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WELL-POSEDNESS OF THE DRUDE–BORN–FEDOROV MODEL FOR CHIRAL MEDIA

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We consider a chiral medium in a bounded domain, enclosed in a perfectly conducting material. We solve the transient Maxwell equations in this domain, when the medium is modeled by the Drude–Born–Fedorov constitutive equations. The input data is located on the boundary, in the form of given surface current and surface charge densities. It is proved that, except for a countable set of chirality admittance values, the problem is mathematically well-posed. This result holds for domains with non-smooth boundaries.

Keywords: Chiral media; Maxwell's equations; Drude–Born–Fedorov relations.

1. Introduction

Chiral materials, along with other bianisotropic media, have been the subject of many studies in the past years and numerous references are available in the literature, dealing both with applications and theoretical works. Indeed, these materials respond with both electric and magnetic polarizations to either electric or magnetic excitations, which makes them of particular interest for optics and many other electromagnetic applications. They can be characterized by different sets of constitutive relations in which the electric and magnetic fields are coupled, the strength of this coupling being measured by the chirality admittance. In the present work, we make use of the so-called Drude–Born–Fedorov constitutive equations. Recent investigations, dealing with the propagation of time-harmonic electromagnetic waves in chiral media assuming such relations, are the subject of articles by Ammari *et al.*[1–4] and also by Athanasiadis *et al.*[6–8] The time domain case has not been as extensively treated, but mathematical studies for problems set in unbounded domains can nevertheless be found in Refs. 9, 21 and 31.

We deal here with the well-posedness of the transient Maxwell equations in a homogeneous, isotropic chiral media surrounded by a perfect conductor, assuming a very general geometry. The interface conditions we consider are the ones coming from the physics. We show that there exists a unique solution to this problem for all but possibly a discrete set of values of the chirality admittance. The proof of this result mainly relies on the existence and uniqueness of linear Beltrami (or force-free) fields for one part and on classical semigroup theory applied to evolution problems for the other part.

The outline of the paper is as follows: Section 2 is devoted to the formulation of the physical problem, which relies on the well-known Maxwell's equations. Its study is then reduced in Sec. 3 to that of a model evolution problem: it appears that there exist several possibilities to close the set of interior equations, by choosing more or less constraining boundary conditions. This problem is subsequently solved in the next three sections. In Sec. 4, the more constraining boundary conditions are kept, together with divergence-free data assumption. Then, in Sec. 5, the condition on the divergence of the data is dropped and less constraining boundary conditions are considered in Sec. 6. Some concluding remarks are made in the last section, and among others the case of a domain with a non-connected boundary is discussed.

2. Modeling of the Problem

2.1. *Maxwell's equations and transmission conditions*

Maxwell's equations in the absence of volume sources read

$$\frac{\partial \mathbf{D}}{\partial t} - \mathbf{curl}\,\mathbf{H} = \mathbf{0} \quad \text{in } \mathbb{R}^3,\ t > 0, \tag{2.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \quad \text{in } \mathbb{R}^3,\ t > 0. \tag{2.2}$$

In the above, $\mathbf{D}$ is the electric displacement, $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ is the magnetic induction, and $\mathbf{E}$ is the electric field. Those vector fields are functions of four variables, namely $(\mathbf{x}, t)$ in $\mathbb{R}^3 \times]0, +\infty[$.

In addition, they satisfy some conditions at interfaces between different media. Consider such an interface (a surface), and let $\mathbf{n}$ denote a unit normal vector to it. Then, the four jump conditions satisfied on the interface are (see Ref. 25 for instance)

$$[\mathbf{D} \cdot \mathbf{n}] = -\sigma, \quad [\mathbf{B} \cdot \mathbf{n}] = 0, \quad [\mathbf{E} \wedge \mathbf{n}] = \mathbf{0} \quad \text{and} \quad [\mathbf{H} \wedge \mathbf{n}] = \mathbf{J}, \tag{2.3}$$

where $\mathbf{J}$ and σ denote respectively the surface current density and surface charge density on the interface.

In what follows, we consider Maxwell's equations in a bounded domain Ω with boundary $\partial\Omega$, and build a relevant evolution problem whose well-posedness can be mathematically investigated. We assume that the domain Ω is made of a homogeneous isotropic medium. Moreover, we assume that this medium is chiral, and that it is enclosed in a perfect conductor.

2.2. *Interior equations*

Maxwell's equations in Ω read

$$\frac{\partial \mathbf{D}}{\partial t} - \mathbf{curl}\,\mathbf{H} = \mathbf{0} \quad \text{in } \Omega,\ t > 0, \tag{2.4}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \quad \text{in } \Omega,\ t > 0. \tag{2.5}$$

Although the generally accepted constitutive laws in chiral media are nonlocal in time, a local approximation, the so-called *optical response approximation*,[9,21] is considered here. This model assumes instantaneous responses of the system and, as a consequence, it is only valid for certain frequency ranges. Nevertheless, as pointed out in Ref. 9, such a hypothesis is normal for homogeneization studies in electromagnetism. A detailed investigation of the error of the optical response approximation model is provided in Ref. 21. The respective behaviors of the electric displacement and magnetic induction are then given by the Drude–Born–Fedorov relations (see Ref. 27 for instance)

$$\mathbf{D} = \varepsilon\,(\mathbf{E} + \beta\,\mathbf{curl}\,\mathbf{E}) \quad \text{and} \quad \mathbf{B} = \mu\,(\mathbf{H} + \beta\,\mathbf{curl}\,\mathbf{H}), \tag{2.6}$$

the real scalars $\varepsilon > 0$, $\mu > 0$ and $\beta \neq 0$ being respectively the electric permittivity, the magnetic permeability, and the chirality admittance. We supplement Maxwell's equations (2.4)–(2.5) with initial conditions. Since the system is of first order in time, one has to provide

$$\mathbf{E}(\cdot, 0) = \mathbf{E}_0, \quad \mathbf{H}(\cdot, 0) = \mathbf{H}_0 \quad \text{in } \Omega. \tag{2.7}$$

2.3. *Boundary conditions*

The Maxwell system (2.4)–(2.7) has to be closed by a set of boundary conditions in order to define an evolution problem in the bounded domain Ω. So, the goal of this subsection is to "construct" relevant instances of those conditions. To this end, we use the interface conditions (2.3) at the boundary $\partial\Omega$, knowing that the electromagnetic field vanishes inside a perfect conductor: interface conditions thus become "candidate" boundary conditions. From now on, $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$.

Possible boundary conditions for the magnetic field and for the electric displacement are

$$\mathbf{H} \wedge \mathbf{n} = \mathbf{J} \quad \text{and} \quad \mathbf{D} \cdot \mathbf{n} = -\sigma \quad \text{on } \partial\Omega.$$

Data $\mathbf{J}$ and σ are then linked through the trace of the Maxwell–Ampère equation (2.4). To see this, let us first introduce the surface divergence operator div_τ and recall the relationship

$$\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n} = \mathrm{div}_\tau\,(\mathbf{u}\wedge\mathbf{n}) \quad \text{on } \partial\Omega, \tag{2.8}$$

which is valid for (sufficiently smooth vector) fields $\mathbf{u}$ defined in Ω. Thus, one gets

$$\frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{n} = -\frac{\partial\sigma}{\partial t} = \mathbf{curl}\,\mathbf{H}\cdot\mathbf{n} = \mathrm{div}_\tau\,(\mathbf{H}\wedge\mathbf{n}) = \mathrm{div}_\tau\,\mathbf{J} \quad \text{on } \partial\Omega.$$

The well-known surface charge conservation equation follows:

$$\frac{\partial\sigma}{\partial t} + \mathrm{div}_\tau\,\mathbf{J} = 0 \quad \text{on } \partial\Omega. \tag{2.9}$$

In other words, the knowledge of $\mathbf{J}$ allows to determine σ completely, provided its initial value $\sigma(\cdot,0)$ is part of the data. On the other hand, the knowledge of σ allows only to determine $\mathbf{J}$ partially. More precisely, it yields no information on the divergence-free part of the surface current density. Thus, for a problem with only surface data (vs. volume data), when one keeps the boundary condition $\mathbf{H}\wedge\mathbf{n} = \mathbf{J}$ on $\partial\Omega$, the condition on $\mathbf{D}$ appears as a straightforward consequence, if the initial data are compatible, that is if $\mathbf{D}(\cdot,0)\cdot\mathbf{n} = -\sigma(\cdot,0)$ on $\partial\Omega$.

In a similar manner, one infers from the Maxwell–Faraday equation (2.5) that

$$\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{n} = -\,\mathrm{div}_\tau\,(\mathbf{E}\wedge\mathbf{n}) \quad \text{on } \partial\Omega.$$

This time, the boundary condition $\mathbf{E}\wedge\mathbf{n} = \mathbf{0}$ on $\partial\Omega$ implies that $\mathbf{B}\cdot\mathbf{n} = 0$ on $\partial\Omega$, as soon as the initial condition $\mathbf{B}(\cdot,0)\cdot\mathbf{n} = 0$ on $\partial\Omega$ holds. Knowledge of the condition $\mathbf{B}\cdot\mathbf{n} = 0$ on $\partial\Omega$ allows one to determine partially the trace of $\mathbf{E}$, since one gets simply $\mathrm{div}_\tau\,(\mathbf{E}\wedge\mathbf{n}) = 0$ on $\partial\Omega$.

From the above, one can actually select the boundary conditions to close the set of equations. In the next section, we keep the *a priori* more constraining pair of boundary conditions. Surface data includes the value of $\mathbf{J}$, which we first assume to be *divergence-free*, so that $\sigma = 0$, and the conditions on $\partial\Omega$ are

$$\mathbf{H}\wedge\mathbf{n} = \mathbf{J} \quad \text{and} \quad \mathbf{E}\wedge\mathbf{n} = \mathbf{0}.$$

The resulting evolution problem is then solved in Sec. 4.1, when the data is smooth (in a sense to be explained) and the domain Ω is assumed to be simply connected with a connected boundary. The more general cases of non-smooth data (what this encompasses being motivated) and of a non-divergence-free current density $\mathbf{J}$ are studied in Sec. 4.2 and Sec. 5 respectively. Then, the choice of the boundary conditions is revisited in Sec. 6. For instance, the *a priori* less constraining boundary conditions — such as $\mathbf{B}\cdot\mathbf{n} = 0$ on $\partial\Omega$ — are used to close the problem. In all cases, the well-posedness of the optical response approximation model, as an evolution problem, is established. Extensions to more complex settings and an application of these results are considered in Sec. 7.

3. Model Problem and Some Notations

From now on, let Ω be a connected bounded open subset of $\mathbb{R}^3$, with Lipschitz continuous boundary $\partial\Omega$. There is no *a priori* assumption that $\partial\Omega$ is connected and we denote by Γ_i, $0 \le i \le p$, the connected components of $\partial\Omega$, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \backslash \overline{\Omega}$.

Plugging relations (2.6) in Maxwell's equations (2.4) and (2.5), we obtain the following system, supplemented with boundary and initial conditions,

$$\begin{cases} \varepsilon \dfrac{\partial}{\partial t}(\mathbf{E} + \beta\, \mathbf{curl}\, \mathbf{E}) - \mathbf{curl}\, \mathbf{H} = \mathbf{0} & \text{in } \Omega,\ t > 0, \\ \mu \dfrac{\partial}{\partial t}(\mathbf{H} + \beta\, \mathbf{curl}\, \mathbf{H}) + \mathbf{curl}\, \mathbf{E} = \mathbf{0} & \text{in } \Omega,\ t > 0, \\ \operatorname{div} \mathbf{E} = 0,\ \operatorname{div} \mathbf{H} = 0 & \text{in } \Omega,\ t > 0, \\ \mathbf{E} \wedge \mathbf{n} = \mathbf{0},\ \mathbf{H} \wedge \mathbf{n} = \mathbf{J} & \text{on } \partial\Omega,\ t > 0, \\ \mathbf{E}(\cdot, 0) = \mathbf{E}_0,\ \mathbf{H}(\cdot, 0) = \mathbf{H}_0 & \text{in } \Omega. \end{cases} \tag{3.1}$$

We include explicitly the fact that both $\mathbf{E}$ and $\mathbf{H}$ are divergence-free. When the coefficients ε, μ or β depend on the space variable, this is not the case anymore (see Sec. 7 for comments).

Let us define some suitable Sobolev spaces, to derive a sound mathematical basis for the model problem. It is assumed that the reader is familiar with such spaces as $L^2(\Omega)$, $L^2(\Omega)^3$, $H(\mathbf{curl};\Omega)$ and $H(\operatorname{div};\Omega)$ and their subspaces $H_0(\mathbf{curl};\Omega)$ and $H_0(\operatorname{div};\Omega)$.

First, according to the definition of the electromagnetic energy,[25] it follows that the electromagnetic field belongs to $H(\mathbf{curl};\Omega) \times H(\mathbf{curl};\Omega)$ at all times. Second, $\mathbf{E}(\cdot, t)$ and $\mathbf{H}(\cdot, t)$ both belong to $X = H(\operatorname{div};\Omega) \cap H(\mathbf{curl};\Omega)$, since they are divergence-free. Moreover, according to the boundary condition on $\mathbf{E}$, one gets that $\mathbf{E}(\cdot, t)$ is in $X_N = H(\operatorname{div};\Omega) \cap H_0(\mathbf{curl};\Omega)$ for any t. We also introduce the space $X_T = H_0(\operatorname{div};\Omega) \cap H(\mathbf{curl};\Omega)$. Recall (see Ref. 5 for instance) that there holds $X_T \cap X_N = H_0^1(\Omega)^3$. Finally, as noted above, we shall use the space of divergence-free fields: $H(\operatorname{div} 0;\Omega) = \left\{\mathbf{v} \in L^2(\Omega)^3 \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\right\}$ and its subspace with vanishing normal traces $H_0(\operatorname{div} 0;\Omega) = \{\mathbf{v} \in H(\operatorname{div} 0;\Omega) \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$.

In order for the electromagnetic field $(\mathbf{E}, \mathbf{H})$ to be unique when the domain is non-simply connected and/or when its boundary is not connected, one has to add some compatibility conditions to the above system of equations. To this end, let us introduce, when the boundary $\partial\Omega$ is not connected, the space

$$\mathcal{H}(e) = \left\{\mathbf{v} \in L^2(\Omega)^3 \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{curl}\, \mathbf{v} = \mathbf{0} \text{ in } \Omega \text{ and } \mathbf{v} \wedge \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\right\},$$

which usually characterizes the electrostatic behavior around perfectly conducting bodies, with boundaries $\Gamma_0, \ldots, \Gamma_p$. Its dimension is equal to p. Here $p = 0$ if and only if, the boundary $\partial\Omega$ is connected. As a matter of fact, according for instance to Ref. 5, a field $\mathbf{v}$ of $\mathcal{H}(e)$ is uniquely characterized by $\langle \mathbf{v} \cdot \mathbf{n}, 1\rangle_{\Gamma_i}$, $1 \le i \le p$. Similarly, when the domain Ω is non-simply connected, we define the space

$$\mathcal{H}(m) = \left\{\mathbf{v} \in L^2(\Omega)^3 \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{curl}\, \mathbf{v} = \mathbf{0} \text{ in } \Omega \text{ and } \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\}.$$

This second space allows one to characterize the magnetostatic behavior inside a non-simply connected cavity (enclosed in a perfectly conducting material). The dimension of $\mathcal{H}(m)$ is equal to the (minimal) number of cuts Σ_k, $k = 1, \ldots, m$, such that $\Omega \setminus \cup_{k=1,\ldots,m}\Sigma_k$ is simply connected. Thus $m = 0$ when the domain is simply connected. According again to Ref. 5, a field $\mathbf{v}$ of $\mathcal{H}(m)$ is uniquely characterized by $\langle \mathbf{v}\cdot\mathbf{n}, 1\rangle_{\Sigma_k}$, $1 \le k \le m$.

Next, let us equip the spaces X_N and X_T with the graph norm on X, that is:

$$\|\mathbf{v}\|_X = \left(\|\mathbf{v}\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\Omega)^3} + \|\operatorname{div}\mathbf{v}\|^2_{L^2(\Omega)}\right)^{1/2}. \tag{3.2}$$

From Ref. 32, one gets

Theorem 3.1. *The spaces X_N and X_T are compactly imbedded into $L^2(\Omega)^3$.*

Moreover, the semi-norm

$$\mathbf{v} \mapsto \left(\|\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\Omega)^3} + \|\operatorname{div}\mathbf{v}\|^2_{L^2(\Omega)} + \sum_{k=1}^{m}\left|\langle \mathbf{v}\cdot\mathbf{n}, 1\rangle_{\Sigma_k^2}\right|\right)^{1/2}$$

is equivalent to the norm (3.2) over X_T (cf. Corollary 3.16 of Ref. 5), whereas the semi-norm

$$\mathbf{v} \mapsto \left(\|\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\Omega)^3} + \|\operatorname{div}\mathbf{v}\|^2_{L^2(\Omega)} + \sum_{i=1}^{p}\left|\langle \mathbf{v}\cdot\mathbf{n}, 1\rangle_{\Gamma_i^2}\right|\right)^{1/2}$$

is equivalent to the norm (3.2) over X_N (cf. Corollary 3.19 of Ref. 5).
Note that one has to be careful. Indeed, the space X is not compactly imbedded into $L^2(\Omega)^3$.[5] Still, there exists a subspace of X containing both X_T and X_N, for which the compact imbedding into $L^2(\Omega)^3$ result remains valid. It is the subspace made of fields of X with a tangential trace in $\boldsymbol{L}^2_T(\partial\Omega) = \left\{\mathbf{v} \in L^2(\partial\Omega)^3 \mid \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \text{ a.e.}\right\}$.[20]

From now on, we assume that the domain Ω is simply connected with a connected boundary $\partial\Omega$. The more general cases are dealt with in Sec. 7.1.

4. Divergence-Free Current Density

In this section, we assume that the surface current density $\mathbf{J}$ is such that $\operatorname{div}_\tau \mathbf{J} = 0$ on $\partial\Omega$. With the help of (2.8), one then gets

$$\mathbf{E}(\cdot,t)\cdot\mathbf{n} = \mathbf{E}_0\cdot\mathbf{n} \quad\text{and}\quad \mathbf{H}(\cdot,t)\cdot\mathbf{n} = \mathbf{H}_0\cdot\mathbf{n} \quad \text{on } \partial\Omega,\ t > 0.$$

To simplify the presentation, we assume that $\mathbf{E}_0\cdot\mathbf{n} = \mathbf{H}_0\cdot\mathbf{n} = 0$ on $\partial\Omega$, so that $\mathbf{E}$ and $\mathbf{H}$ also have vanishing normal traces. Therefore, a well-suited Sobolev space for magnetic fields is

$$X_T^0 = \{\mathbf{v} \in X_T \mid \operatorname{div}_\tau(\mathbf{v}\wedge\mathbf{n}) = 0 \text{ on } \partial\Omega\}\,.$$

As a matter of fact, the field $\mathbf{H}(\cdot, t)$ belongs to X_T^0 for any t. As far as the electric field is concerned, it belongs to the smaller space $X_T \cap X_N = H_0^1(\Omega)^3$ at all times.

4.1. *Smooth data*

When the boundary $\partial\Omega$ is smooth, i.e. at least $\mathscr{C}^{1,1}$ (see Ref. 22), or when the domain Ω is convex, the space X_T is actually a subspace of $H^1(\Omega)^3$ (see Ref. 5 for instance). In this case, the trace of any of its elements belongs to $H^{1/2}(\partial\Omega)^3$. Therefore, if we denote by $\underline{\mathbf{J}}_{\partial\Omega}$ the trace of $\mathbf{H}$ on $\partial\Omega$, there holds

$$\mathbf{J} = \underline{\mathbf{J}}_{\partial\Omega} \wedge \mathbf{n} \text{ on } \partial\Omega, \text{ with } \underline{\mathbf{J}}_{\partial\Omega} \in H^{1/2}(\partial\Omega)^3 \quad \text{and} \quad \underline{\mathbf{J}}_{\partial\Omega} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{4.1}$$

In this subsection, we assume that the data is *smooth*, in the sense defined above. In other words, even if the domain is non-convex and its boundary is not of $\mathscr{C}^{1,1}$ regularity, there exists a surface field $\underline{\mathbf{J}}_{\partial\Omega}$ such that the data $\mathbf{J}$ can be defined by (4.1). In Sec. 4.2, we investigate some other possibilities concerning the regularity to $\mathbf{J}$ and the implications on the solution of Maxwell's equations.

4.1.1. *An equivalent evolution problem*

The first step is to replace the set of Maxwell's equations (3.1) by an evolution problem, for which boundary conditions are homogeneous. The idea here is to exhibit a suitable lifting for the magnetic field $\mathbf{H}$. This is achieved in the following way. Since $\underline{\mathbf{J}}_{\partial\Omega}$ belongs to $H^{1/2}(\partial\Omega)^3$, there exists a (continuous) lifting $\mathbf{H_J}$ of $H^1(\Omega)^3$ such that $\mathbf{H_J} = \underline{\mathbf{J}}_{\partial\Omega}$ on $\partial\Omega$. In addition, it is chosen to belong to $H(\operatorname{div} 0; \Omega)$: this divergence-free property stems from Corollary 2.4 of Ref. 23. In this way, the vector fields $\mathbf{E}$ and $\widetilde{\mathbf{H}} = \mathbf{H} - \mathbf{H_J}$ satisfy

$$\begin{cases} \varepsilon \dfrac{\partial}{\partial t} (\mathbf{E} + \beta \,\mathbf{curl}\, \mathbf{E}) - \mathbf{curl}\, \widetilde{\mathbf{H}} = \mathbf{curl}\, \mathbf{H_J} & \text{in } \Omega,\ t > 0, \\ \mu \dfrac{\partial}{\partial t} \left(\widetilde{\mathbf{H}} + \beta \,\mathbf{curl}\, \widetilde{\mathbf{H}}\right) + \mathbf{curl}\, \mathbf{E} = -\mu \dfrac{\partial}{\partial t} (\mathbf{H_J} + \beta \,\mathbf{curl}\, \mathbf{H_J}) & \text{in } \Omega,\ t > 0, \\ \operatorname{div} \mathbf{E} = 0,\ \operatorname{div} \widetilde{\mathbf{H}} = 0 & \text{in } \Omega,\ t > 0, \\ \mathbf{E} = \mathbf{0},\ \widetilde{\mathbf{H}} = \mathbf{0} & \text{on } \partial\Omega,\ t > 0, \\ \mathbf{E}(\cdot, 0) = \mathbf{E}_0,\ \widetilde{\mathbf{H}}(\cdot, 0) = \widetilde{\mathbf{H}}_0 = \mathbf{H}_0 - \mathbf{H_J}(\cdot, 0) & \text{in } \Omega. \end{cases} \tag{4.2}$$

The next step is to find a relevant mathematical framework, in which the equivalent evolution problem (4.2) can be efficiently solved. System (4.2) is first written as an abstract evolution problem. To this end, let us introduce the Hilbert space $W = H_0(\operatorname{div} 0; \Omega)$, and define the operator $a_\beta = I + \beta \,\mathbf{curl}$ from W to W, with domain $D(a_\beta) = W \cap H_0^1(\Omega)^3$. This accounts for either the electric field, or the magnetic field. To deal with both of them simultaneously, consider the Hilbert space $H = W \times W$, equipped with the following inner product

$$\left(\begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} \right)_H = \int_\Omega \left(\varepsilon \,\varphi_1 \cdot \overline{\varphi_2} + \mu \,\psi_1 \cdot \overline{\psi_2} \right) d\mathbf{x},$$

and the operators from H to H

$$A_\beta = \begin{pmatrix} a_\beta & 0 \\ 0 & a_\beta \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & -\varepsilon^{-1}\mathbf{curl} \\ \mu^{-1}\mathbf{curl} & 0 \end{pmatrix},$$

with domains $D(A_\beta) = D(C) = \big(W \cap H_0^1(\Omega)^3\big) \times \big(W \cap H_0^1(\Omega)^3\big)$. System (4.2) can be rewritten

$$\begin{cases} \dfrac{d}{dt}\,(A_\beta\,\mathscr{E}) + C\,\mathscr{E} = \mathscr{F}, \\ \mathscr{E}(0) = \mathscr{E}_0, \end{cases} \tag{4.3}$$

with $\mathscr{E} = \begin{pmatrix} \mathbf{E} \\ \widetilde{\mathbf{H}} \end{pmatrix}$, $\mathscr{F} = \begin{pmatrix} \varepsilon^{-1}\mathbf{curl}\,\mathbf{H_J} \\ -\frac{\partial}{\partial t}\,(\mathbf{H_J} + \beta\,\mathbf{curl}\,\mathbf{H_J}) \end{pmatrix}$ and $\mathscr{E}_0 = \begin{pmatrix} \mathbf{E}_0 \\ \widetilde{\mathbf{H}}_0 \end{pmatrix}$.

4.1.2. *Invertibility of the space-domain operator*

We shall prove that the operators a_β and A_β are invertible, except possibly for a discrete set of nonzero values of the chirality parameter β. We hence study the existence and uniqueness of a solution to the following stationary problem: *for any* $\mathbf{f} \in W$*, find* $\mathbf{u} \in H^1(\Omega)^3$ *such that*

$$\begin{cases} \mathbf{u} + \beta\,\mathbf{curl}\,\mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Let us first remark that a possible solution to this problem is unique. Indeed, $\forall\ \beta \neq 0$, any solution to the homogeneous system associated to (4.4) is also a solution to the following problem[a]

$$\begin{cases} \mathbf{u} - \beta^2\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}) = \mathbf{0} & \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{u} \wedge \mathbf{n} = \mathbf{u} \wedge \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

Therefore, the field $\mathbf{u}$ is identically equal to zero owing to Holmgren's theorem.

[a]One could alternately use the method proposed by Boulmezaoud to establish the uniqueness of a solution to (4.4). Indeed, a homogeneous solution to (4.4) also satisfies

$$\mathbf{curl}\,\mathbf{u} \wedge \mathbf{u} = \mathbf{0} \text{ in } \Omega,\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega \text{ and } \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega.$$

In other words, it is a Beltrami, or force-free, field, with vanishing trace. One has $\mathbf{curl}\,\mathbf{u} \wedge \mathbf{u} = \mathbf{u} \cdot \nabla\mathbf{u} - \nabla\left(\frac{|\mathbf{u}|^2}{2}\right)$. Then, let us take the scalar product with the position vector $\mathbf{x}$ and sum over Ω. By using the relations

$$\int_\Omega \mathbf{x} \cdot (\mathbf{u} \cdot \nabla\mathbf{u})\; d\mathbf{x} = \sum_{i=1}^{3} \int_\Omega x_i\,(\mathbf{u} \cdot \nabla u_i)\; d\mathbf{x} = -\sum_{i=1}^{3} \int_\Omega \operatorname{div}(x_i\,\mathbf{u})\,u_i\,d\mathbf{x} = -\int_\Omega |\mathbf{u}|^2\,d\mathbf{x}$$

and

$$\int_\Omega \mathbf{x} \cdot \nabla\left(\frac{|\mathbf{u}|^2}{2}\right) d\mathbf{x} = -\int_\Omega (\operatorname{div}\mathbf{x})\,\frac{|\mathbf{u}|^2}{2}\,d\mathbf{x} = -\frac{3}{2}\int_\Omega |\mathbf{u}|^2\,d\mathbf{x},$$

we reach $\|\mathbf{u}\|^2_{L^2(\Omega)^3} = 0$, i.e. $\mathbf{u} = \mathbf{0}$ in Ω.

To establish the existence, we note that a solution to problem (4.4) verifies automatically

$$\begin{cases} \mathbf{u} + 2\beta\,\mathbf{curl}\,\mathbf{u} + \beta^2\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}) = \mathbf{f} + \beta\,\mathbf{curl}\,\mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

Let us introduce the bilinear form $a(\mathbf{u},\mathbf{v}) = (\mathbf{u} + \beta\,\mathbf{curl}\,\mathbf{u}, \mathbf{v} + \beta\,\mathbf{curl}\,\mathbf{v})_{L^2(\Omega)^3}$. The above equations imply that $\mathbf{u}$ solves the variational problem: *find* $\mathbf{u} \in W \cap H_0^1(\Omega)^3$ *such that*

$$a(\mathbf{u},\mathbf{v}) = \langle \mathbf{f} + \beta\,\mathbf{curl}\,\mathbf{f}, \mathbf{v}\rangle_{H_0(\mathbf{curl}\,,\Omega)', H_0(\mathbf{curl}\,,\Omega)}, \quad \forall\, \mathbf{v} \in W \cap H_0^1(\Omega)^3. \tag{4.5}$$

The form $a(\cdot\,,\cdot)$ is trivially sesquilinear, symmetric, continuous on $W \cap H_0^1(\Omega)^3$, definite (due to the uniqueness of a solution to (4.4)) and positive. It is also coercive, which is proved by contradiction.

Assume there exists a sequence $(\mathbf{v}_k)_{k\in\mathbb{N}}$ of $W \cap H_0^1(\Omega)^3$ such that $a(\mathbf{v}_k,\mathbf{v}_k) \underset{k\to+\infty}{\longrightarrow} 0$ and $|\mathbf{v}_k|_{H^1(\Omega)^3} = 1$, $\forall\, k \in \mathbb{N}$. Due to the compactness of the imbedding of $H^1(\Omega)$ into $L^2(\Omega)$, we can extract a subsequence, still denoted $(\mathbf{v}_k)_{k\in\mathbb{N}}$, which converges to a limit $\mathbf{v}$ in $L^2(\Omega)^3$. We immediately deduce that $\mathbf{v} \in W$. We, moreover, have $\beta\,\mathbf{curl}\,\mathbf{v}_k \underset{k\to+\infty}{\longrightarrow} -\mathbf{v}$ in $L^2(\Omega)^3$ and also $\beta\,\mathbf{curl}\,\mathbf{v}_k \underset{k\to+\infty}{\longrightarrow} \beta\,\mathbf{curl}\,\mathbf{v}$ in $H^{-1}(\Omega)^3$. By the uniqueness of the limit, we find that $\mathbf{v} + \beta\,\mathbf{curl}\,\mathbf{v} = \mathbf{0}$. Since $\mathbf{v}$ belongs to $L^2(\Omega)^3$, $\mathbf{v} + \beta\,\mathbf{curl}\,\mathbf{v} = \mathbf{0}$ holds in $L^2(\Omega)^3$. We finally deduce that $\beta\,\mathbf{curl}\,\mathbf{v}_k \underset{k\to+\infty}{\longrightarrow} \beta\,\mathbf{curl}\,\mathbf{v}$ in $L^2(\Omega)^3$, so that $(\mathbf{v}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{v}$ in $H(\mathbf{curl}\,,\Omega)$: $\mathbf{v} \wedge \mathbf{n} = \mathbf{0}$, and $\mathbf{v} \in W \cap H_0^1(\Omega)^3$. Therefore, $\mathbf{v}$ is the unique solution to problem (4.4) with zero right-hand side, i.e. $\mathbf{v} = \mathbf{0}$. This contradicts the initial assumption.

Problem (4.5) consequently admits a unique solution due to Lax–Milgram's lemma. We now have to check that, conversely, a solution to (4.5) is also a solution to (4.4). Let us define $\mathbf{w} = \mathbf{u} + \beta\,\mathbf{curl}\,\mathbf{u} - \mathbf{f}$, where $\mathbf{u}$ is a solution to problem (4.5). By construction, the field $\mathbf{w}$ belongs to W and verifies

$$\beta\,\mathbf{curl}\,\mathbf{w} = \beta\,\mathbf{curl}\,\mathbf{u} + \beta^2\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}) - \beta\,\mathbf{curl}\,\mathbf{f} = -\mathbf{u} - \beta\,\mathbf{curl}\,\mathbf{u} + \mathbf{f} = -\mathbf{w} \quad \text{in } \Omega,$$

hence $\mathbf{w} \in W \cap H(\mathrm{rot};\Omega) \subset X_T$. It therefore satisfies the following homogeneous problem

$$\begin{cases} \mathbf{w} + \beta\,\mathbf{curl}\,\mathbf{w} = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{w} = 0 & \text{in } \Omega, \\ \mathbf{w}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Existence and uniqueness of such linear force-free fields is discussed in detail in Ref. 14 and we make use of some of the results in this reference, which are still valid for fields in X_T, to conclude on the existence — or the lack of it — of a solution to problem (4.4). Keeping the notations of Theorem 1 of Ref. 14, we have the following alternative (below, the $(\alpha_i)_{i\in\mathbb{N}}$ belong to $\mathbb{R}$, as proved in Ref. 33):

- if $-\frac{1}{\beta} \notin \{\alpha_i,\ i \in \mathbb{N}\}$, problem (4.6) admits the trivial solution as its unique solution;
- if $-\frac{1}{\beta} = \alpha_i$, $i \in \mathbb{N}$, problem (4.6) admits nontrivial solutions of the form $\mathbf{w} = \mathbf{curl}\,\psi$, where the field ψ verifies the following homogeneous problem
$$\begin{cases} \mathbf{curl}\,(\mathbf{curl}\,\psi) - \alpha_i\,\mathbf{curl}\,\psi = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\psi = 0 & \text{in } \Omega, \\ \psi \wedge \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$
the solutions of which constitute a finite-dimensional vector space.

The conclusion on the original problem (4.4) is then as follows:

- if $-\frac{1}{\beta} \notin \{\alpha_i,\ i \in \mathbb{N}\}$, the solution to problem (4.6) is $\mathbf{w} = \mathbf{0}$, so that $\mathbf{u}$ is a solution to problem (4.4): the existence property holds;
- if $-\frac{1}{\beta} = \alpha_i$, $i \in \mathbb{N}$, there exist right-hand sides such that problem (4.4) admits no solution. Let us proceed again by contradiction. For instance, let $\mathbf{f}_0 = \mathbf{curl}\,\psi$, with ψ a nonzero solution to the above problem. Then the solution $\mathbf{u}$ to problem (4.4), if it exists, is not equal to zero, according to the uniqueness property. But, it is also a solution to problem (4.5), the right side of which is zero, since $\mathbf{f}_0 + \beta\,\mathbf{curl}\,\mathbf{f}_0 = \mathbf{0}$! So, problem (4.4) admits no solution in this case.... The existence property does not hold.

4.1.3. *Solvability of the set of Maxwell's equations*

The operator A_β being invertible on its domain (which is denoted by D from now on) if $\beta \neq -\frac{1}{\alpha_i}$, $\forall\ i \in \mathbb{N}$, we consider its inverse A_β^{-1} from H to D ($\subset H$) and rewrite the evolution problem (4.3) in the following equivalent form: *find $\mathscr{E}$ such that*
$$\begin{cases} \dfrac{d\mathscr{E}}{dt} + A_\beta^{-1} C\,\mathscr{E} = A_\beta^{-1}\mathscr{F}, \\ \mathscr{E}(0) = \mathscr{E}_0. \end{cases} \tag{4.7}$$
Following Ref. 29, we shall prove that this problem has a unique solution using Lumer–Phillips' theorem. We first show the

Lemma 4.1. *The operators $A_\beta^{-1}C$ and $-A_\beta^{-1}C$ are maximal dissipative.*

Proof. By definition, showing the dissipativeness of $\pm A_\beta^{-1}C$ amounts to proving that
$$\operatorname{Re}\left(A_\beta^{-1}C\,\mathscr{E},\mathscr{E}\right)_D = 0,\ \forall\ \mathscr{E} \in D. \tag{4.8}$$
Recall that $D = (W \cap H_0^1(\Omega)^3) \times (W \cap H_0^1(\Omega)^3)$. Owing to the remarks of Sec. 3 concerning equivalent norms, we can use the following scalar product on D
$$\left(\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\right)_D = \int_\Omega \left(\varepsilon\,\mathbf{curl}\,\varphi_1 \cdot \mathbf{curl}\,\overline{\varphi_2} + \mu\,\mathbf{curl}\,\psi_1 \cdot \mathbf{curl}\,\overline{\psi_2}\right)\,d\mathbf{x}.$$

There follows

$$\left(A_\beta^{-1}B\,\mathscr{E},\mathscr{E}\right)_D = \int_\Omega \left(\mathbf{curl}\, a_\beta^{-1}\mathbf{curl}\,\mathbf{E}\cdot\mathbf{curl}\,\overline{\widetilde{\mathbf{H}}} - \mathbf{curl}\, a_\beta^{-1}\mathbf{curl}\,\widetilde{\mathbf{H}}\cdot\mathbf{curl}\,\overline{\mathbf{E}}\right)\,d\mathbf{x}.$$

We prove then

Lemma 4.2. *For all* $\mathbf{u}$ *of* $W\cap H_0^1(\Omega)^3$, *there holds*: $a_\beta{}^{-1}\mathbf{curl}\,\mathbf{u} = \mathbf{curl}\, a_\beta{}^{-1}\mathbf{u}$.

Proof. Let $\mathbf{u}$ be an element of $W\cap H_0^1(\Omega)^3$, and let $\mathbf{v}$ belong to W. With the help of the density of $W\cap H_0^1(\Omega)^3$ in W (see Theorem 2.8 of Ref. 23), we can consider $\mathbf{v}\in W\cap H_0^1(\Omega)^3$ and get

$$\begin{aligned}\int_\Omega a_\beta^{-1}\mathbf{curl}\,\mathbf{u}\cdot\overline{\mathbf{v}}\,d\mathbf{x} &= \int_\Omega a_\beta^{-1}\left(\mathbf{curl}\,\mathbf{u}+\frac{1}{\beta}\mathbf{u}\right)\cdot\overline{\mathbf{v}}\,d\mathbf{x} - \int_\Omega \frac{1}{\beta}a_\beta^{-1}\mathbf{u}\cdot\overline{\mathbf{v}}\,d\mathbf{x}\\
&= \int_\Omega \frac{1}{\beta}\,\mathbf{u}\cdot\overline{\mathbf{v}}\,d\mathbf{x} - \int_\Omega \frac{1}{\beta}a_\beta^{-1}\mathbf{u}\cdot\overline{\mathbf{v}}\,d\mathbf{x}\\
&= \int_\Omega \frac{1}{\beta}(I+\beta\,\mathbf{curl}\,)a_\beta^{-1}\mathbf{u}\cdot\overline{\mathbf{v}}\,d\mathbf{x} - \int_\Omega \frac{1}{\beta}a_\beta^{-1}\mathbf{u}\cdot\overline{\mathbf{v}}\,d\mathbf{x}\\
&= \int_\Omega \mathbf{curl}\, a_\beta^{-1}\mathbf{u}\cdot\overline{\mathbf{v}}\,d\mathbf{x}.\end{aligned}$$

□

If the brackets $\langle\cdot\,,\cdot\rangle$ stand for the duality product between $H_0(\mathbf{curl}\,;\Omega)$ and its dual, one finds, using Lemma 4.2,

$$\begin{aligned}&\int_\Omega \mathbf{curl}\, a_\beta^{-1}\mathbf{curl}\,\widetilde{\mathbf{H}}\cdot\mathbf{curl}\,\overline{\mathbf{E}}\,d\mathbf{x}\\
&\quad= \left\langle \mathbf{curl}\,\mathbf{curl}\,\overline{\mathbf{E}}, a_\beta^{-1}\mathbf{curl}\,\widetilde{\mathbf{H}}\right\rangle\\
&\quad= \left\langle \mathbf{curl}\,\mathbf{curl}\,\overline{\mathbf{E}}, a_\beta^{-1}\left(\mathbf{curl}\,\widetilde{\mathbf{H}}+\frac{1}{\beta}\widetilde{\mathbf{H}}\right)\right\rangle - \left\langle \mathbf{curl}\,\mathbf{curl}\,\overline{\mathbf{E}}, \frac{1}{\beta}a_\beta^{-1}\widetilde{\mathbf{H}}\right\rangle\\
&\quad= \left\langle \mathbf{curl}\,\mathbf{curl}\,\overline{\mathbf{E}}, \frac{1}{\beta}\widetilde{\mathbf{H}}\right\rangle - \left\langle \mathbf{curl}\,\mathbf{curl}\,\overline{\mathbf{E}}, \frac{1}{\beta}a_\beta^{-1}\widetilde{\mathbf{H}}\right\rangle\\
&\quad= \int_\Omega \frac{1}{\beta}\,\mathbf{curl}\,\overline{\mathbf{E}}\cdot\mathbf{curl}\,\widetilde{\mathbf{H}}\,d\mathbf{x} - \int_\Omega \frac{1}{\beta}\,\mathbf{curl}\,\overline{\mathbf{E}}\cdot\mathbf{curl}\, a_\beta^{-1}\widetilde{\mathbf{H}}\,d\mathbf{x}\\
&\quad= \int_\Omega \frac{1}{\beta}\,\mathbf{curl}\,\overline{\mathbf{E}}\cdot\mathbf{curl}\,\widetilde{\mathbf{H}}\,d\mathbf{x} - \int_\Omega \frac{1}{\beta}\left(a_\beta^{-1}\mathbf{curl}\,\overline{\mathbf{E}}+\beta\,\mathbf{curl}\, a_\beta^{-1}\mathbf{curl}\,\overline{\mathbf{E}}\right)\\
&\qquad\cdot a_\beta^{-1}\mathbf{curl}\,\widetilde{\mathbf{H}}\,d\mathbf{x}\\
&\quad= \int_\Omega \frac{1}{\beta}\,\mathbf{curl}\,\overline{\mathbf{E}}\cdot\mathbf{curl}\,\widetilde{\mathbf{H}}\,d\mathbf{x} - \int_\Omega \frac{1}{\beta}a_\beta^{-1}\mathbf{curl}\,\overline{\mathbf{E}}\\
&\qquad\cdot\left(a_\beta^{-1}\mathbf{curl}\,\widetilde{\mathbf{H}}+\beta\,\mathbf{curl}\, a_\beta^{-1}\mathbf{curl}\,\widetilde{\mathbf{H}}\right)\,d\mathbf{x}\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \frac{1}{\beta}\,\mathbf{curl}\,\overline{\mathbf{E}}\cdot\mathbf{curl}\,\widetilde{\mathbf{H}}\,d\mathbf{x} - \int_\Omega \frac{1}{\beta}\,a_\beta^{-1}\mathbf{curl}\,\overline{\mathbf{E}}\cdot\mathbf{curl}\,\widetilde{\mathbf{H}}\,d\mathbf{x}\\
&= \int_\Omega \frac{1}{\beta}\left(\mathbf{curl}\,\overline{\mathbf{E}} - \mathbf{curl}\,a_\beta^{-1}\overline{\mathbf{E}}\right)\cdot\mathbf{curl}\,\widetilde{\mathbf{H}}\,d\mathbf{x}\\
&= \left\langle \mathbf{curl}\,\mathbf{curl}\,\widetilde{\mathbf{H}}, \frac{1}{\beta}\overline{\mathbf{E}}\right\rangle - \left\langle \mathbf{curl}\,\mathbf{curl}\,\widetilde{\mathbf{H}}, \frac{1}{\beta}a_\beta^{-1}\overline{\mathbf{E}}\right\rangle\\
&= \left\langle \mathbf{curl}\,\mathbf{curl}\,\widetilde{\mathbf{H}}, a_\beta^{-1}\left(\mathbf{curl}\,\overline{\mathbf{E}} + \frac{1}{\beta}\overline{\mathbf{E}}\right)\right\rangle - \left\langle \mathbf{curl}\,\mathbf{curl}\,\widetilde{\mathbf{H}}, \frac{1}{\beta}a_\beta^{-1}\overline{\mathbf{E}}\right\rangle\\
&= \left\langle \mathbf{curl}\,\mathbf{curl}\,\widetilde{\mathbf{H}}, a_\beta^{-1}\mathbf{curl}\,\overline{\mathbf{E}}\right\rangle\\
&= \frac{\int_\Omega \mathbf{curl}\,\widetilde{\mathbf{H}}\cdot\mathbf{curl}\,a_\beta^{-1}\mathbf{curl}\,\overline{\mathbf{E}}\,d\mathbf{x}}{\int_\Omega \mathbf{curl}\,\overline{\widetilde{\mathbf{H}}}\cdot\mathbf{curl}\,a_\beta^{-1}\mathbf{curl}\,\mathbf{E}\,d\mathbf{x}}.
\end{aligned}
$$

Observe that the real part of this identity yields directly the relation (4.8). It remains to prove the maximality of the operators $\pm A_\beta^{-1}C$, that is, for all $\begin{pmatrix}\mathbf{f}\\ \mathbf{g}\end{pmatrix}$ in D, there exists $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ in D such that

$$
(I \pm A_\beta^{-1}C)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}\mathbf{f}\\ \mathbf{g}\end{pmatrix}, \tag{4.9}
$$

or, equivalently,

$$
\psi = \mathbf{g} \mp \mu^{-1}a_\beta^{-1}\mathbf{curl}\,\varphi, \tag{4.10}
$$

and

$$
\varphi + c^2 a_\beta^{-1}\mathbf{curl}\,a_\beta^{-1}\mathbf{curl}\,\varphi = \mathbf{f} \pm \varepsilon^{-1}a_\beta^{-1}\mathbf{curl}\,\mathbf{g}, \tag{4.11}
$$

where $c = (\varepsilon\mu)^{-1/2}$ denotes the light velocity. Problem (4.11) has a unique solution φ in $W \cap H_0^1(\Omega)^3$. Indeed, it admits the following variational formulation: *find* $\varphi \in W \cap H_0^1(\Omega)^3$ *such that*

$$
\begin{aligned}
&\left(\varphi, \overline{\theta}\right)_{L^2(\Omega)^3} + c^2\left(a_\beta^{-1}\mathbf{curl}\,\varphi, a_\beta^{-1}\mathbf{curl}\,\overline{\theta}\right)_{L^2(\Omega)^3}\\
&\quad = \left(\mathbf{f}, \overline{\theta}\right)_{L^2(\Omega)^3} \pm \varepsilon^{-1}\left(a_\beta^{-1}\mathbf{g}, \mathbf{curl}\,\overline{\theta}\right)_{L^2(\Omega)^3}, \quad \forall\, \theta \in W \cap H_0^1(\Omega)^3,
\end{aligned}
$$

which has a unique solution according to Lax–Milgram's lemma, the sesquilinear form defined above being coercive on $W \cap H_0^1(\Omega)^3$ by virtue of the injectivity of the operator a_β^{-1} on this space.

We have yet to show that the field ψ given by (4.10) belongs to $W \cap H_0^1(\Omega)^3$. We immediately deduce from this equality that $\psi = \mathbf{0}$ on $\partial\Omega$ and that $\operatorname{div}\psi = \mathbf{0}$ in Ω. Using (4.11), we have also

$$
a_\beta^{-1}\mathbf{curl}\,\psi = \mp\varepsilon\left(\mathbf{f} - \varphi\right),
$$

that is

$$\mathbf{curl}\,\psi = \mp\varepsilon\left(\mathbf{f} - \varphi + \beta\,\mathbf{curl}\,\left(\mathbf{f} - \varphi\right)\right),$$

hence $\mathbf{curl}\,\psi \in L^2(\Omega)^3$, since φ and $\mathbf{f}$ both belong to $H^1(\Omega)^3$.

Lumer–Phillips' theorem (see Ref. 30, Chap. 1, Theorem 4.3) then ensures that the operator $A_\beta^{-1}C$ is the infinitesimal generator of a contraction C_0-semigroup, thus implying the existence of a solution $(\mathbf{E}, \mathbf{H})$ to the set of Maxwell's equations (3.1). This result is stated as follows:

Theorem 4.1. *Assume that the chirality admittance β is in $\mathbb{R}\backslash\{-1/\alpha_i,\ i \in \mathbb{N}\}$. Consider a data set $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{J})$, such that*

- $\mathbf{E}_0 \in H_0^1(\Omega)^3 \cap H(\mathrm{div}\,0;\Omega)$, $\mathbf{H}_0 \in H^1(\Omega)^3 \cap X_T^0 \cap H(\mathrm{div}\,0;\Omega)$;
- $\mathbf{J}$ *is divergence-free, and there exists a lifting such that* $\mathbf{H_J} \wedge \mathbf{n}_{|\partial\Omega} = \mathbf{J}$, *with regularity* $\mathbf{H_J} \in C^0\big([0, +\infty[, H^1(\Omega)^3 \cap X_T^0 \cap H(\mathrm{div}\,0;\Omega)\big)$.

Then, there exists one, and only one, solution $(\mathbf{E}, \mathbf{H})$ to problem (3.1) *such that*

$$\begin{aligned}
&\mathbf{E} \in C^0\left([0, +\infty[, H_0^1(\Omega)^3 \cap H(\mathrm{div}\,0;\Omega)\right),\\
&\mathbf{H} \in C^0\left([0, +\infty[, H^1(\Omega)^3 \cap X_T^0 \cap H(\mathrm{div}\,0;\Omega)\right).
\end{aligned}$$

4.2. *General case*

Recall[5] that when the boundary $\partial\Omega$ is not $\mathscr{C}^{1,1}$ and when the domain Ω is non-convex, one has $X_T^0 \not\subset H^1(\Omega)^3$. Since the magnetic field is *a priori* only $X_T^0 \cap H(\mathrm{div}\,0;\Omega)$-regular, we introduce the operator $K_\beta = I + \beta\,\mathbf{curl}$, from W to W, with domain $D(K_\beta) = X_T^0 \cap H(\mathrm{div}\,0;\Omega)$. In this way, the operator a_β can be viewed as the restriction of K_β to $H_0^1(\Omega)^3 \cap H(\mathrm{div}\,0;\Omega)$.

In what follows, we postulate the existence of a lifting of the surface current density $\mathbf{J}$, and see that the theory of Sec. 4.1 is still useful. Establishing the existence of such a lifting amounts to giving a full mathematical characterization of the space of tangential traces on $\partial\Omega$ of elements of X_T^0. We begin by formulating necessary conditions for the existence, which can be expressed as requirements on the $H^s(\partial\Omega)$-regularity of the surface field $\mathbf{J}$. Then, we check whether or not these conditions are sufficient.

4.2.1. *Solvability of the set of Maxwell's equations*

To be more precise, to postulate that a lifting $\mathbf{H_J}$ of $\mathbf{J}$ exists, means that there exists $\mathbf{H_J} \in X_T^0$ such that $\mathbf{H_J} \wedge \mathbf{n} = \mathbf{J}$ on $\partial\Omega$. The fact that it is automatically divergence-free, stems again from Corollary 2.4 of Ref. 23: thus, $\mathbf{H_J}$ is in $X_T^0 \cap H(\mathrm{div}\,0;\Omega)$. We remark then that the difference $\widetilde{\mathbf{H}} = \mathbf{H} - \mathbf{H_J}$ is actually, by construction, an element of $H_0^1(\Omega)^3 \cap H(\mathrm{div}\,0;\Omega)$, which satisfies the evolution problem (4.2). According to the results of Sec. 4.1.2, we know that when β is not among $\{-1/\alpha_i,\ i \in \mathbb{N}\}$, then the restriction of the operator K_β, i.e. a_β, is onto. Therefore, the field $\widetilde{\mathbf{H}}$ exists (and is unique). In this manner, we can build a solution to the set of Maxwell's equations (3.1), by adding the solution $(\mathbf{E}, \widetilde{\mathbf{H}})$ to problem (4.2) to the lifting $(\mathbf{0}, \mathbf{H_J})$. Thus, we can still solve the set of Maxwell's equations in this more general setting, when $\beta \notin \{-1/\alpha_i,\ i \in \mathbb{N}\}$, and so extend the range of Theorem 4.1 accordingly, by relaxing the condition on the regularity of $\mathbf{J}$, i.e. provided the lifting $\mathbf{H_J}$ exists in $X_T^0 \cap H(\operatorname{div} 0; \Omega)$. To summarize, we have the

Theorem 4.2. *Assume that the chirality admittance β is in $\mathbb{R}\backslash\{-1/\alpha_i,\ i \in \mathbb{N}\}$. Consider a data set $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{J})$, such that*

- $\mathbf{E}_0 \in H_0^1(\Omega)^3 \cap H(\operatorname{div} 0; \Omega)$, $\mathbf{H}_0 \in X_T^0 \cap H(\operatorname{div} 0; \Omega)$;
- $\mathbf{J}$ *is divergence-free, and there exists a lifting such that $\mathbf{H_J} \wedge \mathbf{n}_{|\partial\Omega} = \mathbf{J}$, with regularity $\mathbf{H_J} \in C^0\left([0, +\infty[, X_T^0 \cap H(\operatorname{div} 0; \Omega)\right)$.*

Then, there exists one, and only one, solution $(\mathbf{E}, \mathbf{H})$ to problem (3.1) such that

$$\mathbf{E} \in C^0\left([0, +\infty[, H_0^1(\Omega)^3 \cap H(\operatorname{div} 0; \Omega)\right),$$
$$\mathbf{H} \in C^0\left([0, +\infty[, X_T^0 \cap H(\operatorname{div} 0; \Omega)\right).$$

What is more, we can also infer some interesting results about the operator K_β. As a matter of fact, one has $K_\beta \mathbf{H_J} \in W$, so there exists — as soon as $\beta \notin \{-1/\alpha_i,\ i \in \mathbb{N}\}$ — a unique $\mathbf{H_R} \in W \cap H_0^1(\Omega)^3$ such that $a_\beta \mathbf{H_R} = -K_\beta \mathbf{H_J}$. Setting $\mathbf{H}'_\mathbf{J} = \mathbf{H_J} + \mathbf{H_R}$, one obtains the splitting $\mathbf{H} = (\widetilde{\mathbf{H}} - \mathbf{H_R}) + \mathbf{H}'_\mathbf{J}$, where $\mathbf{H}'_\mathbf{J}$ satisfies

$$\begin{cases} \mathbf{H}'_\mathbf{J} + \beta\, \mathbf{curl}\, \mathbf{H}'_\mathbf{J} = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div} \mathbf{H}'_\mathbf{J} = 0 & \text{in } \Omega, \\ \mathbf{H}'_\mathbf{J} \wedge \mathbf{n} = \mathbf{J} & \text{on } \partial\Omega. \end{cases}$$

In particular, $\mathbf{H}'_\mathbf{J}$ belongs to the kernel of K_β. Thus, one reaches the decomposition

$$D(K_\beta) = \operatorname{Ker}(K_\beta) \oplus \left(H_0^1(\Omega)^3 \cap H(\operatorname{div} 0; \Omega)\right).$$

The sum is direct, since any element of the intersection is equal to zero, according to the uniqueness property of Sec. 4.1.2.

4.2.2. *Necessary regularity assumptions on the surface current density*

We now address the question of the minimal regularity of the surface current density $\mathbf{J}$ with respect to the regularity of Ω. To begin with, recall that $\boldsymbol{L}_T^2(\partial\Omega) = \left\{\mathbf{v} \in L^2(\partial\Omega)^3 \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \text{ a.e.}\right\}$ and let us introduce the scale of Sobolev spaces, with $0 < s \leq 1/2$,

$$\boldsymbol{H}_\perp^s(\partial\Omega) = \left\{\mathbf{u} \in \boldsymbol{L}_T^2(\partial\Omega) \mid \exists\ \mathbf{v} \in H^{s+1/2}(\Omega)^3,\ \mathbf{v} \wedge \mathbf{n}_{|\partial\Omega} = \mathbf{u}\right\}. \tag{4.12}$$

Then, one can categorize the regularity of the domain as follows (for instance):

(1) The boundary $\partial\Omega$ is $\mathscr{C}^{1,1}$, or Ω is convex.
- One has $X_T \subset H^1(\Omega)^3$: necessarily $\mathbf{J} \in \boldsymbol{H}_\perp^{1/2}(\partial\Omega)$.

(2) The boundary $\partial\Omega$ is not $\mathscr{C}^{1,1}$ and Ω is non-convex.

- If the domain is a Lipschitz polyhedron, i.e. $\partial\Omega$ is Lipschitz and Ω is a polyhedron (this class encompasses most polyhedra, with a few exceptions): $\exists\, \eta \in]0, \frac{1}{2}[$ such that $X_T \subset \cap_{\epsilon>0} H^{1/2+\eta-\epsilon}(\Omega)^3$, and $X_T \not\subset H^{1/2+\eta}(\Omega)^3$ (see Remark 3.8 of Ref. 5). Then, one necessarily has $\mathbf{J} \in \cap_{\epsilon>0} \boldsymbol{H}_{\perp}^{\eta-\epsilon}(\partial\Omega)$.
- If $\partial\Omega$ is only Lipschitz: $\forall\, \mathbf{u} \in X_T$, $\mathbf{u}\wedge\mathbf{n}_{|\partial\Omega} \in \boldsymbol{L}_T^2(\partial\Omega)$ according to Theorem 2 of Ref. 18. Then, one necessarily has $\mathbf{J} \in \boldsymbol{L}_T^2(\partial\Omega)$.

These necessary regularity conditions have to be supplemented with the initial divergence-free assumption, that is $\mathrm{div}_\tau\, \mathbf{J} = 0$ on $\partial\Omega$. In all instances,[17] there exists a surface field $\lambda_\tau \in H^1(\partial\Omega)$ such that $\mathbf{J} = \mathbf{curl}_\tau \lambda_\tau$.

4.2.3. *Sufficient regularity assumptions on the surface current density*

To check whether the necessary conditions of Sec. 4.2.2 are sufficient, let us consider successively a domain with a $\mathscr{C}^{1,1}$-boundary, then a convex polyhedron, and finally a non-convex (Lipschitz) polyhedron.

In the latter cases, we denote by $(F_k)_k$ the faces of $\partial\Omega$, and by e_{ij} the edge between the faces F_i and F_j, when it exists (in this case, τ_{ij} is a unit vector parallel to e_{ij}, and τ_i is such that $(\tau_i, \tau_{ij}, \mathbf{n}_{|F_i})$ is an orthonormal basis of $\mathbb{R}^3$). Elements of $H^{1/2}(\partial\Omega)$ then satisfy compatibility conditions[10,15] at edges between faces (see Theorem 2.5 in Ref. 15). One can write:

$$f \in H^{1/2}(\partial\Omega) \iff f_{|F_k} \in H^{1/2}(F_k),\ \forall\, k \quad \text{and} \quad f_{|F_i} \overset{1/2}{=} f_{|F_j} \ \text{at } e_{ij},\ \forall\, e_{ij}. \tag{4.13}$$

One then characterizes the space $\boldsymbol{H}_{\perp}^{1/2}(\partial\Omega)$ as (Ref. 15, Proposition 2.7):

$$\boldsymbol{H}_{\perp}^{1/2}(\partial\Omega) = \Big\{ \mathbf{u} \in \boldsymbol{L}_T^2(\partial\Omega) \mid \mathbf{u}_{|F_k} \in H^{1/2}(F_k)^3,\ \forall\, k,$$
$$\mathbf{u}_{|F_i} \cdot \tau_i \overset{1/2}{=} \mathbf{u}_{|F_j} \cdot \tau_j \ \text{at } e_{ij},\ \forall\, e_{ij} \Big\}.$$

(1) The boundary $\partial\Omega$ is $\mathscr{C}^{1,1}$.

- Since $\mathbf{J}$ is divergence-free, there exists a surface field λ_τ such that $\mathbf{J} = \mathbf{curl}_\tau \lambda_\tau$. Moreover, as $\mathbf{J}$ belongs to $H^{1/2}(\partial\Omega)$ component by component (cf. (4.12) with a "smooth" $\mathbf{n}$), one has $\lambda_\tau \in H^{3/2}(\partial\Omega)$. Now, it is common knowledge that the trace mapping $T_{0,1} : v \mapsto (v_{|\partial\Omega}, \partial_n v_{|\partial\Omega})$, is onto, when considered from $H^2(\Omega)$ to $H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$. Therefore, given $(\lambda_\tau, 0) \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, there exists $v \in H^2(\Omega)$ such that $T_{0,1} v = (\lambda_\tau, 0)$. Then, $\mathbf{v} = \nabla v$ is an admissible lifting, since $\mathbf{v} \wedge \mathbf{n}_{|\partial\Omega} = \mathbf{J}$, and it belongs to X_T^0 by construction. One can then recover a divergence-free lifting by the usual procedure.

(2) The domain Ω is a convex polyhedron.

- Preliminarily, we note that given any $\mathbf{v}$ of X_T, one has $\mathbf{v}\cdot\mathbf{n}_{|\partial\Omega}=0$; this implies that $\mathbf{u}=\mathbf{v}_{|\partial\Omega}$ is such that

$$\mathbf{u}_{|F_i}\cdot\mathbf{n}_j \overset{1/2}{=} 0 \quad\text{and}\quad 0 \overset{1/2}{=} \mathbf{u}_{|F_j}\cdot\mathbf{n}_i \text{ at } e_{ij},\ \forall\, e_{ij}.$$

The space of admissible surface current density fields $\mathbf{J}$ is a subset of a strict (closed) subspace of $\boldsymbol{H}_{\perp}^{1/2}(\partial\Omega)$, namely

$$\mathcal{J}=\left\{\mathbf{u}\in\boldsymbol{H}_{\perp}^{1/2}(\partial\Omega)\mid \mathbf{u}_{|F_i}\cdot\tau_{ij}\overset{1/2}{=}0 \text{ and } 0\overset{1/2}{=}\mathbf{u}_{|F_j}\cdot\tau_{ij} \text{ at } e_{ij},\ \forall\, e_{ij}\right\}.$$

On the other hand, given $\mathbf{J}\in\mathcal{J}$, one can consider that the field $\mathbf{u}=\mathbf{n}\wedge\mathbf{J}$ (with a vanishing normal component) belongs to $H^{1/2}(\partial\Omega)^3$, since all compatibility conditions of (4.13) are fulfilled for its three components. Therefore, there exists $\mathbf{v}$ in $H^1(\Omega)^3$ such that the trace of its tangential components is equal to $\mathbf{n}\wedge(\mathbf{v}\wedge\mathbf{n})_{|\partial\Omega}=\mathbf{u}$, and $\mathbf{v}\cdot\mathbf{n}_{|\partial\Omega}=0$, so that $\mathbf{v}$ belongs to X_T^0. Evidently, one has $\mathbf{v}\wedge\mathbf{n}_{|\partial\Omega}=\mathbf{J}$. Recovering the divergence-free lifting as usual, one concludes that the space of admissible surface current density fields is exactly $\mathcal{J}$.

(3) The domain Ω is a non-convex (Lipschitz) polyhedron.

- Here, we shall use the previous result, together with a splitting of the space X_T^0. We need to introduce some notations. First, consider

$$\Psi=\left\{\psi\in H^1(\Omega)\mid \Delta\psi\in L^2(\Omega),\ \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega}=0\right\}.$$

According to Grisvard,[24] $\Psi^{\mathrm{reg}}=\Psi\cap H^2(\Omega)$ is a closed subspace of Ψ, so one can write $\Psi=\Psi^{\mathrm{reg}}\oplus\Psi^{\mathrm{sing}}$ (with a suitable subspace Ψ^{sing}). Then, consider the (regular) subspace $X_T^{\mathrm{reg}}=X_T\cap H^1(\Omega)^3$ of X_T and the continuous splitting[11] $X_T=X_T^{\mathrm{reg}}+\nabla\Psi$: one infers that X_T^{reg} is closed in X_T and the splitting $X_T=X_T^{\mathrm{reg}}\oplus\nabla\Psi^{\mathrm{sing}}$ follows.[12]
In our case, we recall that we are interested in X_T^0. Simply, we observe that, for $\psi\in\Psi$, there holds $\mathrm{div}_\tau(\nabla_\tau\psi|_{\partial\Omega}\wedge\mathbf{n})=\mathrm{div}_\tau(\mathbf{curl}_\tau\psi|_{\partial\Omega})=0$. From this, we infer the relevant splitting

$$X_T^0=X_T^{0,\mathrm{reg}}\oplus\nabla\Psi^{\mathrm{sing}},\tag{4.14}$$

with $X_T^{0,\mathrm{reg}}=X_T^0\cap H^1(\Omega)^3$. Finally, we introduce the trace space of Ψ^{sing}, that is $\Lambda^{\mathrm{sing}}=\left\{\lambda\in H^1(\partial\Omega)\mid \exists\,\psi\in\Psi^{\mathrm{sing}},\ \lambda=\psi|_{\partial\Omega}\right\}$, and we define

$$\mathcal{J}^{\mathrm{sing}}=\mathcal{J}\oplus\mathbf{curl}_\tau\Lambda^{\mathrm{sing}}.$$

It turns out that $\mathcal{J}^{\mathrm{sing}}$ is exactly the space of admissible surface current density fields in the case of a non-convex Lipschitz polyhedron. Indeed, it is clear according to (4.14) that any surface current density belongs to $\mathcal{J}^{\mathrm{sing}}$. Conversely, given an element of $\mathcal{J}^{\mathrm{sing}}$, one finds a suitable lifting, as the sum of a regular part in $X_T^{0,\mathrm{reg}}$ (we follow the same steps as in item (2)) and of $\nabla\psi$, with ψ in Ψ^{sing}.

5. On the Removal of the Divergence-Free Assumption

In this section, we have a second look at problem (3.1), this time without assuming that the given surface current density $\mathbf{J}$ is divergence-free.

First, recall that, from Sec. 2.3, one has $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\Omega$ at all times as soon as the initial condition $\mathbf{B}(\cdot, 0) \cdot \mathbf{n} = 0$ on $\partial\Omega$ is verified. This leads, using the Drude–Born–Fedorov constitutive laws (2.6) and relation (2.8), to

$$\mathbf{H} \cdot \mathbf{n} + \beta \operatorname{div}_\tau (\mathbf{H} \wedge \mathbf{n}) = 0 \quad \text{on } \partial\Omega,\ t > 0, \tag{5.1}$$

or equivalently,[b] using the prescribed boundary condition on the magnetic field,

$$\mathbf{H} \cdot \mathbf{n} = -\beta \operatorname{div}_\tau \mathbf{J} \quad \text{on } \partial\Omega. \tag{5.2}$$

Boundary condition (5.2) is no longer homogeneous. Hence, the magnetic field naturally belongs to the space

$$X^+_{\min} = \{\mathbf{v} \in X \mid \mathbf{v} \cdot \mathbf{n} + \beta \operatorname{div}_\tau (\mathbf{v} \wedge \mathbf{n}) = 0 \text{ on } \partial\Omega\}.$$

Classically, the surface current density is an element of the space of tangential traces from $H(\mathbf{curl}, \Omega)$, that is

$$\boldsymbol{H}_{\|}^{-1/2}(\operatorname{div}_\tau; \partial\Omega) = \{\mathbf{u} \mid \exists\, \mathbf{v} \in H(\mathbf{curl}; \Omega),\ \mathbf{v} \wedge \mathbf{n} = \mathbf{u} \text{ on } \partial\Omega\}.$$

Then, in order to lift $\mathbf{J}$, one has to solve: *find* $\mathbf{H_J} \in X^+_{\min}$ *such that*

$$\mathbf{H_J} \wedge \mathbf{n} = \mathbf{J} \quad \text{and} \quad \mathbf{H_J} \cdot \mathbf{n} = -\beta \operatorname{div}_\tau \mathbf{J} \ \text{ on } \partial\Omega.$$

Observe that, by integration by parts, we have

$$\begin{aligned}\int_\Omega \operatorname{div} \mathbf{H_J}\, d\mathbf{x} &= \langle \mathbf{H_J} \cdot \mathbf{n}, 1\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \\ &= -\beta \langle \operatorname{div}_\tau \mathbf{J}, 1\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = 0,\end{aligned}$$

so that $\operatorname{div} \mathbf{H_J} \in L^2_0(\Omega)$, and $\mathbf{H_J}$ can be chosen to be divergence-free as usual. Assuming $\mathbf{H_J}$ exists, the difference $\widetilde{\mathbf{H}} = \mathbf{H} - \mathbf{H_J} \in X^+_{\min}$ then satisfies

$$\operatorname{div} \widetilde{\mathbf{H}} = 0 \ \text{ in } \Omega, \quad \widetilde{\mathbf{H}} \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega \quad \text{and} \quad \widetilde{\mathbf{H}} \wedge \mathbf{n} = \mathbf{0} \ \text{ on } \partial\Omega,$$

and therefore belongs to $H(\operatorname{div} 0; \Omega) \cap H^1_0(\Omega)^3$.

Observe that a more "practical" functional setting can be obtained if one chooses to add a supplementary regularity assumption by imposing that $\mathbf{H} \cdot \mathbf{n}_{|\partial\Omega} \in L^2(\partial\Omega)$, which amounts (see Ref. 18) to $\mathbf{H} \wedge \mathbf{n}_{|\partial\Omega} \in \boldsymbol{L}^2_T(\partial\Omega)$, and one sets instead the problem for $\mathbf{H}$ in the space

$$X^+ = \left\{\mathbf{v} \in X \mid \mathbf{v} \cdot \mathbf{n}_{|\partial\Omega} \in L^2(\partial\Omega),\ \mathbf{v} \cdot \mathbf{n} + \beta \operatorname{div}_\tau (\mathbf{v} \wedge \mathbf{n}) = 0 \text{ on } \partial\Omega\right\},$$

with a datum $\mathbf{J}$ in $\boldsymbol{L}^2_T(\partial\Omega)$ verifying $\operatorname{div}_\tau \mathbf{J} \in L^2(\partial\Omega)$.

[b] Note that constraint (5.1) and boundary condition (5.2) hold *a priori* in the $H^{-1/2}(\partial\Omega)$ sense, since $\mathbf{H}$ always belongs to the space X. Indeed, recall that for any field $\mathbf{v}$ in $H(\operatorname{div}; \Omega)$, one has $\mathbf{v} \cdot \mathbf{n}_{|\partial\Omega} \in H^{-1/2}(\partial\Omega)$. Fortunately, we find by application of Green's formula that $\operatorname{div}_\tau \left(\mathbf{w} \wedge \mathbf{n}_{|\partial\Omega}\right) \in H^{-1/2}(\partial\Omega)$ for any field $\mathbf{w}$ in $H(\mathbf{curl}; \Omega)$.

The knowledge of $\operatorname{div}_\tau \mathbf{J}$ on $\partial\Omega$ allows, using the initial conditions, to determine the surface charge density σ from Eq. (2.9). As a consequence, it stems from the condition $\mathbf{D}\cdot\mathbf{n}_{|\partial\Omega} = -\sigma$ and relations (2.6) that $\mathbf{E}\cdot\mathbf{n}_{|\partial\Omega} = -\varepsilon^{-1}\sigma$, where σ is known.

It remains to lift the normal trace of $\mathbf{E}$ to obtain a field in X_T^0. We proceed *explicitly* (i.e. without having to assume the lifting exists) as follows. We first define the field $\mathbf{E}_\sigma = \nabla\phi$, the function ϕ being the unique solution to the problem: *find* $\phi \in H^1(\Omega)\backslash\mathbb{R}$ *such that*

$$\Delta\phi = 0 \text{ in } \Omega, \quad \frac{\partial\phi}{\partial\mathbf{n}} = -\varepsilon^{-1}\sigma \text{ on } \partial\Omega.$$

Note again that the minimal regularity for the datum σ is $H^{-1/2}(\partial\Omega)$: it holds according to the formula $\sigma(\cdot,t) = -\int_0^t \operatorname{div}_\tau \mathbf{J}(\cdot,s)\,ds + \sigma(\cdot,0)$ (see (2.9)), provided that $\sigma(\cdot,0) \in H^{-1/2}(\partial\Omega)$. The difference $\mathbf{E}' = \mathbf{E} - \mathbf{E}_\sigma$ then verifies

$$\mathbf{curl}\,\mathbf{E}' = \mathbf{curl}\,\mathbf{E} \in L^2(\Omega)^3, \quad \operatorname{div}\mathbf{E}' = \operatorname{div}\mathbf{E} \in L^2(\Omega), \quad \mathbf{E}'\cdot\mathbf{n} = -\varepsilon^{-1}\sigma - \frac{\partial\phi}{\partial\mathbf{n}} = 0$$

on $\partial\Omega$.

We also introduce $\mathbf{D}_\sigma = \varepsilon\,\nabla\phi$, which yields, for $\mathbf{D}' = \mathbf{D} - \mathbf{D}_\sigma$,

$$\mathbf{curl}\,\mathbf{D}' = \mathbf{curl}\,\mathbf{D} \in L^2(\Omega)^3, \ \operatorname{div}\mathbf{D}' = 0 \text{ in } \Omega, \quad \mathbf{D}'\cdot\mathbf{n} = -\sigma - \varepsilon\frac{\partial\phi}{\partial\mathbf{n}} = 0 \text{ on } \partial\Omega.$$

By subtraction, we moreover see that $\mathbf{D}' = \varepsilon\,(\mathbf{E}' + \beta\,\mathbf{curl}\,\mathbf{E}')$. We thus have $\mathbf{curl}\,\mathbf{E}'\cdot\mathbf{n} = \operatorname{div}_\tau(\mathbf{E}'\wedge\mathbf{n}) = 0$ on $\partial\Omega$, hence $\mathbf{E}'$ belongs to X_T^0. The trace $\mathbf{E}'\wedge\mathbf{n}_{|\partial\Omega}$ is equal to $-\nabla\phi\wedge\mathbf{n}_{|\partial\Omega}$ and as such is known. In this way, we can apply the results of Sec. 4.2 and finally conclude that one is able to further extend the range of Theorem 4.1 to the case when the surface current density $\mathbf{J}$ is not divergence-free, provided the lifting $\mathbf{H_J}$ exists in $X^+_{min} \cap H(\operatorname{div}0;\Omega)$. We state the final

Theorem 5.1. *Assume that the chirality admittance* β *is in* $\mathbb{R}\backslash\{-1/\alpha_i,\ i\in\mathbb{N}\}$. *Consider a data set* $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J})$, *such that*

- $\mathbf{E}_0 \in X_N \cap H(\operatorname{div}0;\Omega)$, $\mathbf{H}_0 \in X^+_{(\min)} \cap H(\operatorname{div}0;\Omega)$;
- *there exists a lifting such that* $\mathbf{H_J}\wedge\mathbf{n}_{|\partial\Omega} = \mathbf{J}$, *with regularity* $\mathbf{H_J} \in C^0\left([0,+\infty[, X^+_{(\min)} \cap H(\operatorname{div}0;\Omega)\right)$.

Then, there exists one, and only one, solution $(\mathbf{E},\mathbf{H})$ *to problem* (3.1) *such that*

$$\mathbf{E} \in C^0\left([0,+\infty[, X_N \cap H(\operatorname{div}0;\Omega)\right),$$
$$\mathbf{H} \in C^0([0,+\infty[, X^+_{(\min)} \cap H(\operatorname{div}0;\Omega)).$$

When $\sigma = 0$, the electric field $\mathbf{E}$ belongs to the smaller space $X_N \cap X_T \cap H(\operatorname{div}0;\Omega) = H_0^1(\Omega)^3 \cap H(\operatorname{div}0;\Omega)$ (cf. Theorems 4.1 and 4.2).

6. Other Choices of Boundary Conditions

We consider in this section all other possible sets of boundary conditions for the problem, which seem apparently less restrictive than our initial choice. First, one may choose to impose

$$\mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \mathbf{H} \wedge \mathbf{n} = \mathbf{J} \ \text{ on } \partial\Omega, \ t > 0.$$

Then, the normal trace of the magnetic field $\mathbf{H}$ can be recovered from both above conditions and constitutive relations (2.6): $\mathbf{H} \cdot \mathbf{n} = -\beta \operatorname{div}_\tau \mathbf{J}$ on $\partial\Omega$. However, as already seen in Sec. 2, the first boundary condition allows to recover the tangential trace of the electric field $\mathbf{E}$ only partially. Indeed, using the trace of the Maxwell–Ampère equation (2.4), it yields $\mathbf{curl}\,\mathbf{E} \cdot \mathbf{n} = \operatorname{div}_\tau(\mathbf{E} \wedge \mathbf{n}) = 0$ on $\partial\Omega$. Introducing the surface charge density σ that satisfies (2.9) along with the datum $\mathbf{J}$ and using initial conditions, we have $\mathbf{D} \cdot \mathbf{n} = -\sigma$ on $\partial\Omega$ and subsequently recover, using the Drude–Born–Fedorov equations (2.6), $\mathbf{E} \cdot \mathbf{n} = -\varepsilon^{-1}\sigma$ on $\partial\Omega$. In order to completely determine the tangential trace of $\mathbf{E}$ on the boundary, and according to the results[16] on the Hodge decomposition of elements of $\boldsymbol{H}_{\|}^{-1/2}(\operatorname{div}_\tau; \partial\Omega)$, the rotational part of $\mathbf{E} \wedge \mathbf{n}$ on $\partial\Omega$, determined by the value of $\operatorname{curl}_\tau\left(\mathbf{E} \wedge \mathbf{n}_{|\partial\Omega}\right)$, has to be provided as well. From there, we consider the following scenarios:

(i) if $\operatorname{div}_\tau \mathbf{J} = 0$, then the density σ is zero and the electromagnetic field $(\mathbf{E}, \mathbf{H})$ belongs to $X_T^0 \times X_T^0$, their tangential traces being known. Each of these fields is then lifted as done in Sec. 4.2.

(ii) if $\operatorname{div}_\tau \mathbf{J} \neq 0$, the density σ does not vanish anymore. In this case, the normal traces of $\mathbf{H}$, $\mathbf{E}$ and $\mathbf{D}$ on the boundary do not vanish either, but they are known since they are related to the given datum $\mathbf{J}$. We proceed as in Sec. 5.

Second, another possible choice of boundary conditions is the following:

$$\mathbf{E} \wedge \mathbf{n} = \mathbf{0} \ \text{ on } \partial\Omega \quad \text{and} \quad \mathbf{D} \cdot \mathbf{n} = -\sigma \ \text{ on } \partial\Omega, \ t > 0,$$

the datum being the surface charge density σ. Following the same procedure, one sees that the rotational part of the tangential trace $\mathbf{H} \wedge \mathbf{n}$ must be imposed in order to completely recover this trace.

Third, the final choice is to consider

$$\mathbf{B} \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega \quad \text{and} \quad \mathbf{D} \cdot \mathbf{n} = -\sigma \ \text{ on } \partial\Omega, \ t > 0.$$

This set of boundary conditions needs to be supplemented by two additional conditions on the rotational part of the tangential traces of $\mathbf{E}$ and $\mathbf{H}$.

7. Miscellaneous Remarks

We end this paper by a number of remarks, of various scopes.

7.1. *The topology of the domain*

Up to now, we have assumed that the domain Ω was simply connected with a connected boundary. It is, however, possible to extend all the previous results to more complex geometries, by using the framework developed in Ref. 5.

When the boundary $\partial\Omega$ is not connected, one has to provide the quantities[c] $\langle \mathbf{E}(\cdot,t)\cdot\mathbf{n},1\rangle_{\Gamma_i}$ and $\langle \mathbf{H}(\cdot,t)\cdot\mathbf{n},1\rangle_{\Gamma_i}$, $1\le i\le p$, for any $t>0$, which are easily determined from the data. Indeed, for the electric field $\mathbf{E}$, one has

$$\varepsilon\,\mathbf{E}(\cdot,t)\cdot\mathbf{n}=\sigma(t)=\left(\int_0^t \operatorname{div}_\tau \mathbf{J}(\cdot,s)\,ds-\sigma(\cdot,0)\right)\quad \text{on } \partial\Omega,\ t>0,$$

hence

$$\langle\varepsilon\,\mathbf{E}(\cdot,t)\cdot\mathbf{n},1\rangle_{\Gamma_i}=\int_0^t \langle\operatorname{div}_\tau \mathbf{J}(\cdot,s),1\rangle_{\Gamma_i}\,ds-\langle\sigma(\cdot,0),1\rangle_{\Gamma_i}\,,\ t>0,\ \forall\, i\in\{1,\dots,p\}.$$

Likewise, for the magnetic field $\mathbf{H}$, one gets

$$\langle\mu\,\mathbf{H}(\cdot,t)\cdot\mathbf{n},1\rangle_{\Gamma_i}=-\,\langle\mu\beta\operatorname{div}_\tau \mathbf{J}(\cdot,t),1\rangle_{\Gamma_i}\,,\quad t>0,\ \forall\, i\in\{1,\dots,p\}.$$

Thanks to the above, the orthogonal projections on $\mathcal{H}(e)$ of $\mathbf{E}$ and $\mathbf{H}$ are explicitly known.

On the other hand, when the domain is non-simply connected, we check in Ref. 14 that the existence and uniqueness result used to solve problem (4.6) in Sec. 4.1.2 remains valid provided that the projection of the field $\mathbf{w}$ on $\mathcal{H}(m)$ is known (in our case, zero). Indeed, the idea is that the "natural" space of right-hand sides $\mathbf{f}$ is now

$$W=H_0(\operatorname{div}0;\Omega)\cap\mathcal{H}(m)^\perp.$$

Then, if $\mathbf{u}$ belongs to the domain of the operator $H_0^1(\Omega)^3\cap W$, its $\mathbf{curl}$ is orthogonal to $\mathcal{H}(m)$, according to Theorem 3.17 of Ref. 5 (since in particular, one has $\mathbf{u}\in X_N\cap\mathcal{H}(e)^\perp$). Thus $\mathbf{w}=\mathbf{u}+\beta\,\mathbf{curl}\,\mathbf{u}-\mathbf{f}$ belongs to W by construction, which ensures uniqueness — by orthogonality — of the solution to problem (4.6).

7.2. *Heterogeneous medium*

The case of a heterogeneous electromagnetic chiral medium, with an electric permittivity ε, a magnetic permeability μ and a chiral admittance β that depend on the space variable $\mathbf{x}$, leads to serious difficulties from the angle of the functional spaces brought into play. For instance, one has to satisfy $\operatorname{div}\mathbf{D}=0$ in Ω (from Eq. (2.4)), that is, provided ε is smooth, $\operatorname{div}(\varepsilon(\mathbf{E}+\beta\,\mathbf{curl}\,\mathbf{E}))=\operatorname{div}(\varepsilon\mathbf{E})+\nabla(\varepsilon\beta)\cdot\mathbf{curl}\,\mathbf{E}=0$ in Ω. To impose that the electric field $\mathbf{E}$ belongs to a space such as $H(\operatorname{div}\varepsilon 0;\Omega)$ is in no way sufficient to enforce this constraint (a strictly identical problem holds for the magnetic field $\mathbf{H}$). Moreover, the invertibility of the operator a_β in this case (that is, with a nonconstant coefficient β) remains an open question. In particular, the reader is referred to Ref. 13, in which existence, but not uniqueness, of nonlinear force-free fields in a bounded open set of $\mathbb{R}^3$, is proven.

[c]These are simply the respective coefficients of the projections of $\mathbf{E}$ and $\mathbf{H}$ on $\mathcal{H}(e)$ in the basis given in Ref. 5.

7.3. *Boundary controllability of Maxwell's equations in chiral media*

We deal in this subsection with the controllability of Maxwell's equation in chiral media by means of the Hilbert Uniqueness Method (HUM) introduced by Lions.[28] We assume here that the domain Ω is simply connected with a connected boundary and consider the following controllability problem: *given the initial distribution* $(\mathbf{E}_0, \mathbf{H}_0)$ *in an appropriate functional space and a time* $T > 0$, *find* (*if possible*) *a surface current density* $\mathbf{J}$ *in a suitable functional space such that the solution to system* (3.1) *satisfies*

$$\mathbf{E}(\cdot, T) = \mathbf{H}(\cdot, T) = \mathbf{0} \quad \text{in } \Omega.$$

Computations are omitted below for the sake of brevity, but they are very similar to those of Refs. 26 and 29. Following the principles of HUM theory, the control $\mathbf{J}$ should be chosen in the special form $\mathbf{J} = -\left(\psi - \mu\beta\, \frac{\partial \varphi}{\partial t}\right)$, where (φ, ψ) is a solution to the homogeneous adjoint system

$$\begin{cases} \varepsilon \dfrac{\partial}{\partial t}\,(\psi + \beta\,\mathbf{curl}\,\psi) + \mathbf{curl}\,\varphi = \mathbf{0} & \text{in } \Omega,\ t > 0, \\ \mu \dfrac{\partial}{\partial t}\,(\varphi + \beta\,\mathbf{curl}\,\varphi) - \mathbf{curl}\,\psi = \mathbf{0} & \text{in } \Omega,\ t > 0, \\ \operatorname{div}\varphi = 0,\ \operatorname{div}\psi = 0 & \text{in } \Omega,\ t > 0, \\ \left(\psi - \mu\beta\, \dfrac{\partial \varphi}{\partial t}\right) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega,\ t > 0, \\ \varphi(\cdot, 0) = \varphi_0,\ \psi(\cdot, 0) = \psi_0 & \text{in } \Omega, \end{cases} \tag{7.1}$$

where (φ_0, ψ_0) is a prescribed suitable set of initial data.

Let (φ, ψ) be a solution to system (7.1). Then, forming the expression

$$0 = \int_0^T \int_\Omega \left(\left(\frac{\partial \mathbf{D}}{\partial s} - \mathbf{curl}\,\mathbf{H} \right) \cdot \psi - \left(\frac{\partial \mathbf{B}}{\partial s} + \mathbf{curl}\,\mathbf{E} \right) \cdot \varphi \right) d\mathbf{x}\, ds$$

and proceeding formally, we obtain, after integrating by parts with respect to both time and space variables and if the system (3.1) is brought to rest at time T, that

$$\int_\Omega (\mathbf{D}_0 \cdot \psi_0 - \mathbf{B}_0 \cdot \varphi_0)\ d\mathbf{x} = \int_0^T \int_{\partial\Omega} \left| \psi - \mu\beta\, \frac{\partial \varphi}{\partial s} \right|^2 d\sigma\, ds, \tag{7.2}$$

where $\mathbf{D}_0 = \varepsilon(\mathbf{E}_0 + \beta\,\mathbf{curl}\,\mathbf{E}_0)$ and $\mathbf{B}_0 = \mu(\mathbf{H}_0 + \beta\,\mathbf{curl}\,\mathbf{H}_0)$. The space to which $(\mathbf{D}_0, \mathbf{B}_0)$ belongs is $H(\operatorname{div} 0; \Omega) \times H_0(\operatorname{div} 0; \Omega)$. The left-hand side of (7.2) can thus be replaced by a duality product, and one can apply the theory. A sufficient and necessary condition for exact controllability is that, for T large enough, the square root of the right-hand side of (7.2) defines a norm on the set of initial data (φ_0, ψ_0). However, our main result allows to show that this is not the case.

Indeed, choosing the initial data φ_0 and ψ_0 as non-zero elements of $H_0^1(\Omega)^3 \cap H(\operatorname{div} 0; \Omega)$, we know, according to Theorem 4.1, that there exists a solution (φ', ψ') to

$$\begin{cases} \varepsilon \dfrac{\partial}{\partial t}(\psi' + \beta\, \mathbf{curl}\, \psi') + \mathbf{curl}\, \varphi' = \mathbf{0} & \text{in } \Omega,\ t > 0, \\ \mu \dfrac{\partial}{\partial t}(\varphi' + \beta\, \mathbf{curl}\, \varphi') - \mathbf{curl}\, \psi' = \mathbf{0} & \text{in } \Omega,\ t > 0, \\ \operatorname{div} \varphi' = 0,\ \operatorname{div} \psi' = 0 & \text{in } \Omega,\ t > 0, \\ \varphi' = \mathbf{0},\ \psi' = \mathbf{0} & \text{on } \partial\Omega,\ t > 0, \\ \varphi'(\cdot, 0) = \varphi_0,\ \psi'(\cdot, 0) = \psi_0 & \text{in } \Omega, \end{cases}$$

which obviously is also a solution to system (7.1). This solution verifies the property $\left(\psi - \mu\beta \frac{\partial \varphi}{\partial t}\right)_{|\partial\Omega} = \mathbf{0}$ at all times, which yields a vanishing right-hand side of (7.2). The system (7.1) is thus non-observable and we conclude that the above controllability problem is insolvable, as proved independently in Ref. 19.

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