T-coercivity: a practical tool for the study of variational formulations

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Outline

Abstract framework: *Find* $u \in V$ *s.t.* $\forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W$.
Approximate framework: *Find* $u_\delta \in V_\delta$ *s.t.* $\forall w_\delta \in W_\delta,\ a(u_\delta,w_\delta) = {}_{W'}\langle f, w_\delta\rangle_W$.

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1. First, analyse the variational formulation theoretically:
   - prove well-posedness ;
   - existence, uniqueness and continuous dependence of the solution with respect to the data.

What is T-coercivity?

A tool to study variational formulations

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Within the framework of T-coercivity, steps 1 and 2 are very strongly correlated!

Let

- V, W be Hilbert spaces ;
- $a(\cdot,\cdot)$ be a continuous sesquilinear form on $V \times W$;
- f be an element of W', the dual space of W.

Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W .$$

[Banach-Nečas-Babuška] The *inf-sup condition* writes

$$\text{(isc)} \quad \exists \alpha > 0,\ \forall v \in V,\ \sup_{w \in W\setminus\{0\}} \frac{|a(v,w)|}{\|w\|_W} \geq \alpha \, \|v\|_V .$$

What is T-coercivity?

As an abstract tool

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Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W.$$

Definition (T-coercivity)

The form $a(\cdot,\cdot)$ is T-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V,W) \text{ bijective},\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v,\mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2.$$

NB. In other words, the form $a(\cdot,\mathtt{T}\cdot)$ is coercive on $V \times V$.

Let

- V, W be Hilbert spaces ;
- $a(\cdot,\cdot)$ be a continuous sesquilinear form on $V \times W$;
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Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W .$$

Theorem (Well-posedness)

The three assertions below are equivalent:

(i) *the Problem (VF) is well-posed ;*

(ii) *the form $a(\cdot,\cdot)$ satisfies (isc) and $\{w \in W \,|\, \forall v \in V,\ a(v,w) = 0\} = \{0\}$;*

(iii) *the form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive.*

The operator $\mathtt{T}$ realises the inf-sup condition (isc) explicitly: $w = \mathtt{T}u$ works!

Let

- V be a Hilbert space ;
- $a(\cdot,\cdot)$ be a continuous, sesquilinear, *hermitian* form on $V \times V$;
- f be an element of V', the dual space of V.

Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = {}_{V'}\langle f, w \rangle_V .$$

What is T-coercivity?

As an abstract tool (simplified)

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Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = {}_{V'}\langle f, w\rangle_V .$$

Definition (T-coercivity, hermitian case)

The form $a(\cdot,\cdot)$ is T-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V),\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2 .$$

What is T-coercivity?

As an abstract tool (simplified)

Let

- V be a Hilbert space ;
- $a(\cdot,\cdot)$ be a continuous, sesquilinear, *hermitian* form on $V \times V$;
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Solve

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Theorem (Well-posedness, hermitian case)

The three assertions below are equivalent:

(i) *the Problem (VF) is well-posed ;*

(ii) *the form $a(\cdot,\cdot)$ satisfies (isc) ;*

(iii) *the form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive.*

The operator $\mathtt{T}$ realises the inf-sup condition (isc) explicitly.

Let

- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V ;
- $(W_\delta)_\delta$ be a family of finite dimensional subspaces of W.

Assume that $\dim(V_\delta) = \dim(W_\delta)$ for all $\delta > 0$.
Solve

$$(\text{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = {}_{W'}\langle f, w_\delta \rangle_W.$$

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[Banach-Nečas-Babuška] The *uniform discrete inf-sup condition* writes

$$(\mathrm{udisc}) \quad \exists \alpha_\dagger > 0,\ \forall \delta > 0,\ \forall v_\delta \in V_\delta,\ \sup_{w_\delta \in W_\delta \setminus \{0\}} \frac{|a(v_\delta, w_\delta)|}{\|w_\delta\|_W} \geq \alpha_\dagger \|v_\delta\|_V.$$

NB. When (udisc) is fulfilled, $(\mathrm{VF})_\delta$ is well-posed for all $\delta > 0$.

What is T-coercivity?

As an approximation tool

Let

- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V ;
- $(W_\delta)_\delta$ be a family of finite dimensional subspaces of W.

Assume that $\dim(V_\delta) = \dim(W_\delta)$ for all $\delta > 0$.
Solve

$$(\text{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = {}_{W'}\langle f, w_\delta\rangle_W.$$

Definition (uniform $\mathtt{T}_\delta$-coercivity)

The form a is *uniformly* $\mathtt{T}_\delta$*-coercive* if

$$\exists\alpha^\star, \beta^\star > 0,\ \forall\delta > 0,\ \exists\mathtt{T}_\delta \in \mathcal{L}(V_\delta, W_\delta),\ |||\mathtt{T}_\delta||| \leq \beta^\star \text{ and } \forall v_\delta \in V_\delta,\ |a(v_\delta, \mathtt{T}_\delta v_\delta)| \geq \alpha^\star \|v_\delta\|_V^2.$$

NB. When a is uniformly $\mathtt{T}_\delta$-coercive, $(\text{VF})_\delta$ is well-posed for all $\delta > 0$.

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- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V ;
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Assume that $\dim(V_\delta) = \dim(W_\delta)$ for all $\delta > 0$.
Solve

$$(\mathsf{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = {}_{W'}\langle f, w_\delta\rangle_W.$$

Theorem (Convergence)

Assume that the family $(V_\delta)_\delta$ fulfills the basic approximability property in V.
In addition, assume that

(i) *either, the form $a(\cdot,\cdot)$ satisfies (udisc) ;*

(ii) *or, the form $a(\cdot,\cdot)$ is uniformly T_δ-coercive.*

Then, $\lim_{\delta\to 0} \|u - u_\delta\|_V = 0$.

Use the knowledge on operator $\mathtt{T}$ to derive the discrete operators $(\mathtt{T}_\delta)_\delta$!

What is T-coercivity?

Can be applied to various types of variational formulations

1. Coercive plus compact formulations. See for instance:
   - with integral equations: Buffa-Costabel-Schwab'02 (Thm 7, called Θ-coercivity there), Buffa-Christiansen'03 (Cor. 4.2), Buffa-Christiansen'05 (Prop. 3.7), Buffa'05 (§§3-4).
   - with volume equations: Hiptmair'02 (§5, "$(X+S)$-coercivity"), Buffa'05 (§§3-4), PC'12 (elementary proofs...), book by Sayas-Brown-Hassell (2019) (§15.1).
2. Formulations with sign-changing coefficients. See for instance:
   - for scalar models: BonnetBenDhia-PC-Zwölf'10, Nicaise-Venel'11, BonnetBenDhia-Chesnel-PC'12†, Chesnel-PC'13, Carvalho-Chesnel-PC'17, BonnetBenDhia-Carvalho-PC'18.
   - for EM models: BonnetBenDhia-Chesnel-PC'14† (x2), PC'21.

   † Abstract T-coercivity only.
3. Mixed formulations.
   - for the Stokes model: see below!
   - for diffusion models: Jamelot-PC'13, see below!

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   - for the Stokes model: see below!
   - for diffusion models: Jamelot-PC'13, see below!

1. Let Ω be a domain of $\mathbb{R}^3$. The "simplest" Stokes equations write

$$\begin{cases} -\nu\,\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega \\ \boldsymbol{u} = 0 \text{ on } \partial\Omega, \end{cases}$$

for some $\nu > 0$ (viscosity).

1. Assuming that $\boldsymbol{f} \in (\boldsymbol{H}_0^1(\Omega))'$, one analyses mathematically the model

$$\text{(Stokes)} \qquad \left\{ \begin{array}{l} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ -\nu\, \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega. \end{array} \right.$$

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2. The equivalent variational formulation writes

$$\text{(FV-Stokes)} \begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega) \text{ such that} \\ \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0^1(\Omega) \times L^2_{zmv}(\Omega), \\ \nu \displaystyle\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v}\, d\Omega - \int_\Omega p \operatorname{div} \boldsymbol{v}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{u}\, d\Omega = {}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{H}_0^1(\Omega)}. \end{cases}$$

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Question: how to prove well-posedness "easily"?

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Question: how to prove well-posedness "easily"?

Use T-coercivity for the Stokes model!

Let

- $V = \boldsymbol{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$, endowed with the norm $\|(\boldsymbol{v}, q)\|_V = (|\boldsymbol{v}|^2_{1,\Omega} + \|q\|^2)^{1/2}$;
- $a((\boldsymbol{v}, q), (\boldsymbol{w}, r)) = \nu \int_\Omega \boldsymbol{\nabla} \boldsymbol{v} : \boldsymbol{\nabla} \boldsymbol{w} \, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w} \, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v} \, d\Omega$;
- ${}_{V'}\langle f, (\boldsymbol{w}, r)\rangle_V = {}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{H}^1_0(\Omega)} - \int_\Omega r \, g \, d\Omega$ ($g = 0$ for Stokes).

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The first goal is to prove the inf-sup condition, with the help of $\mathbb{T}$-coercivity.
NB. The form a is not coercive, because $a((0, q), (0, q)) = 0$ for $q \in L^2_{zmv}(\Omega)$.

Let

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- $a((\boldsymbol{v}, q), (\boldsymbol{w}, r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w} \, d\Omega - \int_\Omega q \, \mathrm{div} \, \boldsymbol{w} \, d\Omega - \int_\Omega r \, \mathrm{div} \, \boldsymbol{v} \, d\Omega$;
- ${}_{V'}\langle f, (\boldsymbol{w}, r)\rangle_V = {}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{H}^1_0(\Omega)} - \int_\Omega r \, g \, d\Omega$ ($g = 0$ for Stokes).

The first goal is to prove the inf-sup condition, with the help of T-coercivity.
Given $(\boldsymbol{v}, q) \in V \setminus \{(0, 0)\}$, we look for $(\boldsymbol{w}^\star, r^\star) \in V \setminus \{(0, 0)\}$ with linear dependence such that

$$|a((\boldsymbol{v}, q), (\boldsymbol{w}^\star, r^\star))| \geq \alpha \, \|(\boldsymbol{v}, q)\|_V \, \|(\boldsymbol{w}^\star, r^\star)\|_V,$$

with $\alpha > 0$ independent of $(\boldsymbol{v}, q)$. In other words, T is defined by $\mathtt{T}((\boldsymbol{v}, q)) = (\boldsymbol{w}^\star, r^\star)$.

Let

- $V = \boldsymbol{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$, endowed with the norm $\|(\boldsymbol{v}, q)\|_V = (|\boldsymbol{v}|^2_{1,\Omega} + \|q\|^2)^{1/2}$;
- $a((\boldsymbol{v}, q), (\boldsymbol{w}, r)) = \nu \displaystyle\int_\Omega \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$;
- ${}_{V'}\langle f, (\boldsymbol{w}, r)\rangle_V = {}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{H}^1_0(\Omega)} - \displaystyle\int_\Omega r\, g\, d\Omega$ ($g = 0$ for Stokes).

The first goal is to prove the inf-sup condition, with the help of $\mathbb{T}$-coercivity.
Given $(\boldsymbol{v}, q) \in V \setminus \{(0,0)\}$, we look for $(\boldsymbol{w}^\star, r^\star) \in V \setminus \{(0,0)\}$ with linear dependence such that

$$|a((\boldsymbol{v}, q), (\boldsymbol{w}^\star, r^\star))| \geq \alpha \,\|(\boldsymbol{v}, q)\|_V \,\|(\boldsymbol{w}^\star, r^\star)\|_V,$$

with $\alpha > 0$ independent of $(\boldsymbol{v}, q)$. Three steps:

1. $q = 0$;
2. $\boldsymbol{v} = 0$;
3. General case.

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$.

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$: so choosing $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v},0)$ yields

$$|a((\boldsymbol{v},0),(\boldsymbol{w}^\star,r^\star))| = \nu\, \|(\boldsymbol{v},0)\|_V \|(\boldsymbol{w}^\star,r^\star)\|_V.$$

Stokes model

Constructive proof of well-posedness with T-coercivity - 2

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla} \boldsymbol{v} : \boldsymbol{\nabla} \boldsymbol{w} \, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w} \, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v} \, d\Omega$.

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla} \boldsymbol{v} : \boldsymbol{\nabla} \boldsymbol{w} \, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v} \, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v}, 0)$.
2. $a((0,q),(\boldsymbol{w},r)) = -\int_\Omega q \operatorname{div} \boldsymbol{w} \, d\Omega$: according to eg. `Girault-Raviart'86`,

   $\exists C_{\text{div}} > 0, \ \forall q \in L^2_{vmn}(\Omega), \ \exists \boldsymbol{w}_q \in \boldsymbol{H}^1_0(\Omega)$ such that $\operatorname{div} \boldsymbol{w}_q = q$, with $|\boldsymbol{w}_q|_{1,\Omega} \leq C_{\text{div}} \, \|q\|$.

   So choosing $(\boldsymbol{w}^\star, r^\star) = (-\boldsymbol{w}_q, 0)$ yields

$$|a((0,q),(\boldsymbol{w}^\star,r^\star))| \geq \|q\| \frac{|\boldsymbol{w}_q|_{1,\Omega}}{C_{\text{div}}} = \frac{1}{C_{\text{div}}} \|(0,q)\|_V \, \|(\boldsymbol{w}^\star, r^\star)\|_V .$$

Stokes model

Constructive proof of well-posedness with T-coercivity - 2

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \,\mathrm{div}\, \boldsymbol{w}\, d\Omega - \int_\Omega r\, \mathrm{div}\, \boldsymbol{v}\, d\Omega.$

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega r\,\mathrm{div}\,\boldsymbol{v}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v}, 0)$.
2. $a((0,q),(\boldsymbol{w},r)) = -\int_\Omega q\,\mathrm{div}\,\boldsymbol{w}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (-\boldsymbol{w}_q, 0)$.
3. General case: beginning with the linear combination $\boldsymbol{w}^\star = \lambda \boldsymbol{v} - \mu \boldsymbol{w}_q$, $\lambda, \mu > 0$, one finds

$$a((\boldsymbol{v},q),(\boldsymbol{w}^\star,r)) = \lambda\nu\, |\boldsymbol{v}|^2_{1,\Omega} - \mu\nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}_q\, d\Omega - \int_\Omega (\lambda q + r)\,\mathrm{div}\,\boldsymbol{v}\, d\Omega + \mu \|q\|^2.$$

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega.$

1. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}\, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (\boldsymbol{v}, 0)$.
2. $a((0,q),(\boldsymbol{w},r)) = -\int_\Omega q \operatorname{div} \boldsymbol{w}\, d\Omega$: choose $(\boldsymbol{w}^\star, r^\star) = (-\boldsymbol{w}_q, 0)$.
3. General case: $\boldsymbol{w}^\star = \lambda \boldsymbol{v} - \mu \boldsymbol{w}_q$, $\lambda, \mu > 0$. Next, $r^\star = -\lambda q$ leads to

$$a((\boldsymbol{v},q),(\boldsymbol{w}^\star,r^\star)) = \lambda\nu\, |\boldsymbol{v}|^2_{1,\Omega} + \mu \|q\|^2 - \mu\nu \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla w}_q\, d\Omega.$$

Stokes model

Constructive proof of well-posedness with $\mathbb{T}$-coercivity - 2

Recall $a((\boldsymbol{v}, q), (\boldsymbol{w}, r)) = \nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w} \, d\Omega - \int_\Omega q \operatorname{div} \boldsymbol{w} \, d\Omega - \int_\Omega r \operatorname{div} \boldsymbol{v} \, d\Omega.$

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Stokes model

Constructive proof of well-posedness with $\mathtt{T}$-coercivity - 2

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Finally, the last term can be controlled by the first two terms, using Young's inequality. Eg., choose $(\lambda,\mu) = (\nu (C_{\rm div})^2, 1)$: $\mathtt{T}((\boldsymbol{v},q)) = (\nu(C_{\rm div})^2\boldsymbol{v} - \boldsymbol{w}_q, -\nu(C_{\rm div})^2 q)$.

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Eg., choose $(\lambda,\mu) = (\nu(C_{\text{div}})^2, 1)$: $\mathtt{T}((\boldsymbol{v},q)) = (\nu(C_{\text{div}})^2\boldsymbol{v} - \boldsymbol{w}_q, -\nu(C_{\text{div}})^2 q)$.
NB. Playing with Young's inequality, one finds that there is an "admissible" family of coefficients (λ,μ) that yield T-coercivity.

Stokes model

Constructive proof of well-posedness with $\mathtt{T}$-coercivity - 3

Regarding the proof with $\mathtt{T}$-coercivity, one can make several observations:

1. The result of `Girault-Raviart'86` appears as a requirement to derive the inf-sup condition!
2. The $\mathtt{T}$-coercivity approach is flexible, in the sense that one has at hand a family of operators $\mathtt{T}$ (depending on the chosen linear combination). Among others, one may "optimize" the value of the stability constant with respect to ν.
3. The approach is easily transposed to the approximation, see below!

Stokes model

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The second goal is to prove the uniform discrete inf-sup condition, with the help of the uniform $\mathtt{T}_\delta$-coercivity. Given finite dimensional subspaces $(\boldsymbol{V}_\delta)_\delta$ of $\boldsymbol{H}_0^1(\Omega)$, resp. $(Q_\delta)_\delta$ of $L^2_{zmv}(\Omega)$, one can build an approximation of the Stokes model. Question: how to choose them?

Regarding the proof with T-coercivity, one can make several observations:

1. The result of Girault-Raviart'86 appears as a requirement to derive the inf-sup condition!
2. The T-coercivity approach is flexible, in the sense that one has at hand a family of operators T (depending on the chosen linear combination). Among others, one may "optimize" the value of the stability constant with respect to ν.
3. The approach is easily transposed to the approximation, see below!

The second goal is to prove the uniform discrete inf-sup condition, with the help of the uniform $\mathtt{T}_\delta$-coercivity. Given finite dimensional subspaces $(\boldsymbol{V}_\delta)_\delta$ of $\boldsymbol{H}^1_0(\Omega)$, resp. $(Q_\delta)_\delta$ of $L^2_{zmv}(\Omega)$, one can build an approximation of the Stokes model. Question: how to choose them?

Mimic the previous proof to guarantee uniform $\mathtt{T}_\delta$-coercivity for the Stokes model!

The discrete variational formulation writes

$$\text{(FV-Stokes)}_\delta \left\{ \begin{array}{l} \text{Find } (\boldsymbol{u}_\delta, p_\delta) \in \boldsymbol{V}_\delta \times Q_\delta \text{ such that} \\ \forall (\boldsymbol{v}_\delta, q_\delta) \in \boldsymbol{V}_\delta \times Q_\delta, \\ \nu \displaystyle\int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_\delta : \boldsymbol{\nabla} \boldsymbol{v}_\delta \, d\Omega - \int_\Omega p_\delta \operatorname{div} \boldsymbol{v}_\delta \, d\Omega - \int_\Omega q_\delta \operatorname{div} \boldsymbol{u}_\delta \, d\Omega = {}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{f}, \boldsymbol{v}_\delta \rangle_{\boldsymbol{H}^1_0(\Omega)}. \end{array} \right.$$

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Given $(\boldsymbol{v}_\delta, q_\delta) \in \boldsymbol{V}_\delta \times Q_\delta \setminus \{(0,0)\}$, we look for $(\boldsymbol{w}^\star_\delta, r^\star_\delta) \in \boldsymbol{V}_\delta \times Q_\delta \setminus \{(0,0)\}$ such that

$$|a((\boldsymbol{v}_\delta, q_\delta), (\boldsymbol{w}^\star_\delta, r^\star_\delta))| \geq \alpha_\dagger \, \|(\boldsymbol{v}_\delta, q_\delta)\|_V \, \|(\boldsymbol{w}^\star_\delta, r^\star_\delta)\|_V ,$$

with $\alpha_\dagger > 0$ independent of δ and of $(\boldsymbol{v}_\delta, q_\delta)$.

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with $\alpha_\dagger > 0$ independent of δ and of $(\boldsymbol{v}_\delta, q_\delta)$. Mimicking the $\mathtt{T}$-coercivity approach, one chooses

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\boldsymbol{w}^\star = \nu (C_{\text{div}})^2 \boldsymbol{v}_\delta - \boldsymbol{w}_{q_\delta} \text{ and } r^\star = -\nu (C_{\text{div}})^2 q_\delta,
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with $\boldsymbol{w}_{q_\delta} \in \boldsymbol{H}^1_0(\Omega)$ such that $\operatorname{div} \boldsymbol{w}_{q_\delta} = q_\delta$, and $|\boldsymbol{w}_{q_\delta}|_{1,\Omega} \leq C_{\text{div}} \, \|q_\delta\|$.

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Given $(\boldsymbol{v}_\delta, q_\delta) \in \boldsymbol{V}_\delta \times Q_\delta \setminus \{(0,0)\}$, we look for $(\boldsymbol{w}_\delta^\star, r_\delta^\star) \in \boldsymbol{V}_\delta \times Q_\delta \setminus \{(0,0)\}$ such that

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with $\alpha_\dagger > 0$ independent of δ and of $(\boldsymbol{v}_\delta, q_\delta)$. Mimicking the $\mathtt{T}$-coercivity approach, one chooses

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with $\boldsymbol{w}_{q_\delta} \in \boldsymbol{H}_0^1(\Omega)$ such that $\operatorname{div} \boldsymbol{w}_{q_\delta} = q_\delta$, and $|\boldsymbol{w}_{q_\delta}|_{1,\Omega} \leq C_{\text{div}} \, \|q_\delta\|$.
Difficulty: $\boldsymbol{w}_{q_\delta} \notin \boldsymbol{V}_\delta$ in general, whereas $\boldsymbol{v}_\delta \in \boldsymbol{V}_\delta$ and $r^\star \in Q_\delta$.

How to overcome this difficulty to be able to conclude the proof?

Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\operatorname{div} \boldsymbol{w}_\delta^+ = q_\delta$", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \, \|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

Stokes model

Constructive proof of convergence with uniform $\mathtt{T}_\delta$-coercivity - 2

How to overcome this difficulty to be able to conclude the proof?

Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\mathrm{div}\,\boldsymbol{w}_\delta^+ = q_\delta$", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \,\|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

As a matter of fact, choosing $\boldsymbol{w}_\delta^\star = \nu(C^+)^2\boldsymbol{v}_\delta - \boldsymbol{w}_\delta^+$ and $r_\delta^\star = -\nu(C^+)^2 q_\delta$ immediately yields the uniform discrete inf-sup condition!

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As a matter of fact, choosing $\boldsymbol{w}_\delta^\star = \nu(C^+)^2 \boldsymbol{v}_\delta - \boldsymbol{w}_\delta^+$ and $r_\delta^\star = -\nu(C^+)^2 q_\delta$ immediately yields the uniform discrete inf-sup condition! How so? Just add $_\delta$s to the previous computations!

How to overcome this difficulty to be able to conclude the proof?

Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\mathrm{div}\,\boldsymbol{w}_\delta^+ = q_\delta$", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \, \|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

In other words, one is looking for pairs of discrete spaces $(\boldsymbol{V}_\delta, Q_\delta)_\delta$ such that

$\exists C_\pi > 0, \ \forall \delta, \ \exists \pi_\delta \in \mathcal{L}(\boldsymbol{H}_0^1(\Omega), \boldsymbol{V}_\delta)$ with the properties

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad |\pi_\delta \boldsymbol{v}|_{1,\Omega} \leq C_\pi |\boldsymbol{v}|_{1,\Omega}\,;$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \ \forall q'_\delta \in Q_\delta, \quad \int_\Omega q'_\delta \,\mathrm{div}\,(\pi_\delta \boldsymbol{v})\, d\Omega = \int_\Omega q'_\delta \,\mathrm{div}\,\boldsymbol{v}\, d\Omega.$$

to set $\boxed{\boldsymbol{w}_\delta^+ = \pi_\delta \boldsymbol{w}_{q_\delta}}$.

How to overcome this difficulty to be able to conclude the proof?

Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\operatorname{div} \boldsymbol{w}_\delta^+ = q_\delta$", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \, \|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

By browsing the book by `Boffi-Brezzi-Fortin (2013)`, one finds that:

- the MINI finite element of order $k \geq 1$ does the job!

Stokes model

Constructive proof of convergence with uniform $\mathtt{T}_\delta$-coercivity - 2

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Find $\boldsymbol{w}_\delta^+ \in \boldsymbol{V}_\delta$ such that "$\mathrm{div}\, \boldsymbol{w}_\delta^+ = q_\delta$", and $|\boldsymbol{w}_\delta^+|_{1,\Omega} \leq C^+ \, \|q_\delta\|$ with $C^+ > 0$ independent of δ, q_δ.

By browsing the book by `Boffi-Brezzi-Fortin (2013)`, one finds that:

- the MINI finite element of order $k \geq 1$ does the job!
- the Taylor-Hood finite element of order $k \geq 1$ does the job!

Constructive proof of convergence with uniform $\mathtt{T}_\delta$-coercivity - 3

Regarding the proof with uniform $\mathtt{T}_\delta$-coercivity, one can make further observations:

1. The so-called Fortin lemma appears "naturally" in the proof.
2. One needs to have some knowledge of finite element spaces.
3. The proof is "simple"!

Stokes model

Constructive proof of convergence with uniform $\mathtt{T}_\delta$-coercivity - 3

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1. The so-called Fortin lemma appears "naturally" in the proof.
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$\mathtt{T}$-coercivity and uniform $\mathtt{T}_\delta$-coercivity are indeed strongly correlated for the Stokes model!

Outline

Further remarks

1. Let Ω be a domain of $\mathbb{R}^3$. The basic brick of neutron diffusion writes

$$\begin{cases} -\operatorname{div} \mathbb{D}\nabla u + \sigma u = S_f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

or, equivalently, with the additional unknown $\boldsymbol{p} = -\mathbb{D}\nabla u$,

$$\begin{cases} \operatorname{div} \boldsymbol{p} + \sigma u = S_f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

for some uniformly positive symmetric tensor $\boldsymbol{x} \mapsto \mathbb{D}(\boldsymbol{x})$ (diffusion tensor), and uniformly positive $\boldsymbol{x} \mapsto \sigma(\boldsymbol{x})$ (macroscopic absorption cross section).

1. Assuming that $S_f \in L^2(\Omega)$, one analyses mathematically the model

$$\text{(Diffusion)} \qquad \left\{ \begin{array}{l} \text{Find } (u, \boldsymbol{p}) \in H^1_0(\Omega) \times \boldsymbol{H}(\operatorname{div}\,; \Omega) \text{ such that} \\ \operatorname{div} \boldsymbol{p} + \sigma u = S_f \text{ in } \Omega \\ \mathbb{D}^{-1}\boldsymbol{p} + \nabla u = 0 \text{ in } \Omega. \end{array} \right.$$

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2. After elementary manipulations, the equivalent variational formulation writes

$$\text{(FV-Diffusion)} \quad \left\{ \begin{array}{l} \text{Find } (u, \boldsymbol{p}) \in L^2(\Omega) \times \boldsymbol{H}(\mathrm{div}\,; \Omega) \text{ such that} \\ \forall (w, \boldsymbol{r}) \in L^2(\Omega) \times \boldsymbol{H}(\mathrm{div}\,; \Omega), \\ \displaystyle\int_\Omega \Big(-\mathbb{D}^{-1}\boldsymbol{p} \cdot \boldsymbol{r} + u\, \mathrm{div}\, \boldsymbol{r} + w\, \mathrm{div}\, \boldsymbol{p} + \sigma u w \Big) d\Omega = \int_\Omega S_f w\, d\Omega. \end{array} \right.$$

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Question: how to prove well-posedness "easily"?

1. Assuming that $S_f \in L^2(\Omega)$, one analyses mathematically the model

$$(\text{Diffusion}) \quad \left\{ \begin{array}{l} \text{Find } (u, \boldsymbol{p}) \in H^1_0(\Omega) \times \boldsymbol{H}(\operatorname{div}; \Omega) \text{ such that} \\ \operatorname{div} \boldsymbol{p} + \sigma u = S_f \text{ in } \Omega \\ \mathbb{D}^{-1}\boldsymbol{p} + \nabla u = 0 \text{ in } \Omega. \end{array} \right.$$

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Question: how to prove well-posedness "easily"?

Use T-coercivity for the neutron diffusion model!

Let

- $V = L^2(\Omega) \times \boldsymbol{H}(\mathrm{div}\,;\Omega)$, endowed with the norm $\|(v,\boldsymbol{q})\|_V = (\|v\|^2 + \|\boldsymbol{q}\|^2_{\boldsymbol{H}(\mathrm{div}\,;\Omega)})^{1/2}$;
- $a((v,\boldsymbol{q}),(w,\boldsymbol{r})) = -\int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{r}\,d\Omega + \int_\Omega v\,\mathrm{div}\,\boldsymbol{r}\,d\Omega + \int_\Omega w\,\mathrm{div}\,\boldsymbol{q}\,d\Omega + \int_\Omega \sigma v w\,d\Omega$;
- ${}_{V'}\langle f, w\rangle_V = \int_\Omega S_f w\,d\Omega$.

Neutron diffusion model

Constructive proof of well-posedness with T-coercivity - 1

Let

- $V = L^2(\Omega) \times \boldsymbol{H}(\mathrm{div}\,;\Omega)$, endowed with the norm $\|(v,\boldsymbol{q})\|_V = (\|v\|^2 + \|\boldsymbol{q}\|^2_{\boldsymbol{H}(\mathrm{div}\,;\Omega)})^{1/2}$;
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- ${}_{V'}\langle f, w\rangle_V = \int_\Omega S_f w\,d\Omega$.

Again, the first goal is to prove the inf-sup condition, with the help of T-coercivity.

NB. The form a is not coercive, because $|a((0,\boldsymbol{q}),(0,\boldsymbol{q}))| = \int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{q}\,d\Omega$ controls $\|\boldsymbol{q}\|^2$, but not $\|\boldsymbol{q}\|^2_{\boldsymbol{H}(\mathrm{div}\,;\Omega)}$.

Let

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- ${}_{V'}\langle f, w\rangle_V = \int_\Omega S_f w\,d\Omega$.

Again, the first goal is to prove the inf-sup condition, with the help of $\mathbb{T}$-coercivity.
Given $(v,\boldsymbol{q}) \in V\setminus\{(0,0)\}$, we look for $(w^\star,\boldsymbol{r}^\star) \in V\setminus\{(0,0)\}$ with linear dependence such that

$$|a((v,\boldsymbol{q}),(w^\star,\boldsymbol{r}^\star))| \geq \alpha\,\|(v,\boldsymbol{q})\|_V\,\|(w^\star,\boldsymbol{r}^\star)\|_V,$$

with $\alpha > 0$ independent of $(v,\boldsymbol{q})$.

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Again, the first goal is to prove the inf-sup condition, with the help of T-coercivity.
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with $\alpha > 0$ independent of $(v, \boldsymbol{q})$. Again, three steps:

1. $\boldsymbol{q} = 0$;
2. $v = 0$ and $\boldsymbol{q}$ such that $\operatorname{div} \boldsymbol{q} = 0$;
3. General case.

Neutron diffusion model

Constructive proof of well-posedness with $\mathbb{T}$-coercivity - 2

Recall $a((v,\boldsymbol{q}),(w,\boldsymbol{r})) = -\int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{r}\,d\Omega + \int_\Omega v\,\mathrm{div}\,\boldsymbol{r}\,d\Omega + \int_\Omega w\,\mathrm{div}\,\boldsymbol{q}\,d\Omega + \int_\Omega \sigma v w\,d\Omega$.

One finds that (skipping the details)

1. $a((v,0),(w,\boldsymbol{r})) = \int_\Omega v\,\mathrm{div}\,\boldsymbol{r}\,d\Omega + \int_\Omega \sigma v w\,d\Omega$: choose $(w^\star,\boldsymbol{r}^\star) = (v,0)$.
2. $a((0,\boldsymbol{q}),(w,\boldsymbol{r})) = -\int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{r}\,d\Omega$ (with $\mathrm{div}\,\boldsymbol{q}=0$): choose $(w^\star,\boldsymbol{r}^\star) = (0,-\boldsymbol{q})$.
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3. General case: beginning with $\boldsymbol{r}^\star = -\boldsymbol{q}$, one finds

$$a((v,\boldsymbol{q}),(w,\boldsymbol{r}^\star)) = \int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{q}\,d\Omega + \int_\Omega (w-v)\mathrm{div}\,\boldsymbol{q}\,d\Omega + \int_\Omega \sigma v w\,d\Omega.$$

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3. General case: $\boldsymbol{r}^\star = -\boldsymbol{q}$. Next, $w^\star = \alpha(v+\sigma^{-1}\mathrm{div}\,\boldsymbol{q})$, $\alpha>0$ leads to

$$\begin{aligned} a((v,\boldsymbol{q}),(w^\star,\boldsymbol{r}^\star)) &= \int_\Omega \mathbb{D}^{-1}\boldsymbol{q}\cdot\boldsymbol{q}\,d\Omega + \int_\Omega \sigma^{-1}(\mathrm{div}\,\boldsymbol{q})^2\,d\Omega + \alpha\int_\Omega \sigma v^2\,d\Omega \\ &\quad +(2\alpha-1)\int_\Omega v\,\mathrm{div}\,\boldsymbol{q}\,d\Omega. \end{aligned}$$

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So, choosing $(w^\star,\boldsymbol{r}^\star) = (\frac{1}{2}(v+\sigma^{-1}\mathrm{div}\,\boldsymbol{q}),-\boldsymbol{q})$ yields $\mathbb{T}$-coercivity.

We assume that σ is constant (general case, see `PC-Jamelot-Kpadonou'17`).
The second goal is to prove the uniform discrete inf-sup condition, with the help of the uniform $\mathtt{T}_\delta$-coercivity. Given finite dimensional subspaces $(V_\delta)_\delta$ of $L^2(\Omega)$, resp. $(\boldsymbol{Q}_\delta)_\delta$ of $\boldsymbol{H}(\mathrm{div}\,;\Omega)$, one can build an approximation of the neutron diffusion model. Question: how to choose them?

Mimic the previous proof!

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Given $(v_\delta, \boldsymbol{q}_\delta) \in V_\delta \times \boldsymbol{Q}_\delta \setminus \{(0,0)\}$, we look for $(w_\delta^\star, \boldsymbol{r}_\delta^\star) \in V_\delta \times \boldsymbol{Q}_\delta \setminus \{(0,0)\}$ such that

$$|a((v_\delta, \boldsymbol{q}_\delta), (w_\delta^\star, \boldsymbol{r}_\delta^\star))| \geq \alpha_\dagger \, \|(v_\delta, \boldsymbol{q}_\delta)\|_V \, \|(w_\delta^\star, \boldsymbol{r}_\delta^\star)\|_V ,$$

with $\alpha_\dagger > 0$ independent of δ and of $(v_\delta, \boldsymbol{q}_\delta)$.

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Difficulty: $\mathrm{div}\,\boldsymbol{q}_\delta \in V_\delta$? Whereas $v_\delta \in V_\delta$ and $\boldsymbol{q}_\delta \in \boldsymbol{Q}_\delta$.

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By browsing the book by `Boffi-Brezzi-Fortin (2013)`, one finds that:

the Raviart-Thomas finite element of order $k \geq 0$ does the job!

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The proof is again very "simple"!

Further remarks

Possible extensions:

1. T-coercivity still usable with the Strang lemmas (approximate forms).
2. Stokes model: see `Jamelot'22` for a non-conforming discretisation (Crouzeix-Raviart or Fortin-Soulié finite elements); see master's thesis by `MRoueh'22` for DG discretisation ; see `Barré-Grandmont-Moireau'22` for a poromechanics model.
3. diffusion model: see `PC-Jamelot-Kpadonou'17` or `PC-Giret-Jamelot-Kpadonou'18` for Domain Decomposition (DDM + L^2-jumps).
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NB. For the electrostatic model, one recovers the Nédélec finite element.

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Within the framework of T-coercivity, analysing a variational formulation theoretically and solving it numerically are very strongly correlated issues!

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[IN PROGRESS] (with Mathieu Barré) Study of abstract mixed variational formulations, and "simplification/extensions" of results in the book by Boffi-Brezzi-Fortin (2013).

[TO DO] Investigate how T-coercivity could be extended to formulations set in Banach spaces (using eg. Arendt-Chalendar-Eymard'20).

Thank you for your attention!