

$$\vartheta (\boldsymbol{\eta}, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} + (|\mu_2| \underline{\boldsymbol{e}_2}, \underline{\boldsymbol{v}_2})_{0,2} - (\boldsymbol{e}_2, \boldsymbol{curl} \underline{\boldsymbol{v}_2})_{0,2} - \langle \boldsymbol{e}_1 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T \rangle_\Sigma = 0\,.$$

Since $\boldsymbol{\eta}$ and $\boldsymbol{e}_2$ belong to $\boldsymbol{H}(\boldsymbol{curl};\Omega_2)$, we can integrate by parts to find

$$\vartheta(\boldsymbol{curl}\boldsymbol{\eta},\underline{\boldsymbol{v}_2})_{0,2}-\vartheta\left\langle\boldsymbol{\eta}\times\boldsymbol{n}_2,(\underline{\boldsymbol{v}_2})_T\right\rangle_{\partial\Omega_2}+(|\mu_2|\underline{\boldsymbol{e}_2},\underline{\boldsymbol{v}_2})_{0,2}$$
$$-(\boldsymbol{curl}\boldsymbol{e}_2,\underline{\boldsymbol{v}_2})_{0,2}-\langle\boldsymbol{e}_2\times\boldsymbol{n}_2,(\underline{\boldsymbol{v}_2})_T\rangle_{\partial\Omega_2}-\langle\boldsymbol{e}_1\times\boldsymbol{n}_1,(\underline{\boldsymbol{v}_2})_T\rangle_\Sigma=0\,.$$

Then, we can remove a number of terms.

- We already proved that $0=\vartheta\boldsymbol{curl}\boldsymbol{\eta}-\boldsymbol{\tau}=\vartheta\boldsymbol{curl}\boldsymbol{\eta}+|\mu_2|\underline{\boldsymbol{e}_2}-\boldsymbol{curl}\boldsymbol{e}_2$ in Ω_2.
- In addition, there holds $\langle\boldsymbol{e}_2\times\boldsymbol{n}_2,(\underline{\boldsymbol{v}_2})_T\rangle_{\partial\Omega_2}=\langle\boldsymbol{e}_2\times\boldsymbol{n}_2,(\underline{\boldsymbol{v}_2})_T\rangle_\Sigma$, since $\boldsymbol{e}_2\in\boldsymbol{H}_{0,\Gamma_2}(\boldsymbol{curl},\Omega_2)$.
- Finally, $\boldsymbol{e}_2\times\boldsymbol{n}_{2|\Sigma}=-\boldsymbol{e}_1\times\boldsymbol{n}_{1|\Sigma}$, since $(\boldsymbol{e}_1,\boldsymbol{e}_2)\in\mathcal{X}$.

The conclusion is: $\langle\boldsymbol{\eta}\times\boldsymbol{n}_2,(\underline{\boldsymbol{v}_2})_T\rangle_{\partial\Omega_2}=0$, for all $\underline{\boldsymbol{v}_2}\in\boldsymbol{H}(\boldsymbol{curl},\Omega_2)$. Since $\underline{\boldsymbol{v}_2}\mapsto(\underline{\boldsymbol{v}_2})_{T|\partial\Omega_2}$ is onto, this finally leads to

$$\boldsymbol{\eta}\times\boldsymbol{n}_2|_{\partial\Omega_2}=0\,. \tag{3.11}$$

Recalling first (3.10) and then (3.8) and finally (3.11), we reach

$$0=\frac{1}{\vartheta}\left(\boldsymbol{curl}\frac{1}{|\mu_2|}\boldsymbol{\tau},\boldsymbol{\eta}\right)_{0,2}=\left(\boldsymbol{curl}\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{\eta},\boldsymbol{\eta}\right)_{0,2}=\left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{\eta},\boldsymbol{curl}\boldsymbol{\eta}\right)_{0,2}\,.$$

Thus $\boldsymbol{curl}\boldsymbol{\eta}=0$ and using (3.8) once again yields $\boldsymbol{\tau}=0$. Moreover, since $\boldsymbol{\eta}$ belongs to $\boldsymbol{L}^2(\Omega_2)$ and satisfies $\operatorname{div}\boldsymbol{\eta}=0$ and $\boldsymbol{curl}\boldsymbol{\eta}=0$ with a vanishing tangential trace over Ω_2, $\boldsymbol{\eta}$ vanishes over Ω_2 (see for instance Ref. 8). The second Eq. of (1.7) is thus recovered.
In order to conclude the proof, we must also recover the last Eq. of (1.7). To this aim, let us take in (3.5) $\underline{\boldsymbol{v}_2}=0$

$$\left(\frac{1}{\mu_1}\boldsymbol{curl}\boldsymbol{e}_1,\boldsymbol{curl}\boldsymbol{v}_1\right)_{0,1}-\omega^2(\epsilon_1\boldsymbol{e}_1,\boldsymbol{v}_1)_{0,1}-2\langle\boldsymbol{v}_1\times\boldsymbol{n}_1,(\underline{\boldsymbol{e}_2})_T\rangle_\Sigma+$$
$$\left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{e}_2,\boldsymbol{curl}\boldsymbol{v}_2\right)_{0,2}-(\boldsymbol{curl}\underline{\boldsymbol{e}_2},\boldsymbol{v}_2)_{0,2}=-(\boldsymbol{j}_1,\boldsymbol{v}_1)_{0,1}\,,$$

and let us integrate by parts to get

$$\left(\boldsymbol{curl}\left(\frac{1}{\mu_1}\boldsymbol{curl}\boldsymbol{e}_1\right)-\omega^2\epsilon_1\boldsymbol{e}_1+\boldsymbol{j}_1,\boldsymbol{v}_1\right)_{0,1}+\left(\boldsymbol{curl}\left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{e}_2-\underline{\boldsymbol{e}_2}\right),\boldsymbol{v}_2\right)_{0,2}-$$
$$\left\langle\boldsymbol{v}_1\times\boldsymbol{n}_1,\left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{e}_2+\frac{1}{\mu_1}\boldsymbol{curl}\boldsymbol{e}_1\right)_T\right\rangle_\Sigma=0\,.$$

From this last equation, using the first Eq. of (1.7) together with (3.9), it is straightforward to recover the last Eq. of (1.7). ■

3.3. *Finding a well-posed variational setting for the three-field formulation*

Below, we build a splitting of the bilinear form A^ϑ in a two term sum, so that the first term is coercive over $\mathcal{X}\times\boldsymbol{H}(\boldsymbol{curl};\Omega_2)$, and the second is a compact perturbation of the first one. Let us write $A^\vartheta = A^\vartheta_{coer} + A^\vartheta_{comp}$, with

$$\begin{aligned} A^\vartheta_{coer}(U,V) := &\left(\frac{1}{\mu_1}\boldsymbol{curl}\boldsymbol{e}_1, \boldsymbol{curl}\boldsymbol{v}_1\right)_{0,1} + \frac{1}{\mu_1^{max}}(\boldsymbol{e}_1,\boldsymbol{v}_1)_{0,1} + \left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{e}_2, \boldsymbol{curl}\boldsymbol{v}_2\right)_{0,2} \\ &+(\boldsymbol{e}_2,\boldsymbol{v}_2)_{0,2} + \vartheta(\boldsymbol{curl}\underline{\boldsymbol{e}_2}, \boldsymbol{curl}\underline{\boldsymbol{v}_2})_{0,2} + (|\mu_2|\underline{\boldsymbol{e}_2}, \underline{\boldsymbol{v}_2})_{0,2} \\ &-2\langle \boldsymbol{v}_1\times\boldsymbol{n}_1, (\underline{\boldsymbol{e}_2})_T\rangle_\Sigma - \langle \boldsymbol{e}_1\times\boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma \end{aligned} \tag{3.12}$$

$$\begin{aligned} A^\vartheta_{comp}(U,V) := &-\left(\left(\omega^2\epsilon_1 + \tfrac{1}{\mu_1^{max}}\right)\boldsymbol{e}_1, \boldsymbol{v}_1\right)_{0,1} - (\boldsymbol{e}_2 + \boldsymbol{curl}\underline{\boldsymbol{e}_2}, \boldsymbol{v}_2)_{0,2} \\ &-((1-\vartheta\omega^2\epsilon_2)\boldsymbol{e}_2, \boldsymbol{curl}\underline{\boldsymbol{v}_2})_{0,2}\,, \end{aligned} \tag{3.13}$$

where $U = (\boldsymbol{e}_1, \boldsymbol{e}_2, \underline{\boldsymbol{e}_2})$ and $V = (\boldsymbol{v}_1, \boldsymbol{v}_2, \underline{\boldsymbol{v}_2})$ both belong to $\mathcal{X}\times\boldsymbol{H}(\boldsymbol{curl};\Omega_2)$.
Let us prove that A^ϑ_{coer} is coercive under some suitable conditions.
Since μ_2 and μ_2^{-1} both belong to $L^\infty(\Omega_2)$, the two norms

$$\|\cdot\|_{\widetilde{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)} := [(|\mu_2|\cdot,\cdot)_{0,2} + (\boldsymbol{curl}\,\cdot, \boldsymbol{curl}\,\cdot)_{0,2}]^{1/2}$$

$$\|\cdot\|_{\widehat{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)} := \left[(\cdot,\cdot)_{0,2} + (\frac{1}{|\mu_2|}\boldsymbol{curl}\,\cdot, \boldsymbol{curl}\,\cdot)_{0,2}\right]^{1/2}$$

are equivalent to the $\mathcal{X}_2$ usual norm. Then the term $|\langle \boldsymbol{v}_2\times\boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma|$ can be bounded from above by

$$|\langle \boldsymbol{v}_2\times\boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma| \le \|\boldsymbol{v}_2\|_{\widehat{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)}\|\underline{\boldsymbol{v}_2}\|_{\widetilde{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)}\,. \tag{3.14}$$

On the other hand, for interface terms involving fields defined on Ω_1 and on Ω_2, we introduce some constant. Let $c\in\mathbb{R}^+_\star$ be defined as

$$c := \sup\left\{\frac{\left|\langle \boldsymbol{v}_1\times\boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma\right|}{\|\boldsymbol{v}_1\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_1)}\|\underline{\boldsymbol{v}_2}\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_2)}},\ \boldsymbol{v}_1\in\mathcal{X}_1\setminus\{\boldsymbol{0}\}, \underline{\boldsymbol{v}_2}\in\boldsymbol{H}(\boldsymbol{curl};\Omega_2)\setminus\{\boldsymbol{0}\}\right\}. \tag{3.15}$$

Note that we have

$$\begin{aligned} &\forall(\boldsymbol{v}_1, \underline{\boldsymbol{v}_2})\in\mathcal{X}_1\times\boldsymbol{H}(\boldsymbol{curl};\Omega_2)\,, \\ &\left|\langle \boldsymbol{v}_1\times\boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma\right| \le c\|\boldsymbol{v}_1\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_1)}\|\underline{\boldsymbol{v}_2}\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_2)} \end{aligned} \tag{3.16}$$

with an optimal c.

Let us introduce now the first *global contrast in* μ R_1^μ equal to the ratio μ_2^-/μ_1^{max}.

Proposition 3.2. *Assume that*

$$R_1^\mu > (5/4)c^2 \tag{3.17}$$

holds with c *defined by (3.16). Then, for any* $\vartheta \ge \max(1,\mu_2^-)$, A^ϑ_{coer} *is coercive over* $\{\mathcal{X}\times\boldsymbol{H}(\boldsymbol{curl};\Omega_2)\}$.

Proof: Let us first compute the value of $A^\vartheta_{coer}(V,V)$, with $V = ((\boldsymbol{v}_1, \boldsymbol{v}_2), \underline{\boldsymbol{v}_2})$:

$$\begin{aligned} A^\vartheta_{coer}(V,V) = &\left(\frac{1}{\mu_1}\boldsymbol{curl}\boldsymbol{v}_1, \boldsymbol{curl}\boldsymbol{v}_1\right)_{0,1} + \frac{1}{\mu_1^{max}}(\boldsymbol{v}_1, \boldsymbol{v}_1)_{0,1} + \\ &\left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{v}_2, \boldsymbol{curl}\boldsymbol{v}_2\right)_{0,2} + (\boldsymbol{v}_2, \boldsymbol{v}_2)_{0,2} + \\ &\vartheta(\boldsymbol{curl}\underline{\boldsymbol{v}_2}, \boldsymbol{curl}\underline{\boldsymbol{v}_2})_{0,2} + (|\mu_2|\underline{\boldsymbol{v}_2}, \underline{\boldsymbol{v}_2})_{0,2} - \\ &3\langle \boldsymbol{v}_2 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma \,. \end{aligned}$$

Thus, introducing the real parameter $\eta \in [0,3]$, $A^\vartheta_{coer}(V,V)$ may be bounded from below by

$$\begin{aligned} A^\vartheta_{coer}(V,V) \geq &\left(\frac{1}{\mu_1}\boldsymbol{curl}\boldsymbol{v}_1, \boldsymbol{curl}\boldsymbol{v}_1\right)_{0,1} + \frac{1}{\mu_1^{max}}(\boldsymbol{v}_1, \boldsymbol{v}_1)_{0,1} + \\ &\left(\frac{1}{|\mu_2|}\boldsymbol{curl}\boldsymbol{v}_2, \boldsymbol{curl}\boldsymbol{v}_2\right)_{0,2} + (\boldsymbol{v}_2, \boldsymbol{v}_2)_{0,2} + \\ &\vartheta(\boldsymbol{curl}\underline{\boldsymbol{v}_2}, \boldsymbol{curl}\underline{\boldsymbol{v}_2})_{0,2} + (|\mu_2|\underline{\boldsymbol{v}_2}, \underline{\boldsymbol{v}_2})_{0,2} - \\ &(3-\eta)\left|\langle \boldsymbol{v}_1 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma\right| - \eta\left|\langle \boldsymbol{v}_2 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma\right| \,. \end{aligned}$$

Then, the term $\left|\langle \boldsymbol{v}_2 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma\right|$ is bounded by (3.14), whereas $\left|\langle \boldsymbol{v}_1 \times \boldsymbol{n}_1, (\underline{\boldsymbol{v}_2})_T\rangle_\Sigma\right|$ is bounded as in (3.16). Let us introduce β_1 and β_2, two real and strictly positive parameters such that $\beta_1 + \beta_2 = 1$; we deduce

$$\begin{aligned} A^\vartheta{}_{coer}(V,V) \geq &\frac{1}{\mu_1^{max}}\|\boldsymbol{v}_1\|^2_{\boldsymbol{H}(\boldsymbol{curl};\Omega_1)} + \|\boldsymbol{v}_2\|^2_{\widehat{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)} \\ &+(\beta_1+\beta_2)\left[(|\mu_2|\underline{\boldsymbol{v}_2}, \underline{\boldsymbol{v}_2})_{0,2} + \vartheta\|\boldsymbol{curl}\underline{\boldsymbol{v}_2}\|^2_{0,2}\right] \\ &-(3-\eta)c\|\boldsymbol{v}_1\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_1)}\|\underline{\boldsymbol{v}_2}\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_2)} \\ &-\eta\|\boldsymbol{v}_2\|_{\widehat{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)}\|\underline{\boldsymbol{v}_2}\|_{\widetilde{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)} \,. \end{aligned}$$

Since $\vartheta \geq \max(1, \mu_2^-)$, one actually has

$$\begin{aligned} A^\vartheta{}_{coer}(V,V) \geq &\frac{1}{\mu_1^{max}}\|\boldsymbol{v}_1\|^2_{\boldsymbol{H}(\boldsymbol{curl};\Omega_1)} + \beta_1\mu_2^-\|\underline{\boldsymbol{v}_2}\|^2_{\boldsymbol{H}(\boldsymbol{curl};\Omega_2)} \\ &+\|\boldsymbol{v}_2\|^2_{\widehat{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)} + \beta_2\|\underline{\boldsymbol{v}_2}\|^2_{\widetilde{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)} \\ &-(3-\eta)c\|\boldsymbol{v}_1\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_1)}\|\underline{\boldsymbol{v}_2}\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_2)} \\ &-\eta\|\boldsymbol{v}_2\|_{\widehat{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)}\|\underline{\boldsymbol{v}_2}\|_{\widetilde{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)} \,. \end{aligned} \tag{3.18}$$

Now, the idea is to control the negative terms with (a fraction of) the positive ones. Let us recall the standard result

$$\begin{aligned} &\text{given } m, p \in \mathbb{R}^+_\star \text{ such that } m > p^2\,, \\ &\exists\lambda \in \mathbb{R}^+_\star,\ \forall x, y \in \mathbb{R},\ mx^2 + y^2 - 2pxy \geq \lambda(x^2+y^2). \end{aligned} \tag{3.19}$$

Then, let us set

$$\begin{aligned} x_1 &= \|\boldsymbol{v}_1\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_1)}\,, & y_1 &= \|\underline{\boldsymbol{v}_2}\|_{\boldsymbol{H}(\boldsymbol{curl};\Omega_2)} \\ x_2 &= \|\underline{\boldsymbol{v}_2}\|_{\widetilde{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)}\,, & y_2 &= \|\boldsymbol{v}_2\|_{\widehat{\boldsymbol{H}}(\boldsymbol{curl};\Omega_2)} \end{aligned}$$

and

$$m_1 := \frac{1}{\mu_1^{max}\mu_2^- \beta_1}\,, \qquad p_1 := \frac{c(3-\eta)}{2\beta_1\mu_2^-}$$
$$m_2 = \beta_2\,, \qquad\qquad p_2 = \eta/2\,.$$

With this new notations, the inequality (3.18) can be rewritten as

$$A^{\vartheta}_{coer}(V,V) \geq \beta_1\mu_2^- \left(m_1x_1^2 + y_1^2 - 2p_1x_1y_1\right) + \left(m_2x_2^2 + y_2^2 - 2p_2x_2y_2\right). \qquad (3.20)$$

According to (3.19), the form $A^{\vartheta}_{coer}(V,V)$ is coercive if both condition $m_1 > p_1^2$ and $m_2 > p_2^2$ are satisfied. In other words, provided both

$$\frac{\mu_2^-}{\mu_1^{max}} > \frac{c^2(3-\eta)^2}{4\beta_1} \text{ and } \beta_2 > (\eta/2)^2 \text{ hold.} \qquad (3.21)$$

Since $\beta_2 < 1$, we consider from now on $\eta \in [0,2[$. Moreover, since $\beta_1 + \beta_2 = 1$, the second condition in (3.21) is equivalent to $\beta_1^{-1} > (1-\eta^2/4)^{-1}$. Then the first condition in (3.21) is satisfied for some suitable $\beta_1(\eta)$ (depending here on η) if

$$\frac{\mu_2^-}{\mu_1^{max}} > c^2\frac{(3-\eta)^2}{4-\eta^2}\,. \qquad (3.22)$$

Now, $f : \eta \mapsto (3-\eta^2)/(4-\eta^2)$ takes his minimal value at $\eta = 4/3$, and $f(4/3) = 5/4$. For this optimal value, condition (3.22) reduces to (3.17) for some suitable $\beta_1(4/3)$. ■

Theorem 3.1. *The variational formulation (3.5) fits into the coercive plus compact framework, for $\vartheta \geq \max(1,\mu_2^-)$, if the following conditions are met:*

(1) for a constant-sign ϵ, if the global contrast R_1^μ is large enough.
(2) for a sign-shifting ϵ, if the global contrast R_1^μ is large enough, if assumption (2.10) holds, and if one of the global contrasts R_1^ϵ or R_2^ϵ is large enough.

Proof: When ϵ is sign-constant over the whole domain Ω, we already noted that the embedding of $\boldsymbol{X}$ into $\boldsymbol{L}^2(\Omega)$ is compact. Due to this result, A^{ϑ}_{comp} is a compact perturbation of A^{ϑ}_{coer}, which is coercive provided that the condition (3.17) on R_1^μ holds.
In the case of a sign-shifting ϵ, condition (3.17) has to be supplemented with a condition like (2.18) on R_1^ϵ, to ensure that the embedding of $\boldsymbol{X}$ into $\boldsymbol{L}^2(\Omega)$ is compact. ■

Remark 3.1. The constant c depends only on the geometry, so the lower bound in (3.17) is fixed by the geometrical configuration.
To derive a similar result in the case of a big value of the second *global contrast in* μ $R_2^\mu := \mu_1^{min}/\mu_2^+$, one simply builds an alternate three-field formulation by choosing $\underline{\boldsymbol{e}_1} := \mu_1^{-1}\boldsymbol{curl}\boldsymbol{e}_1$ as the auxiliary unknown.

4. Concluding remarks

According to the previous results, one can solve problem

$$\begin{cases} \omega^2 \epsilon \boldsymbol{e} - \boldsymbol{curl}\left(\dfrac{1}{\mu}\boldsymbol{curl}\boldsymbol{e}\right) = \boldsymbol{j} \text{ in } \Omega \\ \operatorname{div}(\epsilon \boldsymbol{e}) = 0 \text{ in } \Omega \\ \boldsymbol{e} \times \boldsymbol{n}|_{\partial\Omega} = 0 \end{cases} .$$

under very weak assumptions on the coefficients, which cover many challenging configurations of practical interest. Recall that when ϵ and μ are constant-sign coefficients, the problem is well-posed under the assumption that $\epsilon, \epsilon^{-1}, \mu, \mu^{-1}$ all belong to $L^\infty(\Omega)$. These assumptions are always implicit below.

- If only ϵ exhibits a sign-shift at an interface: there are two possible ways to achieve well-posedness, under the condition that at least one of the two global contrasts R_1^ϵ or R_2^ϵ is large enough:
  - According to the compactness result of theorem 2.2, the variational formulation (1.6) fits into the coercive plus compact framework, provided the interface is ”smooth” in the sense of assumption (2.10).
  - Or, one can choose a three-field formulation for the *magnetic field* (like (3.5)), along the same lines as those of section 3, under the weaker assumption that the interface is Lipschitz.
- If only μ exhibits a sign-shift at an interface: one can proceed as in the previous case by *reversing* the roles of the electric and magnetic fields.

- If both ϵ and μ exhibit a sign-shift at interfaces that can be different: one can use a three-field formulation ((3.5), or in the magnetic field) together with the compactness result. In this case, it is required that one of the two global contrasts R_1^ϵ or R_2^ϵ is large enough, *and also* that one of the two global contrasts R_1^μ or R_2^μ is large enough. In this configuration, one interface must be ”smooth” in the sense of assumption (2.10), while the other one is Lipschitz.

	ϵ sign-constants	ϵ sign-shifts
μ sign-constants	NF	NF or 3F
μ sign-shifts	NF or 3F	3F

The above table summarizes, for all possible transitions between media, which formulation(s) can be chosen for solving the problem. There, the acronyms ”NF” and ”3F” denote respectively the ”natural” variational formulation and a three-field variational formulation.

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A. Annex

We recall or prove here a series of elementary results, set in an open, bounded and connected set $\mathcal{O}$ with a Lipschitz polyhedral boundary $\partial\mathcal{O}$. Let $\gamma \subset \partial\mathcal{O}$ be connected (also with a Lipschitz boundary $\partial\gamma$), and $\gamma' := \partial\mathcal{O} \setminus \gamma$. Let $\alpha \in L^\infty(\mathcal{O})$ be positive, with $\alpha^{-1} \in L^\infty(\mathcal{O})$.

A.1. *Results on scalar fields*

The first result deals with the lifting of some scalar data that is prescribed on a part of a boundary: for the sake of completeness, we report the proof. We define the *local contrast* $\mathcal{C}_\alpha^{int}$ equal to the ratio $\alpha^{max}/\alpha^{min}$. Recall that

$$H_{00}^{1/2}(\gamma) := \{p \in H^{1/2}(\gamma) \mid \tilde{p} \in H^{1/2}(\partial\mathcal{O})\},$$

where $\tilde{p}$ is the continuation of p by zero to the whole boundary. This Hilbert space is endowed with the "natural" norm $\|p\|_{H_{00}^{1/2}(\gamma)} := \|\tilde{p}\|_{H^{1/2}(\partial\mathcal{O})}$.

Proposition A.1. *Let $h \in H_{00}^{1/2}(\gamma)$ and define u as the solution to*
Find $u \in H^1(\mathcal{O})$ such that

$$\begin{cases} \operatorname{div}(\alpha\nabla u) = 0 \textit{ in } \mathcal{O} \\ u|_\gamma = h \\ u|_{\gamma'} = 0 \end{cases}. \tag{A}$$

Then the inequality $\|\nabla u\|_{0,\mathcal{O}} \le \mathcal{C}_\alpha^{int}\|h\|_{H_{00}^{1/2}(\gamma)}$ holds.

Proof: Integrating by parts, one finds:

$$(\alpha\nabla u, \nabla u)_0 = \langle \alpha\nabla u\cdot\boldsymbol{n}, u\rangle = {}_{(H_{00}^{1/2}(\gamma))'}\langle \alpha\nabla u\cdot\boldsymbol{n}, h\rangle_{H_{00}^{1/2}(\gamma)}.$$

The normal trace mapping $\gamma_n : \boldsymbol{H}(\operatorname{div};\mathcal{O}) \to (H^{1/2}(\partial\mathcal{O}))'$, $\boldsymbol{v} \mapsto \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\mathcal{O}}$, is such that $\|\gamma_n\| = 1$, according to page 28 of Ref. 11. Moreover, given $\boldsymbol{v} \in \boldsymbol{H}(\operatorname{div};\mathcal{O})$:

$$\begin{aligned}\|\boldsymbol{v}\cdot\boldsymbol{n}|_\gamma\|_{(H_{00}^{1/2}(\gamma))'} &= \sup_{p\in H_{00}^{1/2}(\gamma)} \frac{{}_{(H_{00}^{1/2}(\gamma))'}\langle \boldsymbol{v}\cdot\boldsymbol{n}|_\gamma, p\rangle_{H_{00}^{1/2}(\gamma)}}{\|p\|_{H_{00}^{1/2}(\gamma)}} \\ &= \sup_{p\in H_{00}^{1/2}(\gamma)} \frac{\langle \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\mathcal{O}}, \tilde{p}\rangle}{\|\tilde{p}\|_{H^{1/2}(\partial\mathcal{O})}} \le \|\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\mathcal{O}}\|_{(H^{1/2}(\partial\mathcal{O}))'}.\end{aligned}$$

Combining the above yields:

$$\alpha^{min}\|\nabla u\|_0^2 \le \|\alpha\nabla u\|_{\boldsymbol{H}(\operatorname{div};\mathcal{O})}\,\|h\|_{H_{00}^{1/2}(\gamma)} \le \alpha^{max}\|\nabla u\|_0\,\|h\|_{H_{00}^{1/2}(\gamma)},$$

i. e. the expected result. ∎

Remark A.1. This result typically allows us to study the Dirichlet-to-Neumann operator $\mathbb{S} : h \mapsto \alpha\partial_n u|_\gamma$, defined from $H_{00}^{1/2}(\gamma)$ to $(H_{00}^{1/2}(\gamma))'$, since one finds

$$\|\mathbb{S}h\|_{(H_{00}^{1/2}(\gamma))'} \le \alpha^{max}\mathcal{C}_\alpha^{int}\|h\|_{H_{00}^{1/2}(\gamma)},\ \forall h \in H_{00}^{1/2}(\gamma).$$

Now, assume that the domain $\mathcal{O}$ can be partitioned into $\overline{\mathcal{O}} = \overline{\mathcal{O}_1} \cup \overline{\mathcal{O}_2}$, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, with Lipschitz polyhedral boundaries $\partial\mathcal{O}_i$. We let $\Sigma = \partial\mathcal{O}_1 \cap \partial\mathcal{O}_2$ be the interface with a Lipschitz boundary $\partial\Sigma$, and $B_i = \partial\mathcal{O}_i \setminus \Sigma$. On the interface, let us choose to measure elements of $H_{00}^{1/2}(\Sigma)$ thanks to the norm ${}_1\|p\| := \|\tilde{p}\|_{H^{1/2}(\partial\mathcal{O}_1)}$, with a continuation by zero to $\partial\mathcal{O}_1$. It is well-known (cf. for instance page 19 of Ref. 12) that an equivalent (and more intrinsic) norm on $H^{1/2}_{00}(\Sigma)$ is

$$ {}_0\|p\| :=: \left(\|p\|^2_{H^{1/2}(\Sigma)} + \int_\Sigma \frac{p(\boldsymbol{\sigma})^2}{d(\boldsymbol{\sigma},\partial\Sigma)}\, d\boldsymbol{\sigma} \right)^{1/2}, $$

where $\boldsymbol{\sigma} \mapsto d(\boldsymbol{\sigma},\partial\Sigma)$ is the distance to the boundary of Σ. The equivalence constants between the two norms are completely determined by the geometry of $\partial\mathcal{O}_1$ near Σ. On the other hand, the trace mapping $v_2 \mapsto v_2|_\Sigma$ is also continuous from $H^1_{0,B_2}(\mathcal{O}_2)$ to $H^{1/2}_{00}(\Sigma)$. According to the above, if we measure elements of $H^{1/2}_{00}(\Sigma)$ with ${}_1\|\cdot\|$, the norm of the trace mapping depends on *both* geometrical configurations of $\partial\mathcal{O}_1$ and of $\partial\mathcal{O}_2$ near the interface. So, the continuity modulus should be written as $\mathcal{C}_{1/2}$:

$$ {}_1\|v_2|_\Sigma\| \le \mathcal{C}_{1/2}\|\nabla v_2\|_{0,\mathcal{O}_2},\ \forall v_2 \in H^1_{0,B_2}(\mathcal{O}_2). $$

A.2. *Elementary result on vector fields*

The proposed results deal with the well-posedness of regularized problems.
We recall a result on norms in $\boldsymbol{W}_T(\mathcal{O})$, as proven in Proposition 7.4 of Ref. 10 and Corollary 3.16 of Ref. 1. The semi-norm $\|\cdot\|_{\boldsymbol{W}_T(\mathcal{O})} : \boldsymbol{w} \mapsto \left(\|\boldsymbol{curl u}\|^2_{0,\mathcal{O}} + \|\operatorname{div}\boldsymbol{u}\|^2_{0,\mathcal{O}}\right)^{1/2}$ defines a norm, which is equivalent to the full norm of $\boldsymbol{W}_T(\mathcal{O})$, provided that the domain $\mathcal{O}$ is *simply-connected*.

Proposition A.2. *Assume that $\mathcal{O}$ is simply-connected.*
Let $\boldsymbol{f} \in \boldsymbol{L}^2(\mathcal{O})$. Then, the regularized problem below admits one, and only one, solution:
Find $\boldsymbol{u} \in \boldsymbol{W}_T(\mathcal{O})$ such that

$$ \begin{cases} \boldsymbol{curl}\,(\alpha\boldsymbol{curl u}) - \nabla(\operatorname{div}\boldsymbol{u}) = \boldsymbol{f} \text{ in } \mathcal{O} \\ \boldsymbol{u}\cdot\boldsymbol{n}_{|\partial\mathcal{O}} = 0 \\ \alpha\boldsymbol{curl u}\times\boldsymbol{n}_{|\partial\mathcal{O}} = 0 \end{cases}. \quad \text{(B)} $$

Moreover, one has $\operatorname{div}\boldsymbol{u} \in H^1(\mathcal{O})$.
Finally, the norms $\|\boldsymbol{u}\|_{\boldsymbol{W}_T(\mathcal{O})}$ and $\|\operatorname{div}\boldsymbol{u}\|_{1,\mathcal{O}}$ depend continuously on $\|\boldsymbol{f}\|_{0,\mathcal{O}}$.

Proof: Classically, an equivalent variational formulation of (B) is
Find $\boldsymbol{u} \in \boldsymbol{W}_T(\mathcal{O})$ such that

$$ (\alpha\boldsymbol{curl u}, \boldsymbol{curl w})_{0,\mathcal{O}} + (\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{w})_{0,\mathcal{O}} = (\boldsymbol{f},\boldsymbol{w})_{0,\mathcal{O}},\ \forall \boldsymbol{w} \in \boldsymbol{W}_T(\mathcal{O}). \quad \text{(C)} $$

Well-posedness follows. In addition, taking $\boldsymbol{w} = \boldsymbol{u}$ in (C) yields

$$ \alpha^{min}\|\boldsymbol{curl u}\|^2_{0,\mathcal{O}} + \|\operatorname{div}\boldsymbol{u}\|^2_{0,\mathcal{O}} \le \|\boldsymbol{f}\|_{0,\mathcal{O}}\|\boldsymbol{u}\|_{0,\mathcal{O}}\,, $$

so $\|\boldsymbol{u}\|_{\boldsymbol{W}_T(\mathcal{O})}$ depend continuously on $\|\boldsymbol{f}\|_{0,\mathcal{O}}$.
To prove that one has $\operatorname{div}\boldsymbol{u} \in H^1(\mathcal{O})$, let us introduce the scalar Neumann problem
Find $d \in H^1(\mathcal{O}) \cap L^2_0(\mathcal{O})$ such that

$$ (\nabla d, \nabla w)_{0,\mathcal{O}} = -(\boldsymbol{f}, \nabla w)_{0,\mathcal{O}},\ \forall w \in H^1(\mathcal{O}) \cap L^2_0(\mathcal{O}). \quad \text{(D)} $$

NB. We recall that $L^2_0(\mathcal{O}) := \{v \in L^2(\mathcal{O}) \mid (v,1)_{0,\mathcal{O}} = 0\}$.
According to the Poincaré-Wirtinger inequality, this problem is well-posed, and $\|d\|_{1,\mathcal{O}}$ depends continuously on $\|\boldsymbol{f}\|_{0,\mathcal{O}}$.
Now, let us compare d to $\operatorname{div}\boldsymbol{u}$. To that aim, we introduce a second scalar Neumann problem. Let $\delta \in \mathbb{R}$ be such that $(\delta,1)_{0,\mathcal{O}} = (\operatorname{div}\boldsymbol{u},1)_{0,\mathcal{O}}$. According to the above, δ depends continuously on $\|\boldsymbol{f}\|_{0,\mathcal{O}}$. The second scalar problem reads
Find $v \in H^1(\mathcal{O}) \cap L^2_0(\mathcal{O})$ such that

$$(\nabla v, \nabla w)_{0,\mathcal{O}} = (d + \delta - \operatorname{div}\boldsymbol{u}, w)_{0,\mathcal{O}},\ \forall w \in H^1(\mathcal{O}) \cap L^2_0(\mathcal{O}). \tag{E}$$

Since by construction $d + \delta - \operatorname{div}\boldsymbol{u}$ is orthogonal to constants, the right-hand side defines a linear form on $H^1(\mathcal{O}) \cap L^2_0(\mathcal{O})$. This problem is also well-posed. One finds easily that $v \in H^1(\mathcal{O}) \cap L^2_0(\mathcal{O})$ can be characterized by the relations $\Delta v = \operatorname{div}\boldsymbol{u} - (d+\delta)$ in $\mathcal{O}$, and $\partial_n v_{|\partial\mathcal{O}} = 0$, so that ∇v belongs to $\boldsymbol{W}_T(\mathcal{O})$. Taking successively $\boldsymbol{w} = \nabla v$ in (C), $w = v$ in (D) and integrating by parts, we find

$$(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u} - (d+\delta))_{0,\mathcal{O}} = (\boldsymbol{f}, \nabla v)_{0,\mathcal{O}} = -(\nabla d, \nabla v)_{0,\mathcal{O}} = (d, \operatorname{div}\boldsymbol{u} - (d+\delta))_{0,\mathcal{O}}.$$

In other words, $(\operatorname{div}\boldsymbol{u} - d, \operatorname{div}\boldsymbol{u} - (d+\delta))_{0,\mathcal{O}} = 0$. But $\operatorname{div}\boldsymbol{u} - (d+\delta)$ is orthogonal to constants: $(\delta, \operatorname{div}\boldsymbol{u} - (d+\delta))_{0,\mathcal{O}} = 0$. It follows $\|\operatorname{div}\boldsymbol{u} - (d+\delta)\|_{0,\mathcal{O}} = 0$, *id est* $\operatorname{div} u = d + \delta$ in $\mathcal{O}$. We conclude that $\|\operatorname{div}\boldsymbol{u}\|_{1,\mathcal{O}}$ depends continuously on $\|\boldsymbol{f}\|_{0,\mathcal{O}}$. ■

Corollary A.1. *Let $(\boldsymbol{f}^k)_k$ be a bounded sequence in $\boldsymbol{L}^2(\mathcal{O})$, and let $(\boldsymbol{u}^k)_k$ be the corresponding sequence of solutions to the regularized problems (B) with $\boldsymbol{f} = \boldsymbol{f}^k$. Then, there exists a subsequence of $(\boldsymbol{u}^k)_k$ that converges in $\boldsymbol{W}_T(\mathcal{O})$.*

Proof: According to Proposition A.2, the sequence $(\boldsymbol{u}^k)_k$ is bounded in $\boldsymbol{W}_T(\mathcal{O})$. Thanks to the Weber embedding Theorem, there exists a subsequence, still denoted by $(\boldsymbol{u}^k)_k$, that converges in $\boldsymbol{L}^2(\mathcal{O})$. Taking the difference of Eq. (C) for two indices k and l with the same test field $\boldsymbol{w} = \boldsymbol{u}^k - \boldsymbol{u}^l$, one finds

$$\alpha^{min}\|\boldsymbol{curl}(\boldsymbol{u}^k - \boldsymbol{u}^l)\|^2_{0,\mathcal{O}} + \|\operatorname{div}(\boldsymbol{u}^k - \boldsymbol{u}^l)\|^2_{0,\mathcal{O}} \le \|\boldsymbol{f}^k - \boldsymbol{f}^l\|_{0,\mathcal{O}}\|\boldsymbol{u}^k - \boldsymbol{u}^l\|_{0,\mathcal{O}}.$$

In other words, the subsequence $(\boldsymbol{u}^k)_k$ is a Cauchy subsequence in $\boldsymbol{W}_T(\mathcal{O})$, so it converges. ■

Let us carry on with a final result on regularized problems with data on the boundary. Let us introduce first

$$\boldsymbol{H}^{1/2}_{00n}(\gamma') := \{\boldsymbol{\varphi} \in \boldsymbol{H}^{1/2}(\gamma') \mid \exists \boldsymbol{\phi} \in \boldsymbol{H}^1(\mathcal{O}),\ \boldsymbol{\phi}\cdot\boldsymbol{n}_{|\gamma} = 0,\ \boldsymbol{\varphi} = \boldsymbol{\phi}_{|\gamma'}\}.$$

Using standard results, we know that

$$\begin{cases} \exists \mathcal{C}_{lift} > 0,\ \forall \boldsymbol{\varphi} \in \boldsymbol{H}^{1/2}_{00n}(\gamma'),\ \exists \boldsymbol{\phi} \in \boldsymbol{H}^1(\mathcal{O}), \\ \boldsymbol{\phi}\cdot\boldsymbol{n}_{|\gamma} = 0, \boldsymbol{\varphi} = \boldsymbol{\phi}_{|\gamma'} \text{ and } \|\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\mathcal{O})} \le \mathcal{C}_{lift}\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^{1/2}(\gamma')} \end{cases}. \tag{F}$$

Proposition A.3. *Assume that $\mathcal{O}$ is simply-connected.*
Let $\boldsymbol{\varphi} \in \boldsymbol{H}^{1/2}_{00n}(\gamma')$. Then, the regularized problem below admits one, and only one, solution:

Find $\boldsymbol{u} \in \boldsymbol{H}(\boldsymbol{curl};\mathcal{O}) \cap \boldsymbol{H}(\mathrm{div}\,;\mathcal{O})$ such that

$$\left\{ \begin{array}{l} \boldsymbol{curl}\,(\alpha \boldsymbol{curl}\boldsymbol{u}) - \nabla(\mathrm{div}\,\boldsymbol{u}) = 0 \text{ in } \mathcal{O} \\ \boldsymbol{u}\cdot\boldsymbol{n}_{|\gamma} = 0 \\ \alpha\boldsymbol{curl}\boldsymbol{u}\times\boldsymbol{n}_{|\gamma} = 0 \\ \boldsymbol{u}_{|\gamma'} = \boldsymbol{\varphi} \end{array} \right. . \tag{G}$$

Moreover, the norms $\|\boldsymbol{curl}\boldsymbol{u}\|_{0,\mathcal{O}}$ *and* $\|\mathrm{div}\,\boldsymbol{u}\|_{0,\mathcal{O}}$ *depend continuously on* $\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^{1/2}(\gamma')}$.

Proof: Let us consider $\boldsymbol{\phi}$ satisfying (F). We remark that $\boldsymbol{u}-\boldsymbol{\phi}$ belongs to

$$\boldsymbol{W}_{T-}(\mathcal{O}) := \{\boldsymbol{w} \in \boldsymbol{W}_T(\mathcal{O}) \mid \boldsymbol{w}_{|\gamma'} = 0\},$$

which is a closed subspace of $\boldsymbol{W}_T(\mathcal{O})$.
Introduce $\boldsymbol{u}' := \boldsymbol{u}-\boldsymbol{\phi}$. Then one can reformulate the system of equations in $\boldsymbol{u}'$ as the equivalent variational formulation
Find $\boldsymbol{u}' \in \boldsymbol{W}_{T-}(\mathcal{O})$ *such that*

$$\begin{aligned}(\alpha\boldsymbol{curl}\boldsymbol{u}', \boldsymbol{curl}\boldsymbol{w})_{0,\mathcal{O}} + (\mathrm{div}\,\boldsymbol{u}', \mathrm{div}\,\boldsymbol{w})_{0,\mathcal{O}} \\ = -(\alpha\boldsymbol{curl}\boldsymbol{\phi}, \boldsymbol{curl}\boldsymbol{w})_{0,\mathcal{O}} - (\mathrm{div}\,\boldsymbol{\phi}, \mathrm{div}\,\boldsymbol{w})_{0,\mathcal{O}},\ \forall \boldsymbol{w} \in \boldsymbol{W}_{T-}(\mathcal{O}).\end{aligned}$$

Since $\boldsymbol{W}_{T-}(\mathcal{O})$ can be endowed with the norm of $\boldsymbol{W}_T(\mathcal{O})$, well-posedness of the variational formulation in $\boldsymbol{u}'$ follows. Existence of $\boldsymbol{u}$ is achieved. To obtain uniqueness and continuity with respect to the data, we proceed as follows.
We know that $\|\boldsymbol{u}'\|_{\boldsymbol{W}_T(\mathcal{O})}$ depends continuously on $\|\boldsymbol{\phi}\|_{\boldsymbol{W}_T(\mathcal{O})}$, which is itself bounded by $\|\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\mathcal{O})}$, and so by $\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^{1/2}(\gamma')}$ according to (F). Then, $\|\boldsymbol{u}'+\boldsymbol{\phi}\|_{\boldsymbol{W}_T(\mathcal{O})}$ depends continuously on $\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^{1/2}(\gamma')}$. We conclude that $\boldsymbol{u}$ is unique and that the norms $\|\boldsymbol{curl}\boldsymbol{u}\|_{0,\mathcal{O}}$ and $\|\mathrm{div}\,\boldsymbol{u}\|_{0,\mathcal{O}}$ depend continuously on $\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^{1/2}(\gamma')}$. ■