T-coercivity: a practical tool for the study of variational formulations

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Outline

What is T-coercivity?

A tool to study variational formulations

Abstract framework: *Find* $u \in V$ *s.t.* $\forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W$.
Approximate framework: *Find* $u_\delta \in V_\delta$ *s.t.* $\forall w_\delta \in W_\delta,\ a(u_\delta,w_\delta) = {}_{W'}\langle f, w_\delta\rangle_W$.

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   - prove well-posedness ;
   - existence, uniqueness and continuous dependence of the solution with respect to the data.

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Within the framework of T-coercivity, steps 1 and 2 are very strongly correlated!

What is T-coercivity?

As an abstract tool

Let

- V, W be Hilbert spaces ;
- $a(\cdot,\cdot)$ be a continuous sesquilinear form on $V \times W$;
- f be an element of W', the dual space of W.

Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W.$$

[Banach-Nečas-Babuška] The *inf-sup condition* writes

$$\text{(isc)} \quad \exists \alpha > 0,\ \forall v \in V,\ \sup_{w \in W\setminus\{0\}} \frac{|a(v,w)|}{\|w\|_W} \geq \alpha \|v\|_V.$$

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Definition (T-coercivity)

The form $a(\cdot,\cdot)$ is T-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V,W) \text{ bijective},\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v,\mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2 .$$

NB. In other words, the form $a(\cdot,\mathtt{T}\cdot)$ is coercive on $V \times V$.

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$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W.$$

Theorem (Well-posedness)

The three assertions below are equivalent:

(i) *the Problem (VF) is well-posed ;*

(ii) *the form $a(\cdot,\cdot)$ satisfies (isc) and $\{w \in W \,|\, \forall v \in V,\ a(v,w) = 0\} = \{0\}$;*

(iii) *the form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive.*

The operator $\mathtt{T}$ realises the inf-sup condition (isc) explicitly: $w = \mathtt{T}u$ works!

What is T-coercivity?
As an abstract tool (simplified)

Let

- V be a Hilbert space ;
- $a(\cdot,\cdot)$ be a continuous, sesquilinear, *hermitian* form on $V \times V$;
- f be an element of V', the dual space of V.

Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = {}_{V'}\langle f, w\rangle_V .$$

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Definition (T-coercivity, hermitian case)

The form $a(\cdot,\cdot)$ is $\mathtt{T}$-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V),\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2.$$

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Let

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Theorem (Well-posedness, hermitian case)

The three assertions below are equivalent:

(i) *the Problem (VF) is well-posed ;*

(ii) *the form $a(\cdot,\cdot)$ satisfies (isc) ;*

(iii) *the form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive.*

The operator $\mathtt{T}$ realises the inf-sup condition (isc) explicitly.

Let

- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V ;
- $(W_\delta)_\delta$ be a family of finite dimensional subspaces of W.

Assume that $\dim(V_\delta) = \dim(W_\delta)$ for all $\delta > 0$.
Solve

$$(\mathrm{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = {}_{W'}\langle f, w_\delta \rangle_W.$$

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`[Banach-Nečas-Babuška]` The *uniform discrete inf-sup condition* writes

$$\text{(udisc)} \quad \exists \alpha_\dagger > 0,\ \forall \delta > 0,\ \forall v_\delta \in V_\delta,\ \sup_{w_\delta \in W_\delta \setminus \{0\}} \frac{|a(v_\delta, w_\delta)|}{\|w_\delta\|_W} \geq \alpha_\dagger \|v_\delta\|_V.$$

NB. When (udisc) is fulfilled, (VF)_δ is well-posed for all $\delta > 0$.

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Definition (uniform T_δ-coercivity)

The form a is *uniformly* T_δ*-coercive* if

$$\exists \underline{\alpha}_\dagger, \underline{\beta}_\dagger > 0,\ \forall\delta > 0,\ \exists \mathrm{T}_\delta \in \mathcal{L}(V_\delta, W_\delta),\ |||\mathrm{T}_\delta||| \le \underline{\beta}_\dagger \text{ and } \forall v_\delta \in V_\delta,\ |a(v_\delta, \mathrm{T}_\delta v_\delta)| \ge \underline{\alpha}_\dagger \|v_\delta\|_V^2.$$

NB. When a is uniformly T_δ-coercive, $(\mathsf{VF})_\delta$ is well-posed for all $\delta > 0$.

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Theorem (Céa's lemma)

Assume that the family $(V_\delta)_\delta$ fulfills the basic approximability property in V.
In addition, assume that

(i) *either, the form $a(\cdot,\cdot)$ satisfies (udisc) ;*

(ii) *or, the form $a(\cdot,\cdot)$ is uniformly $\mathtt{T}_\delta$-coercive.*

Then, $\lim_{\delta\to 0} \|u - u_\delta\|_V = 0$.

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- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V ;
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(VF)$_\delta$ *Find* $u_\delta \in V_\delta$ *s.t.* $\forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = {}_{W'}\langle f, w_\delta\rangle_W.$

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(i) *either, the form* $a(\cdot,\cdot)$ *satisfies (udisc) ;*

(ii) *or, the form* $a(\cdot,\cdot)$ *is uniformly* $\mathtt{T}_\delta$*-coercive.*

Then, $\lim_{\delta\to 0} \|u - u_\delta\|_V = 0$. *And error estimates whenever possible...*

Use the knowledge on operator $\mathtt{T}$ to derive the discrete operators $(\mathtt{T}_\delta)_\delta$!

What is T-coercivity?

Can be applied to various types of variational formulations

† = Abstract T-coercivity only.

1. Coercive plus compact formulations. See for instance:
   - integral equations: Buffa-Costabel-Schwab'02 [Θ-coercivity]; Buffa-Christiansen'03; Buffa-Christiansen'05; Buffa'05; Unger'21; Levadoux (2022, HAL report) [τ-coercivity].
   - volume equations: Hiptmair'02 ["$(X+S)$-coercivity"]; Buffa'05; PC'12 ["elementary" proofs]; Hohage-Nannen'15 [S-coercivity]; Sayas-Brown-Hassell (2019)†; Halla'21a ["generalized" proofs].
2. Formulations with sign-changing coefficients. See for instance:
   - for scalar models: BonnetBenDhia-PC-Zwölf'10; Nicaise-Venel'11; BonnetBenDhia-Chesnel-PC'12†; Chesnel-PC'13; Bunoiu-Ramdani'16†; Carvalho-Chesnel-PC'17; BonnetBenDhia-Carvalho-PC'18; Bunoiu-Ramdani-Timofte'21-'22†.
   - for EM models: BonnetBenDhia-Chesnel-PC'14† (2D-3D); Halla'21b (2D); PC'22 (3D).
3. Mixed formulations.
   - for the Stokes model: see below!
   - for diffusion models: Jamelot-PC'13, see below!
   - for the magnetic quasi-static model: Barré-PC (2022, HAL report).

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Outline

NeutronDiffusion

1. Let Ω be a domain of $\mathbb{R}^3$. The "simplest" Stokes equations write

$$\begin{cases} -\nu\,\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega \\ \operatorname{div} \boldsymbol{u} = g \text{ in } \Omega \\ \boldsymbol{u} = 0 \text{ on } \partial\Omega, \end{cases}$$

for some $\nu > 0$ (viscosity). For "classical" Stokes, $g = 0$.

1. Assuming that $\boldsymbol{f} \in (\boldsymbol{H}_0^1(\Omega))'$ and $g \in L^2_{zmv}(\Omega)$, one analyses mathematically the model

$$\text{(Stokes)} \qquad \begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ -\nu\, \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega \\ \operatorname{div} \boldsymbol{u} = g \text{ in } \Omega. \end{cases}$$

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2. The equivalent variational formulation writes

$$\text{(FV-Stokes)} \begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega), \quad \nu \displaystyle\int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v}\, d\Omega \\ \quad - \displaystyle\int_\Omega p \operatorname{div} \boldsymbol{v}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{u}\, d\Omega = {}_{(\boldsymbol{H}_0^1(\Omega))'}\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{H}_0^1(\Omega)} - \int_\Omega g q\, d\Omega. \end{cases}$$

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Question: how to prove well-posedness "easily"?

1. Assuming that $\boldsymbol{f} \in (\boldsymbol{H}^1_0(\Omega))'$ and $g \in L^2_{zmv}(\Omega)$, one analyses mathematically the model

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Question: how to prove well-posedness "easily"?

Use T-coercivity for the Stokes model!

Let

- $V = \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$, endowed with the norm $\|(\boldsymbol{v}, q)\|_V = (|\boldsymbol{v}|^2_{1,\Omega} + \|q\|^2)^{1/2}$;
- $a((\boldsymbol{v}, q), (\boldsymbol{w}, r)) = \nu \int_\Omega \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{w} \, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w} \, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v} \, d\Omega$;
- ${}_{V'}\langle f, (\boldsymbol{w}, r)\rangle_V = {}_{(\boldsymbol{H}_0^1(\Omega))'}\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{H}_0^1(\Omega)} - \int_\Omega r \, g \, d\Omega.$

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The first goal is to prove the inf-sup condition, with the help of T-coercivity.
NB. The form a is not coercive, because $a((0, q), (0, q)) = 0$ for $q \in L^2_{zmv}(\Omega)$.

Let

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The first goal is to prove the inf-sup condition, with the help of $\mathtt{T}$-coercivity.
Given $(\boldsymbol{v}, q) \in V \setminus \{(0,0)\}$, we look for $(\boldsymbol{w}^\star, r^\star) \in V \setminus \{(0,0)\}$ with linear dependence such that

$$|a((\boldsymbol{v}, q), (\boldsymbol{w}^\star, r^\star))| \geq \underline{\alpha} \, \|(\boldsymbol{v}, q)\|^2_V,$$

with $\underline{\alpha} > 0$ independent of $(\boldsymbol{v}, q)$. In other words, $\mathtt{T}$ is defined by $\mathtt{T}((\boldsymbol{v}, q)) = (\boldsymbol{w}^\star, r^\star)$.

Let

- $V = \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$, endowed with the norm $\|(\boldsymbol{v}, q)\|_V = (|\boldsymbol{v}|^2_{1,\Omega} + \|q\|^2)^{1/2}$;
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The first goal is to prove the inf-sup condition, with the help of T-coercivity.
Given $(\boldsymbol{v}, q) \in V \setminus \{(0,0)\}$, we look for $(\boldsymbol{w}^\star, r^\star) \in V \setminus \{(0,0)\}$ with linear dependence such that

$$|a((\boldsymbol{v}, q), (\boldsymbol{w}^\star, r^\star))| \geq \underline{\alpha} \, \|(\boldsymbol{v}, q)\|_V^2,$$

with $\underline{\alpha} > 0$ independent of $(\boldsymbol{v}, q)$. Three steps:

1. $q = 0$;
2. $\boldsymbol{v} = 0$;
3. General case.

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w} \, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w} \, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v} \, d\Omega$.

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w} \, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v} \, d\Omega$: so choosing $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v},0)$ yields

$$|a((\boldsymbol{v},0),(\boldsymbol{w}^\star,r^\star))| = \nu \int_\Omega |\nabla \boldsymbol{v}|^2 \, d\Omega = \nu \, \|(\boldsymbol{v},0)\|_V^2.$$

Stokes model

Constructive proof of well-posedness with T-coercivity - 2

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w}\, d\Omega - \int_\Omega q \,\mathrm{div}\, \boldsymbol{w}\, d\Omega - \int_\Omega r \,\mathrm{div}\, \boldsymbol{v}\, d\Omega.$

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w}\, d\Omega - \int_\Omega r \,\mathrm{div}\, \boldsymbol{v}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v}, 0)$.
2. $a((0,q),(\boldsymbol{w},r)) = -\int_\Omega q \,\mathrm{div}\, \boldsymbol{w}\, d\Omega$: according to eg. `Girault-Raviart'86`,

   $\exists C_{\mathrm{div}} > 0,\ \forall q \in L^2_{zmv}(\Omega),\ \exists \boldsymbol{w}_q \in \boldsymbol{H}^1_0(\Omega)$ such that $\mathrm{div}\, \boldsymbol{w}_q = q,$ with $|\boldsymbol{w}_q|_{1,\Omega} \leq C_{\mathrm{div}} \|q\|$.

   So choosing $(\boldsymbol{w}^\star, r^\star) = (-\boldsymbol{w}_q, 0)$ yields

$$|a((0,q),(\boldsymbol{w}^\star,r^\star))| = \int_\Omega q^2\, d\Omega = \|(0,q)\|_V^2.$$

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega.$

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3. General case: beginning with the linear combination $\boldsymbol{w}^\star = \lambda \boldsymbol{v} - \mu \boldsymbol{w}_q$, $\lambda, \mu > 0$, one finds

$$a((\boldsymbol{v},q),(\boldsymbol{w}^\star,r)) = \lambda\nu\, |\boldsymbol{v}|^2_{1,\Omega} - \mu\nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}_q\, d\Omega - \int_\Omega (\lambda q + r) \operatorname{div} \boldsymbol{v}\, d\Omega + \mu \|q\|^2.$$

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NB. Playing with Young's inequality, one finds that there is an "admissible" family of coefficients (λ,μ) that yield T-coercivity.

Stokes model

Constructive proof of well-posedness with T-coercivity - 3

Regarding the proof with T-coercivity, one can make several observations:

1. The result of Girault-Raviart'86 appears as a requirement to derive the inf-sup condition!
2. The T-coercivity approach is flexible, in the sense that one has at hand a family of operators T (depending on the chosen linear combination). Among others, one may "optimize" the value of the stability constant with respect to ν.
3. The approach is easily transposed to the approximation, see below!

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The second goal is to prove the uniform discrete inf-sup condition, with the help of the uniform $\mathtt{T}_\delta$-coercivity. Given finite dimensional subspaces $(\boldsymbol{V}_\delta)_\delta$ of $\boldsymbol{H}_0^1(\Omega)$, resp. $(Q_\delta)_\delta$ of $L^2_{zmv}(\Omega)$, one can build an approximation of the Stokes model. Question: how to choose them?

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Mimic the previous proof to guarantee uniform $\mathtt{T}_\delta$-coercivity for the Stokes model!

The discrete variational formulation writes

$$
(\text{FV-Stokes})_\delta \left\{ \begin{array}{l} \text{Find } (\boldsymbol{u}_\delta, p_\delta) \in \boldsymbol{V}_\delta \times Q_\delta \text{ such that} \\ \forall (\boldsymbol{v}_\delta, q_\delta) \in \boldsymbol{V}_\delta \times Q_\delta, \\ \nu \displaystyle\int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_\delta : \boldsymbol{\nabla} \boldsymbol{v}_\delta \, d\Omega - \int_\Omega p_\delta \operatorname{div} \boldsymbol{v}_\delta \, d\Omega - \int_\Omega q_\delta \operatorname{div} \boldsymbol{u}_\delta \, d\Omega = {}_{V'}\langle f, (\boldsymbol{v}_\delta, q_\delta) \rangle_V . \end{array} \right.
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Given $(\boldsymbol{v}_\delta, q_\delta) \in \boldsymbol{V}_\delta \times Q_\delta \setminus \{(0,0)\}$, we look for $(\boldsymbol{w}_\delta^\star, r_\delta^\star) \in \boldsymbol{V}_\delta \times Q_\delta \setminus \{(0,0)\}$ such that

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$$\boldsymbol{w}^\star = \nu (C_{\operatorname{div}})^2 \boldsymbol{v}_\delta - \boldsymbol{w}_{q_\delta} \text{ and } r^\star = -\nu (C_{\operatorname{div}})^2 q_\delta,$$

with $\boldsymbol{w}_{q_\delta} \in \boldsymbol{H}_0^1(\Omega)$ such that $\operatorname{div} \boldsymbol{w}_{q_\delta} = q_\delta$, and $|\boldsymbol{w}_{q_\delta}|_{1,\Omega} \leq C_{\operatorname{div}} \, \|q_\delta\|$.

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with $\boldsymbol{w}_{q_\delta} \in \boldsymbol{H}_0^1(\Omega)$ such that $\operatorname{div} \boldsymbol{w}_{q_\delta} = q_\delta$, and $|\boldsymbol{w}_{q_\delta}|_{1,\Omega} \leq C_{\text{div}} \, \|q_\delta\|$.
Difficulty: $\boldsymbol{w}_{q_\delta} \notin \boldsymbol{V}_\delta$ in general, whereas $\boldsymbol{v}_\delta \in \boldsymbol{V}_\delta$ and $r^\star \in Q_\delta$.

How to overcome this difficulty to be able to conclude the proof?

Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\operatorname{div} \boldsymbol{w}_\delta^+ = q_\delta$ weakly", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

Stokes model

Constructive proof of convergence with uniform $\mathtt{T}_\delta$-coercivity - 2

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As a matter of fact, choosing $\boldsymbol{w}_\delta^\star = \nu (C^+)^2 \boldsymbol{v}_\delta - \boldsymbol{w}_\delta^+$ and $r_\delta^\star = -\nu (C^+)^2 q_\delta$ immediately yields the uniform discrete inf-sup condition!

Stokes model

Constructive proof of convergence with uniform $\mathtt{T}_\delta$-coercivity - 2

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> Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\operatorname{div} \boldsymbol{w}_\delta^+ = q_\delta$ weakly", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \, \|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

As a matter of fact, choosing $\boldsymbol{w}_\delta^\star = \nu (C^+)^2 \boldsymbol{v}_\delta - \boldsymbol{w}_\delta^+$ and $r_\delta^\star = -\nu (C^+)^2 q_\delta$ immediately yields the uniform discrete inf-sup condition! How so? Just add ${}_{\delta}$s to the previous computations!

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To summarize, one is looking for pairs of discrete spaces $(\boldsymbol{V}_\delta, Q_\delta)_\delta$ such that

$$\exists C^+ > 0, \ \forall \delta, \qquad \forall q_\delta \in Q_\delta, \ \exists \boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta \text{ with the properties}$$

$$\forall q'_\delta \in Q_\delta, \quad \int_\Omega q'_\delta \operatorname{div} \boldsymbol{w}_\delta^+ \, d\Omega = \int_\Omega q'_\delta \, q_\delta \, d\Omega \, ;$$

$$|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \, \|q_\delta\|.$$

How to overcome this difficulty to be able to conclude the proof?

Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\operatorname{div} \boldsymbol{w}_\delta^+ = q_\delta$ weakly", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

In other words, one is looking for pairs of discrete spaces $(\boldsymbol{V}_\delta, Q_\delta)_\delta$ such that

$\exists C_\pi > 0,\ \forall \delta,\ \exists \pi_\delta \in \mathcal{L}(\boldsymbol{H}_0^1(\Omega), \boldsymbol{V}_\delta)$ with the properties

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\ \forall q'_\delta \in Q_\delta, \quad \int_\Omega q'_\delta \operatorname{div}(\pi_\delta \boldsymbol{v})\, d\Omega = \int_\Omega q'_\delta \operatorname{div} \boldsymbol{v}\, d\Omega\,;$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad |\pi_\delta \boldsymbol{v}|_{1,\Omega} \leq C_\pi |\boldsymbol{v}|_{1,\Omega}.$$

Then one chooses $\boxed{\boldsymbol{w}_\delta^+ = \pi_\delta \boldsymbol{w}_{q_\delta}}$ to get the desired properties with $C^+ = C_\pi C_{\text{div}}$.

Stokes model

Constructive proof of convergence with uniform $\mathtt{T}_\delta$-coercivity - 2

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By browsing the book by `Boffi-Brezzi-Fortin (2013)`, one finds that:

- the MINI FE of order $k \geq 1$ does the job!
- the Taylor-Hood FE of order $k \geq 1$ does the job!

Convergence and error estimates follow...

Regarding the proof with uniform $\mathtt{T}_\delta$-coercivity, one can make further observations:

1. The so-called Fortin lemma appears "naturally" in the proof.
2. One needs to have some knowledge of finite element spaces.
3. The proof is "simple"!

Regarding the proof with uniform $\mathtt{T}_\delta$-coercivity, one can make further observations:

1. The so-called Fortin lemma appears "naturally" in the proof.
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$\mathtt{T}$-coercivity and uniform $\mathtt{T}_\delta$-coercivity are indeed strongly correlated for the Stokes model!

Outline

▸ Further remarks

Neutron diffusion model
The model

1. Let Ω be a domain of $\mathbb{R}^3$. The basic brick of neutron diffusion writes

$$\begin{cases} -\operatorname{div} \mathbb{D}\nabla u + \sigma u = S_f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

or, equivalently, with the additional unknown $\boldsymbol{p} = -\mathbb{D}\nabla u$,

$$\begin{cases} \operatorname{div} \boldsymbol{p} + \sigma u = S_f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

for some uniformly positive symmetric tensor $\boldsymbol{x} \mapsto \mathbb{D}(\boldsymbol{x})$ (diffusion tensor), and uniformly positive $\boldsymbol{x} \mapsto \sigma(\boldsymbol{x})$ (macroscopic absorption cross section).

1. Assuming that $S_f \in L^2(\Omega)$, one analyses mathematically the model

$$\text{(Diffusion)} \qquad \begin{cases} \text{Find } (u, \boldsymbol{p}) \in H^1_0(\Omega) \times \boldsymbol{H}(\mathrm{div}\,;\Omega) \text{ such that} \\ \mathrm{div}\, \boldsymbol{p} + \sigma u = S_f \text{ in } \Omega \\ \mathbb{D}^{-1}\boldsymbol{p} + \nabla u = 0 \text{ in } \Omega. \end{cases}$$

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2. After elementary manipulations, the equivalent variational formulation writes

$$\text{(FV-Diffusion)} \quad \left\{ \begin{array}{l} \text{Find } (u, \boldsymbol{p}) \in L^2(\Omega) \times \boldsymbol{H}(\operatorname{div}; \Omega) \text{ such that} \\ \forall (w, \boldsymbol{r}) \in L^2(\Omega) \times \boldsymbol{H}(\operatorname{div}; \Omega), \\ \displaystyle\int_\Omega \Big(-\mathbb{D}^{-1}\boldsymbol{p} \cdot \boldsymbol{r} + u \operatorname{div} \boldsymbol{r} + w \operatorname{div} \boldsymbol{p} + \sigma u w \Big) d\Omega = \int_\Omega S_f w \, d\Omega. \end{array} \right.$$

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Question: how to prove well-posedness "easily"?

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Question: how to prove well-posedness "easily"?

Use $\mathbb{T}$-coercivity for the neutron diffusion model!

Let

- $V = L^2(\Omega) \times \boldsymbol{H}(\mathrm{div}\,;\Omega)$, endowed with the norm $\|(v,\boldsymbol{q})\|_V = (\|v\|^2 + \|\boldsymbol{q}\|^2_{\boldsymbol{H}(\mathrm{div}\,;\Omega)})^{1/2}$;
- $a((v,\boldsymbol{q}),(w,\boldsymbol{r})) = -\int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{r}\,d\Omega + \int_\Omega v\,\mathrm{div}\,\boldsymbol{r}\,d\Omega + \int_\Omega w\,\mathrm{div}\,\boldsymbol{q}\,d\Omega + \int_\Omega \sigma v w\,d\Omega$;
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- ${}_{V'}\langle f,(w,\boldsymbol{r})\rangle_V = \int_\Omega S_f w\,d\Omega$.

The first goal is to prove the inf-sup condition, with the help of $\mathtt{T}$-coercivity.

NB. The form a is not coercive, because $|a((0,\boldsymbol{q}),(0,\boldsymbol{q}))| = \int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{q}\,d\Omega$ controls $\|\boldsymbol{q}\|^2$, but not $\|\boldsymbol{q}\|^2_{\boldsymbol{H}(\mathrm{div}\,;\Omega)}$.

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- ${}_{V'}\langle f,(w,\boldsymbol{r})\rangle_V = \int_\Omega S_f w\,d\Omega.$

The first goal is to prove the inf-sup condition, with the help of $\mathtt{T}$-coercivity.
Given $(v,\boldsymbol{q}) \in V\setminus\{(0,0)\}$, we look for $(w^\star,\boldsymbol{r}^\star) \in V\setminus\{(0,0)\}$ with linear dependence such that

$$|a((v,\boldsymbol{q}),(w^\star,\boldsymbol{r}^\star))| \geq \underline{\alpha}\,\|(v,\boldsymbol{q})\|_V^2,$$

with $\underline{\alpha} > 0$ independent of $(v,\boldsymbol{q})$. $\mathtt{T}$ is defined by $\mathtt{T}((v,\boldsymbol{q})) = (w^\star,\boldsymbol{r}^\star)$.

Let

- $V = L^2(\Omega) \times \boldsymbol{H}(\mathrm{div}\,;\Omega)$, endowed with the norm $\|(v,\boldsymbol{q})\|_V = (\|v\|^2 + \|\boldsymbol{q}\|^2_{\boldsymbol{H}(\mathrm{div}\,;\Omega)})^{1/2}$;
- $a((v,\boldsymbol{q}),(w,\boldsymbol{r})) = -\int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{r}\,d\Omega + \int_\Omega v\,\mathrm{div}\,\boldsymbol{r}\,d\Omega + \int_\Omega w\,\mathrm{div}\,\boldsymbol{q}\,d\Omega + \int_\Omega \sigma v w\,d\Omega$;
- ${}_{V'}\langle f,(w,\boldsymbol{r})\rangle_V = \int_\Omega S_f w\,d\Omega.$

The first goal is to prove the inf-sup condition, with the help of T-coercivity.
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with $\underline{\alpha} > 0$ independent of $(v,\boldsymbol{q})$. Three steps:

1. $\boldsymbol{q} = 0$;
2. $v = 0$ and $\boldsymbol{q}$ such that $\mathrm{div}\,\boldsymbol{q} = 0$;
3. General case.

Recall $a((v,\boldsymbol{q}),(w,\boldsymbol{r})) = -\int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{r}\,d\Omega + \int_\Omega v\,\mathrm{div}\,\boldsymbol{r}\,d\Omega + \int_\Omega w\,\mathrm{div}\,\boldsymbol{q}\,d\Omega + \int_\Omega \sigma v w\,d\Omega$.

One finds that

1. $a((v,0),(w,\boldsymbol{r})) = \int_\Omega v\,\mathrm{div}\,\boldsymbol{r}\,d\Omega + \int_\Omega \sigma v w\,d\Omega$: choose $(w^\star,\boldsymbol{r}^\star) = (v,0)$.

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2. $a((0,\boldsymbol{q}),(w,\boldsymbol{r})) = -\int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{r}\,d\Omega$ (with $\mathrm{div}\,\boldsymbol{q} = 0$): choose $(w^\star,\boldsymbol{r}^\star) = (0,-\boldsymbol{q})$.

Neutron diffusion model

Constructive proof of well-posedness with $\mathbb{T}$-coercivity - 2

Recall $a((v,\boldsymbol{q}),(w,\boldsymbol{r})) = -\int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{r}\,d\Omega + \int_\Omega v\,\mathrm{div}\,\boldsymbol{r}\,d\Omega + \int_\Omega w\,\mathrm{div}\,\boldsymbol{q}\,d\Omega + \int_\Omega \sigma v w\,d\Omega.$

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3. General case: beginning with $\boldsymbol{r}^\star = -\boldsymbol{q}$, one finds

$$a((v,\boldsymbol{q}),(w,\boldsymbol{r}^\star)) = \int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{q}\,d\Omega + \int_\Omega (w-v)\mathrm{div}\,\boldsymbol{q}\,d\Omega + \int_\Omega \sigma v w\,d\Omega.$$

Neutron diffusion model

Constructive proof of well-posedness with $\mathbb{T}$-coercivity - 2

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3. General case: $\boldsymbol{r}^\star = -\boldsymbol{q}$. Next, $w^\star = \eta(v + \sigma^{-1}\mathrm{div}\,\boldsymbol{q})$, $\eta > 0$ leads to

$$a((v,\boldsymbol{q}),(w^\star,\boldsymbol{r}^\star)) = \int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{q}\,d\Omega + \eta\int_\Omega \sigma^{-1}(\mathrm{div}\,\boldsymbol{q})^2\,d\Omega + \eta\int_\Omega \sigma v^2\,d\Omega + (2\eta-1)\int_\Omega v\,\mathrm{div}\,\boldsymbol{q}\,d\Omega.$$

Neutron diffusion model

Constructive proof of well-posedness with $\mathbb{T}$-coercivity - 2

Recall $a((v,\boldsymbol{q}),(w,\boldsymbol{r})) = -\int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{r}\,d\Omega + \int_\Omega v\,\mathrm{div}\,\boldsymbol{r}\,d\Omega + \int_\Omega w\,\mathrm{div}\,\boldsymbol{q}\,d\Omega + \int_\Omega \sigma v w\,d\Omega$.

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$$\begin{aligned} a((v,\boldsymbol{q}),(w^\star,\boldsymbol{r}^\star)) &= \int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{q}\,d\Omega + \eta\int_\Omega \sigma^{-1}(\mathrm{div}\,\boldsymbol{q})^2\,d\Omega + \eta\int_\Omega \sigma v^2\,d\Omega \\ &\quad +(2\eta-1)\int_\Omega v\,\mathrm{div}\,\boldsymbol{q}\,d\Omega. \end{aligned}$$

So, choosing $(w^\star,\boldsymbol{r}^\star) = (\frac{1}{2}(v+\sigma^{-1}\mathrm{div}\,\boldsymbol{q}),-\boldsymbol{q})$ yields $\mathbb{T}$-coercivity.

We assume that σ is constant (general case, see `PC-Jamelot-Kpadonou'17`).
The second goal is to prove the uniform discrete inf-sup condition, with the help of the uniform $\mathtt{T}_\delta$-coercivity. Given finite dimensional subspaces $(V_\delta)_\delta$ of $L^2(\Omega)$, resp. $(\boldsymbol{Q}_\delta)_\delta$ of $\boldsymbol{H}(\mathrm{div}\,;\Omega)$, one can build an approximation of the neutron diffusion model. Question: how to choose them?

Mimic the previous proof!

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$$w^\star = \frac{1}{2}(v_\delta + \sigma^{-1} \mathrm{div}\, \boldsymbol{q}_\delta) \text{ and } \boldsymbol{r}^\star = -\boldsymbol{q}_\delta.$$

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Difficulty: $\mathrm{div}\, \boldsymbol{q}_\delta \in V_\delta$? Whereas $v_\delta \in V_\delta$ and $\boldsymbol{q}_\delta \in \boldsymbol{Q}_\delta$.

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All pairs of discrete spaces $(V_\delta, \boldsymbol{Q}_\delta)_\delta$ such that $\mathrm{div}\,[\boldsymbol{Q}_\delta] \subset V_\delta$ do the job!

By browsing the book by `Boffi-Brezzi-Fortin (2013)`, one now finds that:

> one can choose the Raviart-Thomas FE of order $k \geq 0$ (for $\boldsymbol{Q}_\delta$).

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The second goal is to prove the uniform discrete inf-sup condition, with the help of the uniform $\mathtt{T}_\delta$-coercivity. Given finite dimensional subspaces $(V_\delta)_\delta$ of $L^2(\Omega)$, resp. $(\boldsymbol{Q}_\delta)_\delta$ of $\boldsymbol{H}(\mathrm{div}\,;\Omega)$, one can build an approximation of the neutron diffusion model. Question: how to choose them?
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> one can choose the Raviart-Thomas FE of order $k \geq 0$ (for $\boldsymbol{Q}_\delta$).

The proof is very "simple"! Convergence and error estimates follow...

Outline

Further remarks

1. The domain Ω is split into N disjoint subdomains $(\Omega_i)_{i=1,N}$: $\overline{\Omega} = \cup_{i=1,N}\overline{\Omega_i}$.
   For v defined over Ω, let $v_i = v_{|\Omega_i}$ for $i = 1, N$.
2. Let $\Gamma_{ij} = int(\overline{\Omega_i} \cap \overline{\Omega_j})$ if $\dim_H(\overline{\Omega_i} \cap \overline{\Omega_j}) = 2$, otherwise $\Gamma_{ij} = \emptyset$, for $i \neq j$.
   Let $\Gamma = \cup_{i<j}\overline{\Gamma_{ij}}$ denote the global interface.
   For $\Gamma_{ij} \neq \emptyset$, let $[q]_{ij}$ denote the jump across Γ_{ij}.
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3. Let $M = \prod_{i<j} L^2(\Gamma_{ij})$, with norm $\|v_\Gamma\|_M = \left(\sum_{i<j} \|v_{\Gamma|\Gamma_{ij}}\|^2_{L^2(\Gamma_{ij})}\right)^{1/2}$.
   Let $\boldsymbol{Q} = \{ \boldsymbol{q} = (\boldsymbol{q}_i)_i \in \boldsymbol{L}^2(\Omega) \,|\, \operatorname{div} \boldsymbol{q}_i \in L^2(\Omega_i),\ i = 1, N,$ and $[\boldsymbol{q} \cdot \boldsymbol{n}] \in M \}$, with norm

$$\|\boldsymbol{q}\|_{\boldsymbol{Q}} = \left(\sum_{i=1,N} \|\boldsymbol{q}_i\|^2_{\boldsymbol{H}(\operatorname{div},\Omega_i)} + \|[\boldsymbol{q} \cdot \boldsymbol{n}]\|^2_M \right)^{1/2}.$$

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4. Finally, let $V_{DD} = L^2(\Omega) \times \boldsymbol{Q} \times M$, endowed with the norm

$$\|(v, \boldsymbol{q}, v_\Gamma)\|_{V_{DD}} = (\|v\|^2 + \|\boldsymbol{q}\|^2_{\boldsymbol{Q}} + \|v_\Gamma\|^2_M)^{1/2}.$$

1. Cf. PC-Jamelot-Kpadonou'17, an equivalent variational formulation to the neutron diffusion model with Domain Decomposition writes

$$
\text{(FV-Diff-DD)}\quad \left\{\begin{array}{l}
\text{Find } (u, \boldsymbol{p}, u_\Gamma) \in V_{DD} \text{ such that} \\
\forall (w, \boldsymbol{r}, w_\Gamma) \in V_{DD}, \\
\displaystyle \sum_{i=1,N} \int_{\Omega_i} \Big(- \mathbb{D}^{-1} \boldsymbol{p}_i \cdot \boldsymbol{r}_i + u_i \operatorname{div} \boldsymbol{r}_i + w_i \operatorname{div} \boldsymbol{p}_i + \sigma u_i w_i \Big) d\Omega \\
\qquad\qquad\qquad \displaystyle - \int_\Gamma \Big([\boldsymbol{p} \cdot \boldsymbol{n}] w_\Gamma + [\boldsymbol{r} \cdot \boldsymbol{n}] u_\Gamma \Big) d\Gamma = \int_\Omega S_f w \, d\Omega.
\end{array}\right.
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Question: how to prove well-posedness "easily"?

Neutron diffusion model with Domain Decomposition

The model

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Use T-coercivity for the neutron diffusion model with Domain Decomposition!

- $V_{DD} = L^2(\Omega) \times \boldsymbol{Q} \times M$ is endowed with $\|(v, \boldsymbol{q}, v_\Gamma)\|_{V_{DD}} = (\|v\|^2 + \|\boldsymbol{q}\|_{\boldsymbol{Q}}^2 + \|v_\Gamma\|_M^2)^{1/2}$.
- Let
  - $$a_{DD}((v, \boldsymbol{q}, v_\Gamma), (w, \boldsymbol{r}, w_\Gamma)) = \sum_{i=1,N} \int_{\Omega_i} \Big(-\mathbb{D}^{-1}\boldsymbol{q}_i \cdot \boldsymbol{r}_i + v_i \operatorname{div} \boldsymbol{r}_i + w_i \operatorname{div} \boldsymbol{q}_i + \sigma v_i w_i \Big) d\Omega$$
    $$- \int_\Gamma \Big([\boldsymbol{q} \cdot \boldsymbol{n}] w_\Gamma + [\boldsymbol{r} \cdot \boldsymbol{n}] v_\Gamma \Big) d\Gamma\,;$$
  - $${}_{V_{DD}'}\langle f, (w, \boldsymbol{r}, w_\Gamma)\rangle_{V_{DD}} = \int_\Omega S_f w \, d\Omega.$$

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    $- \int_\Gamma \Big([\boldsymbol{q} \cdot \boldsymbol{n}] w_\Gamma + [\boldsymbol{r} \cdot \boldsymbol{n}] v_\Gamma \Big) d\Gamma$;
  - ${}_{V_{DD}'}\langle f, (w, \boldsymbol{r}, w_\Gamma) \rangle_{V_{DD}} = \int_\Omega S_f w \, d\Omega.$

Again, the first goal is to prove the inf-sup condition, with the help of $\mathtt{T}$-coercivity.

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  - ${}_{V_{DD}'}\langle f, (w, \boldsymbol{r}, w_\Gamma)\rangle_{V_{DD}} = \int_\Omega S_f w \, d\Omega$.

Again, the first goal is to prove the inf-sup condition, with the help of T-coercivity.
Given $(v, \boldsymbol{q}, v_\Gamma) \in V_{DD} \setminus \{(0,0,0)\}$, we look for $(w^\star, \boldsymbol{r}^\star, w_\Gamma^\star) \in V_{DD} \setminus \{(0,0,0)\}$ with linear dependence such that

$$|a_{DD}((v, \boldsymbol{q}, v_\Gamma), (w^\star, \boldsymbol{r}^\star, w_\Gamma^\star))| \geq \underline{\alpha} \, \|(v, \boldsymbol{q}, v_\Gamma)\|_{V_{DD}}^2,$$

with $\underline{\alpha} > 0$ independent of $(v, \boldsymbol{q}, v_\Gamma)$, and $\mathtt{T}$ is defined by $\mathtt{T}((v, \boldsymbol{q}, v_\Gamma)) = (w^\star, \boldsymbol{r}^\star, w_\Gamma^\star)$.

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with $\alpha > 0$ independent of $(v, \boldsymbol{q}, v_\Gamma)$. Two steps (incremental proof):

1. $v_\Gamma = 0$;
2. General case.

Recall $a_{DD}((v,\boldsymbol{q},v_\Gamma),(w,\boldsymbol{r},w_\Gamma)) = \sum_{i=1,N}\int_{\Omega_i}\Big(-\mathbb{D}^{-1}\boldsymbol{q}_i\cdot\boldsymbol{r}_i + v_i\,\mathrm{div}\,\boldsymbol{r}_i + w_i\,\mathrm{div}\,\boldsymbol{q}_i + \sigma v_i w_i\Big)d\Omega$
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   Choose $((w_i^\star)_i,(\boldsymbol{r}_i^\star)_i) = \big(\frac{1}{2}(v_i+\sigma^{-1}\mathrm{div}\,\boldsymbol{q}_i)_i, -(\boldsymbol{q}_i)_i\big)$ "as before", and $w_\Gamma^\star = -[\boldsymbol{q}\cdot\boldsymbol{n}]$!

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Neutron diffusion model with Domain Decomposition

Constructive proof of well-posedness with T-coercivity - 2

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   Choose $((w^\star)_i, (\boldsymbol{r}_i^\star)_i, w_\Gamma^\star) = \big(\frac{1}{2}(v_i + \sigma^{-1} \mathrm{div}\, \boldsymbol{q}_i)_i, -(\boldsymbol{q}_i + \eta \boldsymbol{v}_i(v_\Gamma))_i, v_\Gamma - [\boldsymbol{q} \cdot \boldsymbol{n}]\big)$, and find ad hoc $\eta > 0$ (independent of the lifting) to yield T-coercivity.

The second goal is to prove the uniform discrete inf-sup condition, with the help of the uniform $\mathtt{T}_\delta$-coercivity. Given finite dimensional subspaces $(V_\delta)_\delta$ of $L^2(\Omega)$, $(\boldsymbol{Q}_\delta)_\delta$ of $\boldsymbol{Q}$ and $(M_\delta)_\delta$ of M, one builds an approximation of the neutron diffusion model with Domain Decomposition.

Question: how to choose them?

Mimic the previous proofs!

Neutron diffusion model with Domain Decomposition

Constructive proof of convergence with uniform $\mathtt{T}_\delta$-coercivity

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- In a subdomain Ω_i (cf. step 1.), the Raviart-Thomas FE of order $k \geq 0$ can be used to define the pair $(V_\delta, \boldsymbol{Q}_\delta)$ restricted to Ω_i.
- Then, on the interface Γ_{ij}, one can choose $M_\delta = (\boldsymbol{Q}_\delta \cdot \boldsymbol{n}_i)_{|\Gamma_{ij}} + (\boldsymbol{Q}_\delta \cdot \boldsymbol{n}_j)_{|\Gamma_{ij}}$ (no crosspoint/no regularity issues).

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The proof has now become very "technical"! However it has been made possible by using $\mathtt{T}$-coercivity incrementally (from one to several subdomains ; from the exact to the discrete variational formulations...).

Convergence and error estimates follow.

Further remarks

Some extensions:

1. Stokes model: see Jamelot (2022, HAL report) for a non-conforming discretisation (Crouzeix-Raviart FE or Fortin-Soulié FE); see master's thesis by MRoueh (2022) for DG discretisation ; see Barré-Grandmont-Moireau'22 for a poromechanics model.
2. diffusion model: see PhD thesis by Giret (2018) for a SPN multigroup model.
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5. in Banach spaces, T-coercivity implies Hilbert structure, see `Ern-Guermont II (2021)`.
6. if possible, discretise the variational formulation with operator T, see `Chesnel-PC'13`.
7. T-coercivity still usable with the Strang lemmas (approximate forms).

Thank you for your attention!