Available online at www.sciencedirect.com
SCIENCE DIRECT®
Comput. Methods Appl. Mech. Engrg. 194 (2005) 559–586
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Augmented formulations for solving Maxwell equations

Patrick Ciarlet Jr. *

ENSTA & CNRS UMR 2706 32, Boulevard Victor, 75739 Paris Cedex 15, France

Received 18 May 2004; accepted 18 May 2004

Abstract

We consider augmented variational formulations for solving the static or time-harmonic Maxwell equations. For that, a term is added to the usual **H**(**curl**) conforming formulations. It consists of a (weighted) L^2 scalar product between the divergence of the EM and the divergence of test fields. In this respect, the methods we present are **H**(**curl**, div) conforming. We also build mixed, augmented variational formulations, with either one or two Lagrange multipliers, to dualize the equation on the divergence and, when applicable, the relation on the tangential or normal trace of the field. It is proven that one can derive formulations, which are equivalent to the original static or time-harmonic Maxwell equations. In the latter case, spurious modes are automatically excluded. Numerical analysis and experiments will be presented in the forthcoming paper [Augmented formulations for solving Maxwell equations: numerical analysis and experiments, in preparation].

Keywords: Maxwell equations; Augmented variational formulations; Lagrange multipliers

0. Introduction

In recent years, much attention has been devoted to the computation of electromagnetic fields in bounded *singular* domains, that is with a non-smooth and non-convex boundary. This is in particular true with numerical methods based on finite element techniques. Although the numerical computation is fairly standard in a 3d domain with either a smooth or a convex boundary, i.e. a *regular* domain, a number of problems need to be addressed in the more general case. In particular, it is common knowledge that in a singular domain, there usually exist intense electromagnetic fields near the *geometrical singularities*, that is reentrant corners and/or edges of the boundary.

* Tel.: +33 1 45 52 54 78; fax: +33 1 45 52 52 82.
E-mail address: patrick.ciarlet@ensta.fr

0045-7825/$ - see front matter
doi:10.1016/j.cma.2004.05.021

For instance, with edge finite element methods [28,29], in order to reach an acceptable approximation of the solution, one way is to use suitable mesh refinement techniques near the corners and edges [30]. It is also possible to adapt the mesh and the degree of the local Finite Element, by *hp* techniques [20,31,32]. If one computes instead a continuous approximation of the field based on the P_1 Lagrange FEM (see [7] and Refs. therein), then one can prove that this method, which works well in a regular domain, fails to capture the electromagnetic field even in a 2d singular domain [6,10,5,24,2,3]. It is well-known that this corresponds to a density problem, i.e. that of the subspace of $\mathbf{H}^1$-smooth fields in the space of EM fields (see for instance [10]). In this case, it is advised in the above mentioned Refs to introduce additional basis functions to the usual FE basis functions: the resulting method is called the singular function method, or the singular complement method.

In terms of functional spaces, edge FEM are $\mathbf{H}(\mathbf{curl})$-conforming, whereas the P_1 Lagrange FEM is $\mathbf{H}(\mathbf{curl}, \mathrm{div})$-conforming. In this paper, we study $\mathbf{H}(\mathbf{curl}, \mathrm{div})$-conforming methods for solving Maxwell equations. We investigate three different approaches, from a theoretical point of view:

- With boundary conditions treated as *essential boundary conditions*. This means that they are explicitly included in the definition of the functional spaces. This approach corresponds more or less to the classical one [7].
- With boundary conditions treated as *natural boundary conditions*, in order to overcome the density problem mentioned above. This amounts to solving Maxwell equations in larger functional spaces, in which the subspace of $\mathbf{H}^1$-smooth fields is dense [14,16].
- In weighted Sobolev spaces. By introducing suitable *weights*, which depend on the distance to the singular edges, one can prove a second density result in another class of larger functional spaces [17].

Therefore, from a numerical point of view, the last two approaches do not require a singular complement, which leads to a more straightforward implementation.

We are interested in solving *static-like* Maxwell equations, and the *time-harmonic* Maxwell equations. Also, we consider *augmented*, and *augmented and mixed* Variational Formulations to solve those equations theoretically. The (augmented and) mixed variational formulations, which are introduced here, yield an efficient framework to handle the *time-dependent* Maxwell equations. Numerical analysis, implementation issues and numerical examples will be dealt with in a forthcoming paper [15]. We usually provide detailed proofs for the electric field. For the magnetic field, we only give the proofs, or parts of proof, which cannot be easily inferred from those corresponding to the electric field, or when they lead to different conclusions.

The outline of this paper is as follows. In the next section, we derive the static and time-harmonic models, and we detail the mathematical framework. Then, in Section 2, we validate augmented, and augmented and mixed variational formulations with the boundary conditions treated as essential. This section is split into two subsections: we consider first the solution of the static model, and then the solution of the time-harmonic model. In Section 3, we follow the same framework, with boundary conditions handled as natural boundary conditions. Then, in Section 4, we solve those models with the electric field in a weighted Sobolev space. Finally, we give a few concluding remarks.

Note that, as far as $\mathbf{H}(\mathbf{curl})$-conforming methods are concerned, mixed formulations have already been introduced to solve the static and time-harmonic models (see for instance [25,1]).

1. Derivation of the models and mathematical framework

If we let c, ε_0 and μ_0 be respectively the light velocity, the dielectric permittivity and the magnetic permeability ($\varepsilon_0\mu_0 c^2 = 1$), Maxwell equations in vacuum read

$$\varepsilon_0 \frac{\partial \mathscr{E}}{\partial t} - \mathbf{curl}\,\mathscr{H} = -\mathscr{J}, \tag{1}$$

$$\mu_0 \frac{\partial \mathscr{H}}{\partial t} + \mathbf{curl}\,\mathscr{E} = 0, \tag{2}$$

$$\mathrm{div}\,(\varepsilon_0 \mathscr{E}) = \rho, \tag{3}$$

$$\mathrm{div}\,(\mu_0 \mathscr{H}) = 0, \tag{4}$$

where $\mathscr{E}$ and $\mathscr{H}$ are the electric and magnetic fields, ρ and $\mathscr{J}$ the charge and current densities. These quantities depend on the space variable $\mathbf{x}$ and on the time variable t.

For the moment, we assume that we consider equations (1)–(4) outside a bounded, open perfect conductor $\mathscr{O}$, with a Lipschitz polyhedral boundary $\partial\mathscr{O}$. These equations are then supplemented with the boundary condition

$$\mathscr{E} \times \mathbf{n}_{\mathscr{O}} = 0 \quad \text{on } \partial\mathscr{O} \tag{5}$$

with $\mathbf{n}_{\mathscr{O}}$ a unit outward normal to $\partial\mathscr{O}$. Note that (2) and (5) imply

$$\mu_0 \frac{\partial}{\partial t}(\mathscr{H} \cdot \mathbf{n}_{\mathscr{O}}) = 0 \quad \text{on } \partial\mathscr{O}. \tag{6}$$

The charge conservation equation is a consequence of equations (1) and (3)

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\mathscr{J} = 0. \tag{7}$$

Last, initial conditions are provided (for instance at time $t = 0$)

$$\mathscr{E}(\cdot, 0) = \mathscr{E}_0, \tag{8}$$

$$\mathscr{H}(\cdot, 0) = \mathscr{H}_0, \tag{9}$$

where the couple $(\mathscr{E}_0, \mathscr{H}_0)$ depends only on the variable $\mathbf{x}$. It is convenient to assume that $\mu_0 \mathscr{H}_0 \cdot \mathbf{n}_{\mathscr{O}|\partial\mathscr{O}} = 0$, so that (6) is equivalent to

$$\mu_0 \mathscr{H} \cdot \mathbf{n}_{\mathscr{O}} = 0 \quad \text{on } \partial\mathscr{O}. \tag{10}$$

Then, we *truncate* the domain, to define a *bounded* computational domain. We thus introduce a *bounded*, *open* subset of $\mathbb{R}^3 \setminus \mathscr{O}$, called Ω, with a *Lipschitz polyhedral* boundary $\partial\Omega$. For convenience, we further assume that the domain Ω is *simply connected*, and that its boundary $\partial\Omega$ is *connected*. The goal is to solve Maxwell equations in this domain Ω.

The boundary $\partial\Omega$ is made up of two parts: Γ_{C} and Γ_{A}, with $\bar{\Gamma}_{\mathrm{C}} = \partial\Omega \cap \partial\mathscr{O}$ the *perfect conductor boundary*, and Γ_{A} an *artificial boundary*. On Γ_{C}, conditions (5) and (10) hold, with $\mathbf{n}_{\mathscr{O}}$ replaced by $\mathbf{n}$, the unit outward normal to $\partial\Omega$. We further split the artificial boundary Γ_{A} into $\Gamma_{\mathrm{A}}^{\mathrm{i}}$ and $\Gamma_{\mathrm{A}}^{\mathrm{a}}$. On $\Gamma_{\mathrm{A}}^{\mathrm{i}}$, we model incoming plane waves, whereas we impose on $\Gamma_{\mathrm{A}}^{\mathrm{a}}$ an absorbing boundary condition. Both conditions can be modelled [7] as a *Silver–Müller* boundary condition on Γ_{A}, which reads

$$\left(\mathscr{E} - \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathscr{H} \times \mathbf{n}\right) \times \mathbf{n} = \vec{\mathbf{e}}^{\star} \times \mathbf{n} \quad \text{on } \Gamma_{\mathrm{A}}. \tag{11}$$

By definition, one has $\vec{\mathbf{e}}^{\star}_{|\Gamma_{\mathrm{A}}^{\mathrm{a}}} = 0$, whereas $\vec{\mathbf{e}}^{\star}_{|\Gamma_{\mathrm{A}}^{\mathrm{i}}}$ is linked to the incoming plane waves. Where $\vec{\mathbf{e}}^{\star} = 0$, condition (11) is actually a first order absorbing boundary condition. *Without loss of generality*, it is possible

- to choose the artificial boundary Γ_A such that it does not intersect any of the *geometrical singularities* of $\partial\mathcal{O}$, i.e. there exists a neighborhood $\mathscr{V}$ of the reentrant corners and/or edges such that $\mathscr{V} \cap \Gamma_A = \emptyset$;
- to split Γ_A into two *smooth* subsets Γ_A^i and Γ_A^a;
- to assume that the incoming wave is smooth.

A study of the existence of the time-dependent EM field, with perfect conductor and absorbing boundary conditions ($\Gamma_A^i = \emptyset$), has been carried out in [8]. It can be generalized to our case ($\vec{\mathbf{e}}^\star \neq 0$, $\bar{\Gamma}_A \cap \bar{\Gamma}_C \neq \emptyset$) with no difficulty.

Remark 1. Note that [1], within the $\mathbf{H}(\mathbf{curl}, \mathrm{div})$ framework, the EM field belongs to $\mathbf{H}(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div})$, outside of $\mathcal{O}$, and it satisfies boundary conditions (5) and (10) on $\partial\mathcal{O}$. Under the above assumption that $\bar{\Gamma}_A$ contains no geometrical singularities, there exists a neighborhood $\mathscr{W}_A$ of Γ_A, such that, with $\mathscr{W} = \mathscr{W}_A \cap (\mathbb{R}^3 \setminus \bar{\mathcal{O}})$, the field actually belongs to $\mathbf{H}^1(\mathscr{W}) \times \mathbf{H}^1(\mathscr{W})$ (cf. [1, Theorems 2.9 and 2.12]).

The above equations, considered in Ω and on its boundary $\partial\Omega$, will be referred to as the *time-dependent Maxwell equations*.

As mentioned in the Introduction, the *static model* is build in order to provide a sound mathematical framework to the *time-dependent Maxwell equations*. So, let us detail the steps, which lead to its definition. We give below a formal presentation for $\mathscr{E}$, which can be justified mathematically, see [9,8]. It is well-known that one can replace Ampère's law (1) by a *second-order* Maxwell equation and an additional initial condition, which read

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} + c^2 \mathbf{curl}\,\mathbf{curl}\,\mathscr{E} = -\frac{1}{\varepsilon_0}\frac{\partial \mathscr{J}}{\partial t}, \tag{12}$$

$$\frac{\partial \mathscr{E}}{\partial t}(0) = \frac{1}{\varepsilon_0}(\mathbf{curl}\,\mathscr{H}_0 - \mathscr{J}(0)). \tag{13}$$

Now, let us consider a test field in

$$\mathscr{T}_E := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega) : \mathbf{v} \times \mathbf{n}_{|\partial\Omega} \in \mathbf{L}^2(\partial\Omega), \quad \mathbf{v} \times \mathbf{n}_{|\Gamma_C} = 0\}.$$

There holds, by integration by parts

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathscr{E}, \mathbf{v})_0 + c^2(\mathbf{curl}\,\mathscr{E}, \mathbf{curl}\ \mathbf{v})_0 - c^2(\mathbf{curl}\,\mathscr{E} \times \mathbf{n}, \mathbf{v}_T)_{0,\partial\Omega} = -\frac{1}{\varepsilon_0}\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{J}, \mathbf{v})_0,$$

where $\mathbf{v}_T$ stands for the tangential trace components, i.e. $\mathbf{v}_T = \mathbf{n} \times (\mathbf{v} \times \mathbf{n})_{|\partial\Omega}$. But, according to Faraday's law (2) and the boundary condition (11) on Γ_A, one finds

$$c^2 \mathbf{curl}\,\mathscr{E} \times \mathbf{n}_{|\Gamma_A} = c\frac{\partial}{\partial t}\vec{\mathbf{e}}_T^\star - c\frac{\partial}{\partial t}\mathscr{E}_T,$$

so that one gets

$$\begin{aligned} &\text{find } \mathscr{E} \in \mathscr{T}_E \\ &\text{s.t. } \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathscr{E}, \mathbf{v})_0 + c^2(\mathbf{curl}\,\mathscr{E}, \mathbf{curl}\,\mathbf{v})_0 + c\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{E}_T, \mathbf{v}_T)_{0,\Gamma_A} \\ &\qquad = -\frac{1}{\varepsilon_0}\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{J}, \mathbf{v})_0 + c\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\mathbf{e}}_T^\star, \mathbf{v}_T)_{0,\Gamma_A} \quad \forall \mathbf{v} \in \mathscr{T}_E. \end{aligned} \tag{14}$$

[1] Sobolev spaces of vector-valued fields are written in boldface.

This suggests strongly that one defines the following *static model* for the electric field: look for a static-like field $\mathscr{E}$, solution to

$$\mathbf{curl}\,\mathscr{E} = \mathbf{f}_E \quad \text{in } \Omega, \tag{15}$$

$$\operatorname{div}\mathscr{E} = g_E \quad \text{in } \Omega, \tag{16}$$

$$\mathscr{E} \times \mathbf{n} = \vec{\mathbf{e}} \times \mathbf{n} \quad \text{on } \partial\Omega. \tag{17}$$

Its a priori regularity is (without forgetting the $\mathbf{H}(\text{div})$-conforming requirement),

$$\mathscr{E} \in \mathscr{T}_E, \quad \operatorname{div}\mathscr{E} \in \mathbf{L}^2(\Omega) \tag{18}$$

with the corresponding assumptions on the data

$$\mathbf{f}_E \in \mathbf{L}^2(\Omega), \quad \operatorname{div}\mathbf{f}_E = 0, \quad g_E \in L^2(\Omega), \quad \vec{\mathbf{e}} \in \mathbf{H}^{1/2}(\partial\Omega). \tag{19}$$

NB. The extra regularity on $\vec{\mathbf{e}}$ is a consequence of Remark 1.

And, finally, $\mathbf{f}$ and $\vec{\mathbf{e}}$ need to fufill a compatibility condition, which reads, according to [11, p. 23]

$$\mathbf{f}_E \cdot \mathbf{n}_{|\partial\Omega} = \operatorname{div}_\Gamma(\vec{\mathbf{e}} \times \mathbf{n}). \tag{20}$$

One can proceed similarly for $\mathscr{H}$: replace Faraday's law by a *second-order* Maxwell equation, and additional boundary and initial conditions, which read

$$\frac{\partial^2 \mathscr{H}}{\partial t^2} + c^2\,\mathbf{curl}(\mathbf{curl}\,\mathscr{H} - \mathscr{J}) = 0, \tag{21}$$

$$\frac{1}{\varepsilon_0}(\mathbf{curl}\,\mathscr{H} - \mathscr{J}) \times \mathbf{n}_{|\Gamma_C} = 0, \tag{22}$$

$$\frac{\partial \mathscr{H}}{\partial t}(0) = -\frac{1}{\mu_0}\mathbf{curl}\,\mathscr{E}_0. \tag{23}$$

Now, let us consider a test field in

$$\mathscr{T}_H := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega) : \mathbf{v} \times \mathbf{n}_{|\partial\Omega} \in \mathbf{L}^2(\partial\Omega)\}.$$

There holds, by integration by parts

$$\frac{d^2}{dt^2}(\mathscr{H}, \mathbf{v})_0 + c^2(\mathbf{curl}\,\mathscr{H} - \mathscr{J}, \mathbf{curl}\,\mathbf{v})_0 - c^2((\mathbf{curl}\,\mathscr{H} - \mathscr{J}) \times \mathbf{n}, \mathbf{v}_T)_{0,\partial\Omega} = 0.$$

According to Ampère's law and the boundary condition (11) on Γ_A, one finds

$$c^2(\mathbf{curl}\,\mathscr{H} - \mathscr{J}) \times \mathbf{n}_{|\Gamma_A} = \frac{1}{\mu_0}\frac{\partial}{\partial t}(\vec{\mathbf{e}}^\star \times \mathbf{n}) - c\frac{\partial}{\partial t}\mathscr{H}_T,$$

so that one gets

$$\begin{aligned} &\text{find } \mathscr{H} \in \mathscr{T}_H \\ &\text{s.t. } \frac{d^2}{dt^2}(\mathscr{H}, \mathbf{v})_0 + c^2(\mathbf{curl}\,\mathscr{H}, \mathbf{curl}\,\mathbf{v})_0 + c\frac{d}{dt}(\mathscr{H}_T, \mathbf{v}_T)_{0,\Gamma_A} \\ &\quad = c^2(\mathscr{J}, \mathbf{curl}\,\mathbf{v})_0 + \frac{1}{\mu_0}\frac{d}{dt}(\vec{\mathbf{e}}^\star \times \mathbf{n}, \mathbf{v}_T)_{0,\Gamma_A} \quad \forall \mathbf{v} \in \mathscr{T}_H. \end{aligned} \tag{24}$$

This suggests that one solves a *mixed* problem, imposing $\mathscr{H} \cdot \mathbf{n}$ on Γ_C and $\mathscr{H} \times \mathbf{n}$ on Γ_A. This is actually what one has to do to solve the time-dependent Maxwell equations. However, we are mainly interested here in resolving problems related to geometrical singularities and the corresponding (numerical) approximation of intense fields. Thanks to Remark 1, the magnetic field is $\mathbf{H}^1$-smooth in a neighborhood of Γ_A, which means it can only be intense in a neighborhood of, and on, Γ_C. This is the reason why we impose a *single* boundary condition on the whole of $\partial\Omega$, the one on Γ_C. We thus obtain the following *static model* for the magnetic field: look for a static-like field $\mathscr{H}$, solution to

$$\mathbf{curl}\,\mathscr{H} = \mathbf{f}_H \quad \text{in } \Omega, \tag{25}$$

$$\operatorname{div}\mathscr{H} = g_H \quad \text{in } \Omega, \tag{26}$$

$$\mathscr{H}\cdot\mathbf{n} = h \quad \text{on } \partial\Omega. \tag{27}$$

Its a priori regularity is

$$\mathscr{H} \in \mathscr{T}_H, \quad \operatorname{div}\mathscr{H} \in \mathbf{L}^2(\Omega) \tag{28}$$

with the corresponding assumptions on the data

$$\mathbf{f}_H \in \mathbf{L}^2(\Omega), \quad \operatorname{div}\mathbf{f}_H = 0, \quad g_H \in L^2(\Omega), \quad h \in L^2(\partial\Omega) \tag{29}$$

and a compatibility condition, which reads

$$(g_H, 1)_0 = (h, 1)_{0,\partial\Omega}. \tag{30}$$

Remark 2. Notice that both *static models* include the usual electrostatic and magnetostatic equations.

To introduce the *time-harmonic Maxwell equations*, let us consider the case of a resonator cavity Ω, bounded by a perfect conductor. As before, Ω is a bounded and simply connected open subset of $\mathbb{R}^3$, with a connected Lipschitz polyhedral boundary $\partial\Omega$. The goal is to solve a source problem or to model eigenmodes of electromagnetic oscillations. In these cases, the solutions to and data of system (1)–(4) are harmonic functions of time. Given $\omega \in \mathbb{R}$, one has relations such as $\mathscr{E}(\mathbf{x}, t) = \operatorname{Re}(\mathbf{e}(\mathbf{x})\exp\imath\omega t)$ and $\mathscr{H}(\mathbf{x}, t) = \operatorname{Re}(\mathbf{h}(\mathbf{x})\exp\imath\omega t)$, where the fields $\mathbf{e}$ and $\mathbf{h}$ belong to $\mathbb{C}^3$. The same holds for the current density ($\mathbf{j} \in \mathbb{C}^3$) and the charge density ($r \in \mathbb{C}$). The equations are

$$\imath\omega\varepsilon_0\mathbf{e} - \mathbf{curl}\,\mathbf{h} = -\mathbf{j} \quad \text{in } \Omega, \tag{31}$$

$$\imath\omega\mu_0\mathbf{h} + \mathbf{curl}\,\mathbf{e} = 0 \quad \text{in } \Omega, \tag{32}$$

$$\operatorname{div}(\varepsilon_0\mathbf{e}) = r \quad \text{in } \Omega, \tag{33}$$

$$\operatorname{div}(\mu_0\mathbf{h}) = 0 \quad \text{in } \Omega, \tag{34}$$

$$\mathbf{e}\times\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{35}$$

$$(\mu_0\mathbf{h})\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{36}$$

The charge conservation reads

$$\imath\omega r + \operatorname{div}\mathbf{j} = 0. \tag{37}$$

It is fairly straightforward to replace (31) and (32) by

$$\omega^2\varepsilon_0\mathbf{e} - \mathbf{curl}(\mu_0^{-1}(\mathbf{curl}\,\mathbf{e})) = \imath\omega\mathbf{j} \quad \text{in } \Omega, \tag{38}$$

$$\omega^2\mu_0\mathbf{h} - \mathbf{curl}(\varepsilon_0^{-1}(\mathbf{curl}\,\mathbf{h} - \mathbf{j})) = 0 \quad \text{in } \Omega, \tag{39}$$

$$\varepsilon_0^{-1}(\mathbf{curl}\,\mathbf{h} - \mathbf{j}) \times \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{40}$$

NB. The problems in **e** and **h** are *coupled*, through the data **j**.

In addition, if one solves

$$\begin{aligned} &\text{find } \phi_e \in H_0^1(\Omega) \\ &\text{s.t.} \quad -\operatorname{div}(\varepsilon_0 \nabla\phi_e) = r, \end{aligned} \tag{41}$$

one can replace $\mathbf{j}$ by $\mathbf{j} - \imath\omega\varepsilon_0\nabla\phi_e$, r by 0, and $\mathbf{e}$ by $\mathbf{e} + \nabla\phi_e$.

To summarize, in the time-harmonic case, we solve a source problem in $\mathbf{e}'$

$$\omega^2\varepsilon_0\mathbf{e}' - \mathbf{curl}(\mu_0^{-1}(\mathbf{curl}\,\mathbf{e}')) = \mathbf{j}' \quad \text{in } \Omega, \tag{42}$$

$$\operatorname{div}(\varepsilon_0\mathbf{e}') = 0 \quad \text{in } \Omega, \tag{43}$$

$$\mathbf{e}' \times \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{44}$$

where we set $\mathbf{j}' = \imath\omega\mathbf{j} + \omega^2\varepsilon_0\nabla\phi_e$ and $\mathbf{e}' = \mathbf{e} + \nabla\phi_e$.

We can also solve a source problem in **h**

$$\omega^2\mu_0\mathbf{h} - \mathbf{curl}(\varepsilon_0^{-1}(\mathbf{curl}\,\mathbf{h} - \mathbf{j})) = 0 \quad \text{in } \Omega, \tag{45}$$

$$\operatorname{div}(\mu_0\mathbf{h}) = 0 \quad \text{in } \Omega, \tag{46}$$

$$(\mu_0\mathbf{h}) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{47}$$

$$\varepsilon_0^{-1}(\mathbf{curl}\,\mathbf{h} - \mathbf{j}) \times \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{48}$$

We have some a priori regularities on the solution and on the data

$$\mathbf{e}' \in \mathbf{H}(\mathbf{curl}, \Omega), \quad \mathbf{h} \in \mathbf{H}(\mathbf{curl}, \Omega), \tag{49}$$

$$\mathbf{j}' \in \mathbf{L}^2(\Omega), \quad \operatorname{div}\mathbf{j}' = 0, \quad \mathbf{j} \in \mathbf{L}^2(\Omega), \quad \operatorname{div}\mathbf{j} = 0. \tag{50}$$

Following for instance [25], we note that Eqs. (31)–(34) and (37) naturally split into decoupled problems (real and imaginary parts): one in $(\Re(\mathbf{e}), \Im(\mathbf{h}), \Im(\mathbf{j}), \Re(r))$, and the other in $(\Im(\mathbf{e}), \Re(\mathbf{h}), \Re(\mathbf{j}), \Im(r))$. Furthermore, the boundary conditions (35) and (36) do not yield any coupling between those two quadruples. For these reasons, we shall consider from now on *real* valued data and electromagnetic fields.

In the following Sections, we set ε_0, μ_0 (and c^2) to one.

We now introduce the mathematical framework. First, let us define five Sobolev spaces:

$$\begin{aligned} &\mathbf{L}_t^2(\partial\Omega) = \{\mathbf{z} : \mathbf{z} \in \mathbf{L}^2(\partial\Omega), \quad \mathbf{z}\cdot\mathbf{n} = 0 \text{ a.e.}\}; \\ &\mathscr{X}_{\mathrm{E}} = \{\mathbf{v} : \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega), \quad \mathbf{v}\times\mathbf{n}_{|\partial\Omega} \in \mathbf{L}_t^2(\partial\Omega)\}, \\ &\mathscr{X}_{\mathrm{E}}^0 = \{\mathbf{v} : \mathbf{v} \in \mathscr{X}_{\mathrm{E}}, \quad \mathbf{v}\times\mathbf{n}_{|\partial\Omega} = 0\}; \\ &\mathscr{X}_{\mathrm{H}} = \{\mathbf{v} : \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega), \quad \mathbf{v}\cdot\mathbf{n}_{|\partial\Omega} \in L^2(\partial\Omega)\}, \\ &\mathscr{X}_{\mathrm{H}}^0 = \{\mathbf{v} : \mathbf{v} \in \mathscr{X}_{\mathrm{H}}, \quad \mathbf{v}\cdot\mathbf{n}_{|\partial\Omega} = 0\}. \end{aligned}$$

Thanks to our assumptions on the data (19) and (29), the static-like and time-harmonic electromagnetic fields' a priori regularities are

$$(\mathscr{E}, \mathscr{H}) \in \mathscr{X}_{\mathrm{E}} \times \mathscr{X}_{\mathrm{H}}, \quad (\mathbf{e}, \mathbf{h}) \in \mathscr{X}_{\mathrm{E}}^0 \times \mathscr{X}_{\mathrm{H}}^0. \tag{51}$$

Recall that the domain Ω is a bounded and simply connected open subset of $\mathbb{R}^3$, with a connected Lipschitz polyhedral boundary $\partial\Omega$. Then, according to [33,21,1], the following results hold

Theorem 3. *In* $\mathscr{X}_E^0$ *and* $\mathscr{X}_H^0$,

$$\|\mathbf{v}\|_{X^0} = \{\|\mathbf{curl}\,\mathbf{v}\|_0^2 + \|\mathrm{div}\,\mathbf{v}\|_0^2\}^{1/2}$$

is a norm, which is equivalent to the full $\mathbf{H}(\mathbf{curl}, div, \Omega)$ *norm.*

In $\mathscr{X}_E$,

$$\|\mathbf{v}\|_{X_E} = \{\|\mathbf{curl}\,\mathbf{v}\|_0^2 + \|\mathrm{div}\,\mathbf{v}\|_0^2 + \|\mathbf{v}\times\mathbf{n}\|_{0,\partial\Omega}^2\}^{1/2}$$

is a norm, which is equivalent to the full $\mathbf{H}(\mathbf{curl}, div, \Omega)$ *norm, plus* $\mathbf{L}^2(\partial\Omega)$ *norm of the tangential trace.*

In $\mathscr{X}_H$,

$$\|\mathbf{v}\|_{X_H} = \{\|\mathbf{curl}\,\mathbf{v}\|_0^2 + \|\mathrm{div}\,\mathbf{v}\|_0^2 + \|\mathbf{v}\cdot\mathbf{n}\|_{0,\partial\Omega}^2\}^{1/2}$$

is a norm, which is equivalent to the full $\mathbf{H}(\mathbf{curl}, div, \Omega)$ *norm, plus* $L^2(\partial\Omega)$ *norm of the normal trace.*

Note that $\|\mathbf{v}\times\mathbf{n}\|_{0,\partial\Omega}$ can be replaced by $\|\mathbf{v}_T\|_{0,\partial\Omega}$ in $\|\mathbf{v}\|_{X_E}$.

We denote by $(\cdot,\cdot)_{X^0}, (\cdot,\cdot)_{X_E}$, and $(\cdot,\cdot)_{X_H}$ the associated scalar products.

Remark 4. If the boundary is not connected, there appears an additional term (see for instance [19]) in the norm in $\mathscr{X}_E^0$: from an electrostatic point of view, it corresponds to the constant value of the electrostatic potential, at the surface of each perfect conductor. When the domain is not simply connected, there appears an additional term in the norm in $\mathscr{X}_H^0$ (see [1,22]). For the additional terms in the norms of $\mathscr{X}_E$ and $\mathscr{X}_H$, we refer the reader to [21].

2. Variational formulations with essential boundary conditions

In this section, we focus on Variational Formulations, with spaces including the boundary condition. More precisely, the field is explicitly split into two parts: the first one corresponds to the *incoming part* (cf. the comment after formula (11)), so it is known, and the second one belongs to $\mathscr{X}_{E,H}^0$. In the PDE vocabulary, this means that the boundary conditions are treated as *essential boundary conditions*. This framework was introduced in [7].

2.1. The static equations

Since $\vec{\mathbf{e}}$ belongs to $\mathbf{H}^{1/2}(\partial\Omega)$, there exists $\tilde{\mathbf{e}}$ in $\mathbf{H}^1(\Omega)^3$ such that $\tilde{\mathbf{e}}_{|\partial\Omega} = \vec{\mathbf{e}}$. We then define $\mathbf{f}_E^0 = \mathbf{f}_E - \mathbf{curl}\,\tilde{\mathbf{e}}$ and $g_E^0 = g_E - \mathrm{div}\,\tilde{\mathbf{e}}$. We thus look for $\mathscr{E}^0 = \mathscr{E} - \tilde{\mathbf{e}}$, which satisfies

$$\mathscr{E}^0 \in \mathscr{X}_E^0, \quad \mathbf{curl}\,\mathscr{E}^0 = \mathbf{f}_E^0, \quad \mathrm{div}\,\mathscr{E}^0 = g_E^0. \tag{52}$$

In order to illustrate our framework, based on *variational formulations*, we introduce our first problem (P_E^0)

$$\text{find } \mathscr{E}^0 \in \mathscr{X}_E^0$$
$$\text{s.t. } (\mathscr{E}^0, \mathbf{v})_{X^0} = (\mathbf{f}_E^0, \mathbf{curl}\,\mathbf{v})_0 + (g_E^0, \mathrm{div}\,\mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}_E^0. \tag{53}$$

With respect to (14), the variational formulation (53) is called an *augmented* variational formulation (or AVF later on), since the $(\mathrm{div}\,\cdot, \mathrm{div}\,\cdot)_0$ product appears in addition to $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_0$. According to the Riesz theorem, it is clear that there exists one, and only one, solution to problem ($P^0_{\rm E}$), which is continuous with respect to the data $(\mathbf{f}^0_{\rm E}, g^0_{\rm E})$ in $\mathbf{L}^2(\Omega)\times L^2(\Omega)$.

There holds

Theorem 5. *The field $\mathscr{E}^0$ satisfies (52) iff it is a solution to problem ($P^0_{\rm E}$).*

Proof. It is clear that if $\mathscr{E}^0$ is a solution to (52), it also solves (53).

Let us consider the reciprocal assertion: let $\mathscr{E}^0$ be the solution to (53). Given $g\in L^2(\Omega)$, $\exists!\phi\in H^1_0(\Omega)$ s.t. $\Delta\phi = g$. As $\mathbf{v}=\nabla\phi\in\mathscr{X}^0_{\rm E}$, there holds $({\rm div}\,\mathscr{E}^0, g)_0 = (g^0_{\rm E}, g)_0$; ${\rm div}\,\mathscr{E}^0 = g^0_{\rm E}$ follows.

By construction, ${\rm div}\,\mathbf{f}^0_{\rm E}=0$, and, moreover, thanks to (20)

$$\mathbf{f}^0_{\rm E}\cdot\mathbf{n}_{|\partial\Omega} = \mathbf{f}_{\rm E}\cdot\mathbf{n}_{|\partial\Omega} - \mathbf{curl}\,\tilde{\mathbf{e}}\cdot\mathbf{n}_{|\partial\Omega} = \mathbf{f}_{\rm E}\cdot\mathbf{n}_{|\partial\Omega} - {\rm div}_\Gamma(\vec{\mathbf{e}}\times\mathbf{n}) = 0.$$

According to [23, Theorem 3.6, p. 48], $\exists!\mathbf{w}^0\in\mathscr{X}^0_{\rm E}$ s.t. ${\rm div}\,\mathbf{w}^0=0$, and $\mathbf{curl}\,\mathbf{w}^0=\mathbf{f}^0_{\rm E}$. As $\mathbf{v}=\mathscr{E}^0-\mathbf{w}^0$ belongs to $\mathscr{X}^0_{\rm E}$, (53) yields $\|\mathbf{curl}(\mathscr{E}^0-\mathbf{w}^0)\|_0^2=0$, or $\mathbf{curl}\,\mathscr{E}^0=\mathbf{f}^0_{\rm E}$. □

This allows to prove

Theorem 6. *There exists one, and only one, solution $\mathscr{E}$ to (15)–(17) in $\mathscr{X}_{\rm E}$.*

Proof. Taking $\mathscr{E}=\mathscr{E}^0+\tilde{\mathbf{e}}$ yields existence.

Uniqueness follows from the fact that if $\mathscr{E}$ and $\mathscr{E}'$ are two solutions to (15)–(17), their difference satisfies (53) with vanishing data, i.e. it also vanishes. □

Evidently, the solution $\mathscr{E}$ depends continuously on the data $(\mathbf{f}_{\rm E}, g_{\rm E}, \mathbf{e})$ in $\mathbf{L}^2(\Omega)\times L^2(\Omega)\times\mathbf{H}^{1/2}(\partial\Omega)$.

For the problem (25)–(27) in $\mathscr{H}$, one uses the same procedure. Indeed, one solves

$$\text{find } \psi\in H^1(\Omega)\cap L^2_0(\Omega)$$
$$\text{s.t. } \Delta\psi = \frac{1}{|\,\Omega\,|}(g_{\rm H},1)_0,\quad \frac{\partial\psi}{\partial n}_{|\partial\Omega} = h.$$

NB. It is a Neumann problem, and the data fulfills the usual compatibility condition, thanks to (30).

Then, $\tilde{\mathbf{h}}=\mathbf{grad}\psi$ is an admissible lifting, together with $\mathbf{f}^0_{\rm H}=\mathbf{f}_{\rm H}$ and $g^0_{\rm H}=g_{\rm H}-(g_{\rm H},1)_0/\,|\,\Omega\,|$. We now look for $\mathscr{H}^0=\mathscr{H}-\tilde{\mathbf{h}}$, solution to

$$\mathscr{H}^0\in\mathscr{X}^0_{\rm H},\quad \mathbf{curl}\,\mathscr{H}^0=\mathbf{f}^0_{\rm H},\quad {\rm div}\,\mathscr{H}^0=g^0_{\rm H}. \tag{54}$$

We then define the problem ($P^0_{\rm H}$)

$$\text{find } \mathscr{H}^0\in\mathscr{X}^0_{\rm H}$$
$$\text{s.t. } (\mathscr{H}^0,\mathbf{v})_{X^0} = (\mathbf{f}^0_{\rm H},\mathbf{curl}\,\mathbf{v})_0 + (g^0_{\rm H},{\rm div}\,\mathbf{v})_0\quad \forall\mathbf{v}\in\mathscr{X}^0_{\rm H}, \tag{55}$$

our second so-called AVF (augmented w.r.t. (24) this time). According to the Riesz theorem, this formulation admits one, and only one, solution, continuous w.r.t. $\mathbf{f}^0_{\rm H}$ and $g^0_{\rm H}$. There follows:

Theorem 7. *The field $\mathscr{H}^0$ satisfies (54) iff it is a solution to problem ($P^0_{\rm H}$).*

Proof. If $\mathscr{H}^0$ is a solution to (54), it also solves (55).

Now, let $\mathscr{H}^0$ be the solution to (55). Given $g\in L^2_0(\Omega)$, $\exists!\phi\in H^1(\Omega)\cap L^2_0(\Omega)$ s.t. $\Delta\phi=g$ and $\partial_n\phi_{|\partial\Omega}=0$. As $\mathbf{v}=\nabla\phi\in\mathscr{X}^0_{\rm H}$, there holds $({\rm div}\,\mathscr{H}^0, g)_0=(g^0_{\rm H}, g)_0$. Since ${\rm div}\,\mathscr{H}^0-g^0_{\rm H}$ belongs to $L^2_0(\Omega)$, ${\rm div}\,\mathscr{H}^0=g^0_{\rm H}$ follows.

By construction, ${\rm div}\,\mathbf{f}^0_{\rm H}=0$, so, according to [23, Theorem 3.5 p. 47], $\exists\mathbf{w}^0\in\mathscr{X}^0_{\rm H}$ s.t. $\mathbf{curl}\,\mathbf{w}^0=\mathbf{f}^0_{\rm H}$. Putting $\mathbf{v}=\mathscr{H}^0-\mathbf{w}^0$ in (53) yields $\|\mathbf{curl}\,\mathscr{H}^0-\mathbf{f}^0_{\rm H}\|_0^2=0$. □

And, with a proof similar to the one given in the electric case,

Theorem 8. *There exists one, and only one, solution* $\mathscr{H}$ *to* (25)–(27) *in* $\mathscr{X}_{\mathrm{H}}$.

Now that existence, uniqueness (and continuous dependence w.r.t. the data) is proven for the static-like fields, let us investigate a dualization of the divergence of the fields. In other words, what happens if one considers (16) and (26) as *constraints*? Note that, in order to solve the time-dependent Maxwell equations, it is important to enforce those relationships, so that one avoids a drift as one iterates in time, if for instance the charge conservation equation is not enforced numerically. This is the approach originally investigated by Assous et al. in [7].

So, let us define a new problem (Q^0_{E})

$$\text{find } (\mathscr{E}^0, p) \in \mathscr{X}^0_{\mathrm{E}} \times L^2(\Omega)$$

$$\text{s.t. } (\mathscr{E}^0, \mathbf{v})_{X^0} + (p, \operatorname{div}\mathbf{v})_0 = (\mathbf{f}^0_{\mathrm{E}}, \mathbf{curl}\,\mathbf{v})_0 + (g^0_{\mathrm{E}}, \operatorname{div}\mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}^0_{\mathrm{E}}, \tag{56}$$

$$(\operatorname{div}\mathscr{E}^0, q)_0 = (g^0_{\mathrm{E}}, q)_0 \quad \forall q \in L^2(\Omega). \tag{57}$$

This is our first *mixed*, *augmented* variational formulation (or MAVF later on). According to Proposition 3.5 of [4], there holds $\operatorname{div}\mathscr{X}^0_{\mathrm{E}} = L^2(\Omega)$. So, a *necessary* condition for (Q^0_{E}) to yield the expected $\mathscr{E}^0$ (i.e. the solution to (P^0_{E}) and (52)), is that p vanishes.

Proposition 9. *In* (Q^0_{E}), *one has* $p = 0$.

Proof. $\exists!\phi_\star \in H^1_0(\Omega)$ s.t. $\Delta\phi_\star = p$. Taking $\mathbf{v}_\star = \nabla\phi_\star$ in (56) yields

$$(\operatorname{div}\mathscr{E}^0, p)_0 + \|p\|^2_0 = (g^0_{\mathrm{E}}, p)_0.$$

Since $(\operatorname{div}\mathscr{E}^0, p)_0 = (g^0_{\mathrm{E}}, p)_0$ according to (57), one has $p = 0$. □

Quoting [20], one calls sometimes p the *dummy variable*.

Remark 10. Notice that one reaches a similar conclusion (in the continuous case), provided the charge conservation equation is true, when one solves the time-dependent Maxwell equations with an MAVF. The well-posedness of such MAVFs is alluded to in Section 5 and Appendix A.

Theorem 11. *Problem* (Q^0_{E}) *admits one, and only one, solution* $(\mathscr{E}^0, p)$. *Moreover*, $\mathscr{E}^0$ *is the solution to problem* (P^0_{E}).

Proof. The existence and uniqueness of the solution to problem (Q^0_{E}) stem from the *Babuska–Brezzi theory* (cf. for instance [23]). The so called V-ellipticity condition is automatic, since the bilinear form involved is the scalar product of $\mathscr{X}^0_{\mathrm{E}}$. The *inf–sup* condition (cf. Appendix A) is proved as follows: $\forall q \in L^2(\Omega)$, $\exists!\phi \in H^1_0(\Omega)$ s.t. $\Delta\phi = q$. As $\mathbf{v} = \nabla\phi$ belongs to $\mathscr{X}^0_{\mathrm{E}}$ with $\|\mathbf{v}\|_{X^0} = \|q\|_0$, there holds $(\operatorname{div}\mathbf{v}, q)_0/\|\mathbf{v}\|_{X^0} = \|q\|_0$, so the *inf–sup* condition follows with a unit constant β. This proves the first point.

To conclude, it is enough to note that problem (Q^0_{E}) reduces to problem (P^0_{E}), since $p = 0$. □

NB. As emphasized in the proof, (Q^0_{E}) satisfies an *inf–sup* condition.

Similarly, we introduce an MAVF for $\mathscr{H}^0$, i.e. a problem (Q^0_{H}),

$$\text{find } (\mathscr{H}^0, p) \in \mathscr{X}^0_{\mathrm{H}} \times L^2_0(\Omega)$$

$$\text{s.t. } (\mathscr{H}^0, \mathbf{v})_{X^0} + (p, \operatorname{div}\mathbf{v})_0 = (\mathbf{f}^0_{\mathrm{H}}, \mathbf{curl}\,\mathbf{v})_0 + (g^0_{\mathrm{H}}, \operatorname{div}\mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}^0_{\mathrm{H}}, \tag{58}$$

$$(\operatorname{div}\mathscr{H}^0, q)_0 = (g^0_{\mathrm{H}}, q)_0 \quad \forall q \in L^2_0(\Omega). \tag{59}$$

We know from [23, Corollary 2.4, p. 24] that $\operatorname{div}\mathbf{H}_0^1(\Omega) = L_0^2(\Omega)$. Since $\mathbf{H}_0^1(\Omega) \subset \mathscr{X}_{\mathrm{H}}^0$ and since $(\operatorname{div}\mathbf{v}, 1)_0 = 0$ for all $\mathbf{v}$ in $\mathscr{X}_{\mathrm{H}}^0$, we derive $\operatorname{div}\mathscr{X}_{\mathrm{H}}^0 = L_0^2(\Omega)$. Again, a *necessary* condition for (Q_{H}^0) to provide the same solution as (P_{H}^0) and (54) is that p vanishes. Following the proof of Proposition 9 (with a Neumann problem in $\phi_\star$).

Proposition 12. *In* (Q_{H}^0), *one has* $p = 0$.

Finally, following the proof of Theorem 11, one easily proves

Theorem 13. *Problem* (Q_{H}^0) *admits one, and only one, solution* $(\mathscr{H}^0, p)$. *Moreover,* $\mathscr{H}^0$ *is the solution to problem* (P_{H}^0).

NB. Problem (Q_{H}^0) satisfies an *inf–sup* condition.

So far, we defined one augmented variational formulation, and one mixed augmented variational formulation, for each of the static-like fields $\mathscr{E}$ and $\mathscr{H}$. More precisely, for the part which belongs to $\mathscr{X}_{\mathrm{E}}^0$ or $\mathscr{X}_{\mathrm{H}}^0$, i.e. $\mathscr{E}^0$ and $\mathscr{H}^0$. We shall see, in the next subsection, that one can try successfully the same mathematical approach, in order to solve the time-harmonic Maxwell equations.

2.2. The time-harmonic equations

In this subsection, we concentrate first on solving the system of equations (42)–(44) in $\mathbf{e}' \in \mathbf{H}(\mathbf{curl}, \Omega)$, given $\mathbf{j}'$ and ω. Then, we derive equivalent variational formulations, set in $\mathscr{X}_{\mathrm{E}}^0$ for the electric field. Let us introduce

$$\mathscr{V}_{\mathrm{E}}^0 := \{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) : \operatorname{div}\mathbf{v} = 0\}.$$

(The set $\mathscr{V}_{\mathrm{E}}^0$ is either a subspace of $\mathbf{H}_0(\mathbf{curl}, \Omega)$ or of $\mathscr{X}_{\mathrm{E}}^0$.)

One can prove easily that the two formulations below are equivalent to the problem (42)–(44). The obvious one,

$$\begin{aligned} &\text{find } \mathbf{e}' \in \mathbf{H}_0(\mathbf{curl}, \Omega) \\ &\text{s.t. } (\mathbf{curl}\,\mathbf{e}', \mathbf{curl}\,\mathbf{v})_0 = \omega^2(\mathbf{e}', \mathbf{v})_0 - (\mathbf{j}', \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega), \qquad (60) \\ &\qquad \operatorname{div}\mathbf{e}' = 0. \qquad (61) \end{aligned}$$

One can also choose to solve an AVF, such as

$$\begin{aligned} &\text{find } \mathbf{e}' \in \mathscr{V}_{\mathrm{E}}^0 \\ &\text{s.t. } (\mathbf{e}', \mathbf{v})_{X^0} = \omega^2(\mathbf{e}', \mathbf{v})_0 - (\mathbf{j}', \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{V}_{\mathrm{E}}^0. \qquad (62) \end{aligned}$$

The reason why a solution $\mathbf{e}' \in \mathscr{V}_{\mathrm{E}}^0$ to (62) solves (60) is simple (why it solves (61) is obvious). Indeed, given $\mathbf{v}$ in $\mathbf{H}_0(\mathbf{curl}, \Omega)$, one considers first $\phi \in H_0^1(\Omega)$ such that $\Delta\phi = \operatorname{div}\mathbf{v}$. Then $\mathbf{v} - \nabla\phi$ belongs to $\mathscr{V}_{\mathrm{E}}^0$, and used as a test function in (62), it yields (60) for $\mathbf{v}$, since

$$(\omega^2\mathbf{e}' - \mathbf{j}', \nabla\phi)_0 = -(\omega^2 \operatorname{div}\mathbf{e}' - \operatorname{div}\mathbf{j}', \phi)_0 = 0.$$

The difference between the two formulations is that $\mathscr{X}_{\mathrm{E}}^0$—and so $\mathscr{V}_{\mathrm{E}}^0$ as a subset of $\mathscr{X}_{\mathrm{E}}^0$—is compactly imbedded into $\mathbf{L}^2(\Omega)$ [33], whereas $\mathbf{H}_0(\mathbf{curl}, \Omega)$ is not. It is therefore possible to use the Fredholm theory to solve problem (62). Indeed, let us define $\eta_{\mathrm{E}} : \mathbf{L}^2(\Omega) \to \mathscr{V}_{\mathrm{E}}^0$ by $(\eta_{\mathrm{E}}\mathbf{f}, \mathbf{v})_{X^0} = (\mathbf{f}, \mathbf{v})_0$ for all $\mathbf{v} \in \mathscr{V}_{\mathrm{E}}^0$, and i_{E} the compact imbedding of $\mathscr{V}_{\mathrm{E}}^0$ into $\mathbf{L}^2(\Omega)$. By construction, $\mathbb{T}_{\mathrm{E}} = \eta_{\mathrm{E}} \circ i_{\mathrm{E}}$ is a (symmetric) compact operator of $\mathscr{V}_{\mathrm{E}}^0$, and (62) now amounts to

$$(\mathbb{I} - \omega^2\mathbb{T}_{\mathrm{E}})\mathbf{e}' = -\eta_{\mathrm{E}}\mathbf{j}' \quad \text{in } \mathscr{V}_{\mathrm{E}}^0.$$

There follows:

Theorem 14. *Assume* $1/\omega^2$ *is not an eigenvalue of* $\mathbb{T}_E$: *problem* (42)–(44) *admits one, and only one, solution. Assume* $1/\omega^2$ *is an eigenvalue of* $\mathbb{T}_E$: *let* $\mathbf{v}_1, \ldots, \mathbf{v}_p$ *denote a basis of* $\ker(\mathbb{I} - \omega^2\mathbb{T}_E)$, *then*

- *either* $\exists l \in \{1, \ldots, p\}$ *st* $(\mathbf{j}, \mathbf{v}_l)_0 \neq 0$: *problem* (42)–(44) *admits no solution*;
- *or* $\forall l \in \{1, \ldots, p\}$, $(\mathbf{j}, \mathbf{v}_l)_0 = 0$: *problem* (42)–(44) *admits an affine space of solutions, in the form* $\mathbf{e}' = \mathbf{e}'_0 + \sum_l \beta_l \mathbf{v}_l$, $(\beta_l)_l \in \mathbb{R}^p$.

Remark 15. The orthogonality condition of the Fredholm theory says that $(\eta_E \mathbf{j}', \mathbf{v}_l)_{X^0} = 0$. Or, according to the definition of operator η_E, $(\mathbf{j}', \mathbf{v}_l)_0 = 0$. Now, since $\mathbf{v} \in \mathscr{V}^0_E$ and $\phi_e \in H^1_0(\Omega)$, there follows automatically $(\nabla\phi_e, \mathbf{v}_l)_0 = 0$ (see (41)), so that $(\mathbf{j}', \mathbf{v}_l)_0 = 0$ corresponds to $(\mathbf{j}, \mathbf{v}_l)_0 = 0$ (it is indeed equivalent, since the orthogonality condition is relevant only when $\omega \neq 0$).

Now that problem (42)–(44) has been solved, we turn to the derivation of equivalent formulations. Let us define our starting point. As a matter of fact, one can consider, in-between (60)–(62), the equivalent formulation, called problem (p^0_E)

$$\text{find } \mathbf{e}' \in \mathscr{X}^0_E$$
$$\text{s.t. } (\mathbf{e}', \mathbf{v})_{X^0} = \omega^2(\mathbf{e}', \mathbf{v})_0 - (\mathbf{j}', \mathbf{v})_0, \quad \forall \mathbf{v} \in \mathscr{X}^0_E, \tag{63}$$
$$\operatorname{div} \mathbf{e}' = 0. \tag{64}$$

Then, we turn to an MAVF, with (64) handled as a constraint, as in the static case: problem (q^0_E),

$$\text{find } \mathbf{e}', p \in \mathscr{X}^0_E \times L^2(\Omega)$$
$$\text{s.t. } (\mathbf{e}', \mathbf{v})_{X^0} + (p, \operatorname{div} \mathbf{v})_0 = \omega^2(\mathbf{e}', \mathbf{v})_0 - (\mathbf{j}', \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}^0_E, \tag{65}$$
$$(\operatorname{div} \mathbf{e}', q)_0 = 0 \quad \forall q \in L^2(\Omega). \tag{66}$$

NB. A similar, well-known, mixed VF, set in $\mathbf{H}_0(\mathbf{curl}, \Omega) \times L^2(\Omega) \times \mathbb{R}$ (with $\mathbf{j}' = 0$), had already been considered to solve the related eigenvalue problem by Kikuchi [25].

There holds

Proposition 16. *In* (q^0_E), *one has* $p = 0$.

Proof. $\exists! \phi_\star \in H^1_0(\Omega)$ s.t. $\Delta\phi_\star = p$. Taking $\mathbf{v}_\star = \nabla\phi_\star$ in (65) yields

$$(\operatorname{div} \mathbf{e}', p)_0 + \|p\|_0^2 = 0.$$

Since $(\operatorname{div} \mathbf{e}', p)_0 = 0$ according to (66), one has $p = 0$. □

According to Theorem 55, thanks to the *inf–sup* condition proved for problem (Q^0_E), one gets finally the

Theorem 17. $(\mathbf{e}', 0)$ *is a solution to problem* (q^0_E) *iff* $\mathbf{e}'$ *is a solution to* (42)–(44).

Second, let us consider the system (45)–(48) in $\mathbf{h} \in \mathbf{H}(\mathbf{curl}, \Omega)$, given $\mathbf{j}$ and ω. We introduce

$$\mathscr{V}^0_H := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega) : \operatorname{div} \mathbf{v} = 0, \ \mathbf{v} \cdot \mathbf{n}_{|\partial\Omega} = 0\}.$$

As in the electric case, it is easily inferred that the two formulations below are equivalent to the original problem (45)–(48)

$$\text{find } \mathbf{h} \in \mathbf{H}(\mathbf{curl}, \Omega)$$
$$\text{s.t. } (\mathbf{curl}\, \mathbf{h}, \mathbf{curl}\, \mathbf{v})_0 = \omega^2(\mathbf{h}, \mathbf{v})_0 + (\mathbf{j}, \mathbf{curl}\, \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega), \tag{67}$$
$$\operatorname{div} \mathbf{h} = 0. \tag{68}$$

Or, one can choose to solve equivalently

$$\text{find } \mathbf{h} \in \mathscr{V}^0_{\mathrm{H}}$$
$$\text{s.t. } (\mathbf{h}, \mathbf{v})_{X^0} = \omega^2(\mathbf{h}, \mathbf{v})_0 + (\mathbf{j}, \mathbf{curl}\,\mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{V}^0_{\mathrm{H}}. \tag{69}$$

Since the imbedding of $\mathscr{X}^0_{\mathrm{H}}$ into $\mathbf{L}^2(\Omega)$ is compact [33], one can utilize the Fredholm theory again. Let $\mathbb{T}_{\mathrm{H}} = \eta_{\mathrm{H}} \circ i_{\mathrm{H}}$, with obvious notations, be the (symmetric) compact operator of $\mathscr{V}^0_{\mathrm{H}}$ into itself. With $\mathbf{j}_{\mathrm{H}} \in \mathscr{V}^0_{\mathrm{H}}$ such that $(\mathbf{j}_{\mathrm{H}}, \mathbf{v})_{X^0} = (\mathbf{j}, \mathbf{curl}\,\mathbf{v})_0$ for all $\mathbf{v} \in \mathscr{V}^0_{\mathrm{H}}$, (69) is equivalent to

$$(\mathbb{I} - \omega^2\mathbb{T}_{\mathrm{H}})\mathbf{h} = \mathbf{j}_{\mathrm{H}} \quad \text{in } \mathscr{V}^0_{\mathrm{H}}.$$

There follows:

Theorem 18. *Assume* $1/\omega^2$ *is not an eigenvalue of* $\mathbb{T}_{\mathrm{H}}$: *problem* (45)–(48) *admits one, and only one, solution. Assume* $1/\omega^2$ *is an eigenvalue of* $\mathbb{T}_{\mathrm{H}}$: *let* $\mathbf{w}_1, \ldots, \mathbf{w}_p$ *denote a basis of* $\ker(\mathbb{I} - \omega^2\mathbb{T}_{\mathrm{H}})$, *then*

- *either* $\exists l \in \{1, \ldots, p\}$ *s.t.* $(\mathbf{j}, \mathbf{curl}\,\mathbf{w}_l)_0 \neq 0$: *problem* (45)–(48) *admits no solution*;
- *or* $\forall l \in \{1, \ldots, p\}$, $(\mathbf{j}, \mathbf{curl}\,\mathbf{w}_l)_0 = 0$: *problem* (45)–(48) *admits an affine space of solutions, in the form* $\mathbf{h} = \mathbf{h}_0 + \sum_l \beta_l \mathbf{w}_l$, $(\beta_l)_l \in \mathbb{R}^p$.

Remark 19. It is well known that the operators $\mathbb{T}_{\mathrm{E}}$ and $\mathbb{T}_{\mathrm{H}}$ have the same eigenvalues (with the same multiplicity). Indeed, one can actually prove that $\mathbf{v}_l$ is an eigenvector of $\mathbb{T}_{\mathrm{E}}$ with eigenvalue λ if, and only if, $\mathbf{w}_l = \mathbf{curl}\mathbf{v}_l$ is an eigenvector of $\mathbb{T}_{\mathrm{H}}$ with eigenvalue λ. Finally, the orthogonality conditions are identical for both the electric and the magnetic problems, so that the Fredholm classifications of the solutions in $\mathbf{e}$ and $\mathbf{h}$ are compatible, as expected.

As far as equivalent formulations of (45)–(48) are concerned, let us begin by the following formulation, called problem (p^0_{H}),

$$\text{find } \mathbf{h} \in \mathscr{X}^0_{\mathrm{H}}$$
$$\text{s.t. } (\mathbf{h}, \mathbf{v})_{X^0} = \omega^2(\mathbf{h}, \mathbf{v})_0 + (\mathbf{j}, \mathbf{curl}\,\mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}^0_{\mathrm{H}}, \tag{70}$$
$$\operatorname{div} \mathbf{h} = 0. \tag{71}$$

With (71) handled as a constraint, we define further the MAVF (q^0_{H}),

$$\text{find } (\mathbf{h}, p) \in \mathscr{X}^0_{\mathrm{H}} \times L^2_0(\Omega)$$
$$\text{s.t. } (\mathbf{h}, \mathbf{v})_{X^0} + (p, \operatorname{div} \mathbf{v})_0 = \omega^2(\mathbf{h}, \mathbf{v})_0 + (\mathbf{j}, \mathbf{curl}\,\mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}^0_{\mathrm{H}}, \tag{72}$$
$$(\operatorname{div} \mathbf{h}, q)_0 = 0 \quad \forall q \in L^2_0(\Omega). \tag{73}$$

There holds

Proposition 20. *In* (q^0_{H}), *one has* $p = 0$.

According to Theorem 55 (*inf–sup* valid for problem (Q^0_{H})), one gets the

Theorem 21. $(\mathbf{h}, 0)$ *is a solution to problem* (q^0_{H}) *iff* $\mathbf{h}$ *is a solution to* (45)–(48).

3. Variational formulations with natural boundary conditions

In this section, we focus on Variational Formulations, with boundary conditions treated as *natural boundary conditions*, i.e. which stem from the formulation. The obvious advantage is computational, since, according to the density of smooth fields in $\mathscr{X}_{\rm E}$ and $\mathscr{X}_{\rm H}$ [14,16], one can (theoretically) use the continuous approximation of the field based on the P_1 Lagrange FEM. Still, one has to validate those formulations, i.e. prove that they are well-posed! This is the topic we address in this section.

3.1. The static equations

As a starting point, we consider the AVFs and MAVFs of Section 2.1. The first difference with these Variational Formulations is that we now treat the boundary conditions as *natural*. This requires that one includes those in a variational form. Consequently, and this is the second difference, we look directly for the *total* field $\mathscr{E}$ or $\mathscr{H}$, i.e. we solve the variational problems in $\mathscr{X}_{\rm E}$ and $\mathscr{X}_{\rm H}$. Let us proceed for $\mathscr{E}$: we introduce the linear form

$$l_{\rm E}(\mathbf{v}) = (\mathbf{f}_{\rm E}, \mathbf{curl}\,\mathbf{v})_0 + (g_{\rm E}, \operatorname{div}\mathbf{v})_0 + (\vec{\mathbf{e}} \times \mathbf{n}, \mathbf{v} \times \mathbf{n})_{0,\partial\Omega}$$

and define the problem $(P_{\rm E})$

$$\text{find } \mathscr{E} \in \mathscr{X}_{\rm E} \quad \text{s.t.} \quad (\mathscr{E}, \mathbf{v})_{X_{\rm E}} = l_{\rm E}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathscr{X}_{\rm E}. \tag{74}$$

With respect to (14), this falls in the AVF category. We infer from the Riesz theorem, that there exists one, and only one, solution to (74), which is continuous w.r.t. the data (cf. (19)). Moreover, one can prove

Theorem 22. *$\mathscr{E}$ is a solution to (15)–(17) iff it solves (P_E).*

Proof. It is clear that if $\mathscr{E}$ is a solution to (15)–(17), it also solves (74).

Let us consider the reciprocal assertion: let $\mathscr{E}$ be the solution to (74). Following the proof of Theorem 5, we find $\operatorname{div}\mathscr{E} = g_{\rm E}$, i.e. (16). Furthermore, according to the same proof, one can write

$$\mathbf{f}_{\rm E} = \mathbf{curl}\,\mathbf{w} \quad \text{with } \mathbf{w} = \mathbf{w}^0 + \tilde{\mathbf{e}}.$$

As $\mathbf{v} = \mathscr{E} - \mathbf{w}$ belongs to $\mathscr{X}_{\rm E}$, with $\mathbf{v} \times \mathbf{n}_{|\partial\Omega} = \mathscr{E} \times \mathbf{n}_{|\partial\Omega} - \vec{\mathbf{e}} \times \mathbf{n}$, (74) yields $\|\mathbf{curl}\,\mathscr{E} - \mathbf{f}_{\rm E}\|_0^2 + \|\mathscr{E} \times \mathbf{n}_{|\partial\Omega} - \vec{\mathbf{e}} \times \mathbf{n}\|_{0,\partial\Omega}^2 = 0$: both (15) and (17) hold. □

By construction, $\mathscr{E}$ depends continuously on the data $(\mathbf{f}_{\rm E}, g_{\rm E}, \vec{\mathbf{e}})$ in $\mathbf{L}^2(\Omega) \times L^2(\Omega) \times \mathbf{L}_t^{1/2}(\partial\Omega)$.

Remark 23. From a numerical point of view, we recall that the difference between $(P_{\rm E})$ and $(P_{\rm E}^0)$ is that $\mathbf{H}^1(\Omega)$ is always *dense* in $\mathscr{X}_{\rm E}$, whereas $\mathbf{H}^1(\Omega) \cap \mathscr{X}_{\rm E}^0$ is *not dense* in $\mathscr{X}_{\rm E}^0$, when Ω is not convex [14,16]. There is no need for a Singular Complement, when one uses a continuous P_1 FE approximation to solve $(P_{\rm E})$ (see [15] for some numerical illustrations).

For the problem in $\mathscr{H}$ (25)–(27), we introduce the linear form

$$l_{\rm H}(\mathbf{v}) = (\mathbf{f}_{\rm H}, \mathbf{curl}\,\mathbf{v})_0 + (g_{\rm H}, \operatorname{div}\mathbf{v})_0 + (h, \mathbf{v} \cdot \mathbf{n})_{0,\partial\Omega},$$

and consider the AVF $(P_{\rm H})$

$$\text{find } \mathscr{H} \in \mathscr{X}_{\rm H} \quad \text{s.t.} \quad (\mathscr{H}, \mathbf{v})_{X_{\rm H}} = l_{\rm H}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathscr{X}_{\rm H}. \tag{75}$$

This is again a formulation, which admits one, and only one, solution, continuous w.r.t. the data (cf. (29)), thanks to the often cited Riesz theorem. In addition, the following theorem can be proven.

Theorem 24. *$\mathscr{H}$ is a solution to (25)–(27) iff it solves (P_H).*

Proof. If $\mathscr{H}$ is a solution to (25)–(27), it also solves (75).

Now, let $\mathscr{H}$ be the solution to (75). We prove in a single shot that both $\operatorname{div}\mathscr{H} = g_{\mathrm{H}}$ and $\mathscr{H}\cdot\mathbf{n}_{|\partial\Omega} = h$ hold, provided the compatibility condition (30) is enforced.

We first deal with *constants*: $\exists!\phi_c \in H^1(\Omega)\cap L_0^2(\Omega)$ s.t. $\Delta\phi_c = |\partial\Omega|$ and $\partial_n\phi_{c|\partial\Omega} = |\Omega|$. Since $\nabla\phi_c \in \mathscr{X}_{\mathrm{H}}$, one finds

$$|\,\partial\Omega\,|\,(\operatorname{div}\mathscr{H},1)_0 + |\,\Omega\,|\,(\mathscr{H}\cdot\mathbf{n},1)_{0,\partial\Omega} = |\,\partial\Omega\,|\,(g_{\mathrm{H}},1)_0 + |\,\Omega\,|\,(h,1)_{0,\partial\Omega}.$$

According to the identity $(\operatorname{div}\mathscr{H},1)_0 = (\mathscr{H}\cdot\mathbf{n},1)_{0,\partial\Omega}$ and to the compatibility condition (30), this yields

$$(\operatorname{div}\mathscr{H},1)_0 = (g_{\mathrm{H}},1)_0. \tag{76}$$

Now, we proceed with the general case. Given $g\in L^2(\Omega)$ and $\mu\in L^2(\partial\Omega)$, let $c\equiv c_{g,\mu}\in\mathbb{R}$ such that $(g+c,1)_0 = (\mu,1)_{0,\partial\Omega}$. Then, $\exists!\phi\in H^1(\Omega)\cap L_0^2(\Omega)$ s.t. $\Delta\phi = (g+c)$ and $\partial_n\phi_{|\partial\Omega} = \mu$. As $\nabla\phi\in\mathscr{X}_{\mathrm{H}}$, there holds, according to (76)

$$(\operatorname{div}\mathscr{H},g)_0 + (\mathscr{H}\cdot\mathbf{n},\mu)_{0,\partial\Omega} = (g_{\mathrm{H}},g)_0 + (h,\mu)_{0,\partial\Omega}.$$

Taking $g = \operatorname{div}\mathscr{H} - g_{\mathrm{H}}$ and $\mu = \mathscr{H}\cdot\mathbf{n}_{|\partial\Omega} - h$ yields $\|\operatorname{div}\mathscr{H} - g_{\mathrm{H}}\|_0^2 + \|\mathscr{H}\cdot\mathbf{n}_{|\partial\Omega} - h\|_{0,\partial\Omega}^2 = 0$, so (26) and (27) hold.

The formulation (P_{H}) reduces to $(\mathbf{curl}\,\mathscr{H},\mathbf{curl}\,\mathbf{v})_0 = (\mathbf{f}_{\mathrm{H}},\mathbf{curl}\,\mathbf{v})_0$, for all $\mathbf{v}$ in $\mathscr{X}_{\mathrm{H}}$. One concludes that (25) is true as in the proof of Theorem 7. □

One notices a difference between the two proofs given for problems (P_{E}) and (P_{H}). For the first one, in $\mathscr{E}$, one deals first with the divergence condition, and then simultaneously with the curl and boundary conditions. For the second one, in $\mathscr{H}$, one deals first with the divergence and boundary conditions with a stone, and then with the curl condition.

As in Section 2.1, it is possible to consider an MAVF, with the divergence condition on the field treated as a constraint. But it is not the only possibility, since one can also investigate another MAVF, with *two constraints*. For $\mathscr{E}$, one on the divergence, and one on the *tangential trace*. For $\mathscr{H}$, one on the divergence, and one on the *normal trace*.

Remark 25. For the time-dependent Maxwell equations, the MAVF with a single constraint is not relevant. As a matter of fact, it seems unlikely that the boundary condition is resolved, when only the divergence condition is treated as a constraint. On the other hand, the MAVF with two constraints is a good candidate to resolve the time-dependent Maxwell equations. This is actually very similar in the case of the time-harmonic equations, as we shall see in Section 3.2.

We begin by the MAVF with a single constraint, on the electric field $\mathscr{E}$. We propose to solve the problem (Q_{E})

$$\text{find } (\mathscr{E},p)\in\mathscr{X}_{\mathrm{E}}\times L^2(\Omega)$$

$$\text{s.t. } (\mathscr{E},\mathbf{v})_{X_{\mathrm{E}}} + (p,\operatorname{div}\mathbf{v})_0 = l_{\mathrm{E}}(\mathbf{v}) \quad \forall\mathbf{v}\in\mathscr{X}_{\mathrm{E}}, \tag{77}$$

$$(\operatorname{div}\mathscr{E},q)_0 = (g_{\mathrm{E}},q)_0 \quad \forall q\in L^2(\Omega). \tag{78}$$

Once more, as $\operatorname{div}\mathscr{X}_{\mathrm{E}} = L^2(\Omega)$, a *necessary* condition for (Q_{E}) to yield the desired solution is that $p = 0$. Actually, with proofs identical to that of Proposition 9 and Theorem 11, one finds

Proposition 26. *In* (Q_E), *one has* $p = 0$.

Theorem 27. *Problem* (Q_E) *admits one, and only one, solution* $(\mathscr{E},p)$. *Moreover,* $\mathscr{E}$ *is the solution to problem* (P_E).

Accordingly, we propose a single constraint MAVF for $\mathscr{H}$, problem $(Q_{\rm H})$

$$\text{find } (\mathscr{H},p)\in\mathscr{X}_{\rm H}\times L^2_0(\Omega)$$
$$\text{s.t. } (\mathscr{H},\mathbf{v})_{X_{\rm H}}+(p,\operatorname{div}\mathbf{v})_0=l_{\rm H}(\mathbf{v})\quad\forall\mathbf{v}\in\mathscr{X}_{\rm H},\tag{79}$$
$$(\operatorname{div}\mathscr{H},q)_0=(g_{\rm H},q)_0\quad\forall q\in L^2_0(\Omega).\tag{80}$$

Since $\operatorname{div}\mathscr{X}_{\rm H}=L^2(\Omega)$, the usual *necessary* condition is $p = 0$.

Proposition 28. *In (Q_H), one has $p = 0$.*

Theorem 29. *Problem (Q_H) admits one, and only one, solution $(\mathscr{H},p)$. Moreover, $\mathscr{H}$ is the solution to problem (P_H).*

NB. Both problems $(Q_{\rm E})$ and $(Q_{\rm H})$ satisfy an *inf–sup* condition.

The (M)AVFs $(P_{E,H})$ and $(Q_{E,H})$ allow to solve the static-like problems (15)–(17) and (25)–(27). In order to deal with the time-dependent Maxwell equations, one has to consider the boundary conditions as constraints, as mentioned before. The MAVF in $\mathscr{E}$, called $(R_{\rm E})$, reads

$$\text{find } (\mathscr{E},p,\vec{\lambda}_{\rm E})\in\mathscr{X}_{\rm E}\times L^2(\Omega)\times\mathbf{L}^2_t(\partial\Omega)$$
$$\text{s.t. } (\mathscr{E},\mathbf{v})_{X_{\rm E}}+(p,\operatorname{div}\mathbf{v})_0+(\vec{\lambda}_{\rm E},\mathbf{v}_{\rm T})_{0,\partial\Omega}=l_{\rm E}(\mathbf{v})\quad\forall\mathbf{v}\in\mathscr{X}_{\rm E},\tag{81}$$
$$(\operatorname{div}\mathscr{E},q)_0=(g_{\rm E},q)_0\quad\forall q\in L^2(\Omega),\tag{82}$$
$$(\mathscr{E}_{\rm T},\vec{\mu})_{0,\partial\Omega}=(\vec{\mathbf{e}}_{\rm T},\vec{\mu})_{0,\partial\Omega}\quad\forall\vec{\mu}\in\mathbf{L}^2_t(\partial\Omega).\tag{83}$$

To go back to (15)–(17), the method of proof is different than the ones we utilized up to now. Let us begin by a preliminary result

Lemma 30. *The space spanned by the tangential trace of elements of $\mathscr{X}_{\rm E}$ is*

$$\Lambda=\{\vec{\mu}\in\mathbf{L}^2_t(\partial\Omega):\operatorname{curl}_\Gamma\vec{\mu}\in H^{-1/2}(\partial\Omega)\}.$$

Proof. Let $\mathbf{v}\in\mathscr{X}_{\rm E}$. Then, one has $\mathbf{v}_{\rm T}\in\mathbf{L}^2_t(\partial\Omega)$ by definition. Also, since $\mathbf{v}$ belongs to $\mathbf{H}(\mathbf{curl},\Omega)$, one has $\operatorname{curl}_\Gamma\mathbf{v}_{\rm T}\in H^{-1/2}(\partial\Omega)$, according to [11, Theorem 3.10]. So $\mathbf{v}_{\rm T}\in\Lambda$.

Conversely, let $\vec{\mu}\in\Lambda$. Thanks to [12, Theorem 5.4], there exists $\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega)$ s.t. $\mathbf{v}_{\rm T}=\vec{\mu}$. Now, $\operatorname{div}\mathbf{v}$ belongs to $H^{-1}(\Omega)$, so $\exists!\phi\in H^1_0(\Omega)$ s.t. $\Delta\phi = \operatorname{div}\mathbf{v}$. Then, $\mathbf{w} = \mathbf{v} - \nabla\phi$ is such that

$$\mathbf{w}\in\mathbf{H}(\mathbf{curl},\Omega),\quad \operatorname{div}\mathbf{w}=0,\quad \mathbf{w}_{\rm T}=\vec{\mu}.$$

Since $\vec{\mu}$ is an element of $\mathbf{L}^2_t(\partial\Omega)$, we conclude that $\mathbf{w}$ is in $\mathscr{X}_{\rm E}$. □

In terms of potentials, note that (see [12, Theorem 3.4])

$$\mathbf{L}^2_t(\partial\Omega)=\nabla_\Gamma H^1(\partial\Omega)\overset{\perp}{\oplus}\mathbf{curl}_\Gamma H^1(\partial\Omega).$$

The additional condition on the tangential curl of elements of Λ means that

$$\Lambda=\nabla_\Gamma H^1(\partial\Omega)\overset{\perp}{\oplus}\mathbf{curl}_\Gamma\mathscr{H}(\partial\Omega)$$

with $\mathscr{H}(\partial\Omega)=\{q\in H^1(\partial\Omega):\Delta_\Gamma q\in H^{-1/2}(\partial\Omega)\}$. Thus $\Lambda\neq\mathbf{L}^2_t(\partial\Omega)$. As a consequence, one has the following "negative result".

Corollary 31. *Problem (R_E) does not satisfy the inf–sup condition.*

Proof. It is based on the Corollary 4.1 of Chapter I of [23]: this corollary is concerned with mixed problems, for which the V-ellipticity condition is true. Applied to our case, it says that problem $(R_{\rm E})$ is well-posed, i.e. that

$$\begin{cases} \mathcal{X}_{\mathrm{E}} \times L^2(\Omega) \times \mathbf{L}_t^2(\partial\Omega) \rightarrow \mathcal{X}'_{\mathrm{E}} \times L^2(\Omega) \times \mathbf{L}_t^2(\partial\Omega), \\ (\mathscr{E}, p, \vec{\lambda}_{\mathrm{E}}) \mapsto (l_{\mathrm{E}}, g_{\mathrm{E}}, \vec{\mathbf{e}}_{\mathrm{T}}) \end{cases}$$

is an isomorphism, iff the *inf–sup* condition holds. Now, if one takes $\vec{\mathbf{e}}_{\mathrm{T}} \in \mathbf{L}_t^2(\partial\Omega) \setminus \Lambda$, it is clear that problem (R_{E}) admits no solution: otherwise, (83) would imply $\mathscr{E}_{\mathrm{T}} = \vec{\mathbf{e}}_{\mathrm{T}}$ in $\mathbf{L}_t^2(\partial\Omega)$, with $\mathscr{E} \in \mathcal{X}_{\mathrm{E}}$, a contradiction. □

However, one can still prove a density result, as below.

Proposition 32. *The space Λ is dense in $\mathbf{L}_t^2(\partial\Omega)$.*

Proof. Let $\varepsilon > 0$ and $\vec{\mu} \in \mathbf{L}_t^2(\partial\Omega)$ be given. By definition, $\mathbf{L}_t^2(\partial\Omega)$ is a subset of $\mathbf{L}^2(\partial\Omega)$, and $\mathbf{H}^{1/2}(\partial\Omega)$ is dense in $\mathbf{L}^2(\partial\Omega)$. So, $\exists \vec{\mu}^\varepsilon \in \mathbf{H}^{1/2}(\partial\Omega)$ s.t. $\|\vec{\mu}^\varepsilon - \vec{\mu}\|_{0,\partial\Omega} \leqslant \varepsilon$. Using the $H^{1/2}(\partial\Omega) - H^1(\Omega)$ lifting operator, one can choose $\mathbf{v}^\varepsilon \in \mathbf{H}^1(\Omega)$ s.t. $\mathbf{v}^\varepsilon_{|\partial\Omega} = \vec{\mu}^\varepsilon$. Now, $\mathbf{v}^\varepsilon$ is an element of $\mathcal{X}_{\mathrm{E}}$, so $\vec{\mu}^\varepsilon_{\mathrm{T}} = \mathbf{v}^\varepsilon_{\mathrm{T}}$ is an element of Λ, and

$$\|\vec{\mu}^\varepsilon_{\mathrm{T}} - \vec{\mu}\|_{\mathbf{L}_t^2(\partial\Omega)} \leqslant \|\vec{\mu}^\varepsilon - \vec{\mu}\|_{0,\partial\Omega} \leqslant \varepsilon,$$

which yields the result. □

This allows to prove

Theorem 33. *Problem (R_{E}) admits one, and only one, solution $(\mathscr{E}, p, \vec{\lambda}_{\mathrm{E}})$. In addition, $(p, \vec{\lambda}_{\mathrm{E}}) = (0, 0)$, so that $\mathscr{E}$ is the solution to (15)–(17).*

Proof. Since the usual Babuska–Brezzi theory is not usable here, we revert to simpler means to prove the existence and uniqueness.

Existence of the solution: let $\mathscr{E}$ be the solution to problem (15)–(17), then $(\mathscr{E}, 0, 0)$ is a solution to problem (R_{E}).

Uniqueness of the solution: let $(\mathscr{E}, p, \vec{\lambda}_{\mathrm{E}})$ be a solution to problem (R_{E}) with $(l_{\mathrm{E}}, g_{\mathrm{E}}, \vec{\mathbf{e}}) = (0, 0, 0)$. In particular, (83) yields $\mathscr{E}_{\mathrm{T}} = 0$, so $\mathscr{E} \in \mathcal{X}^0_{\mathrm{E}}$. Then, with test fields $\mathbf{v}$ in $\mathcal{X}^0_{\mathrm{E}}$ only, (81) and (82) lead to (56) and (57), with zero right-hand sides: $(\mathscr{E}, p)$ satisfies the homogeneous problem (Q^0_{E}), and it is therefore equal to $(0, 0)$. Finally, (81) now reduces to

$$(\vec{\lambda}_{\mathrm{E}}, \mathbf{v}_{\mathrm{T}})_{0,\partial\Omega} = 0 \quad \forall \mathbf{v} \in \mathcal{X}_{\mathrm{E}}. \tag{84}$$

This gives $\vec{\lambda}_{\mathrm{E}} = 0$ according to the density result of Proposition 32.

The obvious by-product of the existence and uniqueness results is that $(p, \vec{\lambda}_{\mathrm{E}})$ always vanishes. □

The MAVF on $\mathscr{H}$, called (R_{H}), reads

$$\text{find } (\mathscr{H}, p, \lambda_{\mathrm{H}}) \in \mathcal{X}_{\mathrm{H}} \times L^2_0(\Omega) \times L^2(\partial\Omega)$$

$$\text{s.t.} \quad (\mathscr{H}, \mathbf{v})_{X_{\mathrm{H}}} + (p, \operatorname{div} \mathbf{v})_0 + (\lambda_{\mathrm{H}}, \mathbf{v} \cdot \mathbf{n})_{0,\partial\Omega} = l_{\mathrm{H}}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{X}_{\mathrm{H}}, \tag{85}$$

$$(\operatorname{div} \mathscr{H}, q)_0 = (g_{\mathrm{H}}, q)_0 \quad \forall q \in L^2_0(\Omega), \tag{86}$$

$$(\mathscr{H} \cdot \mathbf{n}, \mu)_{0,\partial\Omega} = (h, \mu)_{0,\partial\Omega} \quad \forall \mu \in L^2(\partial\Omega). \tag{87}$$

We fall back to the often used pattern of proof, [2] with the *Babuska–Brezzi theory*.

Theorem 34. *Problem (R_{H}) admits one, and only one, solution $(\mathscr{H}, p, \lambda_{\mathrm{H}})$. In addition, $(p, \lambda_H) = (0, 0)$, so that $\mathscr{H}$ is the solution to (25)–(27).*

[2] It is possible to choose the formulation with $(\mathscr{H}, p, \lambda_{\mathrm{H}}) \in \mathcal{X}_{\mathrm{H}} \times L^2_0(\Omega) \times L^2_0(\partial\Omega)$, i.e. with two zero mean-value Lagrange multipliers.

Proof. Existence and uniqueness of the solution to problem (R_{H}).

The V-ellipticity condition is automatic, since the bilinear form involved is the scalar product of $\mathscr{X}_{\mathrm{H}}$.

To check the *inf–sup* condition, let $(q,\mu)\in L^2_0(\Omega)\times L^2(\partial\Omega)$. Define $c_\mu=(\mu,1)_{0,\partial\Omega}/|\Omega|$: $\exists!\phi\in H^1(\Omega)\cap L^2_0(\Omega)$ s.t. $\Delta\phi=q+c_\mu$, $\partial_n\phi_{|\partial\Omega}=\mu$. We note that

$$\|c_\mu\|_0^2=|c_\mu|^2\,|\Omega|=(\mu,1)^2_{0,\partial\Omega}/|\Omega|\leqslant C_\Omega\|\mu\|^2_{0,\partial\Omega}\quad\text{with } C_\Omega=|\partial\Omega|/|\Omega|.$$

Then, $\mathbf{v}=\nabla\phi$ is in $\mathscr{X}_{\mathrm{H}}$, and

$$(q,\operatorname{div}\mathbf{v})_0+(\mu,\mathbf{v}\cdot\mathbf{n})_{0,\partial\Omega}=\|q\|_0^2+\|\mu\|^2_{0,\partial\Omega};$$

$$\|\mathbf{v}\|^2_{X_{\mathrm{H}}}=\|q+c_\mu\|_0^2+\|\mu\|^2_{0,\partial\Omega}=\|q\|_0^2+\|c_\mu\|_0^2+\|\mu\|^2_{0,\partial\Omega}\leqslant(1+C_\Omega)\left[\|q\|_0^2+\|\mu\|^2_{0,\partial\Omega}\right].$$

In other words

$$\inf_{(q,\mu)\neq0}\sup_{\mathbf{v}\in\mathscr{X}_{\mathrm{H}}}\frac{(q,\operatorname{div}\mathbf{v})_0+(\mu,\mathbf{v}\cdot\mathbf{n})_{0,\partial\Omega}}{\|\mathbf{v}\|_{X_{\mathrm{H}}}\left(\|q\|_0^2+\|\mu\|^2_{0,\partial\Omega}\right)^{1/2}}\geqslant\frac{1}{\sqrt{1+C_\Omega}},$$

which means that the *inf–sup* condition is fulfilled.

Let us prove that $(p,\lambda_{\mathrm{H}})=(0,0)$. Set $c_{\mathrm{H}}=(\lambda_{\mathrm{H}},1)_{0,\partial\Omega}/|\Omega|$; $\exists!\phi_\star\in H^1(\Omega)\cap L^2_0(\Omega)$ s.t. $\Delta\phi_\star=p+c_{\mathrm{H}}$, $\partial_n\phi_{\star|\partial\Omega}=\lambda_{\mathrm{H}}$. As $\nabla\phi_\star$ is in $\mathscr{X}_{\mathrm{H}}$, one can consider (85) with $\nabla\phi_\star$, minus (86) with p and (87) with λ_{H}, to get

$$c_{\mathrm{H}}(\operatorname{div}\mathscr{H},1)_0+\|p\|_0^2+\|\lambda_{\mathrm{H}}\|^2_{0,\partial\Omega}=c_{\mathrm{H}}(g_{\mathrm{H}},1)_0.$$

Now, $(\operatorname{div}\mathscr{H},1)_0=(\mathscr{H}\cdot\mathbf{n},1)_{0,\partial\Omega}=(h,1)_{0,\partial\Omega}=(g_{\mathrm{H}},1)_0$, according to the compatibility condition (30). Thus p and λ_{H} vanish, so (85) implies that $\mathscr{H}$ is the solution to problem (P_{H}). According to Theorem 24, $\mathscr{H}$ is the solution to (25)–(27). □

NB. Problem (R_{H}) satisfies an *inf–sup* condition.

In this subsection, we defined one augmented variational formulation, and *two* mixed augmented variational formulations for each of static-like fields $\mathscr{E}$ and $\mathscr{H}$. We shall see whether these formulations fare well or not, when applied to the time-harmonic Maxwell equations. In particular, the *vector* vs. *scalar* boundary condition for $\mathscr{E}$ and $\mathscr{H}$ yields differences, consequences of the facts we emphasized in Lemma 30, Corollary 31 and Proposition 32.

3.2. The time-harmonic equations

We consider the time-harmonic source problems (42)–(44) and (45)–(48), given $\mathbf{j}$, $\mathbf{j}'$ and ω with the electromagnetic field set in $\mathscr{X}_{\mathrm{E}}\times\mathscr{X}_{\mathrm{H}}$.

Let us begin by the electric problem, already solved in $\mathscr{V}^0_{\mathrm{E}}$ in Section 2.2. It is tempting to define first, *à la sauce* (63) and (64), a problem (p_{E}),

$$\text{find } \mathbf{e}'\in\mathscr{X}_{\mathrm{E}}$$

$$\text{s.t.}\quad(\mathbf{e}',\mathbf{v})_{X_{\mathrm{E}}}=\omega^2(\mathbf{e}',\mathbf{v})_0-(\mathbf{j}',\mathbf{v})_0\quad\forall\mathbf{v}\in\mathscr{X}_{\mathrm{E}},\tag{88}$$

$$\operatorname{div}\mathbf{e}'=0.\tag{89}$$

What can be said about the formulation (p_{E})? If one considers test fields which are elements of $\mathscr{X}^0_{\mathrm{E}}$, (88) and (89) is similar to (63) and (64), boundary condition $\mathbf{e}'\times\mathbf{n}_{|\partial\Omega}=0$ *excepted*. Actually, this boundary condition is (p_{E})'s major drawback… Assume for instance that $\mathbf{j}'=0$, and $\omega\neq0$ in (88). Then, if in addition $\mathbf{e}'\times\mathbf{n}_{|\partial\Omega}=0$, one could consider $\mathbf{v}=\nabla\phi$, with smooth ϕ, to find

$$0 = (\mathbf{e}', \nabla\phi)_0 = -_{\mathrm{H}^{-1/2}(\partial\Omega)} < \mathbf{e}' \cdot \mathbf{n}, \phi >_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

By density, $\mathbf{e}' \cdot \mathbf{n}_{|\partial\Omega} = 0$ would follow, so that $\mathbf{e}'$ would necessarily belong to $\mathbf{H}^1_0(\Omega)$, according to [1, Theorem 2.5, p. 827], i.e. a rather *unlikely* truth...

We just carried out an a posteriori analysis of (88) and (89). Let us now attempt an a priori analysis of (42)–(44). Eq. (42) reveals that $\mathbf{curl}\,\mathbf{e}'$ belongs to $\mathbf{H}(\mathbf{curl}, \Omega)$. Since $(\mathbf{curl}\,\mathbf{e}') \cdot \mathbf{n}_{|\partial\Omega} = \mathrm{div}_\Gamma(\mathbf{e}' \times \mathbf{n}_{|\partial\Omega}) = 0$, one has moreover $\mathbf{curl}\,\mathbf{e}' \in \mathscr{X}^0_{\mathrm{H}}$. Thus, $\mathbf{curl}\,\mathbf{e}'$ belongs to $\mathbf{H}^{1/2+\sigma}(\Omega)$ for some $\sigma > 0$, thanks to [1, Proposition 3.7, p. 838], so that $(\mathbf{curl}\,\mathbf{e}')_{|\partial\Omega} \in \mathbf{L}^2(\partial\Omega)$. Therefore, taking the $\mathbf{L}^2(\Omega)$ scalar product of $\mathbf{v} \in \mathscr{X}_{\mathrm{E}}$ and (42), and integrating by parts, one finds

$$(\mathbf{curl}\,\mathbf{e}', \mathbf{curl}\,\mathbf{v})_0 - ((\mathbf{curl}\,\mathbf{e}') \times \mathbf{n}, \mathbf{v}_{\mathrm{T}})_{0,\partial\Omega} = \omega^2(\mathbf{e}', \mathbf{v})_0 - (\mathbf{j}', \mathbf{v})_0. \tag{90}$$

With $\mathrm{div}\,\mathbf{e}' = 0$ and $\mathbf{e}' \times \mathbf{n}_{|\partial\Omega} = 0$, we found (88) with an *additional* term...

The preliminary conclusion is that a formulation set in $\mathscr{X}_{\mathrm{E}}$, which does not take into account the boundary condition $\mathbf{e}' \times \mathbf{n}_{|\partial\Omega} = 0$ explicitly, fails to resolve the time-harmonic electric equations (contrarily to the static case, cf. Section 3.1). So, it is *necessary* to enforce this boundary condition. When it is handled as a *natural* boundary condition, we prove below it can be achieved by imposing it as a *constraint*. Let us introduce the problem (q_{E})

$$\text{find } (\mathbf{e}', \vec{\lambda}_{\mathrm{E}}) \in \mathscr{X}_{\mathrm{E}} \times \mathbf{L}^2_t(\partial\Omega)$$

$$\text{s.t.}\quad (\mathbf{e}', \mathbf{v})_{X_{\mathrm{E}}} + (\vec{\lambda}_{\mathrm{E}}, \mathbf{v}_{\mathrm{T}})_{0,\partial\Omega} = \omega^2(\mathbf{e}', \mathbf{v})_0 - (\mathbf{j}', \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}_{\mathrm{E}}, \tag{91}$$

$$\mathrm{div}\,\mathbf{e}' = 0, \tag{92}$$

$$(\mathbf{e}'_{\mathrm{T}}, \vec{\mu})_{0,\partial\Omega} = 0 \quad \forall \vec{\mu} \in \mathbf{L}^2_t(\partial\Omega). \tag{93}$$

Proposition 35. *In* (q_{E}), *one has* $\vec{\lambda}_{\mathrm{E}} = -(\mathbf{curl}\,\mathbf{e}') \times \mathbf{n}_{|\partial\Omega}$ *in* $\mathbf{L}^2_t(\partial\Omega)$.

Proof. First, take $\mathbf{v} \in \mathscr{D}(\Omega)^3$ in (91) and integrate by parts the curl-curl term, to recover (42). Second, choose $\mathbf{v} \in \mathscr{X}_{\mathrm{E}}$ and perform the same integration by parts: there remains

$$(\vec{\lambda}_{\mathrm{E}} + (\mathbf{curl}\,\mathbf{e}') \times \mathbf{n}, \mathbf{v}_{\mathrm{T}})_{0,\partial\Omega} = 0 \quad \forall \mathbf{v} \in \mathscr{X}_{\mathrm{E}}.$$

The result follows by applying Proposition 32. □

It is then straightforward to prove

Theorem 36. $(\mathbf{e}', \vec{\lambda}_{\mathrm{E}})$ *is a solution to problem* (q_{E}) *with* $\vec{\lambda}_{\mathrm{E}} = -(\mathbf{curl}\,\mathbf{e}') \times \mathbf{n}_{|\partial\Omega}$ *iff* $\mathbf{e}'$ *is a solution to* (42)–(44).

Remark 37. We found that $\vec{\lambda}_{\mathrm{E}} = -(\mathbf{curl}\,\mathbf{e}') \times \mathbf{n}_{|\partial\Omega}$ in Proposition 35, i.e. a nonzero Lagrange multiplier. It can be characterized (see [12] for details) as

$$\vec{\lambda}_{\mathrm{E}} = \nabla_\Gamma \alpha + \mathbf{curl}_\Gamma \beta, \quad \alpha, \beta \in H^1(\partial\Omega)/\mathbb{R},$$

$$\Delta_\Gamma \alpha = (\mathbf{j}' - \omega^2 \mathbf{e}') \cdot \mathbf{n}_{|\partial\Omega},$$

$$\Delta_\Gamma \beta = \mathrm{curl}_\Gamma((\mathbf{curl}\,\mathbf{e}') \times \mathbf{n}_{|\partial\Omega}).$$

The final step is to construct an MAVF, with both (43) and (44) handled as constraints. It is called problem (r_{E})

$$\text{find } (\mathbf{e}', p, \vec{\lambda}_{\mathrm{E}}) \in \mathscr{X}_{\mathrm{E}} \times L^2(\Omega) \times \mathbf{L}^2_t(\partial\Omega)$$

$$\text{s.t.} \quad (\mathbf{e}', \mathbf{v})_{X_{\mathrm{E}}} + (p, \operatorname{div} \mathbf{v})_0 + (\vec{\lambda}_{\mathrm{E}}, \mathbf{v}_{\mathrm{T}})_{0,\partial\Omega} = \omega^2 (\mathbf{e}', \mathbf{v})_0 - (\mathbf{j}', \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}_{\mathrm{E}}, \tag{94}$$

$$(\operatorname{div} \mathbf{e}', q)_0 = 0 \quad \forall q \in L^2(\Omega), \tag{95}$$

$$(\mathbf{e}'_{\mathrm{T}}, \vec{\mu})_{0,\partial\Omega} = 0 \quad \forall \vec{\mu} \in \mathbf{L}^2_t(\partial\Omega). \tag{96}$$

We conclude the time-harmonic electric case by the two results below, which are concerned with the MAVF (r_{E}). The first one is established similarly to Proposition 16.

Proposition 38. *In (r_E), one has $p = 0$.*

Theorem 39. *$(\mathbf{e}', 0, \vec{\lambda}_{\mathrm{E}})$ is a solution to problem (r_E) with $\vec{\lambda}_{\mathrm{E}} = -(\mathbf{curl}\, \mathbf{e}') \times \mathbf{n}_{|\partial\Omega}$ iff $\mathbf{e}'$ is a solution to (42)–(44).*

Proof. It is easily established that $(\mathbf{e}', 0, \vec{\lambda}_{\mathrm{E}})$ is a solution to problem (r_{E}) if, and only if, $(\mathbf{e}', \vec{\lambda}_{\mathrm{E}})$ is a solution to problem (q_{E}), so the conclusion follows from Theorem 36. □

Let us turn now to (45)–(48), solved in $\mathscr{V}^0_{\mathrm{H}}$ in Section 2.2. We introduce a problem (p_{H})

$$\text{find } \mathbf{h} \in \mathscr{X}_{\mathrm{H}}$$

$$\text{s.t.} \quad (\mathbf{h}, \mathbf{v})_{X_{\mathrm{H}}} = \omega^2 (\mathbf{h}, \mathbf{v})_0 + (\mathbf{j}, \mathbf{curl}\, \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}_{\mathrm{H}}, \tag{97}$$

$$\operatorname{div} \mathbf{h} = 0. \tag{98}$$

Contrarily the the electric case (set in $\mathscr{X}_{\mathrm{E}}$), we can prove that problem (p_{H}) is indeed equivalent to the original magnetic time-harmonic equations. To that aim, we establish a preliminary result. Let

$$\Psi := \left\{ \psi \in H^1(\Omega) : \Delta\psi \in L^2(\Omega), \quad \frac{\partial\psi}{\partial n}_{|\partial\Omega} \in L^2(\partial\Omega) \right\}.$$

Proposition 40. *Given $(g, \omega) \in L^2(\partial\Omega) \times \mathbb{R}$, there exists $\psi \in \Psi$ s.t.*

$$\left(\frac{\partial\psi}{\partial n} - \omega^2 \psi \right) \Big|_{\partial\Omega} = g.$$

Proof. Let $f \in L^2(\Omega)$, and solve the problem

$$\text{find } \psi \in H^1(\Omega)$$
$$\text{s.t.} \quad (\nabla\psi, \nabla\psi')_0 - \omega^2 (\psi, \psi')_{0,\partial\Omega} = (f, \psi')_0 + (g, \psi')_{0,\partial\Omega} \quad \forall \psi' \in H^1(\Omega).$$

If this problem admits a solution, it satisfies $-\Delta\psi = f$ and $(\partial_n \psi - \omega^2 \psi)_{|\partial\Omega} = g$, so the Proposition is proved. Now, it is easily checked (for instance by contradiction), that the norm associated to the scalar product

$$(\psi, \psi')_{1,\star} = (\nabla\psi, \nabla\psi')_0 + (\psi, \psi')_{0,\partial\Omega},$$

defines a norm, which is equivalent to $\| \cdot \|_1$ in $H^1(\Omega)$. We rewrite the problem in ψ as

$$\text{find } \psi \in H^1(\Omega)$$
$$\text{s.t.} \quad (\psi, \psi')_{1,\star} - (\omega^2 + 1)(\psi, \psi')_{0,\partial\Omega} = (f, \psi')_0 + (g, \psi')_{0,\partial\Omega} \quad \forall \psi' \in H^1(\Omega).$$

It can be resolved by the Fredholm theory. In order to define a (symmetric) compact operator $\mathbb{T}_1$ of $H^1(\Omega)$, remember that the imbedding of $H^1(\Omega)_{|\partial\Omega} = H^{1/2}(\partial\Omega)$ into $L^2(\partial\Omega)$ is compact. Then, since $\exists! G \in H^1(\Omega)$ s.t. $(G, v)_{1,\star} = (f, v)_0 + (g, v)_{0,\partial\Omega}$, $\forall v \in H^1(\Omega)$, one reaches

$$(\mathbb{I} - (\omega^2 + 1)\mathbb{T}_1)\psi = G \quad \text{in } H^1(\Omega).$$

If $1/(\omega^2 + 1)$ is not an eigenvalue of $\mathbb{T}_1$, ψ exists. Otherwise, let $(\psi_1, \ldots, \psi_p)$ be a basis of $\ker(\mathbb{I} - (\omega^2 + 1)\mathbb{T}_1)$. In order for ψ to exist, it is enough to have $(f, \psi_k)_0 + (g, \psi_k)_{0,\partial\Omega} = 0$, for $1 \leqslant k \leqslant p$. Since g and $(\psi_k)_k$ are fixed and f is not, one simply chooses f such that $(f, \psi_k)_0 = -(g, \psi_k)_{0,\partial\Omega}$, for $1 \leqslant k \leqslant p$. This concludes the proof. □

Theorem 41. $\mathbf{h}$ *is a solution to* (p_H) *iff* $\mathbf{h}$ *is a solution to* (45)–(48).

Proof. If $\mathbf{h}$ solves (45)–(48), it is also a solution to problem (p_{H}). Reciprocally, if $\mathbf{h}$ belongs to $\mathscr{X}_{\mathrm{H}}$ and satisfies (97) and (98), then it also satisfies (70) and (71). To reach the result, one has to check that $\mathbf{h} \cdot \mathbf{n}_{|\partial\Omega} = 0$. We know from the previous Proposition that $\exists \psi \in \Psi$ s.t. $(\partial_n \psi - \omega^2 \psi)_{|\partial\Omega} = \mathbf{h} \cdot \mathbf{n}_{|\partial\Omega}$. Since $\nabla\psi$ belongs to $\mathscr{X}_{\mathrm{H}}$, (97) with $\nabla\psi$ and (98) yield

$$(\mathbf{h} \cdot \mathbf{n}, \partial_n \psi)_{0,\partial\Omega} = \omega^2 (\mathbf{h} \cdot \mathbf{n}, \psi)_{0,\partial\Omega}, \text{ or } \|\mathbf{h} \cdot \mathbf{n}_{|\partial\Omega}\|^2_{0,\partial\Omega} = 0. \quad □$$

Let us now successively consider two formulations with Lagrange multipliers. First, the problem (q_{H}),

$$\text{find } (\mathbf{h}, \lambda_{\mathrm{H}}) \in \mathscr{X}_{\mathrm{H}} \times L^2(\partial\Omega)$$
$$\text{s.t.} \quad (\mathbf{h}, \mathbf{v})_{X_{\mathrm{H}}} + (\lambda_{\mathrm{H}}, \mathbf{v} \cdot \mathbf{n})_{0,\partial\Omega} = \omega^2 (\mathbf{h}, \mathbf{v})_0 + (\mathbf{j}, \mathbf{curl}\ \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}_{\mathrm{H}}, \tag{99}$$
$$\operatorname{div} \mathbf{h} = 0, \tag{100}$$
$$(\mathbf{h} \cdot \mathbf{n}, \mu)_{0,\partial\Omega} = 0 \quad \forall \mu \in L^2(\partial\Omega). \tag{101}$$

Proposition 42. *In* (q_{H}), *one has* $\lambda_{\mathrm{H}} = 0$.

Proof. Given $\lambda_{\mathrm{H}} \in L^2(\partial\Omega)$, $\exists! \phi_\star \in H^1(\Omega) \cap L^2_0(\Omega)$ s.t. $\Delta \phi_\star = (\lambda_{\mathrm{H}}, 1)_{0,\partial\Omega}/|\Omega|$ and $\partial_n \phi_{\star|\partial\Omega} = \lambda_{\mathrm{H}}$. $\nabla\phi_\star$ belongs to $\mathscr{X}_{\mathrm{H}}$, so it can be utilized as a test field in (99): this yields, thanks to (100) and (101), $\|\lambda_{\mathrm{H}}\|_0^2 = \omega^2 (\mathbf{h}, \nabla\phi_\star)_0 = 0$ (by integration by parts). □

It is then straightforward to prove

Theorem 43. $(\mathbf{h}, 0)$ *is a solution to problem* (q_{H}) *iff* $\mathbf{h}$ *is a solution to* (45)–(48).

The final step is to construct an MAVF, with both (46) and (47) handled as constraints: problem (r_{H}),

$$\text{find } (\mathbf{h}, p, \lambda_{\mathrm{H}}) \in \mathscr{X}_{\mathrm{H}} \times L^2_0(\Omega) \times L^2(\partial\Omega)$$
$$\text{s.t.} \quad (\mathbf{h}, \mathbf{v})_{X_{\mathrm{H}}} + (p, \operatorname{div} \mathbf{v})_0 + (\lambda_{\mathrm{H}}, \mathbf{v} \cdot \mathbf{n})_{0,\partial\Omega} = \omega^2 (\mathbf{h}, \mathbf{v})_0 + (\mathbf{j}, \mathbf{curl}\, \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}_{\mathrm{H}}, \tag{102}$$
$$(\operatorname{div} \mathbf{h}, q)_0 = 0 \quad \forall q \in L^2_0(\Omega), \tag{103}$$
$$(\mathbf{h} \cdot \mathbf{n}, \mu)_{0,\partial\Omega} = 0 \quad \forall \mu \in L^2(\partial\Omega). \tag{104}$$

There holds

Proposition 44. *In* (r_H), *one has* $(p, \lambda_H) = (0, 0)$.

Proof. Let us check first that $p = 0$ in $(r_{\rm H})$: $\exists! \phi_\star \in H^1(\Omega) \cap L^2_0(\Omega)$ s.t. $\Delta \phi_\star = p$ and $\partial_n \phi_{\star|\partial\Omega} = 0$. Let us plug $\mathbf{v}_\star = \nabla \phi_\star$ into (102): $\|p\|_0^2 = \omega^2(\mathbf{h}, \nabla\phi_\star)_0 = 0$, by integration by parts.

Second, we note that $(r_{\rm H})$ with $p = 0$ is equivalent to $(q_{\rm H})$, provided that $(r_{\rm H})$ yields $\operatorname{div} \mathbf{h} = 0$. But this is a straightforward consequence of (103) and of $(\operatorname{div} \mathbf{h}, 1)_0 = (\mathbf{h} \cdot \mathbf{n}, 1)_{0,\partial\Omega} = 0$.

Finally, Proposition 42 allows to conclude. □

According to Theorem 55 (*inf–sup* valid for problem $(R_{\rm H})$), one gets the

Theorem 45. $(\mathbf{h}, 0, 0)$ *is a solution to problem* (r_H) *iff* $\mathbf{h}$ *is a solution to* (45)–(48).

If we focus on the magnetic time-harmonic equations with $\mathbf{j} = 0$, and $(\mathbf{h}, \omega)$ as unknowns, i.e. the eigen value problem we note that our approach is different from the one adopted by Costabel, Dauge and Martin in [18]. In [18], a penalization method was investigated, which resulted in solving, with $s > 0$ and $\lambda > 0$:

$$\text{find } (\mathbf{h}, \omega) \in \mathscr{X}_{\rm H} \times \mathbb{R}$$

$$\text{s.t. } (\mathbf{curl}\, \mathbf{h}, \mathbf{curl}\, \mathbf{v})_0 + s(\operatorname{div} \mathbf{h}, \operatorname{div} \mathbf{v})_0 + \lambda(\mathbf{h} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{n})_{0,\partial\Omega} = \omega^2(\mathbf{h}, \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}_{\rm H}.$$

With this VF, spurious eigenmodes on the divergence of the magnetic field appear, since $\operatorname{div} \mathbf{h} = 0$ is neither enforced explicitly (as in problems $(p_{\rm H})$ and $(q_{\rm H})$), nor dualized (as in problem $(r_{\rm H})$).

To conclude the section on the natural boundary conditions, we emphasize the fact that the *vector vs. scalar* boundary condition for $\mathbf{e}'$ and $\mathbf{h}$ produces major differences. On the one hand, problem $(p_{\rm H})$ solves the magnetic time-harmonic equations, whereas $(p_{\rm E})$ does not solve the electric counterpart! On the other hand, there is an *inf–sup* condition for the magnetic problems $(R_{\rm H})$ and $(r_{\rm H})$ with two Lagrange multipliers, whereas there is none for the electric counterparts $(R_{\rm E})$ and $(r_{\rm E})$.

4. Variational formulations in weighted Sobolev spaces

In this section, we focus on Variational Formulations, in *weighted* Sobolev spaces. Again, the obvious advantage is computational, since, according to the density of smooth fields [17] in the $\mathscr{X}^0_{\rm E}$-like [3] space $\mathscr{X}^0_{{\rm E},\gamma}$ to be described below, one can use the continuous approximation of the field based on the P_1 Lagrange FEM. This breakthrough has been achieved in [17] for the approximation of the time-harmonic electric field, *without* Lagrange multiplier. Note that in this section, we follow closely the ideas of Section 2.

Before we proceed with the Variational Formulations, let us recall a few definitions and results. Since we deal with weighted Sobolev spaces, we begin by the weight functional. Among other possibilities, it can be defined with respect to the following distance.

Definition 46. Let E denote the closure of the set of reentrant edges of $\partial\Omega$, that is edges such that the dihedral angle in Ω is larger than π. Then, let

$$d_0(\mathbf{x}) = d(\mathbf{x}, E).$$

After that, we introduce the weighted Lesbesgue and Sobolev spaces that we shall use throughout this section.

Definition 47. Let $\gamma \in [0, 1]$ and set

$$L^2_\gamma(\Omega) = \{g : g \in \mathscr{D}'(\Omega), \quad d_0^\gamma g \in L^2(\Omega)\} \quad \text{with norm } \|g\|_{0,\gamma} = \|d_0^\gamma g\|_0;$$

[3] The ad hoc density result has not been established for spaces of magnetic fields, so we focused on the electric field. Nevertheless, would this result be proven, one could use the same techniques as the ones described in this paper.

$$\mathscr{X}^0_{E,\gamma} = \{\mathbf{v} : \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega),\ \mathrm{div}\,\mathbf{v} \in L^2_\gamma(\Omega)\}.$$

The upper bound (one) on γ is taken so that $L^2_\gamma(\Omega) \subset H^{-1}(\Omega)$ always holds. In [17, Theorem 5.1], the fundamental result was established.

Theorem 48. *There exists $\gamma_0 \in]0, 1/2[$, which depends only on the geometry of $\partial\Omega$, such that, for all $\gamma \in]\gamma_0, 1]$, the set of regular fields*

$$\{\mathbf{v} : \mathbf{v} \in \mathbf{H}^1(\Omega), \quad \mathbf{v} \times \mathbf{n}_{|\partial\Omega} = 0\}$$

is dense in $\mathscr{X}^0_{E,\gamma}$.

Moreover, one can prove that the reduced norm in $\mathscr{X}^0_{E,\gamma}$ is equivalent to the full norm.

Theorem 49. *In $\mathscr{X}^0_{E,\gamma}$, for $\gamma \in]\gamma_0, 1[$,*

$$\|\mathbf{v}\|_{X^0_\gamma} = \{\|\mathbf{curl}\,\mathbf{v}\|_0^2 + \|\mathrm{div}\,\mathbf{v}\|_{0,\gamma}^2\}^{1/2}$$

is a norm, which is equivalent to the full norm.

Proof. According to [17], for $\gamma \in]\gamma_0, 1[$, $L^2_\gamma(\Omega)$ is compactly imbedded into $H^{-1}(\Omega)$, which implies (cf. Corollary 2.3 *op. cit.*) that $\mathscr{X}^0_{E,\gamma}$ is compactly imbedded into $\mathbf{L}^2(\Omega)$.

Then, one proves easily by contradiction that $\|\cdot\|_{X^0_\gamma}$ is equivalent to the full norm: assume there exists a sequence $(\mathbf{v}_k)_k$ of elements of $\mathscr{X}^0_{E,\gamma}$ such that

$$\lim_k \|\mathbf{v}_k\|_{X^0_\gamma} = 0, \quad \|\mathbf{v}_k\|_0 = 1\ \forall k.$$

In particular, $(\mathbf{v}_k)_k$ is bounded in $\mathscr{X}^0_{E,\gamma}$, so there exists a subsequence, still denoted by $(\mathbf{v}_k)_k$, which converges in $\mathbf{L}^2(\Omega)$, to $\mathbf{v}$ (and $\|\mathbf{v}\|_0 = 1$). Now, as $(\mathbf{curl}\,\mathbf{v}_k)_k$ goes to zero in $\mathbf{L}^2(\Omega)$, we infer that $\mathbf{curl}\,\mathbf{v} = 0$. Thus $(\mathbf{v}_k)_k$ converges to $\mathbf{v}$ in $\mathbf{H}(\mathbf{curl}\,\Omega)$, so $\mathbf{v} \times \mathbf{n}_{|\partial\Omega} = 0$. Also, as $\gamma < 1$, $L^2_\gamma(\Omega) \subset H^{-1}(\Omega)$, therefore $\lim_k \mathrm{div}\,\mathbf{v}_k = 0$ in $H^{-1}(\Omega)$. This implies $\mathrm{div}\,\mathbf{v} = 0$. As a consequence, $\mathbf{v}$ is an element of $\mathscr{X}^0_E$ and, thanks to Theorem 3, $\mathbf{v} = 0$, which contradicts $\|\mathbf{v}\|_0 = 1$. □

These two results form the framework that allows to solve the static model in $\mathscr{E}$ and the time-harmonic Maxwell equations in $\mathbf{e}$.

4.1. The static equations

We introduce a lifting of $\vec{\mathbf{e}}$, called $\tilde{\mathbf{e}}$, in $\mathbf{H}^1(\Omega)^3$, and $\mathscr{E}^0 = \mathscr{E} - \tilde{\mathbf{e}}$, which satisfies (52). We consider the AVF $(\underline{P}^0_E)$

$$\begin{aligned} &\text{find } \mathscr{E}^0 \in \mathscr{X}^0_{E,\gamma} \\ &\text{s.t. } (\mathscr{E}^0, \mathbf{v})_{X^0_\gamma} = (\mathbf{f}^0_E, \mathbf{curl}\,\mathbf{v})_0 + (g^0_E, \mathrm{div}\,\mathbf{v})_{0,\gamma} \quad \forall \mathbf{v} \in \mathscr{X}^0_{E,\gamma}. \end{aligned} \tag{105}$$

According to the Riesz theorem, there exists one, and only one, solution to problem $(\underline{P}^0_E)$, continuous with respect to the data $(\mathbf{f}^0_E, g^0_E)$ in $\mathbf{L}^2(\Omega) \times L^2_\gamma(\Omega)$.

Theorem 50. *The field $\mathscr{E}^0$ satisfies (52) iff it is a solution to problem $(\underline{P}^0_E)$.*

Proof. It is clear that if $\mathscr{E}^0$ is a solution to (52), it also solves (105).

Let us consider the reciprocal assertion: let $\mathscr{E}^0$ be the solution to (105). Since $L^2_\gamma(\Omega) \subset H^{-1}(\Omega)$, given $g \in L^2_\gamma(\Omega)$, $\exists!\phi \in H^1_0(\Omega)$ s.t. $\Delta\phi = g$. As $\mathbf{v} = \nabla\phi \in \mathscr{X}^0_{E,\gamma}$, there holds $(\mathrm{div}\,\mathscr{E}^0, g)_{0,\gamma} = (g^0_E, g)_{0,\gamma}$; $\mathrm{div}\,\mathscr{E}^0 = g^0_E$ follows.

The end of the proof is that of Theorem 5. □

Evidently, the solution $\mathscr{E}$ depends continuously on the data $(\mathbf{f}_E, g_E, \mathbf{e})$ in $\mathbf{L}^2(\Omega) \times L^2_\gamma(\Omega) \times \mathbf{H}^{1/2}(\partial\Omega)$.

NB. Within the framework of weighted Sobolev spaces, it is possible to solve (15)–(17) with only $g_E \in L^2_\gamma(\Omega)$ (a weaker assumption than $g_E \in L^2(\Omega)$).

Next, we define the MAVF $(\underline{Q}^0_E)$

$$\text{find } (\mathscr{E}^0, p) \in \mathscr{X}^0_{E,\gamma} \times L^2_\gamma(\Omega)$$

$$\text{s.t.} \quad (\mathscr{E}^0, \mathbf{v})_{X^0_\gamma} + (p, \operatorname{div} \mathbf{v})_{0,\gamma} = (\mathbf{f}^0_E, \mathbf{curl}\, \mathbf{v})_0 + (g^0_E, \operatorname{div} \mathbf{v})_{0,\gamma} \quad \forall \mathbf{v} \in \mathscr{X}^0_{E,\gamma}, \tag{106}$$

$$(\operatorname{div} \mathscr{E}^0, q)_{0,\gamma} = (g^0_E, q)_{0,\gamma} \quad \forall q \in L^2_\gamma(\Omega). \tag{107}$$

There holds $\operatorname{div} \mathscr{X}^0_{E,\gamma} = L^2_\gamma(\Omega)$. So, it is required that p vanishes. Noticing that the Lagrange multipliers belong to $H^{-1}(\Omega)$, one can follow the pattern of proofs given in Section 2.1 to prove the two results below.

Proposition 51. *In $(\underline{Q}^0_E)$, one has $p = 0$.*

Theorem 52. *Problem $(\underline{Q}^0_E)$ admits one, and only one, solution $(\mathscr{E}^0, p)$. Moreover, $\mathscr{E}^0$ is the solution to problem $(\underline{P}^0_E)$.*

NB. Problem $(\underline{Q}^0_E)$ satisfies an *inf–sup* condition.

4.2. The time-harmonic equations

We turn to the time-harmonic source problem (42)–(44), which we solved in $\mathscr{V}^0_E$ in Section 2.2. It is interesting to note that $\mathscr{V}^0_E$ is not only a subspace of $\mathbf{H}_0(\mathbf{curl}\,\Omega)$ or of $\mathscr{X}^0_E$, but also of $\mathscr{X}^0_{E,\gamma}$. Therefore, we define, in-between (60) and (61), and (62), a new equivalent formulation, called problem $(\underline{p}^0_E)$

$$\text{find } \mathbf{e}' \in \mathscr{X}^0_{E,\gamma}$$

$$\text{s.t.} \quad (\mathbf{e}', \mathbf{v})_{X^0_\gamma} = \omega^2 (\mathbf{e}', \mathbf{v})_0 - (\mathbf{j}', \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}^0_{E,\gamma}, \tag{108}$$

$$\operatorname{div} \mathbf{e}' = 0. \tag{109}$$

If (109) is handled as a constraint, one gets problem $(\underline{q}^0_E)$

$$\text{find } (\mathbf{e}', p) \in \mathscr{X}^0_{E,\gamma} \times L^2_\gamma(\Omega)$$

$$\text{s.t.} \quad (\mathbf{e}', \mathbf{v})_{X^0_\gamma} + (p, \operatorname{div} \mathbf{v})_{0,\gamma} = \omega^2 (\mathbf{e}', \mathbf{v})_0 - (\mathbf{j}', \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathscr{X}^0_{E,\gamma}, \tag{110}$$

$$(\operatorname{div} \mathbf{e}', q)_{0,\gamma} = 0 \quad \forall q \in L^2_\gamma(\Omega). \tag{111}$$

There holds successively

Proposition 53. *In $(\underline{q}^0_E)$, one has $p = 0$.*

According to Theorem 55, thanks to the *inf–sup* condition in problem $(\underline{Q}^0_E)$, there holds the

Theorem 54. $(\mathbf{e}', 0)$ *is a solution to problem $(\underline{q}^0_E)$ iff $\mathbf{e}'$ is a solution to (42)–(44).*

5. Conclusion

In this paper, we focused on variational formulations to solve the static-like or time-harmonic Maxwell equations. To that aim, we derived *augmented* variational formulations, where the additional term is de facto a (weighted) L^2 scalar product between the divergence of the EM and the divergence of test fields. The EM fields can live in three different classes of Sobolev spaces:

- $\mathscr{X}^0_{\rm E}/\mathscr{X}^0_{\rm H}$ (0 for vanishing b.c.);
- $\mathscr{X}_{\rm E}/\mathscr{X}_{\rm H}$;
- $\mathscr{X}^0_{{\rm E},\gamma}$ (0 for vanishing b.c.; γ for a weight on the divergence).

In this respect, all the methods we presented are $\mathbf{H}(\mathbf{curl},\mathrm{div})$-conforming, with a weight on the divergence in $\mathscr{X}^0_{{\rm E},\gamma}$. We also built *mixed*, *augmented* variational formulations, with either one or two Lagrange multipliers. In other words, we considered the equation on the divergence as a *constraint*. When applicable (i.e. in $\mathscr{X}_{\rm E}/\mathscr{X}_{\rm H}$), we took into account the relation on the tangential trace of electric fields or on the normal trace of magnetic fields as a second *constraint*.

Let us make two remarks from a theoretical point of view. First, in the case of the magnetic field, we noted already that a density result of regular fields in the *ad hoc* weighted Sobolev space is missing. Second, it is possible to generalize the study we performed here to Maxwell equations, posed in heterogeneous media. More precisely, assume that the electric permittivity ε and magnetic permeability μ are nonnegative piecewise constants. With boundary conditions treated as essential, the same kind of study can still be carried out. As far as the natural boundary conditions approach is concerned, a density result still holds, under some assumptions on the jumps of ε and μ across interfaces (cf. [27]). To our knowledge, in the case of the weighted approach, no result is available.

As far as numerical applications are concerned, we recall that the main difference between those formulations lies in the fact that P^1 continuous aproximations span a *dense subspace* of $\mathscr{X}_{\rm E}/\mathscr{X}_{\rm H}$ and $\mathscr{X}^0_{{\rm E},\gamma}$: a straightforward implementation yields convergence. On the contrary, the FE span a closed and strict subspace of $\mathscr{X}^0_{\rm E}/\mathscr{X}^0_{\rm H}$, thus requiring a *singular complement*. Also, the *uniform discrete inf–sup* condition has been established for the pairs $(\cdot, p)$ in [13], when the field lives in the spaces $\mathscr{X}^0_{\rm E}/\mathscr{X}^0_{\rm H}$ or $\mathscr{X}_{\rm E}/\mathscr{X}_{\rm H}$. There remains to prove a similar condition for $(\mathscr{E}, p)$ when the electric field lives in $\mathscr{X}^0_{{\rm E},\gamma}$, and also on $(\mathscr{E}, \vec{\lambda}_{\rm E})$ in $\mathscr{X}_{\rm E} \times \mathbf{L}^2_t(\partial\Omega)$ and $(\mathscr{H}, \lambda_{\rm H})$ in $\mathscr{X}_{\rm H} \times L^2(\partial\Omega)$.

For the time-dependent Maxwell equations, the same techniques can be applied. For instance, when these equations are solved within the framework developed by Lions and Magenes [26], one can utilize a result similar to the one proved in Appendix A, i.e. think in terms of operators, when constraints are dualized. Note that these equations have already been numerically approximated with $\mathbf{H}(\mathbf{curl},\mathrm{div})$-conforming FEM, see for instance [7,5,3].

Appendix A. Annex

Let H be a Hilbert space, with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We consider that H is a pivot space, i.e. $H' = H$.

Let X and Q be two Hilbert spaces, with norms $\|\cdot\|_X$ and $\|\cdot\|_Q$; X is a subset of H.

Let a and b be two continuous bilinear forms, a: $X \times X \rightarrow \mathbb{R}$ and b: $X \times Q \rightarrow \mathbb{R}$, $(l,\chi) \in X' \times Q'$ and $\lambda \in \mathbb{R}$. We then define

$$V(\chi) := \{v \in X : b(v,q) = \langle \chi, q\rangle \quad \forall q \in Q\}, \quad V := V(0).$$

We introduce two source problems: a direct one, called (P), and a constrained one, called (Q).

Problem (P)

$$\text{find } u \in V(\chi)$$
$$\text{s.t.} \quad a(u,v) = \lambda(u,v) + \langle l, v\rangle \quad \forall v \in V.$$

Problem (Q)

$$\text{find } (u,p) \in X \times Q$$
$$\text{s.t.} \begin{cases} a(u,v) + b(v,p) = \lambda(u,v) + \langle l,v\rangle & \forall v \in X, \\ b(u,q) = \langle \chi, q\rangle & \forall q \in Q. \end{cases}$$

We prove next that, in terms of u, problems (P) and (Q) have the same solutions, under the assumption that b satisfies an *inf–sup* condition, i.e.

$$\exists \beta > 0 \quad \text{s.t.} \quad \forall q \in Q, \quad \sup_{v\in X} \frac{b(v,q)}{\|v\|_X} \geqslant \beta\|q\|_Q.$$

Theorem 55. *Let $u \in X$ be given: u is a solution to (P) iff there exists $p \in Q$ such that (u,p) is a solution to (Q).*

Proof. Let (u,p) be a solution to (Q).

First, u belongs to $V(\chi)$ by definition. Second, since $b(v,p)$ vanishes for $v \in V$, there follows that $a(u,v) = \lambda(u,v) + \langle l,v\rangle$ for all $v \in V$.

Now, let u be a solution to (P).

We introduce three continuous linear operators $A: X \to X'$, $B': Q \to X'$, and $j: H \to X'$, respectively defined by

$$\langle Av,w\rangle = a(v,w) \quad \forall (v,w) \in X \times X,$$
$$\langle B'q,w\rangle = b(w,q) \quad \forall (q,w) \in Q \times X,$$
$$\langle jh,w\rangle = (h,w) \quad \forall (h,w) \in H \times X.$$

Thus, one gets $\langle (A - \lambda j)u - l, v\rangle = 0$, $\forall v \in V$. In other words, $(A - \lambda j)u - l$ belongs to the polar set of V in X', $V^0 := \{g \in X' : \langle g,v\rangle = 0\ \forall v \in V\}$.

According to [23, Lemma 4.1(ii), p. 58], since the *inf–sup* condition is satisfied, there exists one, and only one, p in Q such that $B'p = -(A - \lambda j)u + l$ in X'. Therefore

$$\langle Au,v\rangle + \langle B'p,v\rangle = \lambda\langle ju,v\rangle + \langle l,v\rangle \quad \forall v \in X$$

or equivalently

$$a(u,v) + b(v,p) = \lambda(u,v) + \langle l,v\rangle \quad \forall v \in X.$$

By construction, u belongs to $V(\chi)$, so $b(u,q) = \langle \chi,q\rangle \forall q \in Q$, which concludes the proof. □

Corollary 56. (Q) *admits no solution iff* (P) *admits no solution.*

Proof. It is enough to utilize the contraposed statement of Theorem 55. □

This means that the eigenvalue problems, which can be derived from (P) and (Q), possess the same eigenpairs.

Let us note that one can use a very similar approach to solve 2nd order time-dependent equations with dualized constraints (for instance within the framework developed by Lions and Magenes [26]), such as, for given $T > 0$

$$\text{find } (u,p) :]0,T[\rightarrow X \times Q$$
$$\text{s.t. } \begin{cases} \langle u''(t), v\rangle + a(u(t),v) + b(v,p(t)) = (l(t),v) \quad \forall v \in X, \quad t > 0, \\ b(u(t),q) = \langle \chi(t), q\rangle \quad \forall q \in Q, \quad t > 0 \end{cases}$$

(with initial conditions : $u(0) = u_0 \in X$ and $u'(0) = u_1 \in H$.)

References

[1] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci. 21 (1998) 823–864.

[2] F. Assous, P. Ciarlet Jr., S. Labrunie, Solution of axisymmetric Maxwell equations, Math. Methods Appl. Sci. 26 (2003) 861–896.

[3] F. Assous, P. Ciarlet Jr., S. Labrunie, J. Segré, Numerical solution to the time-dependent Maxwell equations in axisymmetric singular domains: the singular complement method, J. Comput. Phys. 191 (2003) 147–176.

[4] F. Assous, P. Ciarlet Jr., P.-A. Raviart, E. Sonnendrücker, A characterization of the singular part of the solution to Maxwell's equations in a polyhedral domain, Math. Methods Appl. Sci. 22 (1999) 485–499.

[5] F. Assous, P. Ciarlet Jr., J. Segré, Numerical solution to the time-dependent Maxwell equations in two-dimensional singular domains: the singular complement method, J. Comput. Phys. 161 (2000) 218–249.

[6] F. Assous, P. Ciarlet Jr., E. Sonnendrücker, Resolution of the Maxwell equations in a domain with reentrant corners, Modél. Math. Anal. Numér. 32 (1998) 359–389.

[7] F. Assous, P. Degond, E. Heintzé, P.A. Raviart, J. Segré, On a finite element method for solving the three-dimensional Maxwell equations, J. Comput. Phys. 109 (1993) 222–237.

[8] F. Ben Belgacem, C. Bernardi, Spectral element discretization of the Maxwell equations, Math. Comp. 68 (1999) 1497–1520.

[9] F. Ben Belgacem, C. Bernardi, M. Costabel, M. Dauge, Un résultat de densité pour les équations de Maxwell, C.R. Acad. Sci. Paris, Ser. I 324 (1997) 731–736.

[10] A.-S. Bonnet-Ben Dhia, C. Hazard, S. Lohrengel, A singular field method for the solution of Maxwell's equations in polyhedral domains, SIAM J. Appl. Math. 59 (1999) 2028–2044.

[11] A. Buffa, P. Ciarlet Jr., On traces for functional spaces related to Maxwell's equations. Part I: an integration by parts formula in Lipschitz polyhedra, Math. Methods Appl. Sci. 24 (2001) 9–30.

[12] A. Buffa, P. Ciarlet Jr., On traces for functional spaces related to Maxwell's equations. Part II: Hodge decomposition on the boundary of Lipschitz polyhedra and applications, Math. Methods Appl. Sci. 24 (2001) 31–48.

[13] P. Ciarlet Jr., V. Girault, Condition *inf–sup* pour l'élément fini de Taylor–Hood P_2-iso-P_1, 3-D; application aux équations de Maxwell, C.R. Acad. Sci. Paris, Ser. I 335 (2002) 827–832.

[14] P. Ciarlet Jr., C. Hazard, S. Lohrengel, Les équations de Maxwell dans un polyèdre : un résultat de densité, C.R. Acad. Sci. Paris, Ser. I 326 (1998) 1305–1310.

[15] P. Ciarlet Jr., E. Jamelot, Augmented formulations for solving Maxwell equations: numerical analysis and experiments, in preparation.

[16] M. Costabel, M. Dauge, Un résultat de densité pour les équations de Maxwell régularisées dans un domaine lipschitzien, C.R. Acad. Sci. Paris, Ser. I 327 (1998) 849–854.

[17] M. Costabel, M. Dauge, Weighted regularization of Maxwell equations in polyhedral domains, Numer. Math. 93 (2002) 239–277.

[18] M. Costabel, M. Dauge, D. Martin, Numerical investigation of a boundary penalization method for Maxwell equations, in: P. Neittaanmäki et al. (Eds.), Proceedings of the 3rd European Conference on Numerical Mathematics and Advanced Applications, World Scientific, Singapore, 2000, pp. 214–221.

[19] P. Degond, P.-A. Raviart, An analysis of the Darwin model of approximation to Maxwell's equations, Forum Math. 4 (1992) 13–44.

[20] L. Demkowicz, L. Vardapetyan, Modeling of electromagnetic absorption/scattering problems using *hp*-adaptive finite elements, Comput. Methods Appl. Mech. Engrg. 152 (1998) 103–124.

[21] P. Fernandes, G. Gilardi, Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions, Math. Models Methods Appl. Sci. 7 (1997) 957–991.

[22] P. Fernandes, I. Perugia, Vector potential formulation for magnetostatics and modelling of permanent magnets, IMA J. Appl. Math. 66 (2001) 293–318.

[23] V. Girault, P.-A. Raviart, Finite element methods for Navier-Stokes equations, Volume 5 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, Germany (1986).

[24] C. Hazard, S. Lohrengel, A singular field method for Maxwell's equations: numerical aspects for 2D magnetostatics, SIAM J. Appl. Math 40 (2002) 1021–1040.

[25] F. Kikuchi, Mixed and penalty formulations for finite element analysis of an eigenvalue problem in electromagnetism, Comput. Methods Appl. Mech. Engrg. 64 (1987) 509–521.
[26] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, Dunod, Paris, France, 1968.
[27] S. Lohrengel, S. Nicaise, Singularities and density problems for composite materials in electromagnetism, Commun. Part. Diff. Eq. 27 (2002) 1575–1623.
[28] J.C. Nédélec, Mixed finite elements in $\mathbb{R}^3$, Numer. Math. 35 (1980) 315–341.
[29] J.C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^3$, Numer. Math. 50 (1986) 57–81.
[30] S. Nicaise, Edge elements on anisotropic meshes and approximation of the Maxwell equations, SIAM J. Numer. Anal. 39 (2001) 784–816.
[31] W. Rachowicz, L. Demkowicz, An *hp*-adaptive finite element method for electromagnetics—Part II: a 3D implementation, Int. J. Numer. Math. Engng 53 (2002) 147–180.
[32] L. Vardapetyan, L. Demkowicz, D. Neikirk, *hp*-vector finite element method for eigenmode analysis of waveguides, Comput. Methods Appl. Mech. Engrg. 192 (2003) 185–201.
[33] C. Weber, A local compactness theorem for Maxwell's equations, Math. Methods Appl. Sci. 2 (1980) 12–25.