T-coercivity for the Stokes problem with small viscosity

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Outline

What is T-coercivity?

A tool to study variational formulations [Chesnel-PC'13]

Abstract framework: *Find* $u \in V$ *s.t.* $\forall w \in W,\ a(u,w) = \langle f, w\rangle_W.$

Approximate framework: *Find* $u_\delta \in V_\delta$ *s.t.* $\forall w_\delta \in W_\delta,\ a(u_\delta,w_\delta) = \langle f, w_\delta\rangle_W.$

1. First, analyse the variational formulation theoretically:
   - prove well-posedness ;
   - existence, uniqueness and continuous dependence of the solution with respect to the data.

2. Second, solve the variational formulation numerically:
   - find suitable approximations ;
   - prove convergence.

Within the framework of T-coercivity, steps 1 and 2 are very strongly correlated!

Let

- V, W be Hilbert spaces over $\mathbb{C}$;
- $a(\cdot,\cdot)$ be a bounded sesquilinear form on $V \times W$;
- f be an element of W', the dual space of W.

Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = \langle f, w \rangle_W .$$

[Banach-Nečas-Babuška] The *inf-sup condition* writes

$$\text{(isc)} \quad \exists \alpha > 0,\ \forall v \in V,\ \sup_{w \in W \setminus \{0\}} \frac{|a(v,w)|}{\|w\|_W} \geq \alpha \, \|v\|_V .$$

If in addition $\{w \in W \,|\, \forall v \in V,\ a(v,w) = 0\} = \{0\}$, then the variational formulation (VF) is well-posed.

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Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = \langle f, w\rangle_W.$$

Definition (T-coercivity)

The form $a(\cdot,\cdot)$ is T-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V,W) \text{ bijective},\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v,\mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2.$$

NB. In other words, the form $a(\cdot,\mathtt{T}\cdot)$ is coercive on $V \times V$.

Let

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- $a(\cdot,\cdot)$ be a bounded sesquilinear form on $V \times W$;
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Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = \langle f, w\rangle_W .$$

Theorem (Well-posedness)

The three assertions below are equivalent:

(i) *the variational formulation (VF) is well-posed ;*

(ii) *the form $a(\cdot,\cdot)$ satisfies (isc) and $\{w \in W \mid \forall v \in V,\ a(v,w) = 0\} = \{0\}$;*

(iii) *the form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive.*

The operator $\mathtt{T}$ realises the inf-sup condition (isc) explicitly: $w = \mathtt{T}u$ works!

Let

- V, W be Hilbert spaces over $\mathbb{C}$;
- $a(\cdot,\cdot)$ be a bounded sesquilinear form on $V \times W$;
- f be an element of W', the dual space of W.

Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = \langle f, w\rangle_W .$$

Theorem

For any bijective operator $\mathtt{T} \in \mathcal{L}(V,W)$, the variational formulation (VF) is equivalent to

$$\textit{(VF)}_{\mathtt{T}} \quad \text{Find } u \in V \text{ s.t. } \forall v \in V,\ a(u,\mathtt{T}v) = \langle f, \mathtt{T}v\rangle_W$$

Let

- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V;
- $(W_\delta)_\delta$ be a family of finite dimensional subspaces of W.

Assume that $\dim(V_\delta) = \dim(W_\delta)$ for all $\delta > 0$.
Solve

$$(\mathrm{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = \langle f, w_\delta \rangle_W.$$

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$$(\mathsf{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = \langle f, w_\delta \rangle_W .$$

[Banach-Nečas-Babuška] The *uniform discrete inf-sup condition* writes

$$(\mathsf{udisc}) \quad \exists \alpha_\dagger > 0,\ \forall \delta > 0,\ \forall v_\delta \in V_\delta,\ \sup_{w_\delta \in W_\delta \setminus \{0\}} \frac{|a(v_\delta, w_\delta)|}{\|w_\delta\|_W} \geq \alpha_\dagger \|v_\delta\|_V .$$

NB. When (udisc) is fulfilled, $(\mathsf{VF})_\delta$ is well-posed for all $\delta > 0$.

Let

- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V ;
- $(W_\delta)_\delta$ be a family of finite dimensional subspaces of W.

Assume that $\dim(V_\delta) = \dim(W_\delta)$ for all $\delta > 0$.
Solve

$$(\mathsf{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = \langle f, w_\delta \rangle_W.$$

Definition (uniform $\mathtt{T}_\delta$-coercivity)

The form a is *uniformly* $\mathtt{T}_\delta$*-coercive* if

$$\exists \underline{\alpha}_\dagger, \underline{\beta}_\dagger > 0,\ \forall \delta > 0,\ \exists \mathtt{T}_\delta \in \mathcal{L}(V_\delta, W_\delta),\ |||\mathtt{T}_\delta||| \le \underline{\beta}_\dagger \text{ and } \forall v_\delta \in V_\delta,\ |a(v_\delta, \mathtt{T}_\delta v_\delta)| \ge \underline{\alpha}_\dagger \|v_\delta\|_V^2.$$

NB. When a is uniformly $\mathtt{T}_\delta$-coercive, $(\mathsf{VF})_\delta$ is well-posed for all $\delta > 0$.

Let

- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V ;
- $(W_\delta)_\delta$ be a family of finite dimensional subspaces of W.

Assume that $\dim(V_\delta) = \dim(W_\delta)$ for all $\delta > 0$.

Solve

$$(\mathrm{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = \langle f, w_\delta \rangle_W.$$

Theorem (Céa's lemma)

Assume that

(i) *either, the form $a(\cdot,\cdot)$ satisfies (udisc) ;*

(ii) *or, the form $a(\cdot,\cdot)$ is uniformly T_δ-coercive.*

In addition, assume that the family $(V_\delta)_\delta$ fulfills the basic approximability property in V.
Then, $\lim_{\delta\to 0} \|u - u_\delta\|_V = 0$.

Let

- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V ;
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Assume that $\dim(V_\delta) = \dim(W_\delta)$ for all $\delta > 0$.
Solve

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Theorem (Céa's lemma)

Assume that

(i) *either, the form $a(\cdot,\cdot)$ satisfies (udisc) ;*

(ii) *or, the form $a(\cdot,\cdot)$ is uniformly T_δ-coercive.*

In addition, assume that the family $(V_\delta)_\delta$ fulfills the basic approximability property in V.
Then, $\lim_{\delta\to 0} \|u - u_\delta\|_V = 0$. And error estimates whenever possible...

Two key ideas [Chesnel-PC'13]

[1st Key Idea] Use the knowledge on operator $\mathtt{T}$ to derive the discrete operators $(\mathtt{T}_\delta)_\delta$!

[2nd Key Idea] Discretize the variational formulation with (bijective) operator $\mathtt{T}$:

$$(\mathsf{VF})_{\mathtt{T}} \quad \textit{Find } u \in V \textit{ s.t. } \forall v \in V,\ a(u, \mathtt{T}v) = \langle f, \mathtt{T}v \rangle_W \ !$$

What is T-coercivity?

As an approximation tool (solving the equivalent linear system)

Given $\delta > 0$, let $N = \dim(V_\delta)$. $(\mathrm{VF})_\delta$ is equivalent to
Solve

$$\textit{Find } U \in \mathbb{C}^N \textit{ s.t. } \forall W \in \mathbb{C}^N,\ (\mathbb{A}U|W) = (F|W).$$

$$\textit{Or, find } U \in \mathbb{C}^N \textit{ s.t. } \mathbb{A}U = F.$$

[Discrete T-coercivity] Using $\mathbb{T}$ associated with $\mathtt{T}_\delta$, $(\mathrm{VF})_\delta$ is equivalent to
Solve

$$\textit{Find } U \in \mathbb{C}^N \textit{ s.t. } \forall V \in \mathbb{C}^N,\ (\mathbb{A}U|\mathbb{T}V) = (F|\mathbb{T}V).$$

$$\textit{Or, find } U \in \mathbb{C}^N \textit{ s.t. } \mathbb{T}^*\mathbb{A}U = \mathbb{T}^*F.$$

According to the uniform $\mathtt{T}_\delta$-coercivity assumption

$$\forall V \in \mathbb{C}^N,\ |(\mathbb{T}^*\mathbb{A}V|V)| \geq \underline{\alpha}_\dagger(\mathbb{M}V|V).$$

[Explicit T-coercivity] Use $\mathbb{T}$ associated with $\mathtt{T}$ for the approximation of $(\mathrm{VF})_\mathtt{T}$. Same results...

Outline

1. Let Ω be a domain of $\mathbb{R}^d$, $d = 2, 3$. The "simplest" Stokes equations write

$$\begin{cases} -\nu\,\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega \\ \operatorname{div} \boldsymbol{u} = g \text{ in } \Omega \\ \boldsymbol{u} = 0 \text{ on } \partial\Omega, \end{cases}$$

for some $\nu > 0$ (viscosity). For "classical" Stokes, $g = 0$.

1. Assuming that $\boldsymbol{f} \in (\boldsymbol{H}_0^1(\Omega))'$ and $g \in L^2_{zmv}(\Omega)$, one analyses mathematically the model

$$\text{(Stokes)} \qquad \begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ -\nu\,\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega \\ \operatorname{div}\boldsymbol{u} = g \text{ in } \Omega. \end{cases}$$

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2. The equivalent variational formulation writes

$$\text{(VF-Stokes)} \begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega), \quad \nu \displaystyle\int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v}\, d\Omega \\ \quad - \displaystyle\int_\Omega p \operatorname{div} \boldsymbol{v}\, d\Omega - \displaystyle\int_\Omega q \operatorname{div} \boldsymbol{u}\, d\Omega = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{H}_0^1(\Omega)} - \displaystyle\int_\Omega g q\, d\Omega. \end{cases}$$

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$$\text{(VF-Stokes)} \quad \begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega), \quad \nu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v}\, d\Omega \\ \quad - \int_\Omega p \operatorname{div} \boldsymbol{v}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{u}\, d\Omega = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{H}_0^1(\Omega)} - \int_\Omega g q\, d\Omega. \end{cases}$$

Question: how to prove well-posedness "easily"?

1. Assuming that $\boldsymbol{f} \in (\boldsymbol{H}_0^1(\Omega))'$ and $g \in L^2_{zmv}(\Omega)$, one analyses mathematically the model

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$$\text{(VF-Stokes)} \quad \begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega), \quad \nu \displaystyle\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v}\, d\Omega \\ \quad - \displaystyle\int_\Omega p \operatorname{div} \boldsymbol{v}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{u}\, d\Omega = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{H}_0^1(\Omega)} - \int_\Omega g q\, d\Omega. \end{cases}$$

Question: how to prove well-posedness "easily"?

Prove $\mathbb{T}$-coercivity for the Stokes model!

Stokes model

Basic $\mathbb{T}$-coercivity - 1

Let

- $\mathbb{V} = \boldsymbol{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$, endowed with $\|(\boldsymbol{v}, q)\|_{\mathbb{V},\nu} = (|\boldsymbol{v}|^2_{1,\Omega} + \nu^{-2}\|q\|^2)^{1/2}$;
- $a((\boldsymbol{v}, q), (\boldsymbol{w}, r)) = \nu \displaystyle\int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$;
- $\langle f, (\boldsymbol{w}, r)\rangle_{\mathbb{V}} = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{H}^1_0(\Omega)} - \displaystyle\int_\Omega r\, g\, d\Omega.$

Let

- $\mathbb{V} = \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$, endowed with $\|(\boldsymbol{v}, q)\|_{\mathbb{V},\nu} = (|\boldsymbol{v}|^2_{1,\Omega} + \nu^{-2}\|q\|^2)^{1/2}$;
- $a((\boldsymbol{v}, q), (\boldsymbol{w}, r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \,\mathrm{div}\, \boldsymbol{w}\, d\Omega - \int_\Omega r \,\mathrm{div}\, \boldsymbol{v}\, d\Omega$;
- $\langle f, (\boldsymbol{w}, r)\rangle_{\mathbb{V}} = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{H}_0^1(\Omega)} - \int_\Omega r\, g\, d\Omega$.

Basic T-coercivity: prove well-posedness with T-coercivity.
NB. The form a is not coercive, because $a((0, q), (0, q)) = 0$ for $q \in L^2_{zmv}(\Omega)$.

Let

- $\mathbb{V} = \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$, endowed with $\|(\boldsymbol{v}, q)\|_{\mathbb{V},\nu} = (|\boldsymbol{v}|^2_{1,\Omega} + \nu^{-2}\|q\|^2)^{1/2}$;
- $a((\boldsymbol{v}, q), (\boldsymbol{w}, r)) = \nu \int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$;
- $\langle f, (\boldsymbol{w}, r)\rangle_{\mathbb{V}} = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{H}_0^1(\Omega)} - \int_\Omega r\, g\, d\Omega$.

Basic $\mathtt{T}$-coercivity: prove well-posedness with $\mathtt{T}$-coercivity.
Given $(\boldsymbol{v}, q) \in \mathbb{V}$, we look for $(\boldsymbol{w}^\star, r^\star) \in \mathbb{V}$ with linear dependence such that

$$|a((\boldsymbol{v}, q), (\boldsymbol{w}^\star, r^\star))| \geq \underline{\alpha}\, \|(\boldsymbol{v}, q)\|^2_{\mathbb{V},\nu},$$

with $\underline{\alpha} > 0$ independent of $(\boldsymbol{v}, q)$. In other words, $\mathtt{T}$ is defined by $\mathtt{T}((\boldsymbol{v}, q)) = (\boldsymbol{w}^\star, r^\star)$.

Let

- $\mathbb{V} = \boldsymbol{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$, endowed with $\|(\boldsymbol{v}, q)\|_{\mathbb{V},\nu} = (|\boldsymbol{v}|^2_{1,\Omega} + \nu^{-2}\|q\|^2)^{1/2}$;
- $a((\boldsymbol{v}, q), (\boldsymbol{w}, r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$;
- $\langle f, (\boldsymbol{w}, r)\rangle_{\mathbb{V}} = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{H}^1_0(\Omega)} - \int_\Omega r\, g\, d\Omega.$

Basic T-coercivity: prove well-posedness with T-coercivity.
Given $(\boldsymbol{v}, q) \in \mathbb{V}$, we look for $(\boldsymbol{w}^\star, r^\star) \in \mathbb{V}$ with linear dependence such that

$$|a((\boldsymbol{v}, q), (\boldsymbol{w}^\star, r^\star))| \geq \underline{\alpha}\, \|(\boldsymbol{v}, q)\|^2_{\mathbb{V},\nu},$$

with $\underline{\alpha} > 0$ independent of $(\boldsymbol{v}, q)$. Three steps:

1. $q = 0$;
2. $\boldsymbol{v} = 0$;
3. General case.

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{w}\, d\Omega - \int_\Omega q \,\mathrm{div}\, \boldsymbol{w}\, d\Omega - \int_\Omega r \,\mathrm{div}\, \boldsymbol{v}\, d\Omega$.

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{w}\, d\Omega - \int_\Omega r \,\mathrm{div}\, \boldsymbol{v}\, d\Omega$: so choosing $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v},0)$ yields
$$|a((\boldsymbol{v},0),(\boldsymbol{w}^\star,r^\star))| = \nu \int_\Omega |\boldsymbol{\nabla}\boldsymbol{v}|^2\, d\Omega = \nu\, \|(\boldsymbol{v},0)\|^2_{\mathbb{V},\nu}.$$

Stokes model

Basic $\mathbb{T}$-coercivity - 2

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega.$

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v}, 0)$.
2. $a((0,q),(\boldsymbol{w},r)) = -\int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega$: according to eg. `Girault-Raviart'86`,

   $\exists C_{\mathrm{div}} > 0,\ \forall q \in L^2_{zmv}(\Omega),\ \exists \boldsymbol{w}_q \in \boldsymbol{H}^1_0(\Omega)$ such that $\operatorname{div} \boldsymbol{w}_q = q$, with $|\boldsymbol{w}_q|_{1,\Omega} \leq C_{\mathrm{div}} \|q\|$.

   So choosing $(\boldsymbol{w}^\star, r^\star) = (-\boldsymbol{w}_q, 0)$ yields

$$|a((0,q),(\boldsymbol{w}^\star,r^\star))| = \int_\Omega q^2\, d\Omega = \nu^2 \|(0,q)\|^2_{\mathbb{V},\nu}.$$

NB. From now on, we take $\boldsymbol{w}_q$ in the orthogonal of $\boldsymbol{V}_0 = \{\boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega) \,|\, \operatorname{div} \boldsymbol{w} = 0\}$.

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$.

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v}, 0)$.
2. $a((0,q),(\boldsymbol{w},r)) = -\int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (-\boldsymbol{w}_q, 0)$.
3. General case: beginning with the linear combination $\boldsymbol{w}^\star = \lambda \boldsymbol{v} - \mu \boldsymbol{w}_q$, $\lambda, \mu > 0$, one finds

$$a((\boldsymbol{v},q),(\boldsymbol{w}^\star,r)) = \lambda\nu\, |\boldsymbol{v}|^2_{1,\Omega} - \mu\nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}_q\, d\Omega - \int_\Omega (\lambda q + r) \operatorname{div} \boldsymbol{v}\, d\Omega + \mu \|q\|^2.$$

Stokes model

Basic $\mathbb{T}$-coercivity - 2

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$.

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v}, 0)$.
2. $a((0,q),(\boldsymbol{w},r)) = -\int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (-\boldsymbol{w}_q, 0)$.
3. General case: $\boldsymbol{w}^\star = \lambda \boldsymbol{v} - \mu \boldsymbol{w}_q$, $\lambda, \mu > 0$. Next, $r^\star = -\lambda q$ leads to

$$a((\boldsymbol{v},q),(\boldsymbol{w}^\star,r^\star)) = \lambda\nu\, |\boldsymbol{v}|^2_{1,\Omega} + \mu \|q\|^2 - \mu\nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}_q\, d\Omega.$$

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega.$

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$$a((\boldsymbol{v},q),(\boldsymbol{w}^\star,r^\star)) = \lambda\nu\, |\boldsymbol{v}|^2_{1,\Omega} + \mu\|q\|^2 - \mu\nu \int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{w}_q\, d\Omega.$$

Finally, the last term can be controlled by the first two terms thanks to $|\boldsymbol{w}_q|_{1,\Omega} \le C_{\text{div}}\, \|q\|$, using Young's inequality.

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Finally, the last term can be controlled by the first two terms thanks to $|\boldsymbol{w}_q|_{1,\Omega} \le C_{\mathrm{div}}\, \|q\|$, using Young's inequality.
Eg., choose $(\lambda, \mu) = ((C_{\mathrm{div}})^2, \nu^{-1})$: $\mathtt{T}((\boldsymbol{v},q)) = ((C_{\mathrm{div}})^2\boldsymbol{v} - \nu^{-1}\boldsymbol{w}_q, -(C_{\mathrm{div}})^2 q)$.
The operator $\mathtt{T}$ is bijective (one easily builds its inverse).

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega.$

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v}, 0)$.
2. $a((0,q),(\boldsymbol{w},r)) = -\int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (-\boldsymbol{w}_q, 0)$.
3. General case: $\boldsymbol{w}^\star = \lambda \boldsymbol{v} - \mu \boldsymbol{w}_q$, $\lambda, \mu > 0$. Next, $r^\star = -\lambda q$ leads to

$$a((\boldsymbol{v},q),(\boldsymbol{w}^\star,r^\star)) = \lambda\nu\, |\boldsymbol{v}|^2_{1,\Omega} + \mu \|q\|^2 - \mu\nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}_q\, d\Omega.$$

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NB. Playing with Young's inequality, one finds that there is an "admissible" family of coefficients (λ, μ) that yield T-coercivity: eg. for $\mu = \nu^{-1}$, one needs that $\lambda > \frac{1}{4}(C_{\mathrm{div}})^2$.

Regarding the **basic $\mathtt{T}$-coercivity**, one can make several observations:

1. The result of `Girault-Raviart'86` appears as a requirement to derive the $\mathtt{T}$-coercivity!
2. The $\mathtt{T}$-coercivity approach is flexible, in the sense that one has at hand a family of operators $\mathtt{T}$ (depending on the chosen linear combination). Among others, one may "optimize" the value of the stability constant with respect to ν.
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Discrete $\mathtt{T}$-coercivity: prove the uniform discrete inf-sup condition, with the help of the uniform $\mathtt{T}_\delta$-coercivity.

Stokes model

Basic T-coercivity - 3

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Mimic the previous proof to guarantee uniform $\mathtt{T}_\delta$-coercivity! [1st Key Idea]

Discrete T-coercivity | Explicit T-coercivity

The discrete variational formulation writes

$$(\text{VF-Stokes})_\delta \begin{cases} \text{Find } (\boldsymbol{u}_\delta, p_\delta) \in \boldsymbol{V}_\delta \times Q_\delta \text{ such that} \\ \forall (\boldsymbol{v}_\delta, q_\delta) \in \boldsymbol{V}_\delta \times Q_\delta, \\ \nu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_\delta : \boldsymbol{\nabla} \boldsymbol{v}_\delta \, d\Omega - \int_\Omega p_\delta \operatorname{div} \boldsymbol{v}_\delta \, d\Omega - \int_\Omega q_\delta \operatorname{div} \boldsymbol{u}_\delta \, d\Omega = \langle f, (\boldsymbol{v}_\delta, q_\delta) \rangle_{\mathbb{V}}. \end{cases}$$

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Given $(\boldsymbol{v}_\delta, q_\delta) \in \boldsymbol{V}_\delta \times Q_\delta$, we look for $(\boldsymbol{w}_\delta^\star, r_\delta^\star) \in \boldsymbol{V}_\delta \times Q_\delta$ with linear dependence such that

$$|a((\boldsymbol{v}_\delta, q_\delta), (\boldsymbol{w}_\delta^\star, r_\delta^\star))| \geq \underline{\alpha}_\dagger \, \|(\boldsymbol{v}_\delta, q_\delta)\|_{\mathbb{V}}^2,$$

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$$\boldsymbol{w}^\star = (C_{\text{div}})^2 \boldsymbol{v}_\delta - \nu^{-1} \boldsymbol{w}_{q_\delta} \text{ and } r^\star = -(C_{\text{div}})^2 q_\delta,$$

with $\boldsymbol{w}_{q_\delta} \in \boldsymbol{H}_0^1(\Omega)$ such that $\operatorname{div} \boldsymbol{w}_{q_\delta} = q_\delta$, and $|\boldsymbol{w}_{q_\delta}|_{1,\Omega} \leq C_{\text{div}} \, \|q_\delta\|$.

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Difficulty: $\boldsymbol{w}_{q_\delta} \notin \boldsymbol{V}_\delta$ in general, whereas $\boldsymbol{v}_\delta \in \boldsymbol{V}_\delta$ and $r^\star \in Q_\delta$.

How to overcome this difficulty to be able to conclude the proof?

Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\mathrm{div}\, \boldsymbol{w}_\delta^+ = q_\delta$ weakly", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

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As a matter of fact, choosing $\boldsymbol{w}_\delta^\star = (C^+)^2 \boldsymbol{v}_\delta - \nu^{-1} \boldsymbol{w}_\delta^+$ and $r_\delta^\star = -(C^+)^2 q_\delta$ immediately yields the uniform discrete inf-sup condition!

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As a matter of fact, choosing $\boldsymbol{w}_\delta^\star = (C^+)^2 \boldsymbol{v}_\delta - \nu^{-1} \boldsymbol{w}_\delta^+$ and $r_\delta^\star = -(C^+)^2 q_\delta$ immediately yields the uniform discrete inf-sup condition! How so? Just add $_\delta$s to the previous computations!

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To finish the computations as before, we look for pairs of discrete spaces $(\boldsymbol{V}_\delta, Q_\delta)_\delta$ such that

$$\exists C^+ > 0,\ \forall \delta, \qquad \forall q_\delta \in Q_\delta,\ \exists \boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta \text{ with the properties}$$

$$\forall q'_\delta \in Q_\delta, \quad \int_\Omega q'_\delta \operatorname{div} \boldsymbol{w}_\delta^+ \, d\Omega = \int_\Omega q'_\delta \, q_\delta \, d\Omega\,;$$

$$|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \|q_\delta\|.$$

Stokes model

Discrete $\mathbb{T}$-coercivity - 2

How to overcome this difficulty to be able to conclude the proof?

Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\operatorname{div} \boldsymbol{w}_\delta^+ = q_\delta$ weakly", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \, \|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

In other words, one is looking for pairs of discrete spaces $(\boldsymbol{V}_\delta, Q_\delta)_\delta$ such that

$\exists C_\pi > 0, \ \forall \delta, \ \exists \pi_\delta \in \mathcal{L}(\boldsymbol{H}_0^1(\Omega), \boldsymbol{V}_\delta)$ with the properties

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \ \forall q'_\delta \in Q_\delta, \quad \int_\Omega q'_\delta \operatorname{div}(\pi_\delta \boldsymbol{v}) \, d\Omega = \int_\Omega q'_\delta \operatorname{div} \boldsymbol{v} \, d\Omega \, ;$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad |\pi_\delta \boldsymbol{v}|_{1,\Omega} \leq C_\pi |\boldsymbol{v}|_{1,\Omega}.$$

Then one chooses $\boxed{\boldsymbol{w}_\delta^+ = \pi_\delta \boldsymbol{w}_{q_\delta}}$ to get the desired properties with $C^+ = C_\pi C_{\text{div}}$.

Regarding the proof of **discrete T-coercivity**, we observe that:

1. The so-called Fortin lemma appears "naturally" in the proof.
2. One needs to have some knowledge of finite element spaces.
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Browsing `Boffi-Brezzi-Fortin'13`, one finds that **discrete T-coercivity** is achieved with:

- the MINI FE, or the Taylor-Hood FE $\mathbf{P}^{k+1} - P^k$, of order $k \geq 1$;
- the nonconforming Crouzeix-Raviart $\mathbf{P}^1_{nc} - P^0$ is also possible...

Convergence and error estimates follow.

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Convergence and error estimates follow.

T-coercivity and uniform $\mathtt{T}_\delta$-coercivity are indeed strongly correlated! [1st Key Idea]

Let Ω be a domain of $\mathbb{R}^d$, $d = 2, 3$. We consider the "classical" Stokes equations

$$\begin{cases} -\nu\,\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega \\ \boldsymbol{u} = 0 \text{ on } \partial\Omega, \end{cases}$$

for some small $\nu > 0$ (viscosity).

[2nd Key Idea] The operator $\mathtt{T}((\boldsymbol{w},r)) = (\lambda\boldsymbol{w} - \nu^{-1}\boldsymbol{w}_r, -\lambda r)$ is bijective for all $\lambda > 0$. Consider the bilinear form on $\mathbb{V}\times\mathbb{V}$

$$\begin{aligned} a((\boldsymbol{v},q),\mathtt{T}(\boldsymbol{w},r)) &= \nu\lambda\int_\Omega \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{w}\,d\Omega - \int_\Omega \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{w}_r\,d\Omega \\ &\quad -\lambda\int_\Omega q\,\mathrm{div}\,\boldsymbol{w}\,d\Omega + \nu^{-1}\int_\Omega q\,r\,d\Omega + \lambda\int_\Omega r\,\mathrm{div}\,\boldsymbol{v}\,d\Omega, \end{aligned}$$

and the linear form on $\mathbb{V}$

$$\langle f, \mathtt{T}(\boldsymbol{w},r)\rangle_{\mathbb{V}} = \lambda\langle \boldsymbol{f},\boldsymbol{w}\rangle_{\boldsymbol{H}_0^1(\Omega)} - \nu^{-1}\langle \boldsymbol{f},\boldsymbol{w}_r\rangle_{\boldsymbol{H}_0^1(\Omega)}.$$

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Two difficulties in practice:

1. Explicit expression of the terms involving $\boldsymbol{w}_r \in (\boldsymbol{V}_0)^\perp$?
2. Choice of λ?

Given $\boldsymbol{f} \in (\boldsymbol{H}_0^1(\Omega))'$, solving the "classical" Stokes model with $\nu = 1$,

$$\exists!(\mathbf{w}_{\boldsymbol{f}}, z_{\boldsymbol{f}}) \in \boldsymbol{V}_0 \times L^2_{zmv}(\Omega) \text{ such that } \boldsymbol{f} = -\Delta \mathbf{w}_{\boldsymbol{f}} + \nabla z_{\boldsymbol{f}}.$$

For all $r \in L^2_{zmv}(\Omega)$, one has $\langle \boldsymbol{f}, \boldsymbol{w}_r \rangle_{\boldsymbol{H}_0^1(\Omega)} = -\int_\Omega z_{\boldsymbol{f}}\, r\, d\Omega$.

The linear form on $\mathbb{V}$ is equal to

$$\underline{f}_\lambda : \ (\boldsymbol{w}, r) \mapsto \lambda \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{H}_0^1(\Omega)} + \nu^{-1} \int_\Omega z_{\boldsymbol{f}}\, r\, d\Omega.$$

For the "classical" Stokes equations, the solution $\boldsymbol{u}$ belongs to $\boldsymbol{V}_0$.

For all $r \in L^2_{zmv}(\Omega)$, one has $\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla w}_r\, d\Omega = 0$ by orthogonality.

One may consider the simplified bilinear form

$$\underline{a}_\lambda : \ ((\boldsymbol{v}, q), (\boldsymbol{w}, r)) \ \mapsto \ \nu\lambda \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \lambda \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega$$
$$\lambda \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega \quad + \nu^{-1} \int_\Omega q\, r\, d\Omega.$$

1. **Explicit T-coercivity**: the variational formulation with forms $\underline{a}_\lambda$ and $\underline{f}_\lambda$ is

$$\underline{(\mathrm{VF})}_\lambda \quad \begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad \nu\lambda \displaystyle\int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \, d\Omega - \lambda \int_\Omega p \operatorname{div} \boldsymbol{v} \, d\Omega = \lambda \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{H}_0^1(\Omega)} \\ \forall q \in L^2_{zmv}(\Omega), \quad \lambda \displaystyle\int_\Omega q \operatorname{div} \boldsymbol{u} \, d\Omega + \nu^{-1} \int_\Omega p \, q \, d\Omega = \nu^{-1} \int_\Omega z_{\boldsymbol{f}} \, q \, d\Omega. \end{cases}$$

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Theorem (Well-posedness)

For all $\lambda > 0$, the variational formulation $\underline{(VF)}_\lambda$ is equivalent to the "classical" Stokes equations, and it is well-posed.

NB. The form $\underline{a}_\lambda$ is coercive on $\mathbb{V} \times \mathbb{V}$ for all $\lambda > 0$.

Stokes model with small viscosity

Explicit T-coercivity - 3

1. **Explicit T-coercivity**: the variational formulation with forms $\underline{a}_\lambda$ and $\underline{f}_\lambda$ is

$$\underline{(\mathrm{VF})}_\lambda \begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}^1_0(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega), \quad \nu\lambda \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \, d\Omega - \lambda \int_\Omega p \operatorname{div} \boldsymbol{v} \, d\Omega = \lambda \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{H}^1_0(\Omega)} \\ \forall q \in L^2_{zmv}(\Omega), \qquad \lambda \int_\Omega q \operatorname{div} \boldsymbol{u} \, d\Omega \quad + \nu^{-1} \int_\Omega p \, q \, d\Omega \quad = \nu^{-1} \int_\Omega z_{\boldsymbol{f}} \, q \, d\Omega. \end{cases}$$

2. For the approximation:
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$\boxed{\nu = 1}$ Compute first some approximation $z_{\boldsymbol{f},\delta}$ of $z_{\boldsymbol{f}}$.

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$\boxed{\nu > 0}$ Post-process by solving the discrete VF $\underline{(\mathrm{VF})}_\lambda$ with rhs $z_{\boldsymbol{f},\delta}$: we use the $\boxed{\mathbf{P}^1 - P^0 \text{ FE}}$.

For the numerical experiments: $\Omega = (0,1)^2$, and $\nu = 10^{-6}$.

Manufactured test cases:

1. With a smooth solution.
2. With a singular solution: $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \setminus \boldsymbol{H}^2(\Omega)$.

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Triangular meshes, with meshsize h, $h^{-1} \in (10, 160)$:

1. Smooth solution: $10^3 - 272.10^3$ dof for the $\mathbf{P}^1_{nc} - P^0$ FE (50% less for the $\mathbf{P}^1 - P^0$ FE).
2. Singular solution: $2.10^3 - 464.10^3$ dof for the $\mathbf{P}^1_{nc} - P^0$ FE (50% less for the $\mathbf{P}^1 - P^0$ FE).

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1. With a smooth solution.
2. With a singular solution: $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \setminus \boldsymbol{H}^2(\Omega)$.

Triangular meshes, with meshsize h, $h^{-1} \in (10, 160)$:

1. Smooth solution: $10^3 - 272.10^3$ dof for the $\mathbf{P}^1_{nc} - P^0$ FE (50% less for the $\mathbf{P}^1 - P^0$ FE).
2. Singular solution: $2.10^3 - 464.10^3$ dof for the $\mathbf{P}^1_{nc} - P^0$ FE (50% less for the $\mathbf{P}^1 - P^0$ FE).

Error indicators:

- Velocity: $\varepsilon_0^\nu(\boldsymbol{u}_h) = \|\boldsymbol{u} - \boldsymbol{u}_h\| / \|(\boldsymbol{u}, p)\|_{\mathbb{V},\nu}$.
- Pressure: $\varepsilon_0^\nu(p_h) = \nu^{-1}\|p - p_h\| / \|(\boldsymbol{u}, p)\|_{\mathbb{V},\nu}$.

  Results with the $\mathbf{P}^1 - P^0$ FE for solving $\underline{\text{(VF)}}_{\lambda=1}$ with $z_{\boldsymbol{f}}$ are proposed as a reference.

Post-processing is carried out iteratively (initialization with rhs $z_{\boldsymbol{f},\delta}$; 1 or 8 iterations).

For the post-processing steps, one solves $\underline{\text{(VF)}}_{\lambda=1}$.

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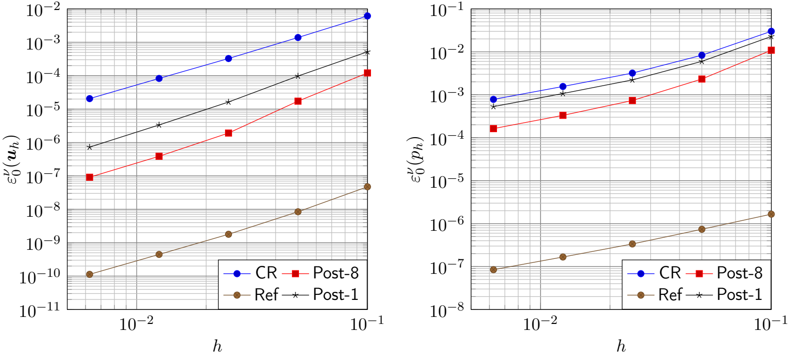

Figure: [Smooth solution] Plots of $\varepsilon_0^\nu(\boldsymbol{u}_h)$ and $\varepsilon_0^\nu(p_h)$ against h.

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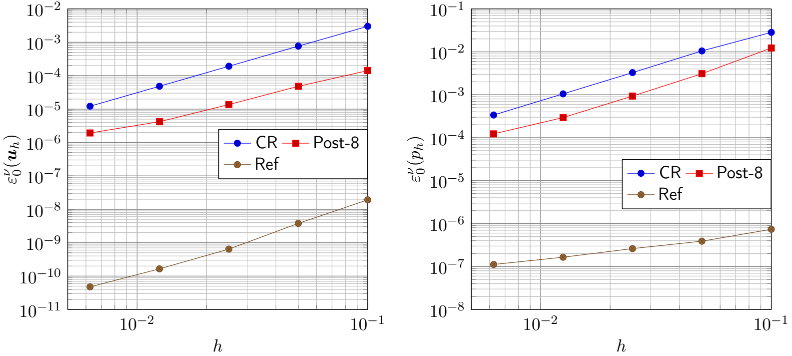

Figure: [Singular solution] Plots of $\varepsilon_0^\nu(\boldsymbol{u}_h)$ and $\varepsilon_0^\nu(p_h)$ against h.

Post-processing is carried out iteratively (initialization with rhs $z_{\boldsymbol{f},\delta}$; 1 or 8 iterations).
[Octave code] overhead cost (CPU time) due to post-processing goes from 125% to 13%.

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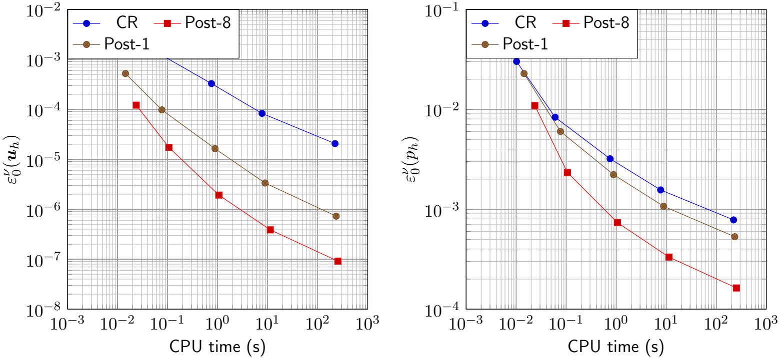

Figure: [Smooth solution] Plots of $\varepsilon_0^\nu(\boldsymbol{u}_h)$ and $\varepsilon_0^\nu(p_h)$ against CPU time.

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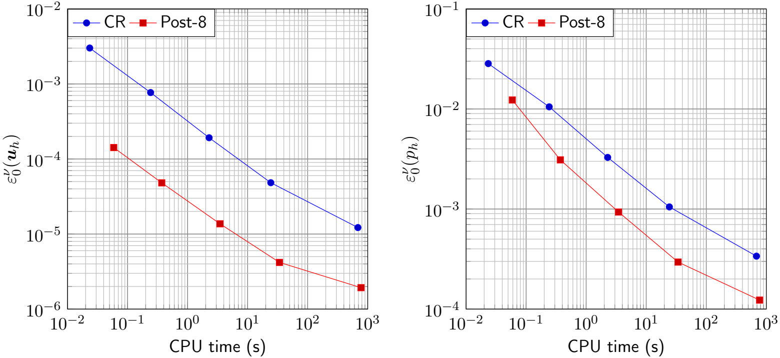

Figure: [Singular solution] Plots of $\varepsilon_0^\nu(\boldsymbol{u}_h)$ and $\varepsilon_0^\nu(p_h)$ against CPU time.

Further remarks

Other uses of $\mathtt{T}$-coercivity:

- Mixed variational formulations:
  - Stokes model: non-conforming discretisation; DG discretisation; poromechanics model.
  - Neutron diffusion model: with Domain Decomposition; SPN multigroup model.
  - Static models in electromagnetism.
- Coercive plus compact formulations.
- Formulations with sign-changing coefficients.

...

From the mathematical side:

- **Weak $\mathtt{T}$-coercivity**: the form $a(\cdot, \mathtt{T}\cdot)$ is coercive + compact on $V \times V$, see PhD thesis by `Chesnel (2012)`, `BonnetBenDhia-Carvalho-PC'18`, `Halla'21`...
- In Banach spaces, $\mathtt{T}$-coercivity implies Hilbert structure, see `Ern-Guermont'21-Vol.II`.

Thank you for your attention!