A Multiscale Approach for Solving Maxwell's Equations in Waveguides with Conical Inclusions

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Abstract. This paper is devoted to the numerical solution of the instationary Maxwell equations in waveguides with metallic conical inclusions on its internal boundary. These conical protuberances are geometrical singularities that generate in their neighborhood, strong electromagnetic fields. Using some recent theoretical and practical results on curl-free singular fields, we have built a method which allows to compute the instationary electromagnetic field. It is based on a splitting of the spaces of solutions into a regular part and a singular one. The singular part is computed with the help of a multiscale representation, written in the vicinity of the geometrical singularities. As an illustration, numerical results in a rectangular waveguide are shown.

Keywords: Maxwell equations; waveguides; conical inclusions; multiscale approach.

1 Introduction

Many practical problems require the computation of electromagnetic fields. In this paper, we are interested in computing the electromagnetic fields which propagate in three-dimensional waveguides with metallic conical inclusions. This can be useful in microwave and millimeter-wave technology, for instance for design purpose. The knowledge of the electromagnetic fields in the vicinity of irregular points of a surface is an old, widely studied problem (cf. [1], [2]). However, it remains a present problem, particularly the influence and the numerical treatment of such irregular points in the wave propagation (see for instance [3] and the references therein).

Within this framework, we developed a numerical method for solving the instationary Maxwell equations [4], with continuous approximations of the electromagnetic field. However in practical examples, the boundary of the computational domain may include reentrant corners and/or edges. They are called geometrical singularities, generate strong fields, and require a careful computation of the electromagnetic field in their neighborhood.

Using some new theoretical and practical results, we have built a method, the singular complement (hereafter *SCM*), which consists in splitting the space of

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solutions into a two-term direct sum (cf. [5]). The first subspace (the regular one) coincides with the whole space of solutions, provided that the domain is either convex, or with a smooth boundary. So, one can compute the regular part of the solution with the help of a classical method such as the finite element or finite difference method. The singular part is computed with the help of a multiscale representation, written in the vicinity of the geometrical singularities.

In this paper, we consider a 3D waveguide with sharp conical metallic protuberances on its internal boundary. Then the singularities consist of conical vertices. We propose an application of the *SCM* for the non-symmetric 3D case. Based on a multiscale representation of the singular part of the solution at the tip of each cone, coupled with a three-dimensional Maxwell formulation, we construct a numerical method of solution. Numerical results are shown. This extends the numerical results obtained in two-dimensional cartesian [5] or axisymetric domains [6]. See also [7] for an extension to prismatic domains based on a Fourier expansion. Following [8], extensions to rounded corners seem to be also possible.

2 Instationary Maxwell Equations

Let Ω be a open, polyhedral subset of $\mathbb{R}^3$, unbounded in one direction (typically a hollow metallic cylinder or a rectangular waveguide) and Γ its boundary. We denote by $\mathbf{n}$ the unit outward normal to Γ. If we let c, ε_0 and μ_0 be respectively the light velocity, the dielectric permittivity and the magnetic permeability ($\varepsilon_0\mu_0 c^2 = 1$), Maxwell's equations in vacuum read

$$\frac{\partial \mathcal{E}}{\partial t} - c^2 \mathbf{curl}\, \mathcal{B} = -\frac{1}{\varepsilon_0}\mathcal{J}, \quad \operatorname{div} \mathcal{E} = \frac{\rho}{\varepsilon_0}, \tag{1}$$

$$\frac{\partial \mathcal{B}}{\partial t} + \mathbf{curl}\, \mathcal{E} = 0, \qquad \operatorname{div} \mathcal{B} = 0, \tag{2}$$

where $\mathcal{E}$ and $\mathcal{B}$ are the electric and magnetic fields, ρ and $\mathcal{J}$ the charge and current densities which depend on the space variable $\mathbf{x}$ and on the time variable t. It is well known that ρ and $\mathcal{J}$ have to verify the charge conservation equation

$$\partial\rho/\partial t + \operatorname{div} \mathcal{J} = 0\,. \tag{3}$$

Waveguide problems are generally set without charges ($\rho = 0$), and this condition expresses that the current density $\mathcal{J}$ is divergence-free.

These equations are supplemented with appropriate boundary conditions. Since the domain of interest is unbounded in one direction, we denote by Γ_C the "real part" of the boundary Γ (i.e. the metallic part), on which one imposes the perfect conducting boundary condition. As it is well known, this is modelled by

$$\mathcal{E} \times \mathbf{n} = 0 \text{ and } \mathcal{B} \cdot \mathbf{n} = 0 \text{ on } \Gamma_C. \tag{4}$$

This condition expresses that the tangential components of $\mathcal{E}$, and the normal component of $\mathcal{B}$ uniformly vanish on Γ_C. Nevetheless, these boundary conditions are not sufficient to model efficiently our problem. Since the domain is not bounded, it has to be "numerically closed" to perform computations. As a consequence, the boundary Γ of Ω is also made up of an artifial part $\Gamma_A = \Gamma \setminus \Gamma_C$, on which a Silver-Müller boundary condition is imposed, namely

$$(\mathcal{E} - c\mathcal{B} \times \mathbf{n}) \times \mathbf{n} = \mathbf{e}^\star \times \mathbf{n} \text{ on } \Gamma_A\,, \tag{5}$$

where the surface field $\mathbf{e}^\star$ is given. In waveguide problems, the artificial boundary Γ_A is often splitted into Γ_A^i and Γ_A^a. On Γ_A^i, we model incoming plane waves by a non-vanishing function $\mathbf{e}^\star$, whereas we impose on Γ_A^a an absorbing boundary condition by choosing $\mathbf{e}^\star = 0$. Without loss of generality, one can choose the location of the artificial boundary Γ_A, in such a way that it does not intersect with the conical inclusions. Moreover, one can also choose a regular shape for Γ_A. Hence, the boundary Γ_A is not an issue for the conical inclusions. For this reason, we will only consider in what follows perfectly conducting boundary (i.e. $\Gamma_A = \emptyset$). The case of Silver-Müller boundary condition will be then introduced.

Writing the Maxwell equations as two second-order in time equations, the electric and the magnetic fields can be handled separately. In this paper, we will focus on the electric field formulation. Let us recall the definitions of the following spaces

$$\mathbf{H}(\mathbf{curl}\,, \Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega), \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\}\,, \tag{6}$$

$$\mathbf{H}(\mathrm{div}\,, \Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega), \mathrm{div}\,\mathbf{u} \in L^2(\Omega)\}\,, \tag{7}$$

$$\mathbf{H}^1(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega), \mathbf{grad}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\}\,. \tag{8}$$

We define the space of electric fields $\mathcal{E}$ with

$$\mathbf{X} = \{\mathbf{x} \in \mathbf{H}(\mathbf{curl}\,, \Omega) \cap \mathbf{H}(\mathrm{div}\,, \Omega) \;:\; \mathbf{x} \times \mathbf{n}_{|\Gamma} = 0\}\,, \tag{9}$$

endowed with the norm $\|\cdot\|_{\mathbf{X}} = [\|\cdot\|_0^2 + \|\mathbf{curl}\,\cdot\|_0^2 + \|\mathrm{div}\,\cdot\|_0^2]^{1/2}$.

When the domain is convex (or with a smooth boundary), the space of electric fields $\mathbf{X}$ is included in $\mathbf{H}^1(\Omega)$. That is not the case anymore in a singular domain (see for instance [9]), in particular in a waveguide with conical inclusions. One thus introduces the regular subspace for electric fields (indexed with $_R$)

$$\mathbf{X}_R = \mathbf{X} \cap \mathbf{H}^1(\Omega), \tag{10}$$

which is actually closed in $\mathbf{X}$ [10]. Hence, one can consider its orthogonal subspace (called singular subspace and indexed with $_S$), and then define a two-part, direct, and orthogonal sum of the space as

$$\mathbf{X} = \mathbf{X}_R \overset{\perp_{\mathbf{X}}}{\oplus} \mathbf{X}_S\,. \tag{11}$$

As a consequence, one can split an element $\mathbf{x}$ into an orthogonal sum of a *regular* part and a *singular* one, namely $\mathbf{x} = \mathbf{x}_R + \mathbf{x}_S$. This is the principle of the *Singular Complement Method*.

As far as numerical computations are concerned, one can also introduce direct, but non-orthogonal two-part sum for the space $\mathbf{X}$. The choice is made with respect to the ease of implementation and depends on the characterization of the singular space. The characterization (and so is the computation) of the elements of $\mathbf{X}_S$ is complicated (see [10]). So, one is able to introduce another (non-orthogonal) singular subspace $\mathbf{L}_S$ and define a two-part, direct sum for the space $\mathbf{X}$ as

$$\mathbf{X} = \mathbf{X}_R \oplus \mathbf{L}_S\,, \tag{12}$$

so that an element $\mathbf{x}$ of $\mathbf{X}$ will be splitted into $\mathbf{x} = \mathbf{x}_R + \mathbf{l}_S$. Following [10], elements of $\mathbf{L}_S$ are the curl-free elements of $\mathbf{X}$ that satisfy:

$$\operatorname{div} \mathbf{l}_S = s_D \quad \text{in } \Omega\,, \tag{13}$$

$$\mathbf{l}_S \times \mathbf{n}_{|\Gamma} = 0\,, \tag{14}$$

where s_D is the (non-vanishing) singular solution (i.e. that belongs to $L^2(\Omega)$) of the Dirichlet homogeneous boundary problem

$$\Delta s_D = 0 \quad \text{in } \Omega\,, \tag{15}$$

$$s_D = 0 \quad \text{on } \Gamma\,. \tag{16}$$

Hence, it appears more convenient to solve a scalar Laplace problem to determine $\mathbf{L}_S$, than a vector one that would appear for the $\mathbf{X}_S$ characterization.

Let us now briefly re-introduce the boundary Γ_A. A first way could be to consider the space of solutions

$$\mathbf{X}^{\Gamma_A} = \{\mathbf{x} \in \mathbf{H}(\mathbf{curl}\,, \Omega) \cap \mathbf{H}(\operatorname{div}\,, \Omega) \;:\; \mathbf{x} \times \mathbf{n}_{|\Gamma_C} = 0\}\,. \tag{17}$$

Then introduce the regular subspace $\mathbf{X}_R^{\Gamma_A} = \mathbf{X}^{\Gamma_A} \cap \mathbf{H}^1(\Omega)$, and construct the ad hoc orthogonal (or non-orthogonal) splitting, in which appears a singular space, say $\mathbf{X}_S^{\Gamma_A}$ (or $\mathbf{L}_S^{\Gamma_A}$ for the non-orthogonal curl-free part). Nevertheless, it is more interesting from a numerical point of view, to consider the (non-orthogonal) splitting

$$\mathbf{X}^{\Gamma_A} = \mathbf{X}_R^{\Gamma_A} \oplus \mathbf{L}_S\,, \tag{18}$$

first since the subspace of singular magnetic fields is $\mathbf{L}_S$, as before (with only a perfect conducting boundary condition). Second, modelling incoming plane waves, or imposing an absorbing boundary condition has no impact, as far as the singular subspace is concerned. It will be sufficient, as soon as Γ_A is not empty, to add in the variational formulation, integral terms on Γ_A as for a regular domain Ω.

3 Multiscale Representation for the Singular Part

In general 3D domains, the main difficulty of such an approach is to take into account accurately singular subspaces. They originate from geometrical singularities, such as reentrant edges, that can meet at reentrant vertices. In addition to generating infinite dimensional singular subspaces, the challenge is to understand and resolve the links between singular edges and singular vertex functions.

In that case, other approaches are possible and very useful, see [11]. Nevertheless, in many cases, one can proceed with this approach. This is the case for 3D prismatic domains, or for 3D domains invariant by rotation (cf. [7,12]). This is also the case for conical vertices that generate a finite dimensional singular subspace ($\mathbf{L}_S$ in our case), even in a full 3D geometry. However, computations retain their 3D character.

To simplify the presentation, we assume in this section that there is only one conical vertex, so that $\mathbf{L}_S$ is of dimension 1. The numerical method consists in first computing numerically the basis of the singular subspace $\mathbf{L}_S$, based on a multiscale representation. Following the characterization (13-14), we shall need s_D to compute $\mathbf{l}_S \in \mathbf{L}_S$. Then using *ad hoc* singular mappings (see [10]), one can associate to s_D a unique scalar potential ψ_S such that

$$-\Delta\psi_S = s_D \quad \text{in } \Omega\,, \tag{19}$$

$$\psi_S = 0 \quad \text{on } \Gamma\,. \tag{20}$$

Then, the singular basis function $\mathbf{l}_S$ will be easily inferred from ψ_S by taking its gradient, $\mathbf{l}_S = \mathbf{grad}\,\psi_S$.

As the keypoint is to compute s_D, let us now examine the behavior of this solution in the vicinity of a sharp cone. This can be done by deriving the asymptotic expansions. As the conical inclusion is locally invariant by rotation, we consider the system of spherical coordinates (r, θ, φ), centered at the tip of the cone. Let Γ_{cone} be the part of the boundary which intersects the cone of equation $\theta = w_0/2$, with w_0 the aperture angle of the cone (*a priori* $\pi \leq w_0 \leq 2\pi$). Hence, one can find s_D in the form of spherical harmonic functions

$$r^\mu\, P_\mu(\cos\theta)\, \sin(m\varphi), \tag{21}$$

where $P_\cdot$ denotes a Legendre function of order 0 and index μ, which is an eigenfunction of the Laplace-Beltrami operator, related to the eigenvalue $-\mu(\mu+1)$.

Proposition 1. *There exists a sequence $(c_l)_{l\in\mathbb{N}^\star}$ such that one can write, for any $N \in \mathbb{N}$, the expansion*

$$s_D(r,\theta,\varphi) = r^{-\lambda-1}\, P_\lambda(\cos\theta) + \sum_{l=1}^{N} c_l r^{\lambda_l}\, P_{\lambda_l}(\cos\theta) + O(r^{\lambda_{N+1}}), \quad r \longrightarrow 0\,. \tag{22}$$

In addition, the coefficient λ can be uniquely determined by solving

$$P_\lambda(\cos(\frac{w_0}{2})) = 0\,. \tag{23}$$

Remark first that, since the boundary Γ_{cone} is locally invariant by rotation, the above expansion does not depend on the variable φ, namely is axisymmetric (see for instance [13]). Moreover, computing the behavior of λ, as a function of the angle w_0, we find that only sharp conical inclusions (i. e. cones for which the aperture is larger than a given value $\beta \simeq 130^o 43'$), generate a singular function.

In other words, $r^{-\lambda-1}$ is singular (i.e. belongs to $L^2(\Omega)$, but not to $H^1(\Omega)$) only for cones for which the aperture is larger than β. This result is in accordance with the ones obtained in invariant by rotation domains [14]. For the other conical inclusions (with an aperture lower than β), no specific treatment will be necessary .

We are ready now to compute s_D. If we denote by $\tilde{s}_D = s_D - r^{-\lambda-1}\, P_\lambda(\cos\theta)$, the regular part of s_D (i.e. that belongs to $H^1(\Omega)$), and recall that $r^{-\lambda-1}P_\lambda(\cos\theta)$ is known as soon as λ was determined, one can compute s_D by solving

$$\Delta \tilde{s}_D = 0 \qquad \text{in } \Omega, \tag{24}$$

$$\tilde{s}_D = -r^{-\lambda-1}\, P_\lambda(\cos\theta) \qquad \text{on } \Gamma. \tag{25}$$

Note that the above equation is homogeneous as $\Delta\,[r^{-\lambda-1}\, P_\lambda(\cos\theta)] = 0$. Moreover, the boundary condition also vanishes on Γ_{cone}, due to the term $P_\lambda(\cos\theta)$. Then, one solves the discrete variational formulation, using continuous, P_1 Lagrange finite elements.

The computation of ψ_S is carried out analogously. In that case, we have to solve a non-homogeneous Dirichlet problem. From a numerical point of view, the term $\int_\Omega s_D\, v\, dx$ appears in the right-hand side of the variational formulation, where v is a test function. It is splitted as $\int_\Omega \tilde{s}_D\, v\, dx + \int_\Omega r^{-\lambda-1}\, P_\lambda(\cos\theta)\, v\, dx$. Then, in order to compute the second term accurately, one uses the analytical knowledge of $r^{-\lambda-1}\, P_\lambda(\cos\theta)$ – up the quasi-exact value of λ – in conjunction with Gauss quadrature formulas, exact up to the 6th order. These formulas require 15 integration points per element of the mesh (in our case, tetrahedron). It is emphasized that these formulas are well-suited, in the sense that no integration point coincides with the tip of the cone, where the (absolute) value of $r^{-\lambda-1}\, P_\lambda(\cos\theta)$ is infinite.

Finally, the computation of $\mathbf{l}_S(= \nabla\psi_S)$ is carried out with the help of the analytical expression in spherical coordinates of $\mathbf{l}_S^P(= \nabla\psi_S^P)$, namely

$$\mathbf{l}_S^P = \lambda r^{\lambda-1} \begin{pmatrix} \dfrac{\cos\varphi}{\sin\theta}\,[P_\lambda(\cos\theta) - \cos\theta P_{\lambda-1}(\cos\theta)] \\ \dfrac{\sin\varphi}{\sin\theta}\,[P_\lambda(\cos\theta) - \cos\theta P_{\lambda-1}(\cos\theta)] \\ P_{\lambda-1}(\cos\theta) \end{pmatrix}. \tag{26}$$

4 Numerical Algorithms

Now, to solve the problem, we have to modify a classical method by handling the above decomposition. We first write Ampère and Faraday's laws as two second-order in time equations, and focus on the electric field problem. To enforce the divergence constraint $\operatorname{div}\mathcal{E} = 0$, we introduce a Lagrange multiplier (say p), to dualize Coulomb's law. In this way, one builds a mixed variational formulation (VF) of Maxwell equations. It is well-posed, as soon as the well-known *inf-sup* (or Babuska-Brezzi [15,16]) condition holds. In addition to begin Mixed,

we use here an Augmented VF (see [4]), which results in a Mixed, Augmented VF. To this end, one adds to the bilinear form $\int_\Omega \mathbf{curl}\,\mathcal{E}\cdot\mathbf{curl}\,\mathbf{F}\,d\mathbf{x}$, the term $\int_\Omega \operatorname{div}\mathcal{E}\operatorname{div}\mathbf{F}\,d\mathbf{x}$. This formulation reads
Find $(\mathcal{E},p)\in\mathbf{X}\times L^2(\Omega)$ *such that*

$$\frac{d^2}{dt^2}\int_\Omega \mathcal{E}\cdot\mathbf{F}\,d\mathbf{x}+c^2\int_\Omega \mathbf{curl}\,\mathcal{E}\cdot\mathbf{curl}\,\mathbf{F}\,d\mathbf{x}+c^2\int_\Omega \operatorname{div}\mathcal{E}\operatorname{div}\mathbf{F}\,d\mathbf{x}+\int_\Omega p\operatorname{div}\mathcal{E}\,d\mathbf{x}$$
$$=-\frac{1}{\varepsilon_0}\frac{d}{dt}\int_\Omega \mathcal{J}\cdot\mathbf{F}\,d\mathbf{x},\qquad \forall\mathbf{F}\in\mathbf{X}, \tag{27}$$
$$\int_\Omega \operatorname{div}\mathcal{E}\,q\,d\mathbf{x}=0\qquad \forall q\in L^2(\Omega)\,. \tag{28}$$

To include the *SCM* in this formulation, the electric field $\mathcal{E}$ is split like $\mathcal{E}(t)=\mathcal{E}_R(t)+\mathcal{E}_S(t)$. One has $\mathcal{E}_S(t)=\kappa(t)\mathbf{l}_S$, where κ is a continous time-dependent function (to be determined). The same splitting is used for the test functions of the variational formulation, which is discretized in time, with the help of the leap-frog scheme. From a practical point of view, we choose the Taylor-Hood, P_2-iso-P_1 finite element. In addition to being well-suited for discretizing saddle-point problems, it allows to build diagonal mass matrices, when suitable quadrature formulas are used. Thus, the solution to the linear system, which involves the mass matrix, is straightforward [4]. This results in a fully discretized VF:

$$\mathbb{M}_\Omega\mathbf{E}_R^{n+1}+\mathbb{M}_{RS}\boldsymbol{\kappa}^{n+1}+\mathbb{L}_\Omega\mathbf{p}^{n+1}=\mathbf{F}^n\,, \tag{29}$$
$$\mathbb{M}_{RS}^T\mathbf{E}_R^{n+1}+\mathbb{M}_S\boldsymbol{\kappa}^{n+1}+\mathbb{L}_S\mathbf{p}^{n+1}=\mathbf{G}^n\,, \tag{30}$$
$$\mathbb{L}_\Omega^T\mathbf{E}_R^{n+1}+\mathbb{L}_S^T\boldsymbol{\kappa}^{n+1}=\mathbf{H}^n\,. \tag{31}$$

Above $\mathbb{M}_\Omega$ denotes the usual mass matrix, and $\mathbb{L}_\Omega$ corresponds to the divergence term involving $\mathbf{l}_R^h$ and the discrete Lagrange multiplier $p_h(t)$. Then, $\mathbb{M}_{RS}$ is a rectangular matrix, which is obtained by taking $\mathbf{L}^2$ scalar products between regular and singular basis functions, $\mathbb{M}_S$ is the "singular" mass matrix, and finally, $\mathbb{L}_S$ corresponds to the divergence term involving $\mathbf{l}_S$ and $p_h(t)$. To solve this system, one first removes the unknown $\boldsymbol{\kappa}^{n+1}$, so that the unknowns $(\mathbf{E}_R^{n+1},\mathbf{p}^{n+1})$ can be obtained with the help of a Uzawa-type algorithm. Finally, one concludes the time-stepping scheme by computing $\boldsymbol{\kappa}^{n+1}$ with the help of (30).

What is added, when compared to the same formulation posed in a waveguide without conical inclusions is, first, equation (30); second, additional terms that appear in (29) and (31), which express the coupling between singular and regular parts. However, these terms are independent of the time variable, so they are computed once and for all, at the initialization stage.

5 Numerical Experiments

We consider a metallic rectangular waveguide with two conical inclusions on its boundary (see Fig. 1). The metallic walls of the guide are assumed to be entirely perfectly conducting and the extremities are absorbing boundary conditions.

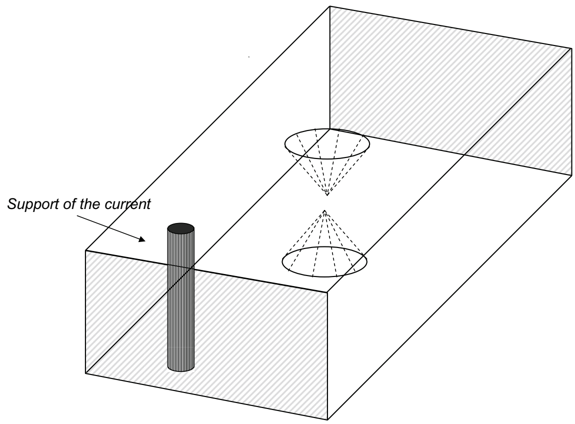

Fig. 1. Rectangular waveguide with conical inclusions

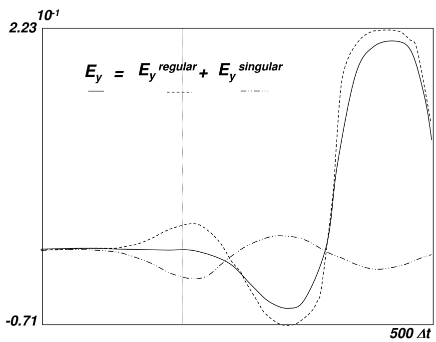

Fig. 2. Comparison of the total field, its singular and regular parts

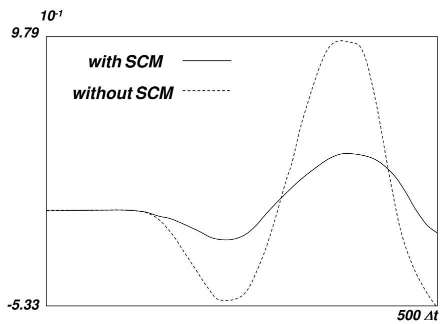

Fig. 3. Comparison of the total field with and without the *SCM*

The initial conditions are set to zero. The electromagnetic wave is generated by a current, with density $\mathcal{J}(t) = \mathcal{J}_z \boldsymbol{e}_z$, the support which is depicted on Fig.1. The value of the z-component is

$$\mathcal{J}_z(x, y, z) = C \sin(\omega t) \,. \tag{32}$$

Above, C is a constant, set to $C = 10^{-5}$, and ω is associated to the frequency $\nu = 5.10^9$ Hz.

We focus on the evolution of the electric field over time, at different nodes of the finite element mesh. First, one can check that the numerical method enforces *causality*, by studying the evolution at node A. By causality, we mean that the value of the electric field at A should be zero as long as the wave generated by the current has not reached A. To that aim, we show, on the same Figure, the – strong – total electric field, $\mathcal{E}_y(A)$, its regular part $\mathcal{E}_y^R(A)$, and its singular part $\mathcal{E}_y^S(A)$ (see Fig. 2.) The singular coefficient $\kappa(t)$ is non-zero as soon as the wave hits the tip of the cone, and it adopts a more or less sinusoidal pattern, with a frequency which depends on ν. Still, the total electric field at A, here $\mathcal{E}_y(A)$ does not vary, until the wave actually reaches A: the regular part 'compensates' for the variations of the singular part before that.

Then, we compare the results at some point B, with or without the *SCM*. In particular, results differ significantly in amplitudes (see the component $\mathcal{E}_y(B)$, Fig. 3): the amplitudes vary with a factor of more than two.

6 Conclusion

In this paper, we were interested in the treatment of conical protuberances in waveguide problems. This is a real 3D problem, even if around each sharp conical vertex, the singular subspace is finite-dimensional. Using a multiscale representation of the electromagnetic field in the vicinity of singular points, we propose an extension of the *SCM* in this non symmetric 3D case. Such approach may be useful in models in which it is possible and reasonable to determine or approximate the behavior of the singular electromagnetic field near irregular points. Extensions to "pseudo-singualrities", namely rounded corners seems also possible (see [8]).

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