# COMPUTING ELECTROMAGNETIC EIGENMODES WITH CONTINUOUS GALERKIN APPROXIMATIONS

Patrick Ciarlet, Jr.

*Laboratoire POEMS, UMR 2706 CNRS/ENSTA/INRIA,*
*ENSTA ParisTech,*
*32, boulevard Victor, 75739 Paris Cedex 15, France*
`patrick.ciarlet@ensta.fr`

Grace Hechme

*Laboratoire POEMS, UMR 2706 CNRS/ENSTA/INRIA,*
*ENSTA ParisTech,*
*32, boulevard Victor, 75739 Paris Cedex 15, France*
`grace.hechme@ensta.fr`

Costabel and Dauge proposed a variational setting to solve numerically the time-harmonic Maxwell equations in $3D$ polyhedral geometries, with a continuous approximation of the electromagnetic field. In order to remove spurious eigenmodes, three computational strategies are then possible. The original method, which requires a parameterization of the variational formulation. The second method, which is based on an *a posteriori* filtering of the computed eigenmodes. And the third method, which uses a mixed variational setting so that all spurious modes are removed *a priori*. In this paper, we discuss the relative merits of the approaches, which are illustrated by a series of 3D numerical examples.

*Keywords*: electromagnetism ; continuous Galerkin discretization ; eigenvalues and eigenvectors computations.

## Introduction

In a recent paper [17], Costabel and Dauge proposed a method, which allowed to discretize the electromagnetic field with a continuous approximation, in a $3D$, non-convex, polyhedron. In a way, they complemented the method earlier developed by Heintzé *et al* [20,2], which relied also on a continuous approximation of the field, but worked only in $3D$, convex polyhedra.

As it is well-known, when solving the Maxwell equations in a non-convex polyhedron with a continuous and conforming discretization, the discretized spaces are always included in a closed, strict subspace – sometimes called the subspace of *regular fields* – of the space of all possible fields. In other words, one cannot hope to approximate the part of the field which belongs to the subspace orthogonal to the subspace of regular fields. Over the past decade, several methods have been devised to address this problem. We refer to Ref. 11 and References therein for an extended discussion on this topic. In Ref. 17 the authors propose to recover density of the dis-

cretized spaces by measuring the electromagnetic fields in *weighted Sobolev spaces.*

In order to solve the time-harmonic Maxwell equations, Costabel and Dauge proceeded by adding a *regularization term*, with a parameter $s$: this resulted in the *parameterized weighted regularization method*. Using this technique, one has to discriminate between two sequences of eigenpairs: one is correct and the other is *spurious*. The spurious eigenvalues vary with the parameter $s$, whereas the correct ones don't. To remove the spurious modes, one has to repeat the computations for various values of the parameter $s$. We propose two alternate methods. The *filter method* (alluded to in Ref. 18), discriminates between the eigenpairs by examining *a posteriori* the divergence of fields. The *mixed method* [11,10,13] imposes *a priori* the divergence free constraint, and spurious modes are automatically excluded. Both methods lead in the end to the correct pairs. However, with the second method, it is expected that the constraint on the divergence of the fields is better taken into account.

The paper is organised as follows. In section 1, we introduce the mathematical framework and develop the variational problem for each of the three methods. In section 2, discretization aspects are analysed and convergence of the discrete pairs is studied. In section 3, we report 3D numerical experiments that illustrate the methods.

## 1. The eigenproblem and related variational formulations

We are interested in computing numerically eigenmodes of electromagnetic oscillations in a resonator cavity, bounded by a perfect conductor. Let $\Omega$ be the computational domain, included in $\mathbb{R}^3$. Mathematically, the domain is assumed to be a bounded, open polyhedron with a Lipschitz, connected, boundary $\partial\Omega$. For simplicity, we assume that $\Omega$ is simply connected. We denote by $\mathbf{n}$ the unit outward normal vector to $\partial\Omega$. Let $c$ be the light velocity in vacuum[a]. Classically, the electromagnetic eigenmodes are non-zero solutions to the time-harmonic Maxwell equations
*Find $\mathcal{E}$ and $\omega$ such that*

$$c^2\mathbf{curl}\,\mathbf{curl}\,\mathcal{E} = \omega^2\mathcal{E} \text{ in } \Omega, \tag{1.1}$$

$$\operatorname{div}\mathcal{E} = 0 \text{ in } \Omega, \tag{1.2}$$

$$\mathcal{E} \times \mathbf{n} = 0 \text{ on } \partial\Omega, \tag{1.3}$$

with $\mathcal{E}$ the electric field and $\omega$ the time-frequency. One can verify[11] that $\omega = 0$ is not an eigenvalue of (1.1)-(1.3). The corresponding magnetic field is given by the formula $\mathcal{B} = \imath\omega^{-1}\mathbf{curl}\,\mathcal{E}$.

To achieve our goal, we would like to discretize the field with a continuous and conforming approximation, based on nodal finite elements. As we mentioned

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[a]The case of a non-homogeneous material is addressed in Ref. 14.

already, different functional spaces will be considered, with regards to the convexity of the domain. We adopt the functional frameworks respectively given by Assous *et al* [2] (case of a convex domain) and Costabel and Dauge [17] (case of a non-convex domain). Let $L^2(\Omega)$ be the Lebesgue space of measurable and square integrable functions over $\Omega$. Its canonical norm and scalar product are respectively denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)_0$. Furthermore, let $(H^s(\Omega))_{s\in\mathbb{R}}$ be the usual scale of Sobolev spaces. We then introduce the following functional spaces for vector fields

$$\mathcal{H}(\mathbf{curl},\Omega) := \{\mathcal{F} \in L^2(\Omega)^3 \,|\, \mathbf{curl}\,\mathcal{F} \in L^2(\Omega)^3\}\,,$$
$$\mathcal{H}_0(\mathbf{curl},\Omega) := \{\mathcal{F} \in \mathcal{H}(\mathbf{curl},\Omega) \,|\, \mathcal{F} \times \mathbf{n}_{|\partial\Omega} = 0\}\,.$$

When $\Omega$ is *convex*, we introduce as well

$$\mathcal{X} := \{\mathcal{F} \in \mathcal{H}_0(\mathbf{curl},\Omega) \,|\, \operatorname{div}\mathcal{F} \in L^2(\Omega)\}.$$

By construction, the electric field $\mathcal{E}$, solution to the time-harmonic equations (1.1)-(1.3), belongs to $\mathcal{X}$. From Ref. 1, we know that $\mathcal{X}$ is a subset of $H^1(\Omega)^3$. In particular, the functional spaces $\mathcal{X} \cap H^1(\Omega)^3$ and $\mathcal{X}$ coincide.

**Remark 1.1.** Why is this property crucial? Let $\mathcal{X}_h$ denote the discrete space obtained with nodal finite elements. Since the intended discretization is continuous, $\mathcal{X}_h$ is included in $H^1(\Omega)^3$. Moreover, a conforming discretization implies that $\mathcal{X}_h$ is also included in $\mathcal{X}$. As a consequence, the density of $\cup_h \mathcal{X}_h$ in $\mathcal{X}$ requires the density of $\mathcal{X} \cap H^1(\Omega)^3$ in $\mathcal{X}$. In other words, the density of $\mathcal{X} \cap H^1(\Omega)^3$ in $\mathcal{X}$ must be seen as a *necessary condition* to achieve a continuous and conforming discretization.

Note that, from Refs. 25, 15, the graph norm and the semi-norm $\|\mathcal{F}\|_{\mathcal{X}} := \left(\|\mathbf{curl}\,\mathcal{F}\|_0^2 + \|\operatorname{div}\mathcal{F}\|_0^2\right)^{1/2}$ are equivalent norms on $\mathcal{X}$.

On the other hand, when $\Omega$ is *non-convex*, $\mathcal{X} \cap H^1(\Omega)^3$ is a strict, closed[1] subspace of $\mathcal{X}$: $\mathcal{X} \cap H^1(\Omega)^3$ can not be dense in $\mathcal{X}$. According to the previous remark, the density condition in $\mathcal{X}$ is not fulfilled. So, another functional space has to be chosen. It turns out that one can address this problem by relaxing the way the norm of the divergence is measured (for another alternative, we refer the interested reader to Ref. 11).

We follow here Ref. 17, Subsection 4.5. Since the domain $\Omega$ is non-convex, its boundary $\partial\Omega$ includes a non-empty set of reentrant edges $E$, with dihedral angles $(\pi/\alpha_e)_{e\in E}$ such that $1/2 < \alpha_e < 1$. Let $d$ denote the distance to $E$: $d(\mathbf{x}) = dist(\mathbf{x}, \cup_{e\in E}\bar{e})$, and introduce the weight $w_\gamma$, a smooth non-negative function of $x$, that depends on a real parameter $\gamma$. It is chosen to behave as $d^\gamma$ in the neighborhood of reentrant edges and corners, and is bounded above and below by a strictly positive constant outside a neighborhood of $E$. Let $L_\gamma^2(\Omega)$ be the weighted functional space below, with $\|.\|_{0,\gamma}$ norm:

$$L_\gamma^2(\Omega) := \{v \in L^2_{\mathrm{loc}}(\Omega) \,|\, w_\gamma\, v \in L^2(\Omega)\}\,, \; \|v\|_{0,\gamma} := \|w_\gamma\, v\|_0.$$

We then define, for $\gamma \geq 0$,

$$\mathcal{X}_\gamma := \{\mathcal{F} \in \mathcal{H}_0(\mathbf{curl},\Omega) \,|\, \operatorname{div}\mathcal{F} \in L_\gamma^2(\Omega)\}.$$

Clearly, $\mathcal{X} \subset \mathcal{X}_\gamma$. Measuring the fields in the weaker norm of $\mathcal{X}_\gamma$ allows us to recover the desired density result. Indeed, there exists $\gamma_{min} \in ]0, 1/2[$ which depends on the geometry of the domain[b], such that for all $\gamma \in ]\gamma_{min}, 1[$, $\mathcal{X}_\gamma \cap H^1(\Omega)^3$ is dense in $\mathcal{X}_\gamma$. Moreover, in $\mathcal{X}_\gamma$, the graph norm and the semi-norm $||\mathcal{F}||_{\mathcal{X}_\gamma} := \left(||\mathbf{curl}\,\mathcal{F}||_0^2 + ||\mathrm{div}\,\mathcal{F}||_{0,\gamma}^2\right)^{1/2}$ are equivalent norms.

**Remark 1.2.** One can choose any $\gamma$ in $]1/2, 1[$, *independently of the geometry.*

From now on, we shall keep the index $_\gamma$ everywhere. In the non-convex case, $\gamma$ belongs to $]1/2, 1[$ and it is fixed. In the convex case, we consider that there is no weight ($\gamma = 0$), so that $\mathcal{X}_0$ stands for $\mathcal{X}$, etc.

We note that the electric field solution to (1.1)-(1.3) naturally belongs to the space

$$\mathcal{K}_\gamma := \{\mathcal{F} \in \mathcal{X}_\gamma \,|\, \mathrm{div}\,\mathcal{F} = 0\}.$$

By setting $\lambda = \omega^2/c^2$, we can rewrite (1.1)-(1.3) as
*Find $(\mathcal{E}, \lambda) \in \mathcal{K}_\gamma \times \mathbb{R}^+$ such that*

$$\mathbf{curl}\,\mathbf{curl}\,\mathcal{E} = \lambda\mathcal{E} \text{ in } \Omega, \tag{1.4}$$

an equivalent variational formulation of which is
*Find $(\mathcal{E}, \lambda) \in \mathcal{K}_\gamma \times \mathbb{R}^+$ such that*

$$(\mathbf{curl}\,\mathcal{E}, \mathbf{curl}\,\mathcal{F})_0 = \lambda(\mathcal{E}, \mathcal{F})_0, \ \forall \mathcal{F} \in \mathcal{K}_\gamma. \tag{1.5}$$

However, building a conforming discretization in $\mathcal{K}_\gamma$ is a difficult task. Therefore, we choose to relax the divergence-free condition on the fields, and solve the eigenproblem in $\mathcal{X}_\gamma$. To that aim, we deal with augmented variational formulations[c], as advocated in Ref. 11. As a matter of fact, let $(\mathcal{E}, \lambda)$ be a solution to (1.4). Consider the $L^2(\Omega)^3$ scalar product between $\mathbf{curl}\,\mathbf{curl}\,\mathcal{E}$ and a test field $\mathcal{F}$ of $\mathcal{X}_\gamma$ and integrate by parts. Then, add the weighted $L_\gamma^2(\Omega)$ scalar product between $\mathrm{div}\,\mathcal{E}$ and the divergence of the test field $\mathrm{div}\,\mathcal{F}$, to reach

$$(\mathbf{curl}\,\mathcal{E}, \mathbf{curl}\,\mathcal{F})_0 + (\mathrm{div}\,\mathcal{E}, \mathrm{div}\,\mathcal{F})_{0,\gamma} = \lambda(\mathcal{E}, \mathcal{F})_0.$$

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[b]More precisely, one has $\gamma_{\min} := 2 - \sigma_\Delta$, where $\sigma_\Delta$ is the *minimum singularity exponent* for the Laplace problem with homogeneous boundary condition. If one introduces the usual space $H_0^1(\Omega) := \{\phi \in H^1(\Omega) \,|\, \phi_{|\partial\Omega} = 0\}$, $\sigma_\Delta$ is such that :

$$\{\phi \in H_0^1(\Omega) \,|\, \Delta\phi \in L^2(\Omega)\} \subset \bigcap_{s < \sigma_\Delta} H^s(\Omega),$$
$$\{\phi \in H_0^1(\Omega) \,|\, \Delta\phi \in L^2(\Omega)\} \not\subset H^{\sigma_\Delta}(\Omega).$$

As can be seen by direct inspection, $\sigma_\Delta \in ]\frac{3}{2}, 2[$.
[c]We shall not consider the popular choice of solving this eigenproblem in $\mathcal{H}_0(\mathbf{curl}, \Omega)$. This approach has been extensively studied, starting with the work of Kikuchi [21]. It relies on a discretization based on edge elements, as introduced by Nédélec [23] (see Bossavit and Rapetti [8] for an illuminating discussion on this topic). One major difficulty has to be addressed within this framework: the removal of *spurious* eigenmodes associated to $\lambda = 0$, with eigenspace equal to $\{\mathcal{F} \in L^2(\Omega)^3 \,|\, \exists\varphi \in H_0^1(\Omega), \ \mathcal{F} = \nabla\varphi\}$, which results in spectral pollution at $\lambda = 0$. Establishing a *discrete compactness* property [21,22,7] allows one to remove this difficulty.

Hence, the eigenpair $(\mathcal{E}, \lambda)$ also solves

*Find $(\mathcal{E}, \lambda) \in \mathcal{X}_\gamma \times \mathbb{R}^+$ such that*

$$(\mathcal{E}, \mathcal{F})_{\mathcal{X}_\gamma} = \lambda(\mathcal{E}, \mathcal{F})_0 \ \forall \mathcal{F} \in \mathcal{X}_\gamma. \tag{1.6}$$

Unfortunately, the reciprocal assertion is not true: in other words, the fact that $(\mathcal{E}, \lambda)$ is a solution to (1.6) does not guarantee that it is an eigenpair of the original problem (1.4), i. e. a "real" eigenpair. The reason is that there exist solutions to (1.6) which are not divergence-free: in this sense we call these eigenpairs *spurious*. Interestingly, provided the spurious eigenvalue is simple, the associated eigenvectors are automatically curl-free[18]. In the case of a multiple eigenvalue, the corresponding eigenvectors can be written as a linear combination of a "real" electric field and of a spurious field.

To remove the spurious pairs, we report below three possible approaches (see Refs. 17, 18, 11, 10).

**The filter approach.** Assume that the eigenproblem (1.6) is discretized with nodal finite elements. Let $(\mathcal{E}_h, \lambda_h)$ denote a discrete eigenpair. As we shall see in the next section, we have convergence of the discrete eigenpairs towards eigenpairs of (1.6). Hence, when the discrete eigenproblem is solved, eigenpairs in both families are computed, that is "real" pairs and spurious pairs. The filter method retains only "real" pairs by monitoring *a posteriori* the value of the *filter ratio*:

$$\frac{\|\text{div } \mathcal{E}_h\|_{0,\gamma}}{\|\mathbf{curl } \ \mathcal{E}_h\|_0}. \tag{1.7}$$

This value is small for "real" pairs since the divergence part of the eigenvector is small, whereas it is large for spurious ones since the **curl** part is small. Thus, one can filter out the spurious pairs. However, when a multiple eigenvalue is dealt with, an additional step must be carried out. Actually, for a multiple eigenvalue, the corresponding eigenvectors are in general linear combinations of vectors in both families. Hence, the eigenspace is first projected onto the subspace of vectors with a null divergence, in order to obtain correct values of the filter ratio, before the filtering process.

**The parameterized approach.** To sort out the pairs, Costabel and Dauge [17] chose to consider instead the *parameterized eigenproblem*

*Find $(\mathcal{E}_s, \lambda_s) \in \mathcal{X}_\gamma \times \mathbb{R}^+$ such that*

$$(\mathbf{curl}\,\mathcal{E}_s, \mathbf{curl}\,\mathcal{F})_0 + s\,(\text{div}\,\mathcal{E}_s, \text{div}\,\mathcal{F})_{0,\gamma} = \lambda_s(\mathcal{E}_s, \mathcal{F})_0 \ \forall \mathcal{F} \in \mathcal{X}_\gamma, \tag{1.8}$$

where $s > 0$ is a parameter. By doing so, the equivalence with (1.1)-(1.3) can be restored: if one lets $s$ vary, the "real" pairs, divergence-free, will be independent of $s$, while the spurious ones will vary with $s$.

In order to discriminate between "real" and spurious pairs, one can use a purely

imaginary parameter instead. In other words, one solves
*Find $(\mathcal{E}'_s, \lambda'_s) \in \mathcal{X}_\gamma \times \mathbb{C}$ such that*

$$(\mathbf{curl}\,\mathcal{E}'_s, \mathbf{curl}\,\mathcal{F})_0 + \imath s\,(\mathrm{div}\,\mathcal{E}'_s, \mathrm{div}\,\mathcal{F})_{0,\gamma} = \lambda'_s(\mathcal{E}'_s, \mathcal{F})_0 \ \forall \mathcal{F} \in \mathcal{X}_\gamma. \qquad (1.9)$$

In this manner, "real" pairs correspond to real eigenvalues ($\lambda'_s \in \mathbb{R}$), whereas spurious pairs correspond to purely imaginary eigenvalues ($\lambda'_s \in \imath\mathbb{R}$). The discrimination is automatic, and there is no need to let the parameter vary anymore. Still, from a numerical point of view, one has to consider complex-valued unknowns, thus increasing the cost of computations. Indeed, the number of real-valued unknowns is multiplied by a factor two. For historical reasons, we still refer to it as the (complex) parameterized approach. We shall use this version of the parameterized approach from now on.

**The mixed approach.** As it is advocated in Refs. 11, 10, 13 by Ciarlet, Jr. *et al*, one can take the constraint on the divergence of the field into account *a priori*, via the introduction of a Lagrange multiplier. In this way, one recovers the equivalence with the original problem. Consider $L^2_{-\gamma}(\Omega)$ the dual space of $L^2_\gamma(\Omega)$ (in the convex case $L^2_{-\gamma}(\Omega) = L^2(\Omega)$). The *mixed eigenproblem* to be solved reads
*Find $(\mathcal{E}, p, \lambda) \in \mathcal{X}_\gamma \times L^2_{-\gamma}(\Omega) \times \mathbb{R}^+$ such that*

$$\begin{cases} (\mathcal{E}, \mathcal{F})_{\mathcal{X}_\gamma} + {}_{L^2_{-\gamma}}\langle p, \mathrm{div}\,\mathcal{F}\rangle_{L^2_\gamma} = \lambda(\mathcal{E}, \mathcal{F})_0 \ \forall \mathcal{F} \in \mathcal{X}_\gamma \\ {}_{L^2_{-\gamma}}\langle q, \mathrm{div}\,\mathcal{E}\rangle_{L^2_\gamma} = 0, \ \forall q \in L^2_{-\gamma}(\Omega). \end{cases} \qquad (1.10)$$

To prove the equivalence with the original problem, one notes that any solution to (1.10) is such that $p = 0$ and $\mathcal{E} \in \mathcal{K}_\gamma$ (see the Annex of Ref. 11). This proves the equivalence with the original problem, since $(\mathcal{E}, \lambda)$ is an eigenpair of (1.4).

## 2. Abstract setting and conforming discretizations

In the following, we examine the convergence of discrete eigenpairs, when we deal with either a convex domain, or a non-convex domain.

### 2.1. *Approaches without multiplier*

First, we briefly recall the abstract theory that can be used to solve the filter and parameterized approaches. Let us introduce $V$, a Hilbert space. Then, let us take $L$, a second Hilbert space, such that $V \subset L$, $V$ is dense in $L$, and $L' \equiv L$. In other words, $L$ plays the role of the 'pivot' space. Next, let $a$ be a bilinear, continuous, symmetric, coercive form on $V \times V$. Last, let $f$ be an element of $L$. Consider the abstract problem
*Find $u \in V$ such that*

$$a(u, v) = (f, v)_L, \ \forall v \in V. \qquad (2.1)$$

One can naturally define the operator $\mathsf{T}: L \to L$, with $u = \mathsf{T}f$.
The associated eigenproblem reads

*Find $(u, \lambda) \in V \times \mathbb{R}$ such that*

$$a(u, v) = \lambda(u, v)_L, \ \forall v \in V, \tag{2.2}$$

or, equivalently,
*Find $(u, \lambda) \in V \times \mathbb{R}$ such that*

$$\lambda \mathsf{T} u = u. \tag{2.3}$$

Next, let us discretize the eigenproblem by defining a series of *finite dimensional subspaces of $V$*, called $(V_h)_h$. To each $V_h$, one can associate the discrete operator $\mathsf{T}_h : \ L \to L$, and $\mathsf{T}_h f = u_h$, with $u_h \in V_h$ such that $a(u_h, v_h) = (f, v_h)_L$, for all $v_h \in V_h$. The discretized eigenproblem reads
*Find $(u_h, \lambda_h) \in V_h \times \mathbb{R}$ such that*

$$\lambda_h \mathsf{T}_h u_h = u_h. \tag{2.4}$$

Classically (cf. Proposition 2.1 of Ref. 5), for a compact operator $\mathsf{T}$ of $\mathcal{L}(L, L)$, uniform convergence of $\mathsf{T}_h$ to $\mathsf{T}$ in $\mathcal{L}(L, L)$ implies convergence of eigenvectors and eigenvalues (note that uniform convergence in either $\mathcal{L}(L, V)$ or $\mathcal{L}(V, V)$ leads to the same conclusion). Moreover, to achieve convergence, it is enough that the three requirements below are fulfilled:

 (i)  the form $a$ is symmetric and coercive over $V$ ;
 (ii)  the operator $\mathsf{T}$ is compact from $L$ to $V$ ;
 (iii)  a *pointwise convergence* result holds:

$$\forall u \in V, \qquad \lim_{h \to 0} \|u - P_h u\|_L = 0,$$

 with $P_h : V \to V_h$ the orthogonal projection operator with respect to $a$.

It is pointed out in Ref. 5 that the compact character of the operator $\mathsf{T}$ in $\mathcal{L}(L, L)$ can be derived from the compact imbedding of $V$ into $L$.

In addition, we remark that in item (ii) – the compact character of $\mathsf{T}$ in $\mathcal{L}(L, V)$ – can also be viewed as a consequence of a second imbedding result. If $A : \ V \to V'$ is defined by $\langle Au, v \rangle = a(u, v)$, for all $u, v$ in $V$, then the solutions to the abstract problem (2.1) belong to

$$V_0 := \{v \in V \mid Av \in L\}$$

by construction. Then, if the imbedding of $V_0$, endowed with the graph norm $\|u\|_{V_0} := \left( \|u\|_V^2 + \|Au\|_L^2 \right)^{1/2}$, into $V$ is compact, item (ii) follows. Also, no *discrete compactness* property is required. Indeed, as one considers a series of conforming approximations $V_h \subset V$, the imbedding of $V_h$ into $L$ is automatically compact.

In order to use this classical theory in our case, we proceed as follows :
Let us introduce a series a regular triangulations $(\mathcal{T}_h)_h$ of $\Omega$, indexed by the mesh-sizes $h$, and made of tetrahedra. We use the conforming, continuous $P_k$ Lagrange

finite element ($k \geq 1$) to discretize, component by component, elements of $\mathcal{X}_\gamma$. The resulting discrete spaces are called $(\mathcal{X}_{\gamma,k,h})_h$. Then, we simply set $V = \mathcal{X}_\gamma$, $L = L^2(\Omega)^3$ and $a(u,v) = (u,v)_{\mathcal{X}_\gamma}$.

Item (i) is straightforward.

The fact that the operator $\mathsf{T}$ is compact in $\mathcal{L}(L,L)$ and that item (ii) holds can be checked directly. As we remarked previously, this can be achieved with the help of imbedding results, which we call the Weber imbedding Theorems [25,17,10]. The first one is the compact imbedding of $\mathcal{X}_\gamma$ into $L^2(\Omega)^3$, whereas the other one is the compact imbedding of the space of magnetic fields $\{\mathcal{F} \in \mathcal{H}(\mathbf{curl}, \Omega) \,|\, \mathrm{div}\,\mathcal{F} \in L^2(\Omega),\ \mathcal{F} \cdot \mathbf{n}_{|\partial\Omega} = 0\}$ into $L^2(\Omega)^3$.

In the convex case, item (iii) is a well-known consequence of the interpolation theory for Lagrange finite elements. In the non-convex case, we refer the reader to the approximation errors provided in Ref. 17.

The convergence of the discrete pairs to the exact pairs thus follows. Recall that the exact pairs include both the "real" and the spurious pairs. The possible ways to discriminate them has been discussed in the previous section. The convergence rates will be analysed at subsection 2.3. After the discretization has been performed, one has to solve a generalized eigenproblem, that writes $\mathbb{K}\vec{u} = \lambda_h \mathbb{M}\vec{u}$. It is solved either in $\mathbb{R}^n$ for the filter approach, or in $\mathbb{C}^n$ for the (complex) parameterized approach. It is beyond the scope of this paper to study the numerical algorithms that can be used to solve this generalized eigenproblem.

### 2.2.  *Approach with a multiplier*

Let us now recall the abstract theory we use for the mixed approach. We keep the same notations as before ($V$, $L$, $a$ on so on). Additionally, let us introduce $Q$, the Hilbert space of the multipliers. Here, $a$ is a bilinear, continuous, symmetric, positive semidefinite form on $V \times V$. Next, let $b$ be a bilinear, continuous form on $V \times Q$. Given $f \in L$, we consider the abstract mixed problem

*Find $(u,p) \in V \times Q$ such that*

$$\begin{cases} a(u,v) + b(v,p) = (f,v)_L, \ \forall v \in V \\ b(u,q) = 0, \ \forall q \in Q \end{cases} . \tag{2.5}$$

Under the assumption that this problem is well-posed, we define the operator $\mathsf{T}$ : $L \to L$, with $u = \mathsf{T}f$.

The associated eigenproblem reads

*Find $(u,p,\lambda) \in V \times Q \times \mathbb{R}$ such that*

$$\begin{cases} a(u,v) + b(v,p) = \lambda(u,v)_L, \ \forall v \in V \\ b(u,q) = 0, \ \forall q \in Q \end{cases} . \tag{2.6}$$

In terms of the field $u$, this eigenproblem can again be formulated as $\lambda\mathsf{T}u = u$, similarly to Eq. (2.3).

We discretize the eigenproblem by defining two series of *finite dimensional sub-spaces*: respectively *of V*, called $(V_h)_h$, and *of Q*, called $(Q_h)_h$. The discretized eigenproblem writes again $\lambda_h \mathsf{T}_h u_h = u_h$, as in Eq. (2.4), with $\mathsf{T}_h$ the discrete counterpart of $\mathsf{T}$. According to Refs. 6, 5, for a compact operator $\mathsf{T}$ of $\mathcal{L}(L, V)$, uniform convergence of $\mathsf{T}_h$ to $\mathsf{T}$ in $\mathcal{L}(L, V)$ implies convergence of eigenvectors and eigenvalues. Moreover, the necessary (see Remark 3.3 of Ref. 5 for a discussion) and sufficient conditions (iv)-(v)-(vi)-(vii) to reach convergence can be listed as below. For that, let us introduce the exact kernel $K := \{v \in V \,|\, b(v, q) = 0, \ \forall q \in Q\}$ and the discrete kernel $K_h := \{v_h \in V_h \,|\, b(v_h, q_h) = 0, \ \forall q_h \in Q_h\}$. We define also the subspaces $V_0$ of $V$ and $Q_0$ of $Q$, made of solutions to the abstract mixed problem (2.5), endowed with their natural norms:

$$V_0 = \{v \in V \,|\, \exists q \in Q \text{ s.t. } (v, q) \text{ solution to } (2.5) \text{ for some } f \in L\},$$
$$Q_0 = \{q \in Q \,|\, \exists v \in V \text{ s.t. } (v, q) \text{ solution to } (2.5) \text{ for some } f \in L\}.$$

To achieve convergence, the four requirements below are necessary and sufficient:

(iv)  the form $a$ is symmetric over $V$ and coercive over the discrete kernel $K_h$ ;
(v)  the operator $\mathsf{T}$ is compact from $L$ to $V$ ;
(vi)  the *weak approximability* of $Q_0$:
  $\exists \omega_1 \,:\, \mathbb{R}^+ \to \mathbb{R}^+$, such that $\lim_{h \to 0^+} \omega_1(h) = 0$ and

$$\sup_{v_h \in K_h} \frac{b(v_h, q_0)}{\|v_h\|_V} \le \omega_1(h) \|q_0\|_{Q_0}, \ \forall q_0 \in Q_0 \,; \tag{2.7}$$

(vii)  the *strong approximability* of $V_0$:
  $\exists \omega_2 \,:\, \mathbb{R}^+ \to \mathbb{R}^+$, such that $\lim_{h \to 0^+} \omega_2(h) = 0$ and

$$\forall v_0 \in V_0, \ \exists v^I \in K_h \text{ s.t. } \|v_0 - v^I\|_V \le \omega_2(h) \|v_0\|_{V_0}. \tag{2.8}$$

How can we use the theory in our case?
Let us introduce again a series a regular triangulations $(\mathcal{T}_h)_h$ of $\Omega$, indexed by the meshsizes $h$, and made of tetrahedra. We use conforming, continuous $P_{k+1}/P_k$ finite elements ($k \ge 1$) to discretize elements of $\mathcal{X}_\gamma \times L^2_{-\gamma}(\Omega)$. If $\Omega$ is convex the Taylor-Hood finite element is used, while a "zero near singularity" Taylor-Hood like finite element (cf. Ref. 13) is applied if $\Omega$ is non-convex. The resulting discrete spaces are called $(\mathcal{X}_{\gamma, k+1, h} \times Q_{k, h})_h$. Then, we simply set $V = \mathcal{X}_\gamma$, $L = L^2(\Omega)^3$, $Q = L^2_{-\gamma}(\Omega)$ and $a(u, v) = (u, v)_{\mathcal{X}_\gamma}$, $b(u, q) = {}_{L^2_{-\gamma}}\langle q, \operatorname{div} u \rangle_{L^2_\gamma}$.
Item (iv) is straightforward.
Item (v) stems again from the Weber imbedding Theorems [10].
The weak approximability (item (vi)) can be established with $\omega_1(h) \simeq h$ in the convex case, and $\omega_1(h) \simeq h^{1-\gamma}$ in the non-convex case (we refer the reader to Ref. 10).
It can be checked [6] that the strong approximability (item (vii)) is actually a conse-

quence of the uniform discrete inf-sup condition[d], which we recall here:

$$\exists \beta > 0 \text{ s.t. } \forall h, \quad \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq \beta. \tag{2.9}$$

In the convex case, this condition has been proven by Stenberg and Boffi [24,4] for the Stokes problem. For this problem, one uses only zero-mean value multipliers, that is $Q_h \subset \{q \in L^2(\Omega) \,|\, (q, 1)_0 = 0\}$. On how to include multipliers with any mean value, as one must for electric fields, we refer the interested reader to Ref. 12.
In the non-convex case, such a result has been obtained very recently for the pairs of discrete spaces $\mathcal{X}_{\gamma, k+1, h} \times Q_{k, h}$ (see Ref. 13).

The convergence of the discrete pairs to the exact pairs thus follows. In this case, no post-treatment is required, since all exact eigenpairs are "real" eigenpairs. The convergence issues are addressed in the next subsection. After discretization, one has to solve a generalized, mixed eigenproblem in $\mathbb{R}^n \times \mathbb{R}^p$, that writes $\mathbb{K}\vec{u} + \mathbb{C}^T \vec{p} = \lambda_h \mathbb{M}\vec{u}$, subject to the constraint $\mathbb{C}\vec{u} = 0$ in $\mathbb{R}^p$. Once again, the numerical algorithms that can be used to solve this generalized mixed eigenproblem are not presented here.

### 2.3.  *Convergence results*

Here, we assume the uniform convergence of $(\mathsf{T}_h)_h$ to $\mathsf{T}$ in either $\mathcal{L}(L, L)$, $\mathcal{L}(L, V)$ or $\mathcal{L}(V, V)$. From this initial result, one can derive error bounds [3] on the eigenvalues and eigenspaces as stated below. For an eigenvalue $\lambda$, let $E_\lambda$ be the associated eigenspace. In the case of a multiple eigenvalue, let $\lambda_h$ be the average of the discrete eigenvalues converging to $\lambda$, and $E_{\lambda_h}$ the sum of the discrete eigenspaces associated to $\lambda_h$. As far as convergence of the eigenvalues is concerned, one wants to measure the error $|\lambda - \lambda_h|$. As far as the eigenspaces are concerned, one wants to bound the *gap* $\hat{\delta}$ between them: given $V_1$ and $V_2$ two subspaces of $V$, we introduce

$$\hat{\delta}(V_1, V_2) = \max(\delta(V_1, V_2), \delta(V_2, V_1)), \text{ where } \delta(V_1, V_2) = \sup_{\substack{v_1 \in V_1 \\ \|v_1\|_V = 1}} \inf_{v_2 \in V_2} \|v_1 - v_2\|_V.$$

We recall here convergence results, the proofs of which can be found in Ref. 3.

**Theorem 2.1.**  *There exists $C > 0$ such that*

$$|\lambda - \lambda_h| < C \, \varepsilon_\lambda(h)^2 \text{ and } \hat{\delta}(E_\lambda, E_{\lambda_h}) < C \, \varepsilon_\lambda(h), \tag{2.10}$$

*with the approximation error*

$$\varepsilon_\lambda(h) = \sup_{\substack{v \in E_\lambda \\ \|v\|_V = 1}} \inf_{v_h \in V_h} \|v - v_h\|_V. \tag{2.11}$$

---

[d]And of the approximation errors (see the upcoming Propositions 2.1 and 2.2).

To be more precise, for a given $N$, the $N$ smallest eigenvalues (with repetition) are approximated in such a way that there exists $h_0$ such that, for all $h \in ]0, h_0[$, the estimates (2.10) hold. In other words, the constant $C$ above depends on $N$.

In the case of a *convex domain*, one has the classical result, which requires an (extra) $H^{1+\sigma}(\Omega)^3$ regularity of the eigenfields (we refer to Ref. 16 for a precise study of this extra regularity).

**Proposition 2.1.** *Assume that one has the imbedding $E_\lambda \subset H^{1+\sigma}(\Omega)^3$, for a given $\sigma > 0$. Then, the bound in the approximation error estimate (2.11) is*

$$\varepsilon_\lambda(h) \simeq h^t, \text{ for } t = \min(\sigma, k).$$

**Remark 2.1.** This extra regularity assumption can be removed by adding weights in convex domains too (cf. Ref. 17), so that the approximation error becomes independent of $\lambda$.

In the case of a *non-convex domain*, one uses the error estimates established in Refs. 17, 18. Recall that the weight exponent $\gamma$ belongs to $]\gamma_{min}, 1[$, and that $\gamma_{min} \in ]0, 1/2[$ depends on the geometry. In particular, one can always choose $\gamma$ so that $(\gamma - \gamma_{min}) > 1/2$.

**Proposition 2.2.** *There exists a uniform bound (independent of $\lambda$) in the approximation error estimate (2.11), which is equal to*

$$\varepsilon^\delta(h) \simeq h^t, \text{ for } t = (\gamma - \gamma_{min}) - \delta, \ \forall \delta > 0.$$

*Assume in addition that one has the imbedding $E_\lambda \subset H^{(1-\gamma_{min})+\sigma}(\Omega)^3$, for a given $\sigma > 0$. Then, the bound in the approximation error estimate (2.11) is*

$$\varepsilon_\lambda^\delta(h) \simeq h^t, \text{ for } t = \min((\gamma - \gamma_{min}) + \sigma - \delta, k), \ \forall \delta > 0.$$

## 3. Numerical experiments

In this section, we highlight the *pros and cons* of the approaches with some numerical results.

### 3.1. *In a convex geometry: the filter approach*

We consider the solution of eigenproblem (1.1)-(1.3) in a three-dimensional cube with edges of unit length. Eigenpairs of this problem are of the form:

$$\omega^2/c^2 := (k_1^2 + k_2^2 + k_3^2)\pi^2, \quad k_1, k_2, k_3 \in \mathbb{N},$$

$$\mathcal{E}(\mathbf{x}) := \begin{pmatrix} \lambda_1 \cos(k_1\pi x) \sin(k_2\pi y) \sin(k_3\pi z) \\ \lambda_2 \sin(k_1\pi x) \cos(k_2\pi y) \sin(k_3\pi z) \\ \lambda_3 \sin(k_1\pi x) \sin(k_2\pi y) \cos(k_3\pi z) \end{pmatrix},$$

subject to

$$k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0,$$

which is the divergence free constraint, with $(x, y, z)$ the cartesian coordinates of a given $\mathbf{x} \in \Omega$.

We compute the smallest 11 Maxwell eigenvalues corresponding to $2\pi^2$ (of multiplicity 3), $3\pi^2$ (of multiplicity 2) and $5\pi^2$ (of multiplicity 6). Computations are carried out on a quasi-uniform mesh with 3072 tetrahedra and 729 vertices. We apply the filter method using vector $P_k$ ($k = 1, 2, 3$) finite element discretizations, leading to problems with respectively 1323, 11475 and 39675 d.o.f., and the mixed method using $P_{k+1} - P_k$ ($k = 1, 2$) Taylor-Hood finite elements, leading to problems with resp. 12204 and 44588 d.o.f. The relative errors on the computed eigenvalues,

$$r_k = |\lambda_{h,k} - \lambda_k|/|\lambda_k|,$$

are reported in Table 1.

| Method | Filter | | | Mixed | |
|---|---|---|---|---|---|
| F E | $P_1$ | $P_2$ | $P_3$ | $P_2 - P_1$ | $P_3 - P_2$ |
| $r_1$ | $3.4 \times 10^{-2}$ | $2.0 \times 10^{-4}$ | $4.8 \times 10^{-7}$ | $2.0 \times 10^{-4}$ | $4.8 \times 10^{-7}$ |
| $r_2$ | $5.1 \times 10^{-2}$ | $2.1 \times 10^{-4}$ | $4.8 \times 10^{-7}$ | $2.1 \times 10^{-4}$ | $4.8 \times 10^{-7}$ |
| $r_3$ | $5.1 \times 10^{-2}$ | $2.1 \times 10^{-4}$ | $5.6 \times 10^{-7}$ | $2.1 \times 10^{-4}$ | $5.6 \times 10^{-7}$ |
| $r_4$ | $7.8 \times 10^{-2}$ | $6.1 \times 10^{-4}$ | $2.4 \times 10^{-6}$ | $6.1 \times 10^{-4}$ | $2.5 \times 10^{-6}$ |
| $r_5$ | $7.9 \times 10^{-2}$ | $6.1 \times 10^{-4}$ | $2.4 \times 10^{-6}$ | $6.2 \times 10^{-4}$ | $2.5 \times 10^{-6}$ |
| $r_6$ | $7.5 \times 10^{-2}$ | $1.0 \times 10^{-3}$ | $5.4 \times 10^{-6}$ | $1.0 \times 10^{-3}$ | $5.4 \times 10^{-6}$ |
| $r_7$ | $7.5 \times 10^{-2}$ | $1.0 \times 10^{-3}$ | $5.4 \times 10^{-6}$ | $1.0 \times 10^{-3}$ | $5.4 \times 10^{-6}$ |
| $r_8$ | $1.0 \times 10^{-1}$ | $1.0 \times 10^{-3}$ | $5.6 \times 10^{-6}$ | $1.0 \times 10^{-3}$ | $5.6 \times 10^{-6}$ |
| $r_9$ | $1.0 \times 10^{-1}$ | $1.0 \times 10^{-3}$ | $5.6 \times 10^{-6}$ | $1.0 \times 10^{-3}$ | $5.6 \times 10^{-6}$ |
| $r_{10}$ | $1.0 \times 10^{-1}$ | $1.0 \times 10^{-3}$ | $6.3 \times 10^{-6}$ | $1.0 \times 10^{-3}$ | $6.3 \times 10^{-6}$ |
| $r_{11}$ | $1.0 \times 10^{-1}$ | $1.0 \times 10^{-3}$ | $6.3 \times 10^{-6}$ | $1.0 \times 10^{-3}$ | $6.3 \times 10^{-6}$ |

Table 1. Relative errors for computations in the unit cube

One can easily remark that both approaches give similar results. This reveals the efficiency of the *a posteriori* filtering. Indeed, the eigenvalue $3\pi^2$ is of multiplicity 3 in the spectrum of the augmented problem (1.6), but only two pairs correspond to "real" pairs. This fact is successfully detected by the filter approach and the spurious pair is excluded as expected.

### 3.2. *In a non-convex geometry: comparisons of the approaches*

In convex domains, computations are fairly standard, and the approaches show analogous behavior. We further investigate the approaches in non-convex domains, when singular eigenfields are handled.

### 3.2.1. *Experiments in the thick L-shaped domain*

First, we solve eigenproblem (1.1)-(1.3) in the thick L-shaped domain [19] having a single reentrant edge, $(]-1,1[^2\backslash[-1,0]^2)\times]0,1[$. The smallest 9 Maxwell eigenvalues [19] up to six digits are listed in Table 2. It is reported that the first, second and fifth eigenvalues exhibit the strongest singularity. The value of $\gamma_{min}$ is equal to $1/3$.

| | |
|---|---|
| $\lambda_1$ | 9.6397 |
| $\lambda_2$ | 11.3452 |
| $\lambda_3$ | 13.4036 |
| $\lambda_4$ | 15.1972 |
| $\lambda_5$ | 19.5093 |
| $\lambda_6$ | 19.7392 |
| $\lambda_7$ | 19.7392 |
| $\lambda_8$ | 19.7392 |
| $\lambda_9$ | 21.2591 |

Table 2. Maxwell eigenvalues in the thick L-shape (Dauge's computations)

Through the experiments, $2D$ triangular meshes of the L-shape are duplicated towards the direction of the reentrant edge so as to generate $3D$ tetrahedral meshes. Two types of $2D$ triangular meshes are used : a quasi-uniform (ungraded) mesh and a graded mesh (graded towards the $2D$ reentrant corner). We obtain $3D$ meshes with 4608 tetrahedra and 1125 vertices in the ungraded case, and 4032 tetrahedra and 1010 vertices in the graded case. The filter and (complex) parameterized ($s = 1$) methods are applied with vector $P_2$ finite elements and the mixed method with "zero near singularity" $P_2 - P_1$ finite elements. These discretizations yield discrete problems resp. with 14843 and 17069 d.o.f. on the ungraded mesh, and, 15818 and 18162 d.o.f. on the graded one. The weight is implemented with $\gamma = 0.95$. The results are shown in Tables 3, 4 and 5. For the (complex) parameterized experiments, we report the values of $r_k = |\Re(\lambda_{h,k}) - \lambda_k|/|\lambda_k|$.

Overall, when compared to the mixed method, the parameterized and the filter methods exhibit smaller relative errors for most of the eigenpairs, and in particular for the singular ones on the graded mesh. Moreover, for the mixed method, appropriate results are only obtained on the graded mesh, since spurious eigenvalues are encountered on the ungraded mesh, as shown in Table 3 ($\lambda_{h,4}$, $\lambda_{h,6}$ and $\lambda_{h,7}$). In fact, for the thick L-shape, the electric field has singularities only in the directions perpendicular to the reentrant edge. Therefore, a grading towards these directions is beneficial to approximate better the singular eigenvalues and improve the convergence rate [e]. Furthermore, for "zero near singularity" $P_2 - P_1$ finite elements (cf.

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[e]This is true for all methods. Compare Tables 4 and 5 for the filter and the parameterized methods.

| | |
|---|---|
| $\lambda_{h,1}$ | 9.6627 |
| $\lambda_{h,2}$ | 10.5974 |
| $\lambda_{h,3}$ | 13.3150 |
| $\lambda_{h,4}$ | 13.6238 |
| $\lambda_{h,5}$ | 15.9957 |
| $\lambda_{h,6}$ | 16.7727 |
| $\lambda_{h,7}$ | 18.6795 |
| $\lambda_{h,8}$ | 19.7434 |
| $\lambda_{h,9}$ | 19.7643 |

Table 3. The smallest 9 eigenvalues given by the mixed method in the thick L-shape with the ungraded mesh

| Method | Filter | Parameterized |
|---|---|---|
| $r_1$ | $2.4 \times 10^{-3}$ | $2.4 \times 10^{-3}$ |
| $r_2$ | $1.7 \times 10^{-2}$ | $3.9 \times 10^{-2}$ |
| $r_3$ | $9.7 \times 10^{-4}$ | $8.1 \times 10^{-4}$ |
| $r_4$ | $1.5 \times 10^{-4}$ | $1.6 \times 10^{-4}$ |
| $r_5$ | $2.6 \times 10^{-3}$ | $1.5 \times 10^{-2}$ |
| $r_6$ | $2.1 \times 10^{-4}$ | $2.1 \times 10^{-4}$ |
| $r_7$ | $1.3 \times 10^{-3}$ | $1.1 \times 10^{-3}$ |
| $r_8$ | $1.4 \times 10^{-3}$ | $1.1 \times 10^{-3}$ |
| $r_9$ | $1.6 \times 10^{-3}$ | $1.2 \times 10^{-3}$ |

Table 4. Relative errors for computations in the thick L-shape with the ungraded mesh

| Method | Filter | Parameterized | Mixed |
|---|---|---|---|
| $r_1$ | $6.1 \times 10^{-4}$ | $6.1 \times 10^{-4}$ | $6.2 \times 10^{-4}$ |
| $r_2$ | $6.5 \times 10^{-3}$ | $1.1 \times 10^{-2}$ | $8.5 \times 10^{-3}$ |
| $r_3$ | $8.1 \times 10^{-4}$ | $7.4 \times 10^{-4}$ | $8.4 \times 10^{-4}$ |
| $r_4$ | $1.1 \times 10^{-4}$ | $1.0 \times 10^{-4}$ | $1.1 \times 10^{-4}$ |
| $r_5$ | $2.0 \times 10^{-3}$ | $4.7 \times 10^{-3}$ | $6.9 \times 10^{-3}$ |
| $r_6$ | $1.8 \times 10^{-4}$ | $1.8 \times 10^{-4}$ | $1.8 \times 10^{-4}$ |
| $r_7$ | $1.2 \times 10^{-3}$ | $1.1 \times 10^{-3}$ | $1.2 \times 10^{-3}$ |
| $r_8$ | $1.2 \times 10^{-3}$ | $1.1 \times 10^{-3}$ | $1.3 \times 10^{-3}$ |
| $r_9$ | $1.3 \times 10^{-3}$ | $1.1 \times 10^{-3}$ | $1.1 \times 10^{-2}$ |

Table 5. Relative errors for computations in the thick L-shape with the graded mesh

Ref. 13), the components of the discrete Lagrange multiplier are set to zero in all tetrahedra having a vertex on the reentrant edge. When the considered $2D$ triangu-

lar mesh is graded towards the reentrant corner, the volume of the aforementioned set of tetrahedra is much smaller than in the case where the $2D$ triangular mesh is ungraded. Hence, in the ungraded case, the constraint is imposed on a smaller volume of the domain $\Omega$, and some spurious eigenvalues might remain in the spectrum of the discrete mixed problem. It is also consistent with the fact that, according to the approximation theory of eigenproblems (see Theorem 2.1 and comments below it), convergence occurs only for sufficiently small $h$.

Note that, after discretisation, the spectrum of the (complex) parameterized problem (1.9) has complex eigenvalues. The spurious pairs have the larger imaginary parts, and the "real" pairs have the larger real parts. Thus, on the one hand, for the parameterized approach, the filtering is uniquely based on the eigenvalues, whereas, on the other hand, for the filter method, the filtering is based on the properties exhibited by the eigenvectors.

### 3.2.2. *Experiments in the Fichera corner domain*

We conclude this section with experiments peformed in the Fichera corner domain, which is the cube $]-1,1[^3$ minus the cube $[-1,0]^3$. The smallest 8 Maxwell eigenmodes given in Ref. 19 are listed in Table 6. The value of $\gamma_{min}$ is equal to $1/3$.

| $\lambda_1$ | 3.2199 |
|---|---|
| $\lambda_2$ | 5.8804 |
| $\lambda_3$ | 5.8804 |
| $\lambda_4$ | 10.6854 |
| $\lambda_5$ | 10.6937 |
| $\lambda_6$ | 10.6937 |
| $\lambda_7$ | 12.3164 |
| $\lambda_8$ | 12.3164 |

Table 6. Maxwell eigenvalues in the Fichera corner

Computations are carried out on a $3D$ mesh having 2688 tetrahedra and 665 vertices, refined towards the reentrant edges and corner. Vector $P_k$ $(k = 2, 3)$ finite elements (filter method), and "zero near singularity" $P_{k+1} - P_k$ $(k = 1, 2)$ finite elements (mixed method) are used, leading to problems with 9894 and 34422 d.o.f., resp. 10461 and 38328 d.o.f. Results are reported in Table 7.

We can see that both methods give quite similar relative errors. Furthermore, the approximations are significantly improved when higher order finite elements are considered. Also, when vector $P_3$ finite elements are used, the first eigenvalue having the highest singularity is better approximated by the filter method.

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| Method | Filter | | Mixed | |
|---|---|---|---|---|
| F E | $P_2$ | $P_3$ | $P_2 - P_1$ | $P_3 - P_2$ |
| $r_1$ | $2.2 \times 10^{-1}$ | $1.1 \times 10^{-3}$ | $1.8 \times 10^{-1}$ | $5.1 \times 10^{-2}$ |
| $r_2$ | $1.5 \times 10^{-2}$ | $6.5 \times 10^{-4}$ | $1.8 \times 10^{-2}$ | $3.9 \times 10^{-4}$ |
| $r_3$ | $1.5 \times 10^{-2}$ | $6.5 \times 10^{-4}$ | $1.8 \times 10^{-2}$ | $3.9 \times 10^{-4}$ |
| $r_4$ | $5.6 \times 10^{-2}$ | $1.8 \times 10^{-3}$ | $3.7 \times 10^{-2}$ | $1.0 \times 10^{-3}$ |
| $r_5$ | $2.5 \times 10^{-2}$ | $6.9 \times 10^{-4}$ | $2.7 \times 10^{-2}$ | $6.4 \times 10^{-4}$ |
| $r_6$ | $2.5 \times 10^{-2}$ | $6.9 \times 10^{-4}$ | $2.7 \times 10^{-2}$ | $6.4 \times 10^{-4}$ |
| $r_7$ | $4.2 \times 10^{-2}$ | $5.8 \times 10^{-4}$ | $3.1 \times 10^{-2}$ | $5.2 \times 10^{-4}$ |
| $r_8$ | $4.2 \times 10^{-2}$ | $5.8 \times 10^{-4}$ | $3.1 \times 10^{-2}$ | $5.2 \times 10^{-4}$ |

Table 7. Relative errors for computations in the Fichera corner

## 4. Discussion: focusing on eigenvalues or eigenvectors?

Up to now, we focused mainly on the convergence of the eigenvalues. What about the convergence of eigenvectors? Except in the particular case of the cube where the exact expression of the eigenvectors is known, no analytic knowledge is available. However, there is – at least – one noticeable exception, since according to Eq. (1.2), all eigenvectors are divergence-free. So, we expect that the divergence of the discrete eigenvectors goes to zero, when measured in $|| \cdot ||_{0,\gamma}$ norm. Recall that it is not exactly zero, since our method is conforming in $\mathcal{X}_\gamma$, but not in the subspace made of divergence-free fields, $\mathcal{K}_\gamma$.

Below, we monitor the evolution of the filter ratios (1.7), that compares (here, for the discrete eigenvectors) the $L^2_\gamma(\Omega)$-norm of the divergence to the $L^2(\Omega)^3$-norm of the curl, for the Fichera corner. On the left side of Figure 1, we report the evolu-
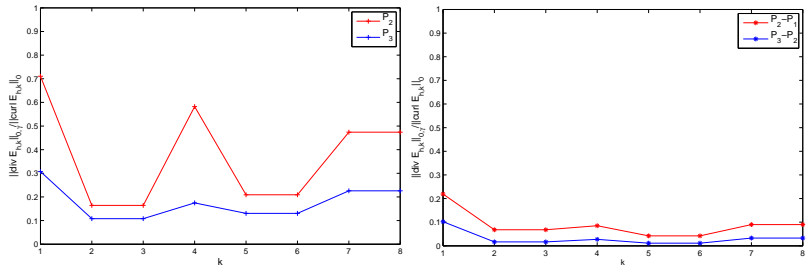


Fig. 1. Filter ratios: for the filter method (left) and for the mixed method (right).

tion of the filter ratios, on a given mesh, when one uses either the vector $P_2$ or the vector $P_3$ finite element to discretize the fields, without a multiplier. Similarly, on right side of Figure 1, we report the same evolution, when a Lagrange multiplier (respectively the scalar $P_1$ or the scalar $P_2$ Finite Element) is used.

One can draw three conclusions from this Figure. The first one is that, the approximation on the divergence is consistently better when one goes from a $P_2$ to a $P_3$ approximation of the fields. The second one is that the mixed method allows us to better control the divergence of the discrete eigenfields, as expected (see also Figure 2 for a specific comparison). We recall that these results on the eigenfields are linked to comparable accuracies for the eigenvalues, as we saw in section 3. The last one is that the threshold one imposes to the ratio (1.7) must be chosen with great care for the filter method. Indeed, if the value were set to 0.5, two "real" eigenvalues would have been filtered out in the $P_2$ computations.
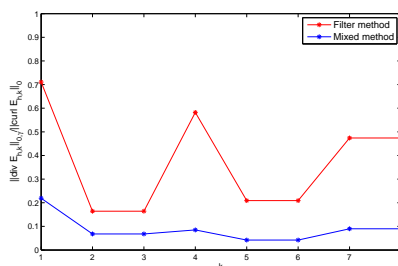


Fig. 2. Filter ratios for both methods, $P_2$ FE.

## 5. Acknowledgements

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