

Two- and three-field formulations for wave transmission between media with opposite sign dielectric constants

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Abstract

We consider a simplified scalar model problem related to Maxwell equations, involving wave transmission between media with opposite sign dielectric and/or magnetic constants. We build two variational formulations equivalent to the model problem. We show that, under some suitable conditions, both formulations are well-posed since they fit into the coercive plus compact framework. Advantages over previous studies is the validity of the formulations in the general case of Lipschitz interface between the two media and L^∞ dielectric and magnetic constants. An interesting feature of these formulations is that they allow a simple finite element numerical implementation.

Key words: wave transmission problem, opposite sign dielectric constants, augmented variational formulation

1 Introduction

Physical models describing the electromagnetic properties of some metamaterials, semiconductors near plasmon resonance [6], plasmas under cyclotron frequency and superconductors (according to London's phenomenologic approach) lead to negative dielectric constant ϵ . In recent years [8] metamaterials, modelled with simultaneously negative dielectric constant ϵ and magnetic permeability μ , have been thoroughly studied, due to their specific electromagnetic behaviour and their wide application range in modern electronics. For practical applications, it is therefore important to be able to capture numerically the electromagnetic field near interfaces between classical dielectric media ($\epsilon > 0$, $\mu > 0$), and superconductors ($\epsilon < 0$, $\mu > 0$) or metamaterials ($\epsilon < 0$, $\mu < 0$).

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Mathematically however, due to the dielectric constant sign shift at the interface, the natural variational formulation of such problems is neither coercive nor coercive plus compact, so it seems not possible to fit straightforwardly the model into a framework leading to a well-posed problem. In this paper we focus on a simplified scalar model problem related to Maxwell equations, which involves similar interface discontinuities. Assume the domain is split in two parts Ω_1 and Ω_2 , with dielectric constant called ϵ_1 , positive on Ω_1 and ϵ_2 negative on Ω_2 , and consider the following equation

$$\operatorname{div} \left(\frac{1}{\epsilon} \nabla u \right) + \omega^2 \mu u = f. \quad (1)$$

This simplified model problem has already been studied in the case of a piecewise constant ϵ , such that $\epsilon_1 \in \mathbb{R}_*^+$ and $\epsilon_2 \in \mathbb{R}_*^-$. In [4] it has been shown, using integral equations, that for a smooth interface $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$, the model problem fits into the Fredholm framework if the contrast $\kappa := \epsilon_1/\epsilon_2$ is not equal to -1 . In [7], using Dirichlet to Neumann operators, it has been shown that the model fits into the Fredholm framework if $|\kappa| \gg 1$ or $|\kappa| \ll 1$ (no required regularity of the interface Σ). The effect of a geometrical singularity on the interface – id est, a non-smooth interface – has been investigated more precisely in [1]. It has been proved there that, for an interface with a right angle, the operator associated with the non-coercive transmission problem is selfadjoint and has compact resolvent if $\kappa \notin [-3, -1/3]$; similar results can be derived for any angle.

Since it is very hard to generalize integral equation methods in the case of a non-smooth interface geometry or in the case of non-constant $(\epsilon_i)_{i=1,2}$, a variational approach is adopted here. By extending a method employed in [2], we introduce two variational formulations allowing both a Lipschitz interface and variable ϵ_1 and ϵ_2 . To that aim, we introduce a new unknown, which is equal to the gradient of u in one of the subdomains. The two-field formulation is valid for interfaces between superconductor and dielectric medium. The three-field formulation is more general: it is valid also for interfaces between metamaterial and dielectric medium, and further it allows to consider a vanishing frequency ω .

For each formulation we derive conditions on ϵ , μ and on the geometry, ensuring that the considered formulation fits into the coercive plus compact framework. One of the main interests of these formulations is that they can be solved numerically with a standard finite element method. As we have mentioned above, the three-field formulation is more general, however it is also more expensive computationally.

2 Regularity assumptions and the model problem

Let Ω be an open, bounded domain of \mathbb{R}^d ($d=2,3$) with Lipschitz boundary $\partial\Omega$. Due to the limited total number of pages, we present here only the 3D case. Nevertheless, the results and proofs can be derived in the same way in 2D.

It is assumed that the domain Ω can be split in two simply connected sub-domains Ω_1 and Ω_2 with Lipschitz boundaries: $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$. Moreover, if we let $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$ be the interface, and define $\Gamma_i = \partial\Omega_i \setminus \Sigma$, it is assumed that Γ_1 and Γ_2 are connected.

Hereafter we adopt the notation, for all quantities v defined on Ω , $v_i := v|_{\Omega_i}$, for $i = 1, 2$. Furthermore, we use the notations

$$\begin{cases} \text{If } v_i > 0 \text{ a. e. in } \Omega_i: v_i^{max} = \sup_{x \in \Omega_i} v_i(x), v_i^{min} = \inf_{x \in \Omega_i} v_i(x). \\ \text{If } v_i < 0 \text{ a. e. in } \Omega_i: v_i^+ = \sup_{x \in \Omega_i} |v_i(x)|, v_i^- = \inf_{x \in \Omega_i} |v_i(x)|. \end{cases}$$

Finally, the outgoing normal from Ω_i ($i=1,2$) is called \mathbf{n}_i .

From now on, we assume that ϵ belongs to $L^\infty(\Omega)$, that it is strictly positive on Ω_1 and strictly negative on Ω_2 and that $\epsilon^{-1} \in L^\infty(\Omega)$. Also, we assume that μ belongs to $L^\infty(\Omega)$.

By setting $\alpha := 1/\epsilon$ and $\beta := \omega^2 \mu$ the scalar model problem (1) for any given $f \in L^2(\Omega)$, may be rewritten as:

find $u \in H^1(\Omega)$ such that

$$\operatorname{div}(\alpha \nabla u) + \beta u = f \quad \text{in } \Omega. \quad (2)$$

We choose to apply, with no loss of generality, an homogeneous Dirichlet condition on $\partial\Omega$: in other words, $u|_{\partial\Omega} = 0$. In this case, the natural variational formulation of (2) supplemented with this boundary condition is:

find $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), (\alpha \nabla u, \nabla v)_{L^2(\Omega)} - (\beta u, v)_{L^2(\Omega)} = -(f, v)_{L^2(\Omega)}.$$

As $(\alpha \nabla u, \nabla v)_{L^2(\Omega)}$ has no specific sign, its coercivity does not hold.

It is easy to prove that problem (2) is equivalent to:

find $(u_1, u_2) \in X_1 \times X_2$ such that

$$\begin{cases} +\operatorname{div}(\alpha_1 \nabla u_1) + \beta_1 u_1 = f_1 & \text{in } \Omega_1 \\ -\operatorname{div}(|\alpha_2| \nabla u_2) + \beta_2 u_2 = f_2 & \text{in } \Omega_2 \\ u_i|_{\Gamma_i} = 0, \quad i = 1, 2 \\ u_1|_{\Sigma} = u_2|_{\Sigma} \\ (\alpha_1 \partial_{\mathbf{n}_1} u_1 + |\alpha_2| \partial_{\mathbf{n}_1} u_2)|_{\Sigma} = 0 \text{ weakly} \end{cases}, \quad (3)$$

with $X_i := \{p \in H^1(\Omega_i) \mid p|_{\Gamma_i} = 0\}$.

In what follows, we will also use the following Sobolev spaces: $\mathbf{H}_{0,\Gamma_i}(\mathbf{curl}; \Omega_i)$, $i\mathbf{H}_{0,\Sigma}(\mathbf{curl}; \Omega_i)$ and \mathbf{X}_i , for $i = 1, 2$, respectively defined by

$$\mathbf{H}_{0,\Gamma_i}(\mathbf{curl}; \Omega_i) := \{\mathbf{p} \mid \mathbf{curl} \mathbf{p} \in \mathbf{L}^2(\Omega_i), \mathbf{p} \times \mathbf{n}|_{\Gamma_i} = 0\}$$

$$i\mathbf{H}_{0,\Sigma}(\mathbf{curl}; \Omega_i) := \{\mathbf{p} \mid \mathbf{curl} \mathbf{p} \in \mathbf{L}^2(\Omega_i), \mathbf{p} \times \mathbf{n}|_{\Sigma} = 0\}$$

$$\mathbf{X}_i := \{\mathbf{p} \in \mathbf{H}(\text{div}; \Omega_i) \mid \mathbf{curl}(\mathbf{p}/|\alpha_i|) \in \mathbf{L}^2(\Omega_i), (\mathbf{p}/|\alpha_i| \times \mathbf{n})|_{\Gamma_i} = 0\}.$$

The spaces $\mathbf{H}_{0,\Gamma_i}(\mathbf{curl}; \Omega_i)$, and $i\mathbf{H}_{0,\Sigma}(\mathbf{curl}; \Omega_i)$ are endowed with the usual norm of $H(\mathbf{curl}; \Omega_i)$, whereas \mathbf{X}_i is endowed with the graph norm. And finally, $(\cdot, \cdot)_{i,j}$ (resp. $\|\cdot\|_{i,j}$) denotes the usual scalar product (resp. norm) on $H^i(\Omega_j)$. Duality brackets on Σ are understood in the sense of the duality $(H_{00}^{1/2}(\Sigma))' - H_{00}^{1/2}(\Sigma)$.

3 Two-field variational formulation

3.1 Derivation of the formulation

As we will see in theorem 3.3, the formulation we are going to derive fits into the coercive plus compact framework when at least in one of two subdomains of Ω , the ratio α/β is negative. The main idea behind the construction of a suitable two-field formulation is to replace, in the subdomain where $\alpha_k/\beta_k < 0$, the scalar unknown u_k by the vector unknown $\mathbf{u}_k := |\alpha_k| \nabla u_k$.

In order to illustrate our approach, assume $\beta_2 > 0$, almost everywhere¹ in Ω_2 , and set $\mathbf{u}_2 := |\alpha_2| \nabla u_2$ (an equivalent choice would be $\mathbf{u}_1 := \alpha_1 \nabla u_1$ provided that $\beta_1 < 0$). Note that the condition $\alpha_2/\beta_2 < 0$ is needed only for the well-posedness of the formulation and not for its derivation.

To build the two-field formulation (4) let us successively

- take the L^2 -scalar product of the first equation of (3) with a test function $v_1 \in X_1$, integrate by parts, and use the second equality of traces in (3):

$$(\alpha_1 \nabla u_1, \nabla v_1)_{0,1} + \langle \mathbf{u}_2 \cdot \mathbf{n}_1, v_1 \rangle_{\Sigma} - (\beta_1 u_1, v_1)_{0,1} = -(f_1, v_1)_{0,1}.$$

- divide the second equation of (3) by β_2 ; take the L^2 -scalar product between the result of the previous operation and the divergence of a vector test function $\mathbf{v}_2 \in \mathbf{X}_2$, integrate by parts, and use the first equality of traces in (3):

$$\left(\frac{\mathbf{u}_2}{|\alpha_2|}, \mathbf{v}_2 \right)_{0,2} + \left(\frac{\text{div} \mathbf{u}_2}{\beta_2}, \text{div} \mathbf{v}_2 \right)_{0,2} + \langle \mathbf{v}_2 \cdot \mathbf{n}_1, u_1 \rangle_{\Sigma} = - \left(\frac{f_2}{\beta_2}, \text{div} \mathbf{v}_2 \right)_{0,2}.$$

¹ Since $\beta_2 := \omega^2 \mu_2$, this corresponds exactly to $\omega \neq 0$ and $\mu_2 > 0$ a. e.

- sum the results obtained at points one and two.

Finally, to recover an augmented variational formulation as proposed in [3], we may add the term $\left(\mathbf{curl} \frac{\mathbf{u}_2}{|\alpha_2|}, \mathbf{curl} \frac{\mathbf{v}_2}{|\alpha_2|} \right)_{L^2(\Omega_2)}$, since $\mathbf{curl} \frac{\mathbf{u}_2}{|\alpha_2|} = 0$ by construction.

The overall result is the two-field formulation below:

find $U = (u_1, \mathbf{u}_2) \in X_1 \times \mathbf{X}_2$ such that

$$\forall V = (v_1, \mathbf{v}_2) \in X_1 \times \mathbf{X}_2, A(U, V) = L(V), \quad (4)$$

where the forms A and L are respectively defined by

$$\begin{aligned} A(U, V) := & \left(\frac{\mathbf{u}_2}{|\alpha_2|}, \mathbf{v}_2 \right)_{0,2} + \left(\frac{\operatorname{div} \mathbf{u}_2}{\beta_2}, \operatorname{div} \mathbf{v}_2 \right)_{0,2} + \left(\mathbf{curl} \frac{\mathbf{u}_2}{|\alpha_2|}, \mathbf{curl} \frac{\mathbf{v}_2}{|\alpha_2|} \right)_{0,2} \\ & + \langle \mathbf{v}_2 \cdot \mathbf{n}_1, u_1 \rangle_\Sigma + \langle \mathbf{u}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma + (\alpha_1 \nabla u_1, \nabla v_1)_{0,1} - (\beta_1 u_1, v_1)_{0,1}, \end{aligned}$$

and

$$L(V) := -(f_1, v_1)_{0,1} - \left(\frac{f_2}{\beta_2}, \operatorname{div} \mathbf{v}_2 \right)_{0,2}.$$

It is important to note that in the definition of the bilinear form A , the two boundary terms $\langle \mathbf{v}_2 \cdot \mathbf{n}_1, u_1 \rangle_\Sigma$ and $\langle \mathbf{u}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma$ are "homogeneous", i. e. without any scaling factor between the two.

N.B. We propose an augmented variational formulation, so that the vector fields can be discretized with the help of a continuous Galerkin method (see [9]).

3.2 Equivalence with the initial problem

Proposition 3.2: The two-field formulation (4) is equivalent to problem (3).

Proof: To begin with, one finds that $u|_{\Gamma_i} = 0$ ($i = 1, 2$), according to the definition of X_1 and X_2 .

Then, let us take in (4) successively $v_1 = 0$ and $\mathbf{v}_2 = 0$: it is straightforward to show that (u_1, \mathbf{u}_2) satisfy (5) and (6).

$$\forall \mathbf{v}_2 \in \mathbf{X}_2, \left(\frac{\operatorname{div} \mathbf{u}_2}{\beta_2}, \operatorname{div} \mathbf{v}_2 \right)_{0,2} + \left(\mathbf{curl} \frac{\mathbf{u}_2}{|\alpha_2|}, \mathbf{curl} \frac{\mathbf{v}_2}{|\alpha_2|} \right)_{0,2} + \left(\frac{\mathbf{u}_2}{|\alpha_2|}, \mathbf{v}_2 \right)_{0,2} + \langle \mathbf{v}_2 \cdot \mathbf{n}_1, u_1 \rangle_\Sigma = - \left(\frac{f_2}{\beta_2}, \operatorname{div} \mathbf{v}_2 \right)_{0,2}. \quad (5)$$

$$\forall v_1 \in X_1, (\alpha_1 \nabla u_1, \nabla v_1)_{0,1} - (\beta_1 u_1, v_1)_{0,1} + \langle \mathbf{u}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma = -(f_1, v_1)_{0,1}. \quad (6)$$

First, we show that (5) implies $\mathbf{curl}(\mathbf{u}_2/|\alpha_2|) = 0$. To that aim, we choose divergence-free test functions in (5). For this, given $\mathbf{f} \in \mathbf{L}^2(\Omega_2)$, we introduce the auxiliary problem (7):

find $\chi \in \mathbf{H}_{0,\Sigma}(\mathbf{curl}; \Omega_2)$ such that

$$\forall \varphi \in \mathbf{H}_{0,\Sigma}(\mathbf{curl}; \Omega_2), (\chi, \varphi)_{0,2} + \left(\frac{\mathbf{curl} \chi}{|\alpha_2|}, \mathbf{curl} \varphi \right)_{0,2} = (\mathbf{f}, \varphi)_{0,2}. \quad (7)$$

Note that such χ is unique and satisfies (according to the Proposition 3.6 of [5] for the trace equality) both

$$\mathbf{curl} \left(\frac{\mathbf{curl} \chi}{|\alpha_2|} \right) = \mathbf{f} - \chi \text{ in } \Omega_2, \quad \left(\frac{\mathbf{curl} \chi}{|\alpha_2|} \right) \times \mathbf{n}|_{\Gamma_2} = 0.$$

Thus, we can choose $\mathbf{v}_2 = \mathbf{curl} \chi$ in (5) and integrate by parts to reach:

$$\left(\mathbf{curl} \frac{\mathbf{u}_2}{|\alpha_2|}, \mathbf{f} \right)_{0,2} = 0.$$

In other words, we get the desired property $\mathbf{curl}(\mathbf{u}_2/|\alpha_2|) = 0$ in $\mathbf{L}^2(\Omega_2)$.

From (5), we now recover the second and fourth equations of (3): Ω_2 being simply connected, there exists $u_2 \in H^1(\Omega_2)$ such that $\nabla u_2 = \mathbf{u}_2/|\alpha_2|$. Recall that $(\mathbf{u}_2/|\alpha_2| \times \mathbf{n})|_{\Gamma_2} = 0$ and α_2 is strictly negative. Thus, Γ_2 being connected, we obtain that $u_2|_{\Gamma_2}$ is actually a constant. The scalar potential u_2 being defined up to a constant, let us choose $u_2|_{\Gamma_2} = 0$. If we integrate (5) by parts, we reach

$$\left(\frac{\operatorname{div} (|\alpha_2| \nabla u_2)}{\beta_2} - u_2 + \frac{f_2}{\beta_2}, \operatorname{div} \mathbf{v}_2 \right)_{0,2} + \langle \mathbf{v}_2 \cdot \mathbf{n}_1, u_1 - u_2 \rangle_{\Sigma} = 0 \quad (8)$$

We want to use in (8) *ad hoc* test functions \mathbf{v}_2 . Given $(p, s) \in L^2(\Omega_2) \times (H_0^{1/2}(\Sigma))'$, let us introduce the auxiliary problem (9):

find $\chi \in X_2$ such that

$$\forall z \in X_2, (|\alpha_2| \nabla \chi, \nabla z)_{0,2} = (p, z)_{0,2} + \langle s, z \rangle_{\Sigma}. \quad (9)$$

Problem (9) is well-posed. Its solution χ is such that $\operatorname{div} |\alpha_2| \nabla \chi = -p$ and $|\alpha_2| \nabla \chi \cdot \mathbf{n}_2|_{\Sigma} = s$. So we can choose $\mathbf{v}_2 = |\alpha_2| \nabla \chi$ in (8). Let us take first $s = 0$, to recover the second equation of (3), and then $p = 0$, to recover the fourth equation of (3).

In order to conclude the proof, we have to recover the first and the last equations of (3). One chooses simply in (6) test functions which span $\mathcal{D}(\Omega_1)$, and then functions, which span X_1 (the trace mapping $X_1 \rightarrow H_0^{1/2}(\Sigma)$ is onto.)

□

3.3 Finding a well-posed variational setting for the form A .

Below, we build a splitting of the bilinear form A in a two term sum, so that the first term is coercive over $\{X_1 \times \mathbf{X}_2\}^2$, and the second one is a compact perturbation of the first one. Let us write $A = A_{coer} + A_{comp}$, with

$$A_{comp} = -((\beta_1 + \alpha_1^{min})u_1, v_1)_{0,1}.$$

Thanks to the compact imbedding of $H^1(\Omega_1)$ into $L^2(\Omega_1)$, A_{comp} is indeed a compact perturbation of A_{coer} . We then prove that the form A_{coer} is coercive under some suitable conditions (cf. theorem 3.3).

We introduce some constant, related to the lifting of the trace of scalar fields and of the normal trace of vector fields. Let the constant $c \in \mathbb{R}_*^+$ be such that (10) holds *optimally* (i. e. the constant c takes the smallest admissible value.)

$$\forall (v_1, \mathbf{v}_2) \in X_1 \times \mathbf{X}_2, |\langle \mathbf{v}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma| \leq c \|\mathbf{v}_2\|_{\mathbf{H}(\text{div}; \Omega_2)} \|v_1\|_{1,1}. \quad (10)$$

Theorem 3.3: Assume that $\alpha_2/\beta_2 < 0$ a. e., and that

$$\frac{\alpha_1^{min}}{\alpha_2^+} > c^2 \left\{ \min \left(1, \frac{\alpha_2^+}{\beta_2^{max}} \right) \right\}^{-1} \quad (11)$$

holds. Then, the form A_{coer} is coercive over $\{X_1 \times \mathbf{X}_2\}^2$.

Proof: Since $\alpha_2/\beta_2 < 0$ and $\alpha_2 < 0$, we have $\beta_2 > 0$ in Ω_2 . Thus, A_{coer} can be bounded from below by

$$\begin{aligned} |A_{coer}(V, V)| &\geq \alpha_1^{min} \|v_1\|_{1,1}^2 + \min \left(\frac{1}{\alpha_2^+}, \frac{1}{\beta_2^{max}} \right) \|\mathbf{v}_2\|_{\mathbf{H}(\text{div}; \Omega_2)}^2 \\ &\quad + \left\| \mathbf{curl} \frac{\mathbf{v}_2}{|\alpha_2|} \right\|_{0,2}^2 - 2c \|\mathbf{v}_2\|_{\mathbf{H}(\text{div}; \Omega_2)} \|v_1\|_{1,1}. \end{aligned}$$

We have to control the term $-2c \|\mathbf{v}_2\|_{\mathbf{H}(\text{div}; \Omega_2)} \|v_1\|_{1,1}$ with (a fraction of) the others. Let us recall that, given $m; p \in \mathbb{R}_*^+$ and $\forall x; y \in \mathbb{R}$ the following equality is true.

$$mx^2 + y^2 - 2pxy = \frac{m+p^2}{2} \left(x - \frac{2p}{m+p^2}y \right)^2 + \frac{m-p^2}{2}x^2 + \frac{m-p^2}{m+p^2}y^2. \quad (12)$$

We find that, if we identify $x := \|v_1\|_{1,1}$, $y := \|\mathbf{v}_2\|_{\mathbf{H}(\text{div}; \Omega_2)}$ and set

$$m := \alpha_1^{min} \left[\min \left(\frac{1}{\alpha_2^+}, \frac{1}{\beta_2^{max}} \right) \right]^{-1}, \quad \text{and} \quad p := c \left[\min \left(\frac{1}{\alpha_2^+}, \frac{1}{\beta_2^{max}} \right) \right]^{-1},$$

the form A_{coer} is coercive as soon as $m > p^2$. By rewriting m and p as defined above, this last inequality leads to (11). □

Remark 3.3: The dependencies on geometrical conditions (shape and regularity of Σ) are implicitly included in the definition of the constant c .

Corollary 3.3: Assume that $\alpha_2/\beta_2 < 0$ a. e., and that (11) holds. Then, the variational formulation (4) fits into the coercive plus compact framework.

Evidently, the knowledge of the vector field \mathbf{u}_2 which solves (4) is enough to recover the scalar field u_2 , both theoretically and numerically.

If one goes back to the original dielectric and magnetic parameters, (11) corresponds precisely to

$$\frac{\epsilon_2^-}{\epsilon_1^{max}} > c^2 \left\{ \min \left(1, \frac{1}{\omega^2 \mu_2^{max} \epsilon_2^-} \right) \right\}^{-1}. \quad (13)$$

Inequality (13) is a *sufficient* condition. Moreover, it implies $\epsilon_2^-/\epsilon_1^{max} > c^2$. In accordance with the literature we find that the model problem fits into the coercive plus compact framework in the case of small contrasts (recall that $\kappa := \epsilon_1/\epsilon_2$.) To recover a similar result in the case of large contrasts – provided that $\beta_1 < 0$ – one could alternatively build a two-field formulation by choosing $\mathbf{u}_1 := \alpha_1 \nabla u_1$ and using vector test functions in \mathbf{X}_1 .

4 Three field variational formulation

As we have already seen, for the two-field formulation to be valid, we had to assume² that at least over one of the two subdomains, we have $\alpha_k/\beta_k < 0$. Moreover, in the case of vanishing β we cannot build the two-field formulation. In order to relax those constraints on β , we derive a three-field variational formulation, which allows to handle any $\beta \in L^\infty(\Omega)$.

4.1 Derivation of the formulation

In this paragraph we propose a more general formulation, which allows to handle a wider set of conditions on the parameters ϵ , μ and ω : the three-field formulation. This time, we keep both scalar unknowns u_1 and u_2 , and we add the vector unknown

² Albeit this assumption may not be optimal.

\mathbf{u}_2 . To begin with, keeping both u_1 and u_2 leads to a reformulated definition of $H_0^1(\Omega)$; we introduce

$$\mathcal{X} := \{(v, w) \in X_1 \times X_2 \mid v|_\Sigma = w|_\Sigma\}.$$

Now, let us

- take the L^2 -scalar product of the first equation of (3) with a test function $v_1 \in X_1$, integrate by parts, and use the second equality of traces in (3):

$$(\alpha_1 \nabla u_1, \nabla v_1)_{0,1} + \langle \mathbf{u}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma - (\beta_1 u_1, v_1)_{0,1} = -(f_1, v_1)_{0,1}.$$

- take the L^2 -scalar product between the second equation of (3) and the divergence of a vector test function $\mathbf{v}_2 \in \mathbf{X}_2$; multiply the resulting equality by a constant factor $\rho > 0$.

$$\rho(\operatorname{div} \mathbf{u}_2, \operatorname{div} \mathbf{v}_2)_{0,2} - \rho(\beta_2 u_2, \operatorname{div} \mathbf{v}_2)_{0,2} = -\rho(f_2, \operatorname{div} \mathbf{v}_2)_{0,2}.$$

- consider, for $(v_2, \mathbf{v}_2) \in X_2 \times \mathbf{X}_2$, the two identities (recall that $\mathbf{u}_2 = |\alpha_2| \nabla u_2$)

$$(|\alpha_2| \nabla u_2, \nabla v_2)_{0,2} + (\operatorname{div} \mathbf{u}_2, v_2)_{0,2} + \langle \mathbf{u}_2 \cdot \mathbf{n}_1, v_2 \rangle_\Sigma = 0; \quad (14)$$

$$(u_2, \operatorname{div} \mathbf{v}_2)_{0,2} + \left(\mathbf{v}_2, \frac{\mathbf{u}_2}{|\alpha_2|} \right)_{0,2} + \langle \mathbf{v}_2 \cdot \mathbf{n}_1, u_2 \rangle_\Sigma = 0. \quad (15)$$

- replace $v_2|_\Sigma$ by $v_1|_\Sigma$ and sum these two equalities with the results obtained following the first two items.

Finally add the term $\left(\operatorname{curl} \frac{\mathbf{u}_2}{|\alpha_2|}, \operatorname{curl} \frac{\mathbf{v}_2}{|\alpha_2|} \right)_{0,2}$ (cf. [3] or paragraph 3.1) in order to build an augmented variational formulation.

To summarize, we introduce the variational formulation (16):

find $U = ((u_1, u_2), \mathbf{u}_2) \in \mathcal{X} \times \mathbf{X}_2$ such that

$$\forall V = ((v_1, v_2), \mathbf{v}_2) \in \mathcal{X} \times \mathbf{X}_2 \quad \mathcal{A}^\rho(U, V) = \mathcal{L}^\rho(V). \quad (16)$$

The forms \mathcal{A}^ρ and \mathcal{L}^ρ are respectively defined by

$$\begin{aligned} \mathcal{A}^\rho(U, V) := & \rho(\operatorname{div} \mathbf{u}_2, \operatorname{div} \mathbf{v}_2)_{0,2} + \left(\operatorname{curl} \frac{\mathbf{u}_2}{|\alpha_2|}, \operatorname{curl} \frac{\mathbf{v}_2}{|\alpha_2|} \right)_{0,2} + \left(\frac{\mathbf{u}_2}{|\alpha_2|}, \mathbf{v}_2 \right)_{0,2} \\ & + (|\alpha_2| \nabla u_2, \nabla v_2)_{0,2} + (\operatorname{div} \mathbf{u}_2, v_2)_{0,2} + (u_2, \operatorname{div} \mathbf{v}_2)_{0,2} - \rho(\beta_2 u_2, \operatorname{div} \mathbf{v}_2)_{0,2} \\ & + (\alpha_1 \nabla u_1, \nabla v_1)_{0,1} - (\beta_1 u_1, v_1)_{0,1} + 2 \langle \mathbf{u}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma + \langle \mathbf{v}_2 \cdot \mathbf{n}_1, u_2 \rangle_\Sigma, \end{aligned}$$

and

$$\mathcal{L}^\rho(V) := -(f_1, v_1)_{0,1} - \rho(f_2, \operatorname{div} \mathbf{v}_2)_{0,2}.$$

Again, it is important to note that in the definition of the bilinear form \mathcal{A}^ρ , the two boundary terms $\langle \mathbf{v}_2 \cdot \mathbf{n}_1, u_1 \rangle_\Sigma$ and $\langle \mathbf{u}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma$ remain "homogeneous". In addition, we remark that this is true for any choice of the factor ρ , which we will fit to some optimal value when we establish the coercivity of \mathcal{A}^ρ .

N.B. Again, the use of an augmented variational formulation allows to use a continuous Galerkin discretisation of the vector fields (cf. [9]).

4.2 Equivalence with the initial problem

Proposition 4.2: The three-field formulation (16) is equivalent to problem (3).

Proof: To begin with, one finds that $u_1|_\Sigma = u_2|_\Sigma$ and $u|_{\Gamma_i} = 0$ ($i = 1, 2$), according to the definition of \mathcal{X} .

Then, one recovers the first equation (3), by choosing in (16) test functions v_1 which span $\mathcal{D}(\Omega_1)$, and $(v_2, \mathbf{v}_2) = (0, 0)$.

Next, we have that $\mathbf{curl}(\mathbf{u}_2/|\alpha_2|) = 0$: this is achieved as in Proposition 3.2, by taking $(v_1, v_2) = (0, 0)$ and $\mathbf{v}_2 = \mathbf{curl} \chi$, χ being the solution to (7).

From there, we establish that $\mathbf{u}_2 = |\alpha_2| \nabla u_2$ and that the second equation of (3) is recovered. Unfortunately, it does not seem possible to carry out the proof "sequentially", so we proceed "in parallel"...

We introduce $\boldsymbol{\tau} = \nabla u_2 - \mathbf{u}_2/|\alpha_2|$, and $\eta = -\operatorname{div} \mathbf{u}_2 + \beta_2 u_2 - f_2$, and prove that both fields vanish over Ω_2 . To start with, we know that $\boldsymbol{\tau} \in H_{0,\Gamma_2}(\mathbf{curl}; \Omega_2)$, $\mathbf{curl} \boldsymbol{\tau} = 0$, and that $\eta \in L^2(\Omega_2)$.

Choose first in (16) $(v_1, v_2) = (0, 0)$, and $\mathbf{v}_2 \in \mathcal{D}(\Omega_2)^3$, to reach $\rho \nabla \eta = \boldsymbol{\tau}$ in the sense of distributions over Ω_2 . Therefore, η belongs to $H^1(\Omega_2)$ and, in addition (since Γ_2 is connected), $\eta|_{\Gamma_2} = c_2 \in \mathbb{R}$.

Then, let us prove that the trace of η is actually equal to c_2 over the whole boundary $\partial\Omega_2$. For that, choose in (16) $(v_1, v_2) = (0, 0)$, and $\mathbf{v}_2 \in \mathbf{X}_2$, and integrate by parts. This yields

$$\forall \mathbf{v}_2 \in \mathbf{X}_2, \langle \mathbf{v}_2 \cdot \mathbf{n}_2, \eta \rangle_{\partial\Omega_2} = 0. \quad (17)$$

Consider then an *ad hoc* test function \mathbf{v}_2 , built in the following way: solve problem (9), with $p = 0$ and $s = c_2 - \eta|_\Sigma$, which belongs to $L^2(\Sigma)$, and set $\mathbf{v}_2 = |\alpha_2| \nabla \chi$. One gets $\operatorname{div} \mathbf{v}_2 = 0$ and $\mathbf{v}_2 \cdot \mathbf{n}_2|_\Sigma = c_2 - \eta|_\Sigma$. We note that since $\operatorname{div} \mathbf{v}_2 = 0$, there holds in particular $\langle \mathbf{v}_2 \cdot \mathbf{n}_2, 1 \rangle_{\partial\Omega_2} = 0$. Using this vector field in (17) leads to

$$0 = \langle \mathbf{v}_2 \cdot \mathbf{n}_2, \eta \rangle_{\partial\Omega_2} = \langle \mathbf{v}_2 \cdot \mathbf{n}_2, \eta - c_2 \rangle_{\partial\Omega_2} = \langle \mathbf{v}_2 \cdot \mathbf{n}_2, \eta - c_2 \rangle_\Sigma = -\|\eta - c_2\|_{L^2(\Sigma)}^2.$$

Therefore, $\eta|_{\partial\Omega_2} = c_2$, so that $\boldsymbol{\tau} \in H_0(\mathbf{curl}; \Omega_2)$ (since $\boldsymbol{\tau} = \rho \nabla \eta$.)

Next, choose in (16) $(v_1, \mathbf{v}_2) = (0, 0)$, and $v_2 \in \mathcal{D}(\Omega_2)$. One finds $\operatorname{div} |\alpha_2| \tau = 0$ in the sense of distributions over Ω_2 . Thus η belongs to $H^1(\Omega_2)$, and it satisfies $\operatorname{div} |\alpha_2| \nabla \eta = 0$ (since $\rho \nabla \eta = \tau$) with a constant trace ($=c_2$) over $\partial\Omega_2$. In other words, $\eta = c_2$ over Ω_2 , and $\tau = 0$, so that $\mathbf{u}_2 = |\alpha_2| \nabla u_2$ holds.

There remains to prove that $c_2 = 0$ to recover the second equation of (3). We choose again in (16) $(v_1, v_2) = (0, 0)$, and $\mathbf{v}_2 \in \mathbf{X}_2$, *without* integrating by parts, to reach

$$\forall \mathbf{v}_2 \in \mathbf{X}_2, c_2(1, \operatorname{div} \mathbf{v}_2)_{0,2} = 0. \quad (18)$$

Since the range of the divergence from \mathbf{X}_2 is exactly $L^2(\Omega_2)$, there follows $c_2 = 0$, our intended target.

In order to conclude the proof, we consider in (16) $(v_1, v_2) \in \mathcal{X}$ and $\mathbf{v}_2 = 0$. By integrating by parts and using the previous results, one reaches easily

$$\forall v_1 \in X_1, \langle (\alpha_1 \partial_{\mathbf{n}_1} u_1 + |\alpha_2| \partial_{\mathbf{n}_1} u_2), v_1 \rangle_{\Sigma} = 0.$$

The last equation of (3) follows. □

4.3 Finding a well-posed variational setting for the form \mathcal{A}^ρ

As for the two-field formulation, we split \mathcal{A}^ρ as $\mathcal{A}^\rho = \mathcal{A}_{coer}^\rho + \mathcal{A}_{comp}^\rho$, with

$$\begin{aligned} \mathcal{A}_{comp}^\rho := & -\frac{1}{\rho} (u_2, v_2)_{0,2} - \rho (\beta_2 u_2, \operatorname{div} \mathbf{v}_2)_{0,2} + (\operatorname{div} \mathbf{u}_2, v_2)_{0,2} \\ & + (u_2, \operatorname{div} \mathbf{v}_2)_{0,2} - \left((\beta_1 + \alpha_1^{min}) u_1, v_1 \right)_{0,1} \end{aligned}$$

Thanks to the compact imbedding of $H^1(\Omega_i)$ into $L^2(\Omega_i)$, $i = 1, 2$, \mathcal{A}_{comp}^ρ is a compact perturbation of \mathcal{A}_{coer}^ρ . Let us prove that the form \mathcal{A}_{coer}^ρ is coercive under some suitable conditions.

Theorem 4.3: Assume that

$$\frac{\alpha_1^{min}}{\alpha_2^+} > 2c^2 \quad (19)$$

holds, with c defined by (10). Then, for $\rho \geq 1/\alpha_2^+$, the form \mathcal{A}_{coer}^ρ is coercive over $\{\mathcal{X} \times \mathbf{X}_2\}^2$.

Proof: Let us first compute the value of $\mathcal{A}_{coer}^\rho(V, V)$,

$$\begin{aligned} \mathcal{A}_{coer}^\rho(V, V) &= \rho \|\operatorname{div} \mathbf{v}_2\|_{0,2}^2 + \left\| \mathbf{curl} \frac{\mathbf{v}_2}{|\alpha_2|} \right\|_{0,2}^2 + \left(\mathbf{v}_2, \frac{\mathbf{v}_2}{|\alpha_2|} \right)_{0,2} + (|\alpha_2| \nabla v_2, \nabla v_2)_{0,2} \\ &\quad + \frac{1}{\rho} \|v_2\|_{0,2}^2 + (\alpha_1 \nabla v_1, \nabla v_1)_{0,1} + \alpha_1^{\min} (v_1, v_1)_{0,1} + 3 \langle \mathbf{v}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma. \end{aligned}$$

Thus, introducing the real parameter $\eta \in [0, 3]$, $|\mathcal{A}_{coer}^\rho(V, V)|$ may be bounded from below by

$$\begin{aligned} \mathcal{A}_{coer}^\rho(V, V) &\geq \rho \|\operatorname{div} \mathbf{v}_2\|_{0,2}^2 + \left\| \mathbf{curl} \frac{\mathbf{v}_2}{|\alpha_2|} \right\|_{0,2}^2 + \left(\mathbf{v}_2, \frac{\mathbf{v}_2}{|\alpha_2|} \right)_{0,2} + (|\alpha_2| \nabla v_2, \nabla v_2)_{0,2} \\ &\quad + \frac{1}{\rho} \|v_2\|_{0,2}^2 + (\alpha_1 \nabla v_1, \nabla v_1)_{0,1} + \alpha_1^{\min} (v_1, v_1)_{0,1} \\ &\quad - (3 - \eta) |\langle \mathbf{v}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma| - \eta |\langle \mathbf{v}_2 \cdot \mathbf{n}_2, v_2 \rangle_\Sigma|. \end{aligned}$$

The term $|\langle \mathbf{v}_2 \cdot \mathbf{n}_1, v_1 \rangle_\Sigma|$ is bounded as in (10), whereas $|\langle \mathbf{v}_2 \cdot \mathbf{n}_2, v_2 \rangle_\Sigma|$ is bounded by

$$\begin{aligned} |\langle \mathbf{v}_2 \cdot \mathbf{n}_2, v_2 \rangle_\Sigma| &\leq |(\rho^{1/2} \operatorname{div} \mathbf{v}_2, \rho^{-1/2} v_2)_{0,2}| + |(|\alpha_2|^{1/2} \nabla v_2, |\alpha_2|^{-1/2} \mathbf{v}_2)_{0,2}| \\ &\leq \frac{1}{2} \left[\rho \|\operatorname{div} \mathbf{v}_2\|_{0,2}^2 + \rho^{-1} \|v_2\|_{0,2}^2 \right. \\ &\quad \left. + (|\alpha_2| \nabla v_2, \nabla v_2)_{0,2} + (|\alpha_2|^{-1} \mathbf{v}_2, \mathbf{v}_2)_{0,2} \right]. \end{aligned}$$

To get coercivity, it is advised to restrict η to $[0, 2[$... We deduce

$$\begin{aligned} |A_{coer}^\rho(V, V)| &\geq \alpha_1^{\min} \|v_1\|_{1,1}^2 + (1 - \eta/2) \min \left(\frac{1}{\alpha_2^+}, \rho \right) \|\mathbf{v}_2\|_{\mathbf{H}(\operatorname{div}; \Omega_2)}^2 \\ &\quad - (3 - \eta) c \|\mathbf{v}_2\|_{\mathbf{H}(\operatorname{div}; \Omega_2)} \|v_1\|_{1,1} \\ &\quad + \left\| \mathbf{curl} \frac{\mathbf{v}_2}{|\alpha_2|} \right\|_{0,2}^2 + (1 - \eta/2) \left[\|\alpha_2\|^{1/2} \nabla v_2\|_{0,2}^2 + \frac{1}{\rho} \|v_2\|_{0,2}^2 \right]. \end{aligned} \tag{20}$$

Since $\rho \geq 1/\alpha_2^+$, one has actually $\min(1/\alpha_2^+, \rho) = 1/\alpha_2^+$. In equation (12) let us identify $x = \|v_1\|_{1,1}$, $y = \|\mathbf{v}_2\|_{\mathbf{X}_2}$ and set

$$m := \alpha_1^{\min} \alpha_2^+ \frac{2}{2 - \eta}, \quad p := c \alpha_2^+ \frac{3 - \eta}{2 - \eta}.$$

The form A_{coer}^ρ is coercive when $m > p^2$, i. e.

$$\frac{\alpha_1^{\min}}{\alpha_2^+} > c^2 \frac{(3 - \eta)^2}{4 - 2\eta}. \tag{21}$$

Now, $f : \eta \mapsto (3 - \eta)^2 / (4 - 2\eta)$ takes its minimal value over $]0, 2[$ at $\eta = 1$, and $f(1) = 2$. For this optimal value, (21) reduces to (19). □

Remark 4.3: If we consider ρ in $]0, 1/\alpha_2^+[$, the right-hand side of (19) changes to $2c^2/(\alpha_2^+\rho)$.

Corollary 4.3: Assume that (19) holds. Then, the variational formulation (16) fits into the coercive plus compact framework, for $\rho \geq 1/\alpha_2^+$.

In the subdomain Ω_2 , the numerical approximation is *overdetermined*, in the sense that both the scalar field u_2 and the vector field \mathbf{u}_2 are computed.

If one goes back to the original dielectric and magnetic parameters, (19) corresponds precisely to

$$\frac{\epsilon_2^-}{\epsilon_1^{max}} > 2c^2. \quad (22)$$

The model problem thus fits into the coercive plus compact framework in the case of small contrasts. To derive a similar result in the case of large contrasts one simply builds a three-field formulation by choosing $\mathbf{u}_1 := \alpha_1 \nabla u_1$ and using vector test functions in \mathbf{X}_1 .

5 Conclusion

In this paper we focused on solving a scalar wave transmission problem between media with opposite sign dielectric and/or magnetic constants. For this, we derived two- and three-field variational formulations. The following table summarizes, for all possible transitions between the two media, which "simplest" formulation can be chosen for solving this problem. Below, N. F., 2 F. and 3 F. denote respectively the natural, the two- and the three-field variational formulations.

| | $\epsilon_2 < 0$ $\mu_2 < 0$ | $\epsilon_2 < 0$ $\mu_2 > 0$ | $\epsilon_2 > 0$ $\mu_2 < 0$ | $\epsilon_2 > 0$ $\mu_2 > 0$ |
|-----------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\epsilon_1 < 0; \mu_1 < 0$ | N.F. | N.F. | 2 F. | 3 F. |
| $\epsilon_1 < 0; \mu_1 > 0$ | N.F. | N.F. | 2 F. ^(*) | 2 F. |
| $\epsilon_1 > 0; \mu_1 < 0$ | 2 F. | 2 F. ^(*) | N.F. | N.F. |
| $\epsilon_1 > 0; \mu_1 > 0$ | 3 F. | 2 F. | N.F. | N.F. |

Nota Bene: As we saw in paragraph 3.1, the two-field formulation is valid when at least over one of two subdomains Ω_1, Ω_2 we have $\epsilon\mu < 0$. In the cases ^(*) both

$\epsilon_1\mu_1$ and $\epsilon_2\mu_2$ are negative: we can build the two-field formulation by arbitrarily choosing where to introduce the vector unknown.

One possible continuation of the present work is to deal with the numerical implementation of the formulations and their comparison. Also of interest is to try and replace the volume vector unknown by an interface unknown in the three-field formulation, and to derive a suitable Domain Decomposition Method to solve the original scalar problem. Finally, one can try and extend the approach followed here to the static and/or harmonic Maxwell equations.

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