The Singular Complement Method

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Introduction

In this paper, we propose a method, called the *Singular Complement Method* (latter referred to as the SCM), which allows to solve PDEs, such as the Laplace problem, Maxwell's equations, etc., in a non-smooth and non-convex domain. In order to define the SCM, let us recall first some basic ingredients of Domain Decomposition Methods (or DDM).

Consider the variational problem (with obvious notations)
find $u \in V$ *such that*

$$a(u, v) = l(v), \quad \forall v \in V. \tag{1}$$

In order to solve it, one can use a DDM, which generally consists in splitting the Hilbert space V into the sum of K subspaces

$$V = V_1 + V_2 + \cdots + V_K, \tag{2}$$

and then getting the solution u of (1), *via* some solves of subproblems such as
find $u_i \in V_i$ *such that*

$$a_i(u_i, v_i) = l_i(v_i), \quad \forall v_i \in V_i, \quad 1 \leq i \leq K. \tag{3}$$

This can be achieved iteratively or not. The aim is primarily to reduce the overall amount of work, necessary to compute a good numerical approximation of the solution. When the discretization of the problem is achieved by a Finite Element Method (or FEM), one usually obtains the splitting (2) with the help of a partition of the mesh.

The philosophy of the SCM is different, although the tools are similar: the idea is still to split the space V, but with respect to *regularity*.
Indeed, elements of V belong to the scale of Sobolev spaces $H^\alpha(\Omega)$, or $H^\alpha(\Omega)^n$, where $\Omega \subset \mathbb{R}^n$ is the computational domain, and $\alpha \in \mathbb{R}^+$. Interestingly, for a given space V, the supremum $\alpha_{\min}$ of all possible values of the exponent α, depends on the convexity of the domain and on the smoothness of its boundary. Let α_0 be the supremum when the domain is convex, or smooth. When the domain is non-convex and non-smooth, $\alpha_{\min} < \alpha_0$ usually holds.
Then, let $V_R = V \cap H^{\alpha_0}(\Omega)$ (or $V_R = V \cap H^{\alpha_0}(\Omega)^n$ for vector fields) be the space of *regular* elements. Assume that V_R is *closed* in V, and let

$$V = V_R \oplus V_S \tag{4}$$

with V_S the space of *singular* elements. The sum is *direct*; in addition, it can be *orthogonal*. When the domain is convex or smooth, one has $V_S = \{0\}$ by definition. Then, regular elements are approximated by a *Lagrange FEM*, whereas elements of V_S are computed in a manner, which depends on the problem to solve: in other words, the idea behind the SCM is to *enlarge* the space of test-functions. Basically, it is designed to achieve the following results:

- Improve the *convergence rate* (for the Laplace problem),
- Capture numerically the *real* solution (for Maxwell's equations).

In what follows, we shall introduce, in Section 1, the SCM for the Laplace problem and for Maxwell's equations in a polyhedron. We describe the main theoretical results that are required to solve electromagnetic problems and, in particular, we emphasize the strong links between the singular elements for both problems. In Sections 2 and 3, we present the theory, and the numerical tools, which we have developed, to solve the static, time-harmonic and time-dependent Maxwell equations in a polygon of $\mathbb{R}^2$, or in an axisymmetric domain of $\mathbb{R}^3$.

1 The problems in a polyhedron

Let Ω be a bounded, simply connected, Lipschitz polyhedron, Γ its connected boundary, $(\Gamma_k)_{1\le k\le F}$ the set of faces, and $\mathbf{n}$ the unit outward normal to Γ. The L^2-scalar product is denoted by $(f,g)_0$, the associated norm by $\|f\|_0$. We shall use the differential operators div, $\mathbf{curl}$ and the related 'non-standard' Sobolev spaces and norms

$$\begin{aligned}
&\mathbf{L}^2(\Omega) := \{\mathbf{v} = (v_1, v_2, v_3)^T :\ v_i \in L^2(\Omega),\ 1 \le i \le 3\},\\
&\|\mathbf{v}\|_0 := \left(\|v_1\|_0^2 + \|v_2\|_0^2 + \|v_3\|_0^2\right)^{1/2};\\
&\mathbf{H}(\mathrm{div}\,,\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) :\ \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\},\\
&\|\mathbf{v}\|_{0,\mathrm{div}} := \left(\|\mathbf{v}\|_0^2 + \|\mathrm{div}\,\mathbf{v}\|_0^2\right)^{1/2};\\
&\mathbf{H}(\mathbf{curl}\,,\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) :\ \mathbf{curl}\,\mathbf{v} \in \mathbf{L}^2(\Omega)\},\\
&\|\mathbf{v}\|_{0,\mathbf{curl}} := \left(\|\mathbf{v}\|_0^2 + \|\mathbf{curl}\,\mathbf{v}\|_0^2\right)^{1/2};\\
&\mathbf{H}(\mathbf{curl}\,,\mathrm{div}\,,\Omega) := \mathbf{H}(\mathbf{curl}\,,\Omega) \cap \mathbf{H}(\mathrm{div}\,,\Omega),\\
&\|\mathbf{v}\|_{0,\mathbf{curl}\,,\mathrm{div}} := \left(\|\mathbf{v}\|_0^2 + \|\mathbf{curl}\,\mathbf{v}\|_0^2 + \|\mathrm{div}\,\mathbf{v}\|_0^2\right)^{1/2}, \text{ and}
\end{aligned}$$

$$|\mathbf{v}|_{\mathbf{curl}\,,\mathrm{div}} := \left(\|\mathbf{curl}\,\mathbf{v}\|_0^2 + \|\mathrm{div}\,\mathbf{v}\|_0^2\right)^{1/2}.$$

In addition, the usual Sobolev spaces for vector fields shall be written $\mathbf{H}^s(\Omega)$, and $\mathbf{H}^s(\Gamma)$. Then, let us recall that fields of $\mathbf{H}(\mathrm{div}\,,\Omega)$ (resp. $\mathbf{H}(\mathbf{curl}\,,\Omega)$) have a normal trace (resp. tangential components) on Γ, which belongs to $H^{-1/2}(\Gamma)$ (resp. $\mathbf{H}^{-1/2}(\Gamma)$); this allows to define the subspaces with the vanishing corresponding trace, and

$$\mathcal{X} := \mathbf{H}_0(\mathbf{curl}\,,\Omega) \cap \mathbf{H}(\mathrm{div}\,,\Omega), \quad \mathcal{Y} := \mathbf{H}(\mathbf{curl}\,,\Omega) \cap \mathbf{H}_0(\mathrm{div}\,,\Omega).$$

Let us state the Weber inequality, which stems from the compact embedding results of Weber [34].

Proposition 1.1 *In $\mathcal{X}$ and $\mathcal{Y}$, the semi-norm $|\cdot|_{\mathbf{curl}\,,\mathrm{div}}$ is a norm, which is equivalent to the full norm.*

Last, let us mention that one can generalize what we state below, to the case of a Lipschitz *curvilinear* polyhedron, by using the work of Costabel *et al.* [19].

1.1 The Laplace problem

The model problem is, given $f \in L^2(\Omega)$, solve
find $\phi \in H^1_0(\Omega)$ *such that*

$$-\Delta\phi = f \text{ in } \Omega. \tag{5}$$

The regularity of the solution depends on the geometry of the domain [21, 20]. Let us call *minimal regularity* of the solution the supremum of the set

$$\{\alpha \in \mathbb{R} : \ \forall f \in L^2(\Omega),\ \phi \text{ solution of (5) belongs to } H^\alpha(\Omega)\}.$$

Theorem 1.2 *If Ω is convex, the minimal regularity is $\alpha_0 = 2$.*
If Ω is non-convex, the minimal regularity is $\alpha_{\min} = 3/2 + \sigma$, with $0 < \sigma < 1/2$ depending on the geometry, i.e. conical angles at reentrant vertices, dihedral angles at reentrant edges.

In the non-convex case, by minimal regularity, we mean that all solutions ϕ belong to $H^{3/2+\sigma-\varepsilon}(\Omega)$, for any $\varepsilon > 0$, and that some do not belong to $H^{3/2+\sigma}(\Omega)$.

If one discretizes (5) with the P_1 Lagrange FEM, with h as the meshsize, there holds by standard analysis

Corollary 1.3 *If Ω is convex, the convergence rate in H^1-norm is in $O(h)$.*
If Ω is non-convex, the convergence rate in H^1-norm is in $O(h^{1/2+\sigma-\varepsilon})$, $\forall\varepsilon$.

Remark 1.4 *Here, it is crucial to impose $f \in L^2(\Omega)$. If f is only in $H^{-1}(\Omega)$, the regularity of ϕ can be as low as $\phi \in H^1(\Omega)$, for Ω convex or not: the convergence rate is undetermined, and there are no methods that allow to improve it.*

To improve the convergence rate, one can think of: mesh refinement, the (Dual) Singular Function Method, multigrid methods [15], the SCM, etc.

The mesh refinement techniques are well-kwown [33]. So are the (Dual) Singular Function Methods (or (D)SFM), which work in 2d domains, see for instance [21, 22]. They are based on the adjunction of test-functions, the (dual) singular functions, to the space of FE.
The SCM is based on the same idea, as mentioned in the Introduction. Its origin (2d case) can be traced back to Moussaoui [29]. More precisely, let

$$\Phi := \{\phi \in H^1_0(\Omega) : \ \Delta\phi \in L^2(\Omega)\}, \text{ and } \Phi_R := \Phi \cap H^2(\Omega).$$

One has the

Theorem 1.5 *In Φ, $\|\phi\|_\Phi := \|\Delta\phi\|_0$ is a norm, which is equivalent to the graph norm $\phi \mapsto \{\|\phi\|_1^2 + \|\Delta\phi\|_0^2\}^{1/2}$. As a consequence, $\|\cdot\|_\Phi$ is equivalent to $\|\cdot\|_2$ in Φ_R.*

Proof : Thanks to the POINCARÉ inequality, the graph norm is equivalent to $\{|\phi|_1^2 + \|\Delta\phi\|_0^2\}^{1/2} = \|\mathbf{grad}\,\phi\|_{0,\mathbf{curl}\,,\mathrm{div}}$, with $\mathbf{grad}\,\phi$ in $\mathcal{X}$. Now, one infers from the WEBER inequality that it is also equivalent to $\|\Delta\phi\|_0$.
To prove the other half, let $\phi \in \Phi_R$. There holds

$$\|\phi\|_\Phi^2 = |\mathbf{grad}\,\phi|_{\mathbf{curl}\,,\mathrm{div}}^2 = \|\mathbf{grad}\,(\mathbf{grad}\,\phi)\|_0^2 = |\phi|_2^2,$$

where the second equality has been obtained by Costabel *et al* [17, 19], as $\mathbf{grad}\,\phi$ belongs to $\mathbf{H}^1(\Omega)$ and has vanishing tangential components on Γ. Finally, one can use the first part of the Theorem to conclude. ∎

Corollary 1.6 *$\Delta\Phi_R$ is closed in $L^2(\Omega)$.*

Starting from this result, one can first define its orthogonal, called N:

$$L^2(\Omega) = \Delta\Phi_R \overset{\perp}{\oplus} N, \tag{6}$$

and then Φ_S, the inverse image of N. By construction, both Φ_R and Φ_S are closed in Φ and so $\Phi = \Phi_R \overset{\perp}{\oplus} \Phi_S$, i.e. (4). Now, following [2], it is possible to characterize elements of N, and, as consequence, elements of Φ_S.

Theorem 1.7 *An element p of $L^2(\Omega)$ belongs to N if and only if*

$$\Delta p = 0 \ \textit{in}\ \Omega, \quad p_{|\Gamma_k} = 0 \ \textit{in}\ H_{00}^{-1/2}(\Gamma_k),\ 1 \le k \le F.$$

(Recall that $H_{00}^{1/2}(\Gamma_k) := \{f \in H^{1/2}(\Gamma_k) :\ \rho_k^{-1/2} f \in L^2(\Gamma_k)\}$, where ρ_k denotes the distance to the boundary of Γ_k; $H_{00}^{-1/2}(\Gamma_k)$ is the dual space of $H_{00}^{1/2}(\Gamma_k)$.)
As for the numerical computation of elements of N and Φ_S, see the next Section for problems in axisymmetric domains and §3.2 for problems in 2d. Let us mention that in the 2d case (see [29]), one gets results similar to those of the DSFM, that is, the recovery of an overall convergence rate in $O(h)$ in H^1-norm.

Let us conclude this Subsection by some extensions.
The first one is the *homogeneous Neumann problem*, for which the same theory can be developed in $\Psi/\mathbb{R}$, where
$\Psi := \{\psi \in H^1(\Omega) :\ \partial_n \psi_{|\Gamma} = 0 \text{ on } \Gamma,\ \Delta\psi \in L^2(\Omega)\}$.
Another one is about the scalar *wave equation* which, given $T > 0$, reads
find $\phi(t) \in H_0^1(\Omega)$ such that

$$\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = f \text{ in } \Omega\times]0,T[, \quad \phi(0) = \phi_0. \tag{7}$$

The theory of Lions and Magenes [27] leads to

Theorem 1.8 *Assume that $f \in L^2(0,T;H_0^1(\Omega))$ and $\phi_0 \in \Phi$. Then, there exists one and only one solution of the wave equation (7), with regularity*

$$\phi \in \mathcal{C}^0(0,T;\Phi) \cap \mathcal{C}^1(0,T;H_0^1(\Omega)).$$

Next, from (4) applied to Φ, there comes the *continuous decomposition in time* of the solution, that is the

Corollary 1.9 *One can write* $\phi(t) = \phi_R(t) + \phi_S(t)$ *for all* t, *with*

$$(\phi_R, \phi_S) \in \mathcal{C}^0(0, T; \Phi_R \times \Phi_S).$$

Finally, one could use the same kind of idea for a *non-homogeneous boundary condition*, provided that the data is smooth enough on Γ, or for problems with jumps.

1.2 Mathematical tools for Maxwell's equations

We consider the electromagnetic fields in vacuum, enclosed by a perfectly conducting material. The electric permittivity and magnetic permeability are set to one. The electromagnetic field is denoted by $(\mathcal{E}, \mathcal{B})$. The sets of equations are :
The *time-dependent* Maxwell equations in $(\mathcal{E}, \mathcal{B})$:

$$\left\{ \begin{array}{l} \partial_t \mathcal{E} - \mathbf{curl}\, \mathcal{B} = -\mathcal{J},\ \partial_t \mathcal{B} + \mathbf{curl}\, \mathcal{E} = 0 \text{ in } \Omega \times]0, T[, \\ \operatorname{div} \mathcal{E} = \rho,\ \operatorname{div} \mathcal{B} = 0 \text{ in } \Omega \times]0, T[, \\ \mathcal{E} \times \mathbf{n} = 0 \text{ on } \Gamma \times]0, T[, \\ \mathcal{E}(0) = \mathcal{E}_0,\ \mathcal{B}(0) = \mathcal{B}_0. \end{array} \right. \tag{8}$$

The *time-harmonic* Maxwell equations on $\mathcal{E}$, a *complex-valued* field:

$$\left\{ \begin{array}{l} \mathbf{curl}\, \mathbf{curl}\, \mathcal{E} - k^2 \mathcal{E} = \mathcal{J} \text{ in } \Omega, \\ \operatorname{div} \mathcal{E} = 0 \text{ in } \Omega, \\ \mathcal{E} \times \mathbf{n} = 0 \text{ on } \Gamma. \end{array} \right. \tag{9}$$

The *static* Maxwell equations, with $\mathcal{U}$ being either the electrostatic or the magnetostatic field:

$$\left\{ \begin{array}{l} \mathbf{curl} \mathcal{U} = \mathcal{F} \text{ in } \Omega, \\ \operatorname{div} \mathcal{U} = \mathcal{G} \text{ in } \Omega, \\ \mathcal{U} \times \mathbf{n} = 0 \text{ on } \Gamma, \text{ or } \mathcal{U} \cdot \mathbf{n} = 0 \text{ on } \Gamma. \end{array} \right. \tag{10}$$

Unless otherwise specified, we consider that (10) is the electrostatic problem.

Let us say a few words on the existence and uniqueness of the solution of each problem (cf. [3, 14, 16], in this order).

Theorem 1.10 The time-dependent problem.
Assume that $\mathcal{J} \in \mathcal{C}^0(0, T; \mathbf{H}(\operatorname{div}, \Omega)) \cap H^1(0, T; \mathbf{L}^2(\Omega))$ *and* $\rho \in \mathcal{C}^1(0, T; L^2(\Omega))$. *Then, there exists one and only one solution* $(\mathcal{E}, \mathcal{B})$ *of (8), with*

$$(\mathcal{E}, \mathcal{B}) \in \mathcal{C}^0(0, T; \mathcal{X} \times \mathcal{Y}) \cap \mathcal{C}^1(0, T; \mathbf{H}(\operatorname{div}, \Omega) \times \mathbf{H}(\operatorname{div}, \Omega)).$$

The time-harmonic problem.
Assume that $\mathcal{J}$ *belongs to* $\mathbf{H}(\operatorname{div}, \Omega)$, $\operatorname{div} \mathcal{J} = 0$, *and* $Im(k) \neq 0$. *Then, there*

exists one and only one solution $\mathcal{E}$ solution of (9) in the 'complexified' $\mathcal{X}$.

The static problem.
Assume that $\mathcal{F}$ belongs to $\mathbf{H}_0(\mathrm{div}\,,\Omega)$, with $\mathrm{div}\,\mathcal{F}=0$, and that $\mathcal{G}$ is in $L^2(\Omega)$. Then, there exists one and only one solution $\mathcal{E}$ solution of (10) in $\mathcal{X}$.

Here, we considered that the data is L^2-regular. Actually, this is *equivalent* to the assumption that we made previously for the Laplace problem, i.e. that the Laplacian of the solution is in $L^2(\Omega)$.

As for the regularity of the solution, one finds again that it depends on the geometry of the domain [1]: let us consider, for instance, the static field $\mathcal{U}$.

Theorem 1.11 *If Ω is convex, the minimal regularity is $\alpha_0 = 1$.
If Ω is non-convex, the minimal regularity is $\alpha_{\min} = 1/2 + \sigma$.*

In the case of Maxwell's equations, we thus let

$$\mathcal{X}_R := \mathcal{X} \cap \mathbf{H}^1(\Omega), \text{ and } \mathcal{Y}_R := \mathcal{Y} \cap \mathbf{H}^1(\Omega).$$

The original idea was to take advantage of the H^1-regularity of the field, when the domain is *convex* [8], to discretize it by the P_1 Lagrange FEM, instead of the 'usual' edge FEM [30, 31]. As a matter of fact, the former includes two key ingredients, which the latter lacks:

- For the time-dependent Maxwell equations, the mass matrix can be lumped, with no loss in precision, thus leading to very inexpensive numerical schemes.
- The numerical electromagnetic field is continuous, so the method can be used in conjunction with a particle-pushing scheme, to solve the coupled Vlasov-Maxwell system of equations.

The question to be answered is: what happens when Ω is a *non-convex domain*? For that, let us begin with the

Theorem 1.12 *$\mathcal{X}_R$ (resp. $\mathcal{Y}_R$) is closed in $\mathcal{X}$ (resp. $\mathcal{Y}$).
Therefore, when Ω is non-convex, $\mathcal{X}_R$ (resp. $\mathcal{Y}_R$) is not dense in $\mathcal{X}$ (resp. $\mathcal{Y}$).*

Proof : The norm in $\mathcal{X}$ is $|\cdot|_{\mathbf{curl}\,,\mathrm{div}}$. With the help of the formula [17, 19]:

$$(\mathbf{grad}\,\mathbf{u}, \mathbf{grad}\,\mathbf{v})_0 = (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})_0,\ \forall \mathbf{u}, \mathbf{v} \in \mathcal{X}_R \times \mathcal{X}_R, \tag{11}$$

one gets that the norm in $\mathcal{X}_R$ is equivalent to the $\mathbf{H}^1$-norm, and thus $\mathcal{X}_R$ is closed in $\mathcal{X}$. As a consequence, $\mathcal{X}_R$ is dense in $\mathcal{X}$ iff $\mathcal{X}_R = \mathcal{X}$. According to Theorem 1.11, this is not the case when Ω is non-convex.
(The proof for $\mathcal{Y}$ and its regular subspace is identical.) ■

The immediate consequence is that one *can not capture numerically* the solution of the above problems, with the help of the Lagrange FEM only, if the

solution is not in the regular space. In particular, mesh refinement techniques *do not work*.
As a matter of fact, let us split $\mathcal{X}$ *à la* (4), $\mathcal{X} = \mathcal{X}_R \overset{\perp}{\oplus} \mathcal{X}_S$, with $\mathcal{X}_S = \mathcal{X}_R^{\perp}$. Is is clear, from the definition of $\mathcal{X}_R$, that any subspace of $\mathcal{X}$ generated by the P_1 Lagrange FEM is actually a subspace of $\mathcal{X}_R$. Thus, with self-explanatory notations, (4) leads to

$$\|\mathcal{E} - \mathcal{E}^h\|_{\mathcal{X}}^2 = \|\mathcal{E}_R - \mathcal{E}^h\|_{\mathcal{X}}^2 + \|\mathcal{E}_S\|_{\mathcal{X}}^2 \geq \|\mathcal{E}_S\|_{\mathcal{X}}^2.$$

Is there a hope of finding an intermediate solution, between the edge FEM, and the P_1 Lagrange FEM? The answer is clearly 'no', if one looks for a piecewise smooth FE (i.e. a FE, whose restriction to each element of the triangulation is smooth), like the edge or Lagrange FEMs. Indeed, it has been remarked by Hazard and Lenoir [24] that any $\mathbf{H}(\mathbf{curl}\,, \mathrm{div}\,)$-conforming FEM, with a piecewise smooth FE, is actually $\mathbf{H}^1$-conforming.

Therefore, it is required that one adds the SCM (or the SFM) to be able to compute an approximation of the solution[1]. One discretizes the regular part with the P_1 Lagrange FEM, which means a P_1 approximation component by component, and taking into account the boundary condition. Evidently, this method can be applied to all three Maxwell problems: time-dependent (8), time-harmonic (9) or static (10).
Now, how can one *approximate the singular part*? One possible idea, that we develop further in the other Sections, is to relate the singular electric fields to singular elements of the Laplace operator, i.e. to elements of Φ_S.
Let us conclude this Subsection by displaying this relationship. For that, we need a result, obtained by Birman and Solomyak [11].

Theorem 1.13 *Let Ω be a bounded Lipschitz domain. Then, for all $\mathbf{u}$ in $\mathcal{X}$, there exist $\mathbf{u}_0$ in $\mathcal{X}_R$ and $\phi \in \Phi$ such that*

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{grad}\,\phi, \quad ||\mathbf{u}_0||_1^2 + ||\Delta\phi||_0^2 \leq C\,||\mathbf{u}||_{0,\mathbf{curl}\,,\mathrm{div}}^2. \tag{12}$$

Here, C denotes a nonnegative constant, which is independent of $\mathbf{u}$.

In $\mathcal{Y}$, they proved the same result, provided that the domain has a piecewise-smooth boundary [11, 12]. As a consequence, one can prove the

Theorem 1.14 *The following decomposition is direct and continuous*

$$\mathcal{X} = \mathcal{X}_R \overset{c}{\oplus} \mathbf{grad}\,\Phi_S.$$

Proof : From (12), it is clear that $\mathcal{X} = \mathcal{X}_R + \mathbf{grad}\,\Phi_S$.
Then, let $\mathbf{v} \in \mathcal{X}_R \cap \mathbf{grad}\,\Phi_S$: by construction, $\mathbf{v} \in \mathbf{H}^1(\Omega) \cap \mathbf{grad}\,\Phi$, i.e. $\mathbf{v} \in \mathbf{grad}\,\Phi_R$. Also, one infers from (4) applied to Φ that $\mathbf{grad}\,\Phi$ can be split (in

[1] Another alternative is to use the edge FEM, possibly with a specifically designed SCM.

$\mathcal{X}$) into $\mathbf{grad}\,\Phi_R \oplus \mathbf{grad}\,\Phi_S$. So, $\mathbf{v} = 0$, and the sum is direct.
Last, the application

$$\begin{array}{l} \mathcal{X}_R \times \mathbf{grad}\,\Phi_S \to \mathcal{X} \\ (\mathbf{v}_R, \mathbf{grad}\,\phi_S) \mapsto \mathbf{v} = \mathbf{v}_R + \mathbf{grad}\,\phi_S \end{array}$$

is linear, continuous and bijective. Now, as $\mathcal{X}_R \times \mathbf{grad}\,\Phi_S$ and $\mathcal{X}$ are Banach spaces, the open mapping Theorem allows to conclude that the inverse of the application is also continuous. ∎

Again, one can prove the same type of result on $\mathcal{Y}$. In other words, the singular electric or magnetic fields are *one-to-one* with the gradients of the singular elements of the Laplacian.

2 Maxwell's equations in an axisymmetric domain

Let Ω be the domain limited by a surface of *revolution* Γ; ω and γ_b their intersections with a meridian half-plane. One has $\gamma := \partial\omega = \gamma_a \cup \gamma_b$, where γ_a is the segment of the axis lying between the extremities of γ_b. ν is its unit outward normal, and τ the unit tangential vector such that (τ, ν) is direct.

Moreover, it is assumed that γ_b is a polygonal line with edges $(\gamma_k)_{1\le k\le F}$. The Γ_k are the corresponding faces of Γ; the off-axis corners of γ_b generate circular edges in Γ, whereas the extremities are conical vertices of Γ.

The natural coordinates for this domain are the cylindrical coordinates (r, θ, z), with the basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$. A meridian half-plane is defined by the equation $\theta = cst$, and (r, z) are cartesian coordinates in this half-plane.

Definition 2.1 *For any vector field, the meridian and azimuthal components of* $\mathbf{u}$ *are resp.* $\mathbf{u}_m := \varpi_m(\mathbf{u}) := u_r\,\mathbf{e}_r + u_z\,\mathbf{e}_z$ *and* $\mathbf{u}_\theta := \varpi_\theta(\mathbf{u}) := u_\theta\,\mathbf{e}_\theta$.

We are interested in the case where the sources of the electromagnetic fields, and hence the fields themselves, possess a *symmetry of revolution.* This fact means that the scalar (resp. vector) fields are entirely characterized by their "trace" in ω, i.e. the datum of their value in a meridian half-plane (resp. by the trace of their cylindrical components). Obviously, this is equivalent to the vanishing of all derivatives with respect to θ of these fields or components. In this Section, it is thus assumed that $\partial_\theta \cdot = 0$.

Proposition 2.2 *For any axisymmetric vector field* $\mathbf{u}$*, the following identities hold:* $\mathbf{curl}\,\mathbf{u}_m = \varpi_\theta(\mathbf{curl}\,\mathbf{u})$, $\mathbf{curl}\,\mathbf{u}_\theta = \varpi_m(\mathbf{curl}\,\mathbf{u})$, $\operatorname{div}\mathbf{u}_m = \operatorname{div}\mathbf{u}$, $\Delta\mathbf{u}_m = \varpi_m(\Delta\mathbf{u})$, $\Delta\mathbf{u}_\theta = \varpi_\theta(\Delta\mathbf{u})$. *Hence, if* $\mathbf{u}$ *is meridian* ($\varpi_\theta(\mathbf{u}) = 0$)*,* $\mathbf{curl}\,\mathbf{u}$ *is azimuthal and* $\Delta\mathbf{u}$ *is meridian; if* $\mathbf{u}$ *is azimuthal* ($\varpi_m(\mathbf{u}) = 0$)*,* $\mathbf{curl}\,\mathbf{u}$ *is meridian,* $\Delta\mathbf{u}$ *is azimuthal and* $\operatorname{div}\mathbf{u} = 0$*,*

A similar property holds for the Jacobian of an axisymmetric vector field: there is a decoupling of the meridian and azimuthal components.

Finally, as the meridian and azimuthal components of vector fields are mutually orthogonal pointwise, the same is true in the sense of the $\mathbf{L}^2(\Omega)$ scalar product: for $(\mathbf{u}, \mathbf{v}) \in [\mathbf{L}^2(\Omega)]^2$, there holds $(\mathbf{u}_\theta, \mathbf{v}_m)_{0,\Omega} = 0$. This property is also true for the curl and the vector Laplace operators, or the Jacobian of a field, provided that they belong to $\mathbf{L}^2(\Omega)$.

2.1 Reduction to two-dimensional problems.

Thanks to Proposition 2.2, it is possible to decouple each of the Maxwell systems (8, 9, 10) into a couple of problems set in $\Omega\times]0,T[$, involving different components of the fields $\mathcal{E}$ and $\mathcal{B}$. Given the expression of differential operators in cylindrical coordinates, these problems read as follows in $\omega\times]0,T[$.

The *time-dependent equations* (8), split into a system of unknowns $(\mathbf{E}_m, B_\theta)$:

$$\left\{ \begin{array}{l} \partial_t \mathbf{E}_m - r^{-1}\, \mathbf{curl}\,(r\, B_\theta) = -\mathbf{J}_m,\ \partial_t B_\theta + \mathrm{curl}\, \mathbf{E}_m = 0 \text{ in } \omega\times]0,T[, \\ r^{-1}\,\mathrm{div}\,(r\,\mathbf{E}_m) = \rho \text{ in } \omega\times]0,T[, \quad \mathbf{E}_m\cdot\tau = 0 \text{ on } \gamma_b\times]0,T[, \\ \mathbf{E}_m(0) = \mathbf{E}_{m0},\ B_\theta(0) = B_{\theta 0}. \end{array} \right. \tag{13}$$

and a system of unknowns $(E_\theta, \mathbf{B}_m)$:

$$\left\{ \begin{array}{l} \partial_t E_\theta - \mathrm{curl}\,\mathbf{B}_m = -J_\theta,\ \partial_t \mathbf{B}_m + r^{-1}\,\mathbf{curl}\,(r\,E_\theta) = 0 \text{ in } \omega\times]0,T[, \\ r^{-1}\,\mathrm{div}\,(r\,\mathbf{B}_m) = 0 \text{ in } \omega\times]0,T[, \quad \mathbf{B}_m\cdot\nu = 0,\ E_\theta = 0 \text{ on } \gamma_b\times]0,T[, \\ E_\theta(0) = E_{\theta 0},\ \mathbf{B}_m(0) = \mathbf{B}_{m0}. \end{array} \right. \tag{14}$$

The *static equations* (10), split into a system of unknown $\mathbf{U}_m$:

$$\left\{ \begin{array}{l} \mathrm{curl}\,\mathbf{U}_m = F_\theta \text{ in } \omega, \\ r^{-1}\,\mathrm{div}\,(r\,\mathbf{U}_m) = \mathcal{G} \text{ in } \omega, \\ \mathbf{U}_m\cdot\tau = 0 \text{ on } \gamma_b,\ \text{ or } \mathbf{U}_m\cdot\nu = 0 \text{ on } \gamma_b. \end{array} \right. \tag{15}$$

and a system of unknown U_θ:

$$\left\{ \begin{array}{l} r^{-1}\,\mathbf{curl}\,(r\,U_\theta) = \mathbf{F}_m \text{ in } \omega, \\ U_\theta = 0 \text{ on } \gamma_b,\ \text{ for the electrostatic problem only.} \end{array} \right. \tag{16}$$

2.2 Sobolev spaces

We denote by a $\breve{}$ the respective subspaces of axisymmetric vector fields in the various Sobolev spaces, e.g. $\breve{\mathbf{L}}^2(\Omega)$, $\breve{\mathbf{H}}^1(\Omega)$, $\breve{\mathbf{H}}(\mathbf{curl}\,, \mathrm{div}\,, \Omega)$, $\breve{\mathcal{X}}$, $\breve{\mathcal{Y}}_R$; by $\|\cdot\|_{s,\Omega}$ the H^s-norm, by $\|\cdot\|_{0,\mathbf{curl}\,,\mathrm{div}\,,\Omega}$ the $H(\mathbf{curl}\,, \mathrm{div}\,)$-norm.

As pointed out earlier, elements of these spaces are characterized by their trace in ω. We refer to [10] for their full description. For now, we only need the

Definition 2.3 *For $\alpha \in \mathbb{R}$, let $L^2_\alpha(\omega)$ be the space of square-integrable functions in ω with respect to the measure $r^\alpha\, dr\, dz$, and $H^s_\alpha(\omega)$, for $s \in \mathbb{R}$, the related scale of Sobolev spaces, with the canonical norms $||\cdot||_{s,\alpha,\omega}$.*

2.3 Closedness results.

The aim of this Subsection is to prove the analogue of Theorem 1.12. Because of the technicalities induced by the geometry [4], it is necessary to distinguish between the *inductive* proof for the electric field and the *constructive* proof for the magnetic field.

We shall now sketch these proofs; details are found in [4].

Lemma 2.4 *The following inequalities are equivalent:*

$$\exists C_1, \quad \forall \mathbf{u} \in \mathcal{X}_R, \quad ||\mathbf{u}||_{1,\Omega} \quad \leq \quad C_1 \, ||\mathbf{u}||_{0,\mathbf{curl}\,,\mathrm{div}\,,\Omega}, \tag{17}$$

$$\exists C_2, \quad \forall \phi \in \Phi_R, \quad ||\phi||_{2,\Omega} \quad \leq \quad C_2 \, ||\Delta\phi||_{0,\Omega}. \tag{18}$$

Proof : For $\mathbf{u} = \mathbf{grad}\,\phi$, (17) implies (18) by Poincaré's and Weber's inequalities. Conversely, (17) stems from (18) and Theorem 1.13. ■

Theorem 2.5 *(17) is satisfied in Ω if and only if all the conical angles at the vertices are different from a prescribed value $\pi/\delta_- \simeq 130°$. This case excluded, $\mathcal{X}_R$ is closed within $\mathcal{X}$.*

Proof : (17) is equivalent to (18). The necessary and sufficient condition for (18) to hold has been established by Dauge [20]. ■

In the following, when considering the *electric* case, we suppose that all conical angles are different from π/δ_-.

Theorem 2.6 *In $\breve{\mathcal{Y}}_R$, the following estimate holds:*

$$\exists K, \quad \forall \mathbf{u} \in \breve{\mathcal{Y}}_R, \quad \|\nabla\mathbf{u}\|^2_{0,\Omega} \leq K \left(\|\mathbf{curl}\,\mathbf{u}\|^2_{0,\Omega} + \|\mathrm{div}\,\mathbf{u}\|^2_{0,\Omega}\right). \tag{19}$$

Hence, by Poincaré and Weber's inequalities, the $\|\cdot\|_{1,\Omega}$ and $\|\cdot\|_{0,\mathbf{curl}\,,\mathrm{div}\,,\Omega}$ norms are equivalent on this space, and $\breve{\mathcal{Y}}_R$ is closed within $\breve{\mathcal{Y}}$.

The equivalent of (11) in Ω reads (cf. [18]): for any $(\mathbf{u},\mathbf{v}) \in \left[\mathbf{H}^2(\Omega)\right]^2$,

$$(\nabla\mathbf{u}, \nabla\mathbf{v})_{0,\Omega} = (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{0,\Omega} + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})_{0,\Omega} - b(\mathbf{u},\mathbf{v}) + d(\mathbf{u},\mathbf{v}), \tag{20}$$

where $b(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are bilinear forms defined on the boundary. The term $d(\mathbf{u},\mathbf{v})$ vanishes when $(\mathbf{u},\mathbf{v}) \in [\mathcal{Y} \cap \mathbf{H}^2(\Omega)]^2$. It is proven in [4] that this space is dense within $\mathcal{Y}_R$. So one can extend d by 0 to $\mathcal{Y}_R$.

All other terms in (20) are meaningful for $(\mathbf{u},\mathbf{v}) \in \left[\mathbf{H}^1(\Omega)\right]^2$: for an axisymmetric domain Ω, the bilinear form $b(\mathbf{u},\mathbf{v})$ is $\displaystyle\int_\Gamma \frac{\nu_r}{r}\,(u_\theta\, v_\theta + u_\nu\, v_\nu)\; d\Gamma$. Hence (20) is valid for $(\mathbf{u},\mathbf{v}) \in [\mathcal{Y}_R]^2$, with $u_\nu = v_\nu = 0$ on the boundary.

The inequality (19) is now equivalent, thanks to (20), to

$$\exists k < 1, \quad -b(\mathbf{u},\mathbf{u}) \leq k\, \|\nabla\mathbf{u}\|^2_{0,\Omega}\,. \tag{21}$$

Since $\nabla\mathbf{u}_m$ and $\nabla\mathbf{u}_\theta$ are $\mathbf{L}^2$-orthogonal, and $b(\mathbf{u},\mathbf{u})$ depends only on $\mathbf{u}_\theta$, it is sufficient to check (21)—or (19)—for $\mathbf{u} \in E^1_\theta = \left\{\mathbf{u} \in \breve{\mathbf{H}}^1(\Omega) : \mathbf{u} \parallel \mathbf{e}_\theta\right\}$.

Lemma 2.7 *The space $H^1_{-1}(\omega)$ is continuously imbedded into $L^2_{-3}(\omega)$, i.e.*

$$\exists K_1, \quad \forall u \in H^1_{-1}(\omega), \quad ||u|||^2_{0,-3,\omega} \leq K_1\, ||\mathbf{grad}\, u||^2_{0,-1,\omega}.$$

This 2d Hardy inequality is obtained by localization and Fubini Theorem.

Proposition 2.8 *The inequality (19) is satisfied for all $\mathbf{u} \in \breve{\mathcal{Y}}_R$.*

Proof : Let $\mathbf{u} \in E^1_\theta$ and $v = r\, u_\theta$. From the expressions of $\mathbf{curl}\,\mathbf{u}$, $\operatorname{div}\mathbf{u}$ and $\nabla\mathbf{u}$ in cylindrical coordinates, it follows: $\|\mathbf{curl}\,\mathbf{u}\|^2_{0,\Omega}+\|\operatorname{div}\mathbf{u}\|^2_{0,\Omega} = 2\pi\, \|\mathbf{grad}\, v\|^2_{0,-1,\omega}$ and $\|\nabla\mathbf{u}\|^2_{0,\Omega} = 2\pi\, \left[\|\mathbf{grad}\, u_\theta\|^2_{0,1,\omega} + \|u_\theta\|^2_{0,-1,\omega}\right]$. The latter norm is equivalent to $\|\mathbf{grad}\, v\|^2_{0,-1,\omega} + \|v\|^2_{0,-3,\omega}$. Thus (19) stems from the above Lemma; it also proves that *any* azimuthal vector field in $\breve{\mathbf{H}}(\mathbf{curl}\,, \operatorname{div}\,, \Omega)$ is in $\breve{\mathbf{H}}^1(\Omega)$. ∎

2.4 A characterization of singular fields

This Subsection describes the relationship between the singular electric and magnetic fields and scalar singularities of Laplace-like operators.

Electric case. Let $\breve{\mathcal{X}}$ be the natural space of electric fields, and $\breve{\mathcal{X}}_R$ its regular subspace. We derive from Theorem 2.5 the direct and continuous decomposition

$$\breve{\mathcal{X}} = \breve{\mathcal{X}}_R \overset{c}{\oplus} \mathbf{grad}\, \breve{\Phi}_S. \tag{22}$$

As the elements of $\breve{\Phi}_S$ are characterized by their Laplacian, we will study $\breve{N} = \Delta\breve{\Phi}_S$. For this purpose, we shall adapt the method of [7, 5] and the references therein, with a specific treatment for the conical vertices. To that end, on any face Γ_k, let $\breve{H}(\Gamma_k)$ be the axisymmetric subspace of $H^{1/2}_{00}(\Gamma_k)$.

Lemma 2.9 *The application γ^k_1, which is the trace on Γ_k of the normal derivative, is continuous and surjective from $\breve{\Phi}_R$ onto $\breve{H}(\Gamma_k)$, and there exists a continuous lifting operator from $\breve{H}(\Gamma_k)$ to $\breve{\Phi}_R$.*

This result allows to prove an integration by parts formula, between elements of $\breve{\Phi}_R$ and elements of the space $D(\Delta,\Omega) := \{g \in L^2(\Omega) :\ \Delta g \in L^2(\Omega)\}$.

Lemma 2.10 *Let $p \in D(\Delta,\Omega)$ and $u \in \breve{\Phi}_R$. There holds*

$$\int_\Omega (p\,\Delta u - u\,\Delta p)\, d\Omega = \sum_{1\leq k\leq F} {}_{\breve{H}(\Gamma_k)'}\left\langle p, \gamma^k_1 u\right\rangle_{\breve{H}(\Gamma_k)}.$$

The first characterization of $\breve{N}$ follows from the above Lemmas.

Theorem 2.11 *Let $p \in \breve{L}^2(\Omega)$: p belongs to $\breve{N}$ if and only if*

$$\Delta p = 0 \text{ in } \Omega, \quad p_{|\Gamma_k} = 0 \text{ in } \breve{H}(\Gamma_k)',\ 1 \leq k \leq F.$$

In a meridian half-plane, the second characterization of elements of $\breve{N}$ is then

Corollary 2.12 *Let $p \in L^2_1(\omega)$: p belongs to $\breve{N}$ if and only if*

$$\begin{aligned} &\Delta^+ p := \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} = 0 \ \textit{in } \omega, \\ &p_{|\gamma_k} = 0,\ 1 \le k \le F, \\ &p \in \mathcal{C}^\infty(\overline{\varpi} \setminus \mathcal{V}_b),\ \textit{for any neighborhood } \mathcal{V}_b \textit{ of } \gamma_b. \end{aligned}$$

The study of the Laplace-like operator Δ^+ is performed in [9]. It extends Grisvard's work [22] to the axisymmetric case:

Theorem 2.13 *The space $\breve{N}$, and consequently $\breve{\mathcal{X}}_S$, is of finite dimension, equal to $K_e + K_v$, with K_e the number of reentrant edges, et K_v the number of vertices with conical angle larger than π/δ_-.*

Magnetic case. The *natural space* of axisymmetric magnetic fields is

$$\breve{\mathcal{W}} = \{\mathbf{v} \in \breve{\mathcal{Y}} : \ \operatorname{div} \mathbf{v} = 0\}, \text{ with norm } ||\mathbf{curl}\,\mathbf{v}||_{0,\Omega}. \tag{23}$$

Then, if $\breve{\mathcal{W}}_R = \breve{\mathcal{W}} \cap \breve{\mathbf{H}}^1(\Omega)$ is the space of regular fields, we infer from Theorem 2.6 that $\breve{\mathcal{W}}_R$ is closed in $\breve{\mathcal{W}}$. Let $\breve{\mathcal{W}}_S$ be its orthogonal, i.e.

$$\breve{\mathcal{W}} = \breve{\mathcal{W}}_R \overset{\perp}{\oplus} \breve{\mathcal{W}}_S. \tag{24}$$

We had remarked in the proof of Proposition 2.8 that an azimuthal field is always regular; hence, the singular fields are meridian. Moreover, elements of $\breve{\mathcal{W}}$ are determined *via* their curl. So, given $\mathcal{B}_S \in \breve{\mathcal{W}}_S$, define $\mathcal{P} = \mathbf{curl}\,\mathcal{B}_S$: $\mathcal{B}_S$ is meridian, therefore $\mathcal{P}$ is azimuthal.

Now, let $\breve{\mathcal{M}}_R$ be the space $\mathbf{curl}^{-1}\breve{\mathcal{W}}_R$ of potentials of elements of $\breve{\mathcal{W}}_R$. The orthogonality, in the sense of (23), of $\mathcal{B}_S$ and elements of $\breve{\mathcal{W}}_R$ becomes

$$(\mathcal{P}, \Delta\mathcal{A})_{0,\Omega} = 0, \quad \forall \mathcal{A} \in \breve{\mathcal{M}}_R$$

as $\Delta = -\mathbf{curl}\,\mathbf{curl} + \mathbf{grad}\,\operatorname{div}$. As $\mathcal{P}$ is azimuthal, it is enough to consider only elements of $\breve{\mathcal{M}}_{\theta R} = \{\mathcal{A} \in \breve{\mathcal{M}}_R : \ \mathcal{A} \parallel \mathbf{e}_\theta\}$. So, we are left with a scalar problem, similar to the electric case, and we obtain the two characterizations of $\mathbf{curl}\,\breve{\mathcal{W}}_S$.

Theorem 2.14 *Let $\mathcal{P} \in \breve{\mathbf{L}}^2(\Omega)$, $\mathcal{P} \parallel \mathbf{e}_\theta$; $\mathcal{P} = P_\theta\,\mathbf{e}_\theta$ belongs to $\mathbf{curl}\,\breve{\mathcal{W}}_S$ iff*

$$\Delta\mathcal{P} = 0 \ \textit{in } \Omega, \quad P_{\theta|\Gamma_k} = 0 \ \textit{in } \breve{H}(\Gamma_k)',\ 1 \le k \le F.$$

Corollary 2.15 *Let $P_\theta = p/r$: $p \in L^2_{-1}(\omega)$ is characterized as a solution of*

$$\begin{aligned} &\Delta^- p := \frac{\partial^2 p}{\partial r^2} - \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} = 0 \ \textit{in } \omega, \\ &p_{|\gamma_k} = 0,\ 1 \le k \le F, \\ &p/r \in \mathcal{C}^\infty(\overline{\varpi} \setminus \mathcal{V}_b),\ \textit{for any neighborhood } \mathcal{V}_b \textit{ of } \gamma_b. \end{aligned}$$

Again, the study of the operator Δ^- (cf. [9]) gives the equivalent of Grisvard's result [22] in this case:

Theorem 2.16 *The space defined by Corollary 2.15, and consequently $\breve{\mathcal{W}}_S$, is of finite dimension, equal to K_e, the number of reentrant edges.*

Now, it is more convenient for numerical computations to use the variable $P = P_\theta$. It satisfies:

$$P \in L^2_1(\omega),\ \Delta' P := \frac{\partial^2 P}{\partial r^2} + \frac{\partial^2 P}{\partial z^2} + \frac{1}{r}\frac{\partial P}{\partial r} - \frac{P}{r^2} = 0 \text{ in } \omega,\ P = 0 \text{ on } \gamma. \quad (25)$$

2.5 Existence and uniqueness results.

If the data and initial conditions are axisymmetric, so are the solutions of (8) and (10), and, under the hypotheses of Theorem 1.10

$$\mathcal{E} \in \mathcal{C}^0(0,T;\breve{\mathcal{X}}), \quad \mathcal{B} \in \mathcal{C}^0(0,T;\breve{\mathcal{W}}).$$

Then it follows from (22) and (24) that the electromagnetic field can be decomposed into regular and singular parts continuously with respect to time:

$$\begin{aligned}
\mathcal{E}(t) &= \mathcal{E}_R(t) + \mathcal{E}_S(t), \quad (\mathcal{E}_R, \mathcal{E}_S) \in \mathcal{C}^0(0,T;\breve{\mathcal{X}}_R \times \breve{\mathcal{X}}_S),\\
\mathcal{B}(t) &= \mathcal{B}_R(t) + \mathcal{B}_S(t), \quad (\mathcal{B}_R, \mathcal{B}_S) \in \mathcal{C}^0(0,T;\breve{\mathcal{W}}_R \times \breve{\mathcal{W}}_S).
\end{aligned}$$

Moreover, as the projections ϖ_m and ϖ_θ are smooth, each of the systems (13–16) admits a unique solution in the relevant space; that of (13) and (14) depend continuously on time. As a consequence, the decomposition (4) can be refined by using three subspaces: meridian regular, meridian singular, azimuthal. (Recall that azimuthal implies regular.) Each of the problems (3) then admits a unique and continuous solution.

2.6 Principle of the numerical method.

The SCM follows from the above decomposition. As the singular parts span a finite-dimensional space, it is sufficient to find an approximation of a basis. The problems (3) amount to a classical FE formulation, for the regular parts, and a low-dimensional linear system, for the singular parts.

Computation of bases of $\mathcal{N}$ and $\breve{N}$. We look for a basis of the spaces $\breve{N}$ and $\mathcal{N} := \{P \text{ satisfying } (25)\}$, whose dimensions are given by Theorems 2.13 and 2.16. We have at hand an approximate knowledge of these bases [9].

- There is one basis function $p_j^- \in \mathcal{N}$ or $p_j^+ \in \breve{N}$ associated to each relevant geometric singularity A_j as follows. In a neighborhood ω_j of A_j, there holds $p_j^\pm|_{\omega_j} = p_j^S + q_j^\pm$, where the *principal part* p_j^S is just in $L^2_1(\omega_j)$, and the remainder $q_j^\pm$ is of H^1-style regularity in ω_j. In $\omega_j' = \omega \setminus \overline{\omega}_j$, $p_j^\pm$ is of H^1-style regularity.

- In ω_j, define local polar coordinates (ρ_j, ϕ_j) centered at A_j.

- If A_j is a reentrant edge of opening $\beta_j = \pi/\alpha_j$, $1/2 < \alpha_j < 1$, one has $p_j^S = \rho_j^{-\alpha_j} \sin(\alpha_j\, \phi_j)$, for the electric and magnetic cases.

- (For the electric case only.) If A_j is a conical vertex of opening π/δ_j, $1/2 < \delta_j < \delta_-$, one finds $p_j^S = \rho_j^{-1-\nu_j} P_{\nu_j}(\cos\phi_j)$, where P_ν denotes the Legendre function and $\nu_j \in]0, 1/2[$ is given by $P_{\nu_j}(\cos(\pi/\delta_j)) = 0$.

In the whole of ω the function $q_j^\pm = p_j^\pm - p_j^S$ satisfies

$$-\Delta^+ q_j^+ = \Delta^+ p_j^S, \text{ resp. } -\Delta' q_j^- = \Delta' p_j^S \text{ in } \omega, \quad q_j^\pm\big|_\gamma = -p_j^S\big|_\gamma \text{ on } \gamma, \qquad (26)$$

but, unlike in the cartesian geometry, it is not possible to compute it variationally: if A_j is an edge, neither p_j^S nor $q_j^\pm$ is of H^1-style regularity near the axis, and the problem (26) is ill-posed. This hindrance can be overcome:

- either by multiplying p_j^S by the 'not-too-noisy' cut-off function $\eta(r) = r/r(A_j)$, i.e. defining $\widehat{q}_j = p_j - \eta\, p_j^S$ which is regular in the whole of ω;

- or by domain decomposition, computing q_j in ω_j and p_j in ω'_j, and enforcing standard transmission conditions between ω_j and ω'_j (*à la* §3.2.2).

Computation of bases of $\breve{\mathcal{W}}_S$ and $\mathbf{grad}\, \breve{\Phi}_S$. Our task is now to compute $\mathcal{B}_j = \mathbf{curl}\,(-\Delta')^{-1} p_j^-$, which is in $\breve{\mathcal{W}}_S$ since $\mathbf{curl}\, \mathcal{B}_j = p_j^-\, \mathbf{e}_\theta$, and $\mathcal{E}_j = \mathbf{grad}\,(-\Delta^+)^{-1} p_j^+$. First, one solves variationally:

$$\begin{aligned} -\Delta' \psi_j &= p_j^- \text{ in } \omega, \quad \psi_j = 0 \text{ on } \gamma, \\ -\Delta^+ \varphi_j &= p_j^+ \text{ in } \omega, \quad \varphi_j = 0 \text{ on } \gamma_b. \end{aligned}$$

One has: $\psi_j = \psi_j^R + \sum_{1\le i\le K_e} c_j^i\, \psi_i^S$, $\varphi_j = \varphi_j^R + \sum_{1\le i\le K_e+K_v} d_j^i\, \varphi_i^S$, where:

- the ψ_j^R and φ_j^R are of H^2-style regularity,
- $\psi_i^S = \varphi_i^S = \rho_i^{\alpha_i} \sin(\alpha_i\, \phi_i)$ near a reentrant edge,
- $\varphi_i^S = \rho_i^{\nu_i} P_{\nu_i}(\cos\phi_i)$ near a vertex of conical angle larger than π/δ_-.

The singularity coefficients c_j^i, d_j^i can be extracted by quadrature formulas [10] or spectral methods. In $\omega_0 = \omega \setminus \bigcup \overline{\omega}_i$, ψ_j and φ_j are regular. The corresponding decompositions of $\mathcal{B}_j$ and $\mathcal{E}_j$ are:

$$\mathcal{B}_j = \mathbf{curl}\, \psi_j^R + \sum_{1\le i\le K_e} c_j^i\, \mathbf{curl}\, \psi_i^S, \quad \mathcal{E}_j = \mathbf{grad}\, \varphi_j^R + \sum_{1\le i\le K_e+K_v} d_j^i\, \mathbf{grad}\, \varphi_i^S.$$

$\mathbf{curl}\, \psi_j^R$ and $\mathbf{grad}\, \varphi_j^R$ are of H^1-style regularity and can be computed variationally, while $\mathbf{curl}\, \psi_i^S$ and $\mathbf{grad}\, \varphi_i^S$ are analytically known. In ω_0, the whole of $\mathcal{B}$ and $\mathcal{E}$ can be computed variationally.

Finally, it is possible to orthogonalize the decomposition (22) by subtracting to $\mathcal{E}_j$ its orthogonal projection on $\breve{\mathcal{X}}_R$. This is no difficulty.

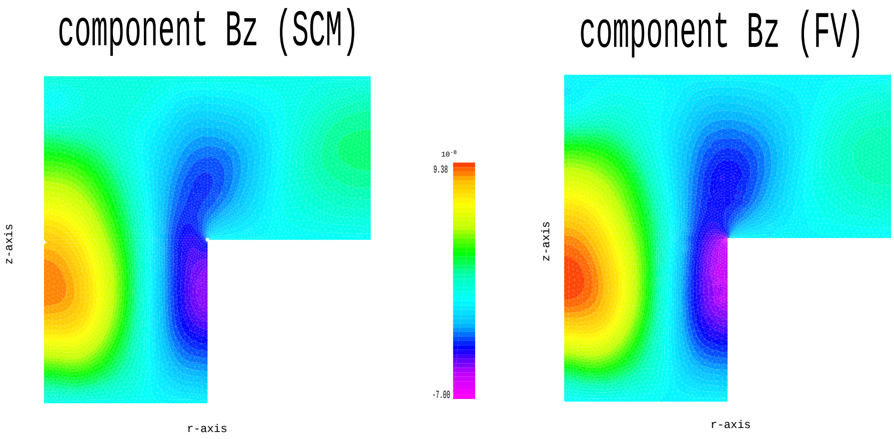

Figure 1: Computed magnetic field: The SCM and Finite Volume techniques.

2.7 Numerical Results

As an illustration of the SCM in the axisymmetric case, one can compute the electromagnetic field generated by a current. A top hat domain Ω (ω L-shaped) is considered, and a perfectly conducting boundary condition is imposed. The initial conditions are set to zero. The electromagnetic wave is generated by a current $\mathcal{J}(\mathbf{x}, t) = J_\theta \mathbf{e}_\theta$, $J_\theta = 10 \sin(\lambda t)$, with a frequency $\lambda/2\pi = 2, 5.10^9$ Hertz. The support of this current is a little disc centered at the middle of the domain. Because it is impossible to provide an analytical solution, we compare our results to the computations made by another code, based on Finite Volume (FV) techniques *à la* Delaunay [26]. The space and time discretizations of the SCM are detailed later on, in Section 3.2.3. Figure 1 shows the isovalues of the magnetic field (B_z component after 1000 time steps), which have been computed by the two methods. The results obtained by both methods are comparable, which shows the feasability of the SCM. Moreover, the SCM provides a numerical solution which is less noisy.

3 Maxwell's equations in a polygon

In what follows, it is assumed that both the data and the initial conditions *do not depend* on the transverse variable z. Then the original problem can be identified with a problem posed in a section of an infinite cylinder, which is a 2d polygon ω, with boundary γ, a set of edges $(\gamma_k)_{1 \le k \le E}$, and a unit outward normal ν. The notations are those of Section 1, except that the Sobolev spaces are based on the scalar curl; also the 2d calligraphic spaces ($\mathcal{X}$, $\mathcal{Y}$) and unknowns ($\mathcal{E}$, etc.) are written in *boldface*, i.e. $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{E} = (E_1, E_2)^T$, etc.

3.1 The time-harmonic Maxwell equations

This Section summarizes the results obtained in [14] and [25], and we refer to these papers for any detail.

We are looking for a numerical approximation of the solution $\mathbf{E}$ to

$$\begin{cases} \mathbf{curl}\,\mathrm{curl}\,\mathbf{E} - k^2\mathbf{E} = \mathbf{J} \text{ in } \omega, \\ \mathbf{E}\cdot\tau = 0 \text{ on } \gamma. \end{cases} \tag{27}$$

For sake of simplicity regarding existence and uniqueness questions, we suppose that k is a complex number with nonzero imaginary part (which means that we are concerned with the electromagnetic problem in a homogeneous and dissipative medium) or, in order to include stationnary problems, that $k = 0$. The vector field $\mathbf{J}$ is a datum that represents the impressed current density. We assume that

$$\mathrm{div}\,\mathbf{J} = 0 \text{ in } \omega,$$

which amounts to saying that the electric charge density vanishes in the whole domain. The *singular field method* is based on the fact that the solution of (27) can be found by solving an equivalent *regularized* problem similar to the vector Helmholtz equation. Formally, the regularized problem is given by

$$\begin{cases} -\Delta\mathbf{E} - k^2\mathbf{E} = \mathbf{J} \text{ in } \omega, \\ \mathbf{E}\cdot\tau = 0 \text{ on } \gamma, \\ \mathrm{div}\,\mathbf{E} = 0 \text{ on } \gamma. \end{cases} \tag{28}$$

Indeed, a solution of (27) is clearly divergence free and, thus, satisfies (28). Conversely, let $\mathbf{E}$ be a solution of (28). Its divergence $\varphi = \mathrm{div}\,\mathbf{E}$ satisfies

$$\begin{cases} -\Delta\varphi - k^2\varphi = 0 \text{ in } \omega, \\ \varphi = 0 \text{ on } \gamma, \end{cases}$$

which yields $\varphi = 0$ (provided φ is assumed regular enough).

The Section is organized as follows: in a first part (§3.1.1) we make precise the functional setting and give the corresponding variational formulation. In particular, we address the question of equivalence between the classical and the regularized formulations of the problem. We show that the latter can be set in two 'neighboring' functional spaces whenever the domain ω has at least one reentrant corner. Of course, only one of them leads to the equivalence with the classical formulation. The key of the method lies in the fact that the 'right' functional space can be written as the direct sum of a space of regular fields completed by a (finite-dimensional) space of singular fields. We give two possible decompositions which lead to the *singular field method* (SFM) and its orthogonal variant (OSFM) described in §3.1.2. In §3.1.3, the analysis of the convergence of these methods is addressed. It turns out that both numerical schemes have the same rate of convergence but the numerical applications presented in §3.1.4 clearly show that OSFM yields far better results: we shall try to explain why.

3.1.1 Classical and regularized formulations

The variational interpretation of the classical problem (27) leads us naturally to seek $\mathbf{E}$ in the space $\mathbf{H}_0(\mathrm{curl}\,)$. If we assume the datum $\mathbf{J}$ to belong to $\mathbf{L}^2(\omega)$, the weak form of (27) is given by

$$\mathcal{P}_0(\mathrm{curl}\,) \qquad \begin{cases} \textit{Find } \mathbf{E} \in \mathbf{H}_0(\mathrm{curl}\,) \textit{ such that} \\ (\mathrm{curl}\,\mathbf{E}, \mathrm{curl}\,\mathbf{E}')_0 - k^2(\mathbf{E}, \mathbf{E}')_0 = (\mathbf{J}, \mathbf{E}')_0 \ \forall \mathbf{E}' \in \mathbf{H}_0(\mathrm{curl}\,). \end{cases}$$

The sesquilinear form $(\mathrm{curl}\,\mathbf{E}, \mathrm{curl}\,\mathbf{E}')_0 - k^2(\mathbf{E}, \mathbf{E}')_0$ being coercive on $\mathbf{H}_0(\mathrm{curl}\,)$ (due to condition $Im(k) \neq 0$), we infer the existence and uniqueness of the solution of $\mathcal{P}_0(\mathrm{curl}\,)$ from Lax-Milgram's theorem.

Let us now consider the regularized problem (28). Its variational formulation involves the functional space $\mathbf{X}$ and thus amounts simply to adding $(\mathrm{div}\,\mathbf{E}, \mathrm{div}\,\mathbf{E}')_0$ in $\mathcal{P}_0(\mathrm{curl}\,)$. We therefore consider the problem

$$\mathcal{P}_0(\mathrm{curl}\,, \mathrm{div}\,) \qquad \begin{cases} \textit{Find } \mathbf{E} \in \mathbf{X} \textit{ such that} \\ a(\mathbf{E}, \mathbf{E}') = (\mathbf{J}, \mathbf{E}')_0 \ \forall \mathbf{E}' \in \mathbf{X}, \end{cases}$$

where $a(\mathbf{E}, \mathbf{E}') := (\mathrm{curl}\,\mathbf{E}, \mathrm{curl}\,\mathbf{E}')_0 + (\mathrm{div}\,\mathbf{E}, \mathrm{div}\,\mathbf{E}')_0 - k^2(\mathbf{E}, \mathbf{E}')_0$.
For the same reason as above, $\mathcal{P}_0(\mathrm{curl}\,, \mathrm{div}\,)$ has a unique solution which coincides with that of $\mathcal{P}_0(\mathrm{curl}\,)$ provided $\mathrm{div}\,\mathbf{J} = 0$. Indeed, choosing $\mathbf{E}' = \mathbf{grad}\,\varphi'$ with $\varphi' \in \mathcal{D}(\omega)$ in $\mathcal{P}_0(\mathrm{curl}\,)$ yields that the solution of $\mathcal{P}_0(\mathrm{curl}\,)$ is divergence-free. It thus belongs to $\mathbf{X}$ and satisfies the variational equation of $\mathcal{P}_0(\mathrm{curl}\,, \mathrm{div}\,)$, in other words it does coincide with the solution of $\mathcal{P}_0(\mathrm{curl}\,, \mathrm{div}\,)$.

We thus deduce that $\mathbf{X}$ is the appropriate functional frame for the regularized problem. But the situation becomes more involved if we consider the following problem given on the subspace of $\mathbf{X}$ of *regular* fields:

$$\mathcal{P}_0(\mathbf{grad}\,) \qquad \begin{cases} \textit{Find } \mathbf{E} \in \mathbf{X}_R \textit{ such that} \\ a(\mathbf{E}, \mathbf{E}') = (\mathbf{J}, \mathbf{E}')_0 \ \forall \mathbf{E}' \in \mathbf{X}_R. \end{cases}$$

As mentioned in Section 1, $\mathbf{X}_R$ is a closed subspace of $\mathbf{X}$ and hence, the form $a(\cdot,\cdot)$ is still coercive on $\mathbf{X}_R$. Whenever ω has at least one reentrant corner, $\mathbf{X}_R$ is strictly contained in $\mathbf{X}$, and the respective solutions to $\mathcal{P}_0(\mathrm{curl}\,, \mathrm{div}\,)$ and $\mathcal{P}_0(\mathbf{grad}\,)$ are in general different. In particular, an H^1-conforming FE discretization can only provide an approximation of the *non-physical* problem $\mathcal{P}_0(\mathbf{grad}\,)$.

In order to perform a method based on nodal (Lagrange) FE, which is able to capture the singular behavior (and thus solves problem $\mathcal{P}_0(\mathrm{curl}\,, \mathrm{div}\,)$), we decompose $\mathbf{X}$ into a regular and a singular part,

$$\mathbf{X} = \mathbf{X}_R \oplus \mathbf{X}_S.$$

Of course, $\mathbf{X}_S$ is not uniquely determined. Hereafter, we will give two possible choices, leading to two different methods.

Notice that the above existence and equivalence results keep true in the stationnary case corresponding to $k = 0$. In order to simplify the presentation, we will consider this case only, and we thus set from now on

$$a(\mathbf{E}, \mathbf{E}') := (\operatorname{curl} \mathbf{E}, \operatorname{curl} \mathbf{E}')_0 + (\operatorname{div} \mathbf{E}, \operatorname{div} \mathbf{E}')_0.$$

Let us set some notations. Without loss of generality, assume that ω has exactly one reentrant corner of measure $\beta = \pi/\alpha$, $1/2 < \alpha < 1$, at the vertex S. We use the local polar coordinates (r, θ), and we fix a regular cut-off function $\eta = \eta(r)$ such that $\eta \equiv 1$ near S and $\eta \equiv 0$ near the other vertices. The function

$$s(r, \theta) = r^\alpha \sin(\alpha\theta)$$

belongs to $H^1(\omega) \setminus H^2(\omega)$ as $\alpha < 1$ and is called *singular function* at S. We finally introduce the subspace of $H_0^1(\omega)$ given by

$$\mathcal{S} = \operatorname{span}\{\eta s\}.$$

Owing to Theorem 1.13 (see also [14]), we have the direct decomposition:

Theorem 3.1

$$\mathbf{X} = \mathbf{X}_R \overset{c}{\oplus} \mathbf{grad}\, \mathcal{S}. \tag{29}$$

An *orthogonal* decomposition can be deduced from (29) solving an auxiliary inhomogeneous variational problem, which is similar to $\mathcal{P}_0(\mathbf{grad}\,)$:

Theorem 3.2

$$\mathbf{X} = \mathbf{X}_R \overset{\perp}{\oplus} \operatorname{span}\{\mathbf{grad}\,(s) + \mathbf{F}\} \tag{30}$$

where $\mathbf{F}$ *is the solution of the problem*
find $\mathbf{F} \in \mathbf{H}^1(\omega)$ such that

$$a(\mathbf{F}, \mathbf{E}') = 0, \ \forall \mathbf{E}' \in \mathbf{X}_R, \quad \mathbf{F} \cdot \tau = -\mathbf{grad}\, s \cdot \tau \ \textit{ on } \gamma. \tag{31}$$

Remark 3.3 *Decomposition (30) is orthogonal in the sense that*

$$a(\mathbf{grad}\,(s) + \mathbf{F}, \mathbf{E}'_R) = 0 \,\forall \mathbf{E}'_R \in \mathbf{X}_R.$$

Notice, however, that the above relation fails whenever $k \neq 0$*. Nevertheless, in this case, the remaining terms are of lower order and involve only the* L^2*-scalar product of the sesquilinear form.*

3.1.2 Description of the method

We give the algorithms of both SFM and OSFM which are based respectively on the decompositions (29) and (30). To this end, let $(\mathcal{T}_h)_{0<h<h_0}$, be a family of regular triangulations of the domain ω. We consider the P_1 Lagrange FEM:

$$\mathbf{Y}_h := \{\mathbf{E}_h \in \mathbf{H}^1(\omega) : \ \mathbf{E}_{h|T_h} \text{ is affine } \forall T_h \in \mathcal{T}_h\}.$$

Let $\{M_I\}$ be the set of nodal points of the triangulation and

$$\mathbf{V}_h := \{\mathbf{E}_h \in \mathbf{Y}_h : \ (\mathbf{E}_h \cdot \tau)(M_I) = 0, \ \forall M_I \in \gamma\}$$

the discretization space of $\mathbf{X}_R$. Let $N_h = \dim \mathbf{V}_h$ and $(\mathbf{w}_I)_{I=1,\dots,N_h}$ be the basis functions. Note that the discrete boundary condition $(\mathbf{E}_h \cdot \tau)(M_I) = 0$ is ambiguous if M_I is a vertex of the polygon; in this case it should be understood as $\mathbf{E}_h(M_I) = 0$ (i.e. both components of $\mathbf{E}_h(M_I)$ vanish.)

The singular field method (SFM) Owing to (29), the discretization space is given by

$$\mathbf{X}_h = \mathbf{V}_h \oplus \mathbf{grad}\, \mathcal{S}.$$

The matrix form of the discrete problem then reads as follows:

$$\begin{pmatrix} \mathbb{A} & C \\ C^T & \mathbb{A}_s \end{pmatrix} \begin{pmatrix} \vec{\mathsf{E}}_R \\ e_s \end{pmatrix} = \begin{pmatrix} \vec{\mathsf{J}} \\ j_s \end{pmatrix}, \tag{32}$$

where

- $\mathbb{A}$ and $\vec{\mathsf{J}}$ respectively denote the stiffness matrix and the right-hand side corresponding to the FE space $\mathbf{V}_h$,

$\mathbb{A}_{IJ} = a(\mathbf{w}_J, \mathbf{w}_I), \ I, J = 1, \dots, N_h$ and $\mathsf{J}_J = (\mathbf{J}, \mathbf{w}_J)_0, \ J = 1, \dots, N_h$.

- $\mathbb{A}_s$ and j_s denote the matrix and the right-hand side of order 1 corresponding to the singular field,

$\mathbb{A}_s = a(\mathbf{grad}\,(\eta s), \mathbf{grad}\,(\eta s))$ and $j_s = (\mathbf{J}, \mathbf{grad}\,(\eta s))_0$.

- C is a matrix of order $N_h \times 1$ coupling the basis functions of FE-type to the singular field,

$C_{I1} = a(\mathbf{grad}\,(\eta s), \mathbf{w}_I), \ 1 \le I \le N_h$.

In order to preserve the advantages of the sparse matrix A_{FE} in the resolution of (32), the SFM consists in solving separately the two linear systems

$$\mathbb{A}\vec{\mathsf{E}}^{\star} = \vec{\mathsf{J}}, \ \mathbb{A}S = C,$$

and taking into account that (32) may be written as

$$\mathbb{A}\vec{\mathsf{E}}_R = \mathbb{A}\left(\vec{\mathsf{E}}^{\star} - e_s S\right), \quad \mathbb{A}_s e_s = j_s - C^T \vec{\mathsf{E}}_R. \tag{33}$$

The left equality clearly implies that $\vec{\mathsf{E}}_R = \vec{\mathsf{E}}^{\star} - e_s S$. Substituting this identity into the right one thus yields the singular coefficient e_s. Thus,

$$\mathbf{E}_h = \sum_{1 \le l \le N_h} \left(\mathsf{E}^{\star}_I - e_s S_I\right) \mathbf{w}_I + e_s \mathbf{grad}\,(\eta s).$$

Notice the similarity of (33) with (3).

The orthogonal singular field method (OSFM) This time, the discretization space is given by

$$\widetilde{\mathbf{X}_h} = \mathbf{V}_h \oplus \mathrm{span}\{\mathbf{grad}\,(s) + \mathbf{F}_h\},$$

where $\mathbf{F}_h$ denotes the FE-approximation of problem (31).

In consequence, the matrix form of the discrete problem is the following:

$$\begin{pmatrix} \mathbb{A} & 0 \\ 0 & \widetilde{\mathbb{A}_s} \end{pmatrix} \begin{pmatrix} \vec{\mathsf{E}}'_R \\ e'_s \end{pmatrix} = \begin{pmatrix} \vec{\mathsf{J}} \\ \widetilde{j_s} \end{pmatrix},$$

where $\mathbb{A}$ and $\vec{\mathsf{J}}$ take the same significance as before, and $\widetilde{\mathbb{A}_s}$ and $\widetilde{j_s}$ are respectively given by

$$\widetilde{\mathbb{A}_s} = a(\mathbf{F}_h, \mathbf{F}_h) \text{ and } \widetilde{j_s} = (\mathbf{J}, \mathbf{grad}\,(s) + \mathbf{F}_h)_0,$$

taking into account that $\mathbf{grad}\,(s)$ is curl- and divergence-free.

The algorithm of the OSFM is then straightforward.

Remark 3.4 *Both methods can be extended to the case of K_c reentrant corners: $\vec{\mathsf{e}}_s$ is a vector of $\mathbb{R}^{K_c}$, and C and S (for the SFM) are matrices of order $N_h \times K_c$. See also §3.2.3, in which algorithms are given in this case.*

3.1.3 Error analysis

We state in this Section the main convergence results. All proofs may be found in [25].

Theorem 3.5 *Let $\mathbf{E}$ be the solution of $\mathcal{P}_0(\mathrm{curl}\,, \mathrm{div}\,)$ and $\mathbf{E}_h$ its approximation by the SFM. Assume that the regular part of $\mathbf{E}$ belongs to $\mathbf{H}^{s+1}(\omega)$ with s in $]0,1]$. Then, we have*

$$\|\mathbf{E} - \mathbf{E}_h\|_{0,\mathrm{curl}\,,\mathrm{div}} = \mathcal{O}(h^s)$$

for the error in the energy norm, and

$$\|\mathbf{E} - \mathbf{E}_h\|_0 = \mathcal{O}(h^\lambda),\ \forall \lambda < s + 2\alpha - 1,$$

in the L^2-norm. Moreover, the error of the OSFM is of the same order as the one of the SFM.

3.1.4 Numerical results

In this Section, we present numerical tests of both methods in the case where the exact solutions are known. The domain is formed by three quarters of a circle with center $\mathbf{0}$ and radius 2, the only reentrant corner being of measure $\beta = 3\pi/2$ ($\alpha = 2/3$). We consider two families of solutions, for $n \in \mathbb{N}$:

$$\mathbf{G}_n(r,\theta) = \mathbf{grad}\,(\eta(r) r^{n\alpha} \sin(n\alpha\theta)) \text{ and } \mathbf{H}_n(r,\theta) = \mathbf{curl}\,(\eta(r) r^{n\alpha} \cos(n\alpha\theta)),$$

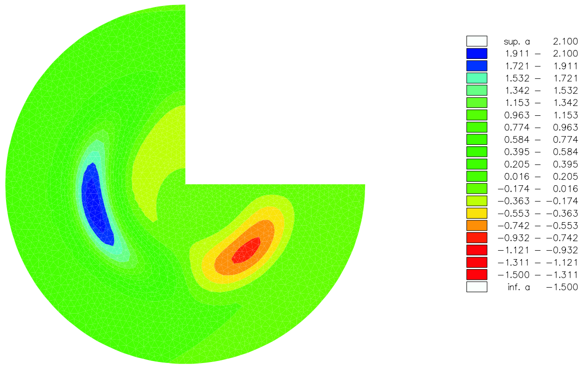

Figure 2: FE-approximation of $\mathbf{G}_1$.

$\mathbf{G}_n$ and $\mathbf{H}_n$ have the same regularity depending on n: $\mathbf{G}_n \in \mathbf{H}^s(\omega)$, $\forall s < n\alpha$. In particular, $\mathbf{G}_n$ is of class $\mathbf{H}^1$ for $n > 1$, whereas $\mathbf{G}_1$ has a *non-zero* component in any complementary space of $\mathbf{X}_R$.

Both methods have been tested on four unstructured grids. The mesh parameter h varies from $h = 2^{-1}$ to $h = 2^{-4}$, the latter corresponding to roughly 2×8.500 degrees of freedom. Notice that no particular mesh refinement has been done near the corner. The cut-off function η is a piecewise polynomial function of class $\mathcal{C}^3$. The coefficients of the terms $\mathbb{A}$, $\vec{\mathrm{J}}$ and C are calculated using a 7-point-quadrature formula (exact for polynomials up to order 5). The coefficients $\mathbb{A}_s$ and j_s are calculated analytically. The implementation of the boundary condition is realized *via* a rotation which maps the canonical basis on a local basis of the normal and tangential vectors; in the latter basis the vector boundary condition is decoupled and standard techniques apply. The linear systems occuring in the algorithms are solved by a direct method based on Cholesky factorization. All tests have been realized with the FE-code MELINA[2].

It may be clearly seen on Figure 2 that the standard FEM fails for a singular solution field (here, we represent the x-component of the FE-approximation of $\mathbf{G}_1$). Indeed, the condition $\mathbf{E}_h \cdot \tau_{|\gamma} = 0$ forces the FE-approximation to vanish at $\mathbf{0}$ whereas the exact solution tends to ∞ at the corner. Hence, we are not faced with an accuracy problem (which could be handled alternatively by a *local* mesh refinement), but with the choice of the appropriated functional frame: the FE-aproximation converges to the solution of $\mathcal{P}_0(\mathbf{grad}\,)$ which is *globally* different from the physical solution.

Figure 3 shows the discrete L^2-error of the SFM,

$$\|\mathbf{E} - \mathbf{E}_h\|_h := \left(\frac{1}{N_h} \sum_{I \in \overset{\circ}{\omega}} |\mathbf{E}(M_I) - \mathbf{E}_h(M_I)|^2 \right)^{1/2},$$

[2]developed by O. Debayser (ENSTA, Paris, France) and D. Martin (IRMAR, University of Rennes 1, France) at SMP, ENSTA, see [28].

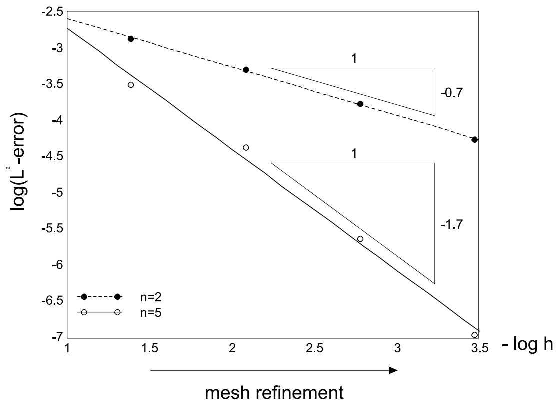

Figure 3: The SFM for $\mathbf{H}_n$, $n = 2, n = 5$.

in logarithmic scale for the regular fields $\mathbf{H}_2$ and $\mathbf{H}_5$.

The numerical values are in good accordance with the theory of §3.1.3. Figure 4 compares the SFM- and OSFM-approximations of the singular field $\mathbf{H}_1$. It turns out that the OSFM yields the better results. This is probably due to the cut-off function η involved in the implementation of SFM. Indeed, this numerical instability is known for *singular function methods* (see for example [13]) and leads to high values of the constant in the error estimates, and thus to poor accuracy.

3.2 The time-dependent Maxwell equations

We are looking now for a numerical approximation of the 2d time-dependent Maxwell equations, (8) being rewritten as two decoupled sets of second order in time equations. In this paper, we focus on the first one (the second one could be written in the same way [7]). It can be written as follows:

$$\begin{cases} \dfrac{\partial^2 \mathbf{E}}{\partial t^2} + \mathbf{curl}\,\mathrm{curl}\,\mathbf{E} = -\dfrac{\partial \mathbf{J}}{\partial t}, \quad \dfrac{\partial^2 B_z}{\partial t^2} - \Delta B_z = \mathrm{curl}\,\mathbf{J} \text{ in } \omega \times]0, T[, \\ \mathrm{div}\,\mathbf{E} = \rho \text{ in } \omega \times]0, T[, \\ \mathbf{E} \cdot \tau = 0, \quad \dfrac{\partial B_z}{\partial \nu} - \mathbf{J} \cdot \tau = 0 \text{ on } \gamma \times]0, T[, \\ \mathbf{E}(0) = \mathbf{E}_0, \quad B_z(0) = B_{z0}, \\ \dfrac{\partial \mathbf{E}}{\partial t}(0) = \mathbf{curl}\,B_{z0} - \mathbf{J}(\cdot, 0), \quad \dfrac{\partial B_z}{\partial t}(0) = -\mathrm{curl}\,\mathbf{E}_0. \end{cases}$$

(The second order in time system of equations is closed with the help of initial conditions on $\partial_t \mathbf{E}$ and $\partial_t B_z$.)

As mentioned in Section 1, the B_z component, as the solution of a wave equation, always belongs to $H^1(\omega)$, even in a non-convex domain. As a consequence, we consider below only the computation of the field $\mathbf{E}$.

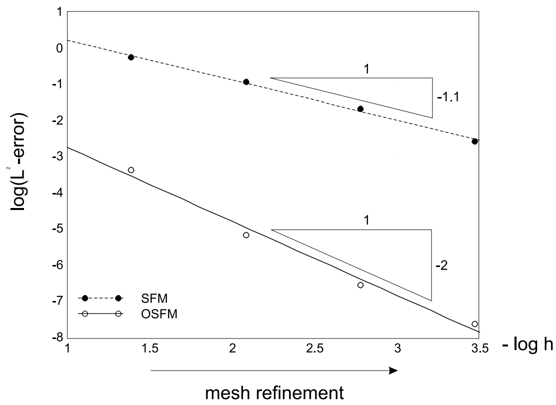

Figure 4: the SFM/OSFM for $\mathbf{H}_1$.

Remark 3.6 *For the sake of simplicity, the problem will be written in the absence of charges:* $\operatorname{div}\mathbf{E} = 0$. *The space of solutions becomes*

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{X} : \operatorname{div}\mathbf{v} = 0\}$$

By using the Helmholtz decomposition, it can be proved that the singular space $\mathbf{X}_S$ *of* $\mathbf{X}$ *is a (strict) subspace of* $\mathbf{curl}\,\Phi_S + \mathbf{grad}\,\Psi_S$*, where* Φ_S *is the space introduced in Section 1, and* Ψ_S *its counterpart for the homogeneous Neumann problem. Hence, the method described here for* $\mathbf{V}$ *can be adapted to* $\mathbf{X}$.

3.2.1 Description of the method

We first introduce a variational form of the equations, i.e.
find $\mathbf{E}(t) \in \mathbf{V}$ *such that*

$$\frac{d^2}{dt^2}(\mathbf{E},\mathbf{F})_0 + (\operatorname{curl}\mathbf{E}, \operatorname{curl}\mathbf{F})_0 = -\frac{d}{dt}(\mathbf{J},\mathbf{F})_0 \quad \forall \mathbf{F} \in \mathbf{V},$$

with the same initial conditions. As in the 3d case (see Theorem 1.10), there exists one and only one solution of this problem. Moreover, we have the following orthogonal decomposition of $\mathbf{V}$, analogous to the one previously obtained in $\mathcal{X}$.

Theorem 3.7 *The space* $\mathbf{V}$ *can be split in the orthogonal sum* $\mathbf{V} = \mathbf{V}_R \overset{\perp}{\oplus} \mathbf{V}_S$.

From this splitting, we obtain a continuous (orthogonal) decomposition in time of the electric field, that is

$$\mathbf{E}(t) = \mathbf{E}_R(t) + \mathbf{E}_S(t)\ .$$

By using again the relation between the singular solutions of Maxwell's equations and those of the Laplace problem, we obtain that the vector space $\mathbf{V}_S$ is finite dimensional, of dimension K_c, the number of reentrant corners, defined by $\operatorname{curl} \mathbf{V}_S = N$ (N introduced at (6)). For $(\mathbf{v}_S^j)_{1\le j\le K_c}$ a basis of $\mathbf{V}_S$, we have

$$\mathbf{E}(t) = \mathbf{E}_R(t) + \sum_{1\le j\le K_c} \kappa_j(t)\,\mathbf{v}_S^j,$$

where $(\kappa_j)_{1\le j\le K_c}$ are K_c functions at least continuous in time. With this decomposition, the variational formulation becomes:
find $\mathbf{E}_R \in \mathbf{V}_R$ *such that*

$$\begin{aligned}\frac{d^2}{dt^2}(\mathbf{E}_R, \mathbf{F}_R)_0 \quad + \quad & (\operatorname{curl} \mathbf{E}_R, \operatorname{curl} \mathbf{F}_R)_0 = -\frac{d}{dt}(\mathbf{J}, \mathbf{F}_R)_0 \\ & - \sum_{1\le j\le K_c} \kappa_j''(t)\,(\mathbf{v}_S^j, \mathbf{F}_R)_0, \quad \forall \mathbf{F}_R \in \mathbf{V}_R\,, \end{aligned} \tag{34}$$

completed with K_c equations, derived by using $(\mathbf{v}_S^i)_{1\le i\le K_c}$ as K_c test functions. Thanks to the orthogonality of regular and singular fields, one gets:

$$\begin{aligned}\frac{d^2}{dt^2}(\mathbf{E}_R, \mathbf{v}_S^i)_0 + \sum_{1\le j\le K_c} \kappa_j''(t)(\mathbf{v}_S^j, \mathbf{v}_S^i)_0 \quad + \quad \kappa_i(t)(\operatorname{curl} \mathbf{v}_S^j, \operatorname{curl} \mathbf{v}_S^i)_0 = \\ -\frac{d}{dt}(\mathbf{J}, \mathbf{v}_S^i)_0\ , 1 \le i \le K_c.\end{aligned} \tag{35}$$

In order to compute numerically the solution, we have first to determine a basis of $\mathbf{V}_S$, and then to solve the time-dependent formulation.

3.2.2 Determination of a basis of $\mathbf{V}_S$

For the sake of simplicity, let us assume that K_c is equal to 1. To compute $\mathbf{v}_S$, a basis of $\mathbf{V}_S$, the isomorphism between $\mathbf{V}_S$ and N is used. The framework of the algorithm is then:

- Compute a basis of N, i.e. a non vanishing element p_S of $L^2_0(\omega)$, such that

$$\Delta p_S = 0 \text{ in } \omega, \quad \frac{\partial p_S}{\partial \nu} = 0 \text{ on } \gamma_k,\ 1 \le k \le E.$$

- Compute $\mathbf{v}_S \in \mathbf{V}$, the solution of

$$\operatorname{curl} \mathbf{v}_S = p_S \text{ in } \omega, \quad \operatorname{div} \mathbf{v}_S = 0 \text{ in } \omega, \quad \mathbf{v}_S \cdot \tau = 0 \text{ on } \gamma. \tag{36}$$

Instead of solving (36), it is more practical to make use of another isomorphism, in the same spirit as in Section 1: to $\mathbf{v}_S \in \mathbf{V}_S$, there corresponds one and only one scalar potential $\phi_S \in H^1(\omega)/\mathbb{R}$ such that

$$-\Delta \phi_S = p_S \text{ in } \omega, \quad \frac{\partial \phi_S}{\partial \nu} = 0 \text{ on } \gamma.$$

Now, as ϕ_S is sufficiently smooth (i.e. with regularity H^1), one can easily solve this problem with the help of a variational formulation. The computation of

$\mathbf{v}_S \in \mathbf{V}_S$ then stems from the identity $\mathbf{v}_S = \mathbf{curl}\, \phi_S$.

Computation of p_S (ϕ_S, $\mathbf{v}_S$): first method
A partition of ω into ω_c and ω_e is introduced, where ω_c stands for an open angular sector of radius R centered at the reentrant corner, with an angle $\beta = \pi/\alpha$, $1/2 < \alpha < 1$, and where ω_e is the open domain such that $\omega_c \cap \omega_e = \emptyset$ and $\overline{\omega}_c \cup \overline{\omega}_e = \overline{\omega}$. Last, Let γ_c (resp. γ_e) denote the boundary of ω_c (resp. ω_e), which is split in $\mathcal{B} \cup \tilde{\gamma_c}$ (resp. $\mathcal{B} \cup \tilde{\gamma_e}$), with the interface $\mathcal{B} = \gamma_c \cap \gamma_e$.

The computation of p_S (for instance) can be divided in three substeps (cf. [7]).

1. The restriction of p_S to ω_c, p_S^c, can be written using the polar coordinates,

$$p_S^c(r,\theta) = \sum_{n \geq -1} A_n r^{n\alpha} \cos(n\alpha\theta), \text{ with } A_{-1} \neq 0 \,.$$

   Every A_n can be written as an integral of $p^c_{S|\mathcal{B}}$ over $\mathcal{B}$.

2. Let ν^c denote the unit outward normal to ω_c. One then defines the capacitance operator $T : p^c_{S|\mathcal{B}} \mapsto \dfrac{\partial p_S^c}{\partial \nu^c}_{|\mathcal{B}}$, by

$$T(p_S^c) = T_1(p_S^c) - 2\alpha \frac{A_{-1}}{R^{\alpha+1}} \cos(\alpha\theta),$$

$$\text{where} \qquad T_1(p_S^c) = \frac{2\alpha^2}{\pi R} \sum_{n \geq 1} n \left\{ \int_0^\beta p_S^c(R,\theta') \cos(n\alpha\theta')\, d\theta' \right\} \cos(n\alpha\theta).$$

3. With the help of the transmission conditions: $p_S^e = p_S^c$ and $\partial_{\nu^c} p_S^e = \partial_{\nu^c} p_S^c$ on $\mathcal{B}$, one gets the missing boundary condition for the exterior problem (on the interface). Let ν^e denote the unit outward normal to ω_e, the exterior problem, written in a variational form, reads
   find $p_S^e \in H^1(\omega_e)/\mathbb{R}$ such that

$$\int_{\omega_e} \nabla p_S^e \cdot \nabla q \, d\omega + \int_{\mathcal{B}} T_1(p_S^e) q \, d\sigma = \frac{2\alpha A_{-1}}{R^{\alpha+1}} \int_{\mathcal{B}} \cos(\alpha\theta) q \, d\sigma, \ \forall q \in H^1(\omega_e)/\mathbb{R}.$$

   Clearly, the bilinear form $(p,q) \mapsto \displaystyle\int_{\mathcal{B}} T_1(p) q \, d\sigma$ is symmetric positive. Thus, for a given A_{-1}, the above exterior problem is well-posed.

The computation of ϕ_S and $\mathbf{v}_S$ can be carried out in the same way.

Computation of p_S (ϕ_S, $\mathbf{v}_S$): second method
Instead of partitioning ω into ω_c and ω_e, one can split p_S the basis of N into

$$p_S = p_S^{reg} + s^\star(r,\theta)$$

where $s^\star(r,\theta) = r^{-\alpha}\cos(\alpha\theta)$ is the *dual singular function* (see Section 3.1) for the Neumann problem, and p_S^{reg} the regular part of the solution (that belongs here in $H^1(\omega)$). To compute p_S, we have only to solve the problem in p_S^{reg}

$$-\Delta p_S^{reg} = \Delta s^\star (=0) \text{ in } \omega,$$
$$\frac{\partial p_S^{reg}}{\partial \nu} = 0 \text{ on } \tilde{\gamma}_c, \quad \frac{\partial p_S^{reg}}{\partial \nu} = -\frac{\partial s^\star}{\partial \nu} \text{ on } \tilde{\gamma}_e.$$

Remark 3.8 *This second one only requires the knowledge of the dual singular function, which is easier to get than the complete local solution, and should carry out to 3d problems. Moreover its implementation is simpler in the case of several reentrant corners.*

3.2.3 Solution to the time-dependent problem

We consider here the case of K_c reentrant corners. One proceeds first a *semi-discretization in space*, by using the P_1 Lagrange FEM. Let $(\mathbf{w}_I)_{I,1,\cdots,N_h}$ be a basis of $\mathbf{V}_R^h$, the FE approximation space of $\mathbf{V}_R$. The formulation (34) can be written equivalently as a linear system, where $'$ stands for the derivative in time:

$$\mathbb{M}_\omega\ \vec{\mathsf{E}}_R'' \ +\ \mathbb{R}_\omega\ \vec{\mathsf{E}}_R\ = -\mathbb{M}_\omega\ \vec{\mathsf{J}}'\ -\sum_{1\le j\le K_c} \kappa_j''(t)\ \vec{\Lambda}_j\ , \tag{37}$$

where $\mathbb{M}_\omega$ is the mass matrix, $\mathbb{R}_\omega$ is the curl matrix, and $\vec{\Lambda}_j$ (for a fixed j) the vector whose components are $(\Lambda_j)_I = (\mathbf{v}_S^j, \mathbf{w}_I)_0$, $1 \le I \le N_h$.
We denote by $\vec{\mathsf{k}}(t)$ the vector of $\mathbb{R}^{K_c}$ whose components are $\kappa_j(t)$. Starting from (35), we obtain

$$(\vec{\mathsf{e}}^s)'' + \mathbb{V}_s\vec{\mathsf{k}}'' + \mathbb{P}_s\vec{\mathsf{k}} = -(\vec{\mathsf{j}}^s)'$$

where $\vec{\mathsf{e}}^s$ and $\vec{\mathsf{j}}^s$ are vectors of $\mathbb{R}^{K_c}$, with components

$$\mathsf{e}_j^s = (\mathbf{E}_R, \mathbf{v}_S^j)_0 = (\vec{\mathsf{E}}_R|\vec{\Lambda}_j) \text{ and } \mathsf{j}_j^s = (\mathbf{J}, \mathbf{v}_S^j)_0 = (\vec{\mathsf{J}}|\vec{\Lambda}_j).$$

$\mathbb{V}_s$ and $\mathbb{P}_s$ are $K_c \times K_c$ matrices, defined by $(\mathbb{V}_s)_{ij} = (\mathbf{v}_S^i, \mathbf{v}_S^j)_0$ and $(\mathbb{P}_s)_{ij} = (p_S^i, p_S^j)_0$. By plugging this expression in (37), one obtains

$$\mathbb{M}_\omega\ \vec{\mathsf{E}}_R'' \ +\ \mathbb{R}_\omega\ \vec{\mathsf{E}}_R\ = -\mathbb{M}_\omega\ \vec{\mathsf{J}}'\ +\sum_{1\le j\le K_c} \{{\mathbb{V}_s}^{-1}((\vec{\mathsf{j}}^s)' + \mathbb{P}_s\vec{\mathsf{k}} + (\vec{\mathsf{e}}^s)'')\}_j\vec{\Lambda}_j\ ,$$

which is implicit in $\vec{\mathsf{E}}_R''$. After a *time discretization* involving a second-order explicit (leap-frog) scheme, the scheme reads

$$\mathbb{M}_\omega\ \vec{\mathsf{E}}_R^{n+1}\ -\sum_{1\le j\le K_c} \{{\mathbb{V}_s}^{-1}(\vec{\mathsf{e}}^s)^{n+1}\}_j\vec{\Lambda}_j\ = \vec{\mathsf{G}}^n\ .$$

Here the superscript ^{n} (resp. $^{n+1}$) stands for a variable at time $t^n = n\Delta t$ (resp. t^{n+1}), and $\vec{\mathsf{G}}^n$ is a set of quantities known at time t^n. After a few elementary algebraic manipulations, this expression can be written as

$$(\mathbb{M}_\omega \; - \sum_{1\le j\le K_c} \vec{\mathsf{U}}_j \vec{\Lambda}_j^T)\; \vec{\mathsf{E}}_R^{n+1} \; = \vec{\mathsf{G}}^n \;, \tag{38}$$

where $\vec{\mathsf{U}}_j$ is a linear combination of the $(\vec{\Lambda}_k)_{1\le k\le K_c}$: $\vec{\mathsf{U}}_j = \sum_{1\le k\le K_c} (\mathbb{V}_s{}^{-1})_{kj} \vec{\Lambda}_k$. It can be solved (for instance) with the help of the following formula (see [23] for a review), $\mathbb{A}$ $N\times N$, $\mathbb{U}$ and $\mathbb{W}$ $N\times K_c$

$$(\mathbb{A} - \mathbb{U}\mathbb{W}^T)^{-1} = \mathbb{A}^{-1} + \mathbb{A}^{-1}\mathbb{U}(\mathbb{I} - \mathbb{W}^T\mathbb{A}^{-1}\mathbb{U})^{-1}\mathbb{W}^T\mathbb{A}^{-1}$$

that only requires (compared to the unmodified system, that is $\mathbb{M}_\omega\; \vec{\mathsf{E}}_R^{n+1} = \vec{\mathsf{G}}^n$) the additional computation of the small $K_c\times K_c$ matrix $(\mathbb{I}-\mathbb{W}^T\mathbb{A}^{-1}\mathbb{U})^{-1}$. This formula is applied here for $\mathbb{A} = \mathbb{M}_\omega$. Recall that the mass matrix $\mathbb{M}_\omega$ is diagonalized thanks to a quadrature formula (see [8]), which preserves the accuracy. In this way, the linear system to solve (38) appears as a slight modification compared to the one obtained without the SCM.

3.2.4 Numerical results

Results of the computation of a basis of $\mathbf{V}_S$ are similar to those shown in §3.1.4 and will not be presented here. We refer the reader to [6] for more detailed numerical examples.

For the first case, one computes the electromagnetic field generated by a current, the space and time characteristics of which are similar to those of a bunched beam of particles. An L-shaped domain ω is considered where perfectly conducting boundary condition is imposed. The initial conditions are set to zero. The electromagnetic wave is generated by a current $\mathbf{J}(\mathbf{x},t) = (J_1, J_2)^T$, the support of which is bounded, with $J_1 = 0$, $J_2 = 10\sin(\lambda t)$, for λ associated to a frequency of $2,5.10^9$ Hertz. This current generates a wave which propagates both on the left and on the right. Physically, as long as the wave has not reached the reentrant corner, the field is smooth. Let t_s be the impact time, then, if one writes $\mathbf{E}(\mathbf{x},t) = \mathbf{E}_R(\mathbf{x},t) + \kappa(t)\mathbf{v}_S(\mathbf{x})$, $\kappa(t)$ is equal to zero for all t lower than t_s, and so $\mathbf{E}_R(\mathbf{x},t)$ and $\mathbf{E}(\mathbf{x},t)$ coincide. Now, on the one hand, for t greater than t_s , $\kappa(t)\neq 0$, and the support of $\mathbf{v}_S$ being non local (in fact, the support of $\mathbf{v}_S$ spans the whole of ω), one has $\kappa(t)\mathbf{v}_S(\mathbf{x})\neq 0$, for all $\mathbf{x}\in\omega$ and $t > t_s$. On the other hand, however, one wishes to reproduce the obvious physical behavior, which is that for any point $\mathbf{x}$ and time t, $\mathbf{E}(\mathbf{x},t) = 0$ if $t < t_{\mathbf{x}}$, where $t_{\mathbf{x}}$ denotes the time at which the electromagnetic wave reaches $\mathbf{x}$. One can check (see Figure 5) that $\mathbf{E}_R(\mathbf{x},t)$ takes non-zero values, and therefore that it 'compensates' for $\kappa(t)$ $\mathbf{v}_S(\mathbf{x})$, i.e. $\mathbf{E}_R(\mathbf{x},t) = -\kappa(t)$ $\mathbf{v}_S(\mathbf{x})$. Thus, $\mathbf{E}(\mathbf{x},t)$ remains equal to zero while $t_s < t < t_{\mathbf{x}}$.

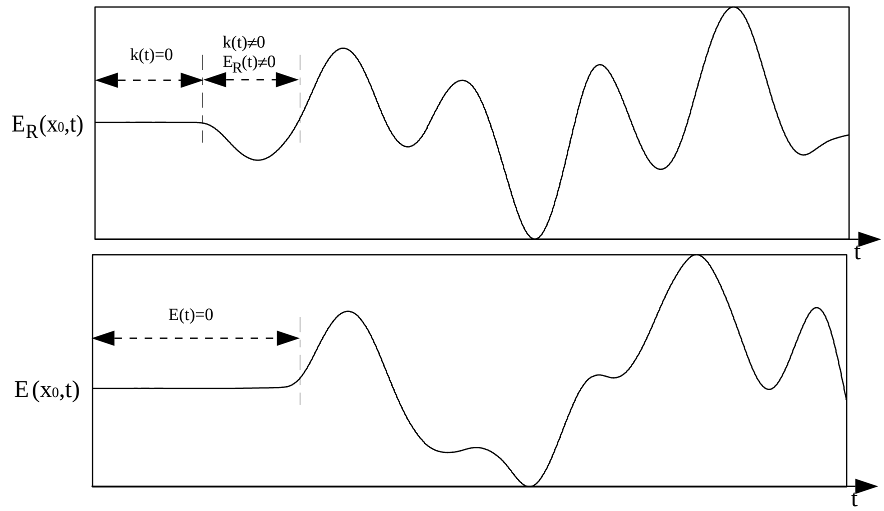

Figure 5: At a given point $\mathbf{x}_0$, comparison of $\mathbf{E}_R(\mathbf{x}_0, t)$ (top) and $\mathbf{E}(\mathbf{x}_0, t)$ (bottom) with t varying.

The second example is a guided wave which propagates in a standard *singular* geometry, as commonly studied devices such as hyperfrequency systems often include waveguides. This case illustrates of the possibilities of the method, when it is used on a more 'complete' formulation, that is with different types of boundary conditions and several reentrant corners. An incident wave enters in a step waveguide through the left boundary, and exits through the right boundary. At the initial time, the electromagnetic field is equal to zero all over the guide.

The Figure 6 depicts the isovalues of the first component of the electric field after 1000 time-steps. the SCM provides a numerical solution which is precise especially in the neighborhood of the corners. The result obtained *via* the classical nodal FE code (without the SCM) shows a most unlikely approximation of the true solution (no singular behavior).

Conclusion

We proposed a method, called the *Singular Complement Method*, to solve PDEs in a non-smooth and a non-convex domain. It is based on a splitting of the space of solutions V with respect to regularity (cf. (4)), in a subspace V_R made of regular elements, which is equal to V when the domain is smooth or convex, and a subspace of singular elements V_S. Regular elements are approximated by the P_1 Lagrange FEM, and test-functions are added to capture numerically the singular part of the solution.

In 3d domains, for the Laplace problem as well as for Maxwell's equations, the theoretical aspects are under control, but there still remains to provide

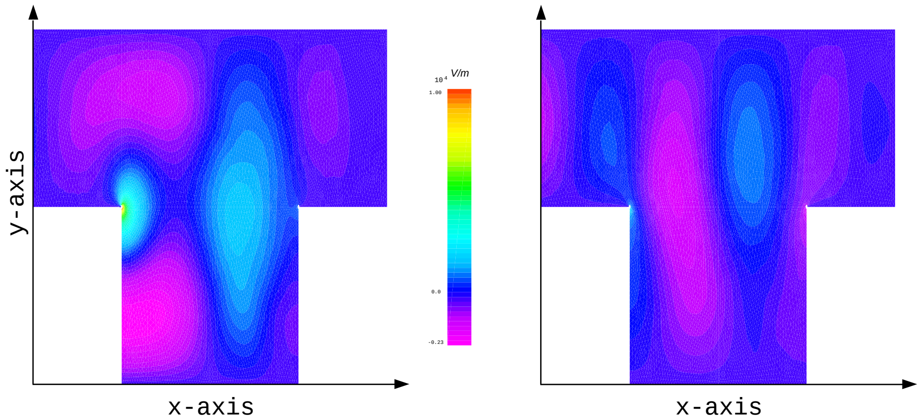

Figure 6: Computed electric field: with and without the SCM.

an effective approximation of the singular part of the solution. Basically, two problems have to be overcome:

- The dimension of V_S is infinite.
- The edge and vertex singularities are linked (cf. [21]).

These difficulties are not really equivalent. Indeed, on the one hand, one usually deals with infinite dimensional vector spaces: for instance, the space V_R is efficiently approximated with the help of the P_1 Lagrange FEM. On the other hand, finding an approximation, which takes into account the links between the two types of geometrical singularities, is much more challenging.

In 2d (or in axisymmetric domains), the situation is well understood theoretically, and numerical experiments are well under way and partial results are satisfactory. Moreover, the SCM is easy to implement, as it can be included in already existing codes, without having to rewrite them in their entirety; also, it generates a reasonable overhead (low additional memory requirements, small cpu costs). So, all's well that ends well, cf. [32].

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