How to solve problems with sign-changing coefficients

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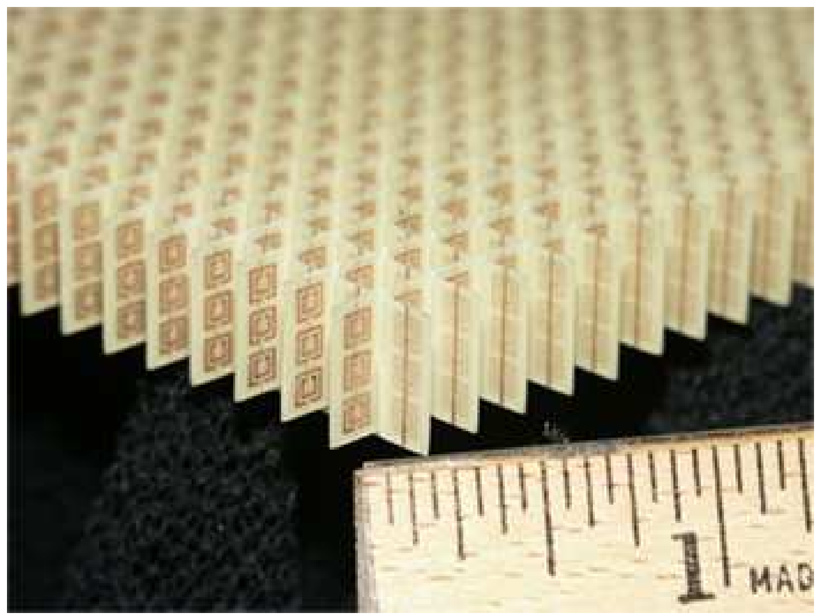
Outline

- Introduction: a model problem with sign-changing coefficients
- Abstract and practical $\mathtt{T}$-coercivity
- Optimality of $\mathtt{T}$-coercivity
- Design of meshes and numerical illustrations
- Extensions

From the physics

- Modelling in electromagnetism: negative materials, for which $\varepsilon(\omega) < 0$ and/or $\mu(\omega) < 0$ in some frequency ranges.
- Two families of negative materials: metals, and metamaterials.



Source: NASA Glenn Research.

From the physics

- Modelling in electromagnetism: negative materials, for which $\varepsilon(\omega) < 0$ and/or $\mu(\omega) < 0$ in some frequency ranges.

- Two families of negative materials: metals, and metamaterials.
- In the presence of negative materials:
  - "Extra-ordinary" applications in physics: NIMs ($\varepsilon(\omega) < 0$ and $\mu(\omega) < 0$)

Source: shutterstock.com.

From the physics

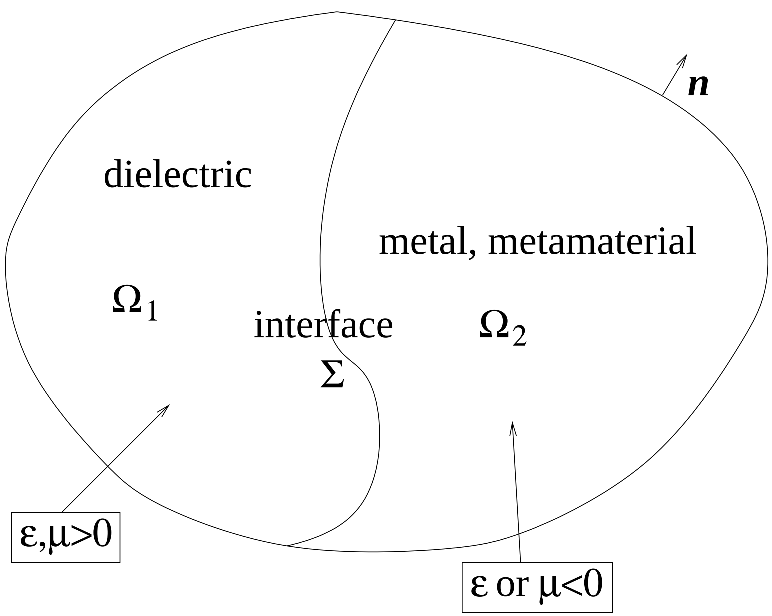
- Modelling in electromagnetism: negative materials, for which $\varepsilon(\omega) < 0$ and/or $\mu(\omega) < 0$ in some frequency ranges.

- Two families of negative materials: metals, and metamaterials.
- In the presence of negative materials:
  - “Extra-ordinary” applications in physics...
  - But mathematical *and* numerical difficulties!

Sign-changing coefficients

- A model scalar *transmission* problem, set in a bounded domain Ω of $\mathbb{R}^d$, $d = 1, 2, 3$.

$$\begin{cases} \textit{Find } u \in H^1(\Omega) \textit{ such that} \\ -\mathrm{div}\,(\sigma \mathbf{grad}\, u) - \omega^2 \eta u = f \text{ in } \Omega \\ + \text{ b.c. on } \partial\Omega. \end{cases}$$



The case of an *inclusion* with $\Sigma \cap \partial\Omega = \emptyset$ is also possible.

Sign-changing coefficients

- A model scalar *transmission* problem, set in a bounded domain Ω of $\mathbb{R}^d$, $d = 1, 2, 3$.

$$\left\{ \begin{array}{l} \textit{Find } u \in H^1(\Omega) \textit{ such that} \\ -\mathrm{div}\,(\sigma \,\mathbf{grad}\, u) - \omega^2 \eta u = f \text{ in } \Omega \\ + \text{ b.c. on } \partial\Omega. \end{array} \right.$$

  - $\sigma \in L^\infty(\Omega)$ is a sign-changing coefficient: $\left\{ \begin{array}{l} \sigma > 0 \text{ in } \Omega_1, \text{ with meas}(\Omega_1) > 0\,; \\ \sigma < 0 \text{ in } \Omega_2, \text{ with meas}(\Omega_2) > 0. \end{array} \right.$
  - $\sigma^{-1} \in L^\infty(\Omega)$.

The parameter σ is discontinuous across the interface Σ.

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Sign-changing coefficients

- A model scalar *transmission* problem, set in a bounded domain Ω of $\mathbb{R}^d$, $d = 1, 2, 3$.

$$\begin{cases} \textit{Find } u \in H^1(\Omega) \textit{ such that} \\ -\mathrm{div}\,(\sigma \,\mathbf{grad}\, u) - \omega^2 \eta u = f \text{ in } \Omega \\ + \text{ b.c. on } \partial\Omega. \end{cases}$$

  - $\sigma \in L^\infty(\Omega)$ is a sign-changing coefficient ; $\sigma^{-1} \in L^\infty(\Omega)$.
  - $\omega \geq 0$, $\eta \in L^\infty(\Omega)$.

- One can consider a Dirichlet or a Neumann b.c., cf. [BonnetBenDhia-Chesnel-PC'12]: we choose a *homogeneous Dirichlet b.c.*.
- The term $-\omega^2 \eta u$ is a compact perturbation, cf. [BonnetBenDhia-PC-Zwölf'10], [BonnetBenDhia-Carvalho-PC'18]: for the ease of exposition we set $\omega = 0$ in this talk.
- Again for the ease of exposition, we assume that $\sigma_{|\Omega_1}$ and $\sigma_{|\Omega_2}$ are *constants*.
- Other (related) models studied by Després et al (PhD of L.-M. Imbert-Gérard (2013), A. Nicolopoulos (2019)) and Labrunie et al (PhD of T. Hattori (2014)).

Sign-changing coefficients

- A model scalar *transmission* problem, set in a bounded domain Ω of $\mathbb{R}^d$, $d = 1, 2, 3$.

$$\left\{ \begin{array}{l} \textit{Find } u \in H_0^1(\Omega) \textit{ such that} \\ -\mathrm{div}\,(\sigma \mathbf{grad}\, u) = f \text{ in } \Omega. \end{array} \right.$$

σ is a piecewise constant, sign-changing, coefficient: $\sigma_1 = \sigma_{|\Omega_1} > 0$, $\sigma_2 = \sigma_{|\Omega_2} < 0$.

- We study the equivalent Variational Formulation

$$\textit{Find } u \in H_0^1(\Omega) \textit{ such that } \quad \forall w \in H_0^1(\Omega), \quad \int_\Omega \sigma\, \mathbf{grad}\, u \cdot \overline{\mathbf{grad}\, w}\, d\Omega = \int_\Omega f \overline{w}\, d\Omega.$$

- The main difficulty is that $(v, w) \mapsto \int_\Omega \sigma\, \mathbf{grad}\, v \cdot \overline{\mathbf{grad}\, w}\, d\Omega$ is *not coercive* in $H_0^1(\Omega)$.

When is this problem well-posed? $\Longrightarrow$ Address this issue with T-coercivity!

Abstract setting

- Let
  - V be a Hilbert space ;
  - $a(\cdot,\cdot)$ be a continuous sesquilinear form on $V \times V$;
  - f be an element of V', the dual space of V.

  Aim: solve the Variational Formulation

$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = \langle f, w\rangle.$$

- [Ladyzhenskaya-Babuska-Brezzi] Recall the *inf-sup condition*

$$(isc) \quad \exists \alpha' > 0,\ \forall v \in V,\ \sup_{w \in V\setminus\{0\}} \frac{|a(v,w)|}{\|w\|_V} \geq \alpha' \|v\|_V.$$

- The form $a(\cdot,\cdot)$ is $\mathtt{T}$-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V) \text{ bijective},\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha} \|v\|_V^2.$$

Abstract setting

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$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = \langle f, w \rangle.$$

- `Theorem (Well-posedness)`

  Assume that $a(\cdot,\cdot)$ is *hermitian*. The three assertions below are equivalent:

  (i) the Problem (VF) is well-posed ;

  (ii) the form $a(\cdot,\cdot)$ satisfies an inf-sup condition ;

  (iii) the form $a(\cdot,\cdot)$ is `T`-coercive.

The operator `T` realizes the inf-sup condition (isc) explicitly.

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Abstract setting

- Let
  - V be a Hilbert space ;
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  Aim: solve the Variational Formulation

$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = \langle f, w \rangle.$$

- Introduce the *weak inf-sup condition*

$$(wisc) \quad \exists \mathtt{C} \in \mathcal{K}(V),\ \alpha', \beta' > 0,\ \forall v \in V,\ \sup_{w \in V \setminus \{0\}} \frac{|a(v,w)|}{\|w\|_V} \geq \alpha' \|v\|_V - \beta' \|\mathtt{C}v\|_V.$$

- The form $a(\cdot,\cdot)$ is *weakly* $\mathtt{T}$*-coercive* if

$$\exists \mathtt{C} \in \mathcal{K}(V),\ \mathtt{T} \in \mathcal{L}(V) \text{ bijective},\ \exists \underline{\alpha}, \underline{\beta} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha} \|v\|_V^2 - \underline{\beta} \|\mathtt{C}v\|_V^2.$$

Abstract setting

- Let
  - V be a Hilbert space ;
  - $a(\cdot,\cdot)$ be a continuous sesquilinear form on $V \times V$;
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  Aim: solve the Variational Formulation

$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = \langle f, w \rangle.$$

- Theorem (Well-posedness in Fredholm sense)
  Assume that $a(\cdot,\cdot)$ is *hermitian*. The three assertions below are equivalent:
  (i) the Problem (VF) is well-posed in the Fredholm sense ;
  (ii) the form $a(\cdot,\cdot)$ satisfies a weak inf-sup condition ;
  (iii) the form $a(\cdot,\cdot)$ is weakly T-coercive.

- When $\omega \neq 0$ in the model problem, the compact perturbation $-\omega^2 \eta u$ can be absorbed in C.

Practical $\mathbb{T}$-coercivity

- In the case of the scalar *transmission* problem:
  - Ω, Ω_1 and Ω_2 are domains of $\mathbb{R}^d$, $d \geq 1$: $\Omega_1 \cap \Omega_2 = \emptyset$, $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$;
  - the *interface* is $\Sigma := \overline{\Omega_1} \cap \overline{\Omega_2}$; the boundaries are $\Gamma_k := \partial\Omega \cap \partial\Omega_k$, $k = 1, 2$;
  - $V := H_0^1(\Omega)$; the form is $a(v, w) := \int_\Omega \sigma \, \mathbf{grad}\, v \cdot \overline{\mathbf{grad}\, w} \, d\Omega$.
- Introduce $V_k := \{v_k \in H^1(\Omega_k) \,|\, v_{k|\Gamma_k} = 0\}$, $k = 1, 2$:

$$\boxed{V = \{v \,|\, v_k := v_{|\Omega_k} \in V_k,\ k = 1, 2,\ \text{Matching}_\Sigma(v_1, v_2) = 0\}},$$

with $\text{Matching}_\Sigma(v_1, v_2) := v_{1|\Sigma} - v_{2|\Sigma}$.

- By construction:

$$a(v, w) := \int_{\Omega_1} \sigma_1 \, \mathbf{grad}\, v_1 \cdot \overline{\mathbf{grad}\, w_1} \, d\Omega - \int_{\Omega_2} |\sigma_2| \, \mathbf{grad}\, v_2 \cdot \overline{\mathbf{grad}\, w_2} \, d\Omega.$$

Practical $\mathtt{T}$-coercivity-2

$$a(v,\mathtt{T}v) := \int_{\Omega_1} \sigma_1\, \mathbf{grad}\, v_1 \cdot \overline{\mathbf{grad}\,(\mathtt{T}v)_1}\, d\Omega - \int_{\Omega_2} |\sigma_2|\, \mathbf{grad}\, v_2 \cdot \overline{\mathbf{grad}\,(\mathtt{T}v)_2}\, d\Omega.$$

Following ideas of [BonnetBenDhia-PC-Zwölf'10], [Nicaise-Venel'11]...

- First attempt: let $\mathtt{R}_{1\to 2} \in \mathcal{L}(V_1, V_2)$ s.t. for all $v_1 \in V_1$, $\text{Matching}_\Sigma(v_1, \mathtt{R}_{1\to 2} v_1) = 0$.

$$\forall v \in V, \quad \mathtt{T}_1\, v := \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 + 2\mathtt{R}_{1\to 2}\, v_1 & \text{in } \Omega_2 \end{cases}.$$

To obtain $\mathtt{T}$-coercivity with $\mathtt{T}_1$ (one uses Young's inequality):
it is sufficient that $\sigma_1/|\sigma_2| > |||\mathtt{R}_{1\to 2}|||^2$, with $|||\mathtt{R}_{1\to 2}||| := \|\mathtt{R}_{1\to 2}\|_{\mathcal{L}(V_1,V_2)}$.

Practical $\mathtt{T}$-coercivity-2

$$a(v,\mathtt{T}v) := \int_{\Omega_1} \sigma_1\, \mathbf{grad}\, v_1 \cdot \overline{\mathbf{grad}\,(\mathtt{T}v)_1}\, d\Omega - \int_{\Omega_2} |\sigma_2|\, \mathbf{grad}\, v_2 \cdot \overline{\mathbf{grad}\,(\mathtt{T}v)_2}\, d\Omega.$$

Following ideas of [BonnetBenDhia-PC-Zwölf'10], [Nicaise-Venel'11]...

- First attempt: let $\mathtt{R}_{1\to 2} \in \mathcal{L}(V_1, V_2)$ s.t. for all $v_1 \in V_1$, Matching$_\Sigma(v_1, \mathtt{R}_{1\to 2} v_1) = 0$.

$$\forall v \in V, \quad \mathtt{T}_1\, v := \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 + 2\mathtt{R}_{1\to 2}\, v_1 & \text{in } \Omega_2 \end{cases}.$$

- Second attempt: let $\mathtt{R}_{2\to 1} \in \mathcal{L}(V_2, V_1)$ s.t. for all $v_2 \in V_2$, Matching$_\Sigma(\mathtt{R}_{2\to 1} v_2, v_2) = 0$.

$$\forall v \in V, \quad \mathtt{T}_2\, v := \begin{cases} v_1 - 2\mathtt{R}_{2\to 1}\, v_2 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{cases}.$$

To obtain $\mathtt{T}$-coercivity with $\mathtt{T}_2$, it is sufficient that:
$|\sigma_2|/\sigma_1 > |||\mathtt{R}_{2\to 1}|||^2$, with $|||\mathtt{R}_{2\to 1}||| := \|\mathtt{R}_{2\to 1}\|_{\mathcal{L}(V_2, V_1)}$.

Practical $\mathtt{T}$-coercivity-2

$$a(v,\mathtt{T}v) := \int_{\Omega_1} \sigma_1 \,\mathbf{grad}\, v_1 \cdot \overline{\mathbf{grad}\,(\mathtt{T}v)_1}\, d\Omega - \int_{\Omega_2} |\sigma_2|\, \mathbf{grad}\, v_2 \cdot \overline{\mathbf{grad}\,(\mathtt{T}v)_2}\, d\Omega.$$

To achieve $\mathtt{T}$-coercivity with $\mathtt{T}_1$ or $\mathtt{T}_2$, it is sufficient that

$$\frac{\sigma_1}{|\sigma_2|} > \left(\inf_{\mathtt{R}_{1\to2}} |||\mathtt{R}_{1\to2}|||\right)^2 \quad \text{or} \quad \frac{|\sigma_2|}{\sigma_1} > \left(\inf_{\mathtt{R}_{2\to1}} |||\mathtt{R}_{2\to1}|||\right)^2 .$$

- How to choose the operators $\mathtt{R}_{1\to2}$, $\mathtt{R}_{2\to1}$?
  - using traces on Σ, liftings, cf. [BonnetBenDhia-PC-Zwölf'10], [Nicaise-Venel'11] ;
  - using geometrical transforms, cf. [Nicaise-Venel'11], [BonnetBenDhia-Chesnel-PC'12], [BonnetBenDhia-Carvalho-PC'18].

Optimality of $\mathtt{T}$-coercivity

- How to find the set of *coefficients* (σ_1, σ_2) that lead to a well-posed problem?

$\longrightarrow$ The relevant parameter is the *contrast* $\sigma_2/\sigma_1 \in]-\infty, 0[$.

$\longrightarrow$ Find the “best” operators $\mathtt{T}$ (or $\mathtt{R}_{1\to 2}$, $\mathtt{R}_{2\to 1}$...).

Optimality of $\mathtt{T}$-coercivity-2

- Study of an elementary setting (PhD of L. Chesnel (2012)):
- Case $\sigma_1 \neq -\sigma_2$, in a symmetric geometry.

  Let $\mathcal{S}^\Sigma$ be the symmetry (geometrical) transform with respect to Σ.
  - Let $\mathtt{R}_{1\to 2} \in \mathcal{L}(V_1, V_2)$ s.t. for all $v_1 \in V_1$, $\mathtt{R}_{1\to 2} v_1 = v_1 \circ \mathcal{S}^\Sigma$, a.e. in Ω_2.

    One finds $|||\mathtt{R}_{1\to 2}||| = 1$.

    If $-1 < \sigma_2/\sigma_1$, one achieves $\mathtt{T}$-coercivity.
  - Let $\mathtt{R}_{2\to 1} \in \mathcal{L}(V_2, V_1)$ s.t. for all $v_2 \in V_2$, $\mathtt{R}_{2\to 1} v_2 = v_2 \circ \mathcal{S}^\Sigma$, a.e. in Ω_1.

    One finds $|||\mathtt{R}_{2\to 1}||| = 1$.

    If $\sigma_2/\sigma_1 < -1$, one achieves $\mathtt{T}$-coercivity.

Optimality of $\mathbb{T}$-coercivity-2

- Study of an elementary setting (PhD of L. Chesnel (2012)):

- Case $\sigma_1 \neq -\sigma_2$, in a symmetric geometry.

  The scalar *transmission* problem is well-posed since $\sigma_2/\sigma_1 \neq -1$.

- Case $\sigma_1 = -\sigma_2$, in a symmetric geometry.

  The scalar *transmission* problem is ill-posed when $\sigma_2/\sigma_1 = -1$ (*Critical case*.)

In a symmetric geometry, the scalar *transmission* problem is well-posed iff $\sigma_2/\sigma_1 \neq -1$

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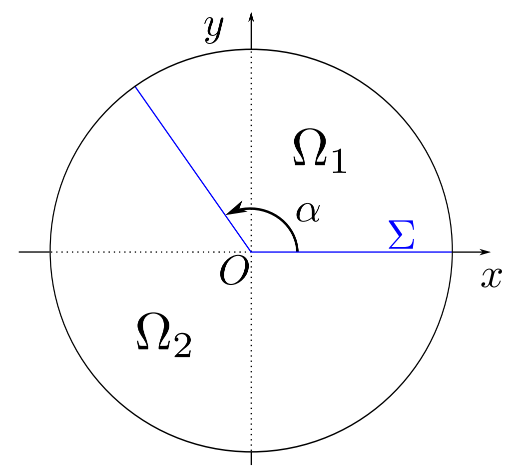
Optimality of $\mathtt{T}$-coercivity-3

- Study of elementary geometries:

1. Symmetric geometry (2D/3D)

2. Interface with an interior corner
   Operators $\mathtt{R}_{1\to 2}$, $\mathtt{R}_{2\to 1}$ combine rotation + angle dilation geometrical transforms:
   $(\mathtt{R}_{1\to 2}\, v_1)(\rho,\theta) = v_1(\rho, \frac{\alpha}{2\pi-\alpha}\,(2\pi-\theta))$;
   $(\mathtt{R}_{2\to 1}\, v_2)(\rho,\theta) = v_2(\rho, 2\pi - \frac{2\pi-\alpha}{\alpha}\theta)$.
   (cf. [BonnetBenDhia-Chesnel-PC'12]).
   $|||\mathtt{R}_{1\to 2}|||^2 = \max(\frac{2\pi-\alpha}{\alpha}, \frac{\alpha}{2\pi-\alpha})$;
   $|||\mathtt{R}_{2\to 1}|||^2 = \max(\frac{2\pi-\alpha}{\alpha}, \frac{\alpha}{2\pi-\alpha})$.



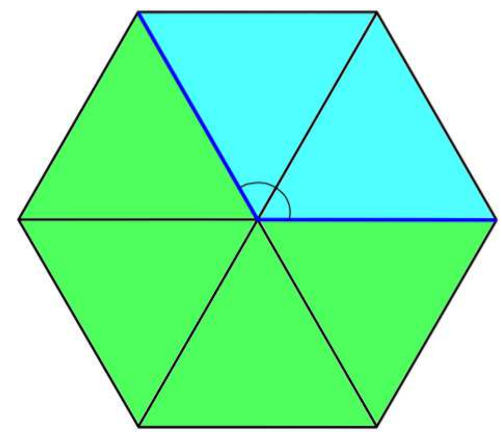
Optimality of $\mathtt{T}$-coercivity-3

- Study of elementary geometries:

1. Symmetric geometry (2D/3D)

2. Interface with an interior corner

   Operators $\mathtt{R}_{1\to2}$, $\mathtt{R}_{2\to1}$ combine rotation + symmetry geometrical transforms:
   (cf. [BonnetBenDhia-Carvalho-PC'18]).



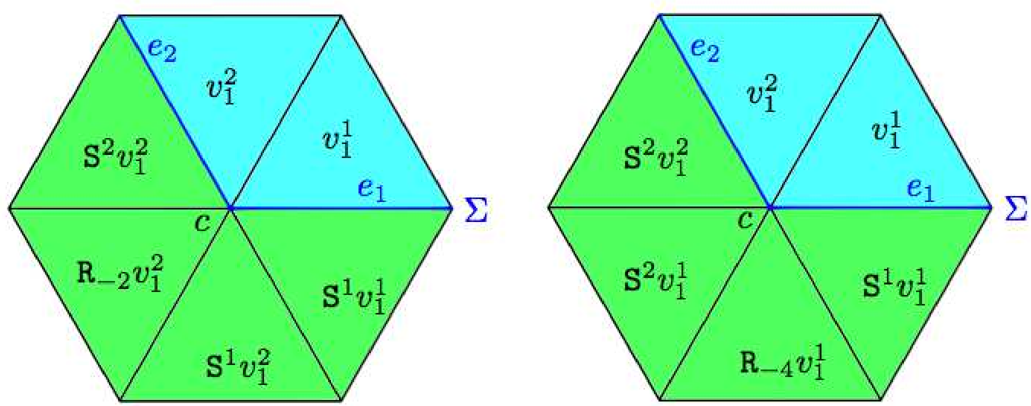
Optimality of $\mathtt{T}$-coercivity-3

- Study of elementary geometries:

1. Symmetric geometry (2D/3D)

2. Interface with an interior corner

Admissible operators $\mathtt{R}_{1\to 2}$:



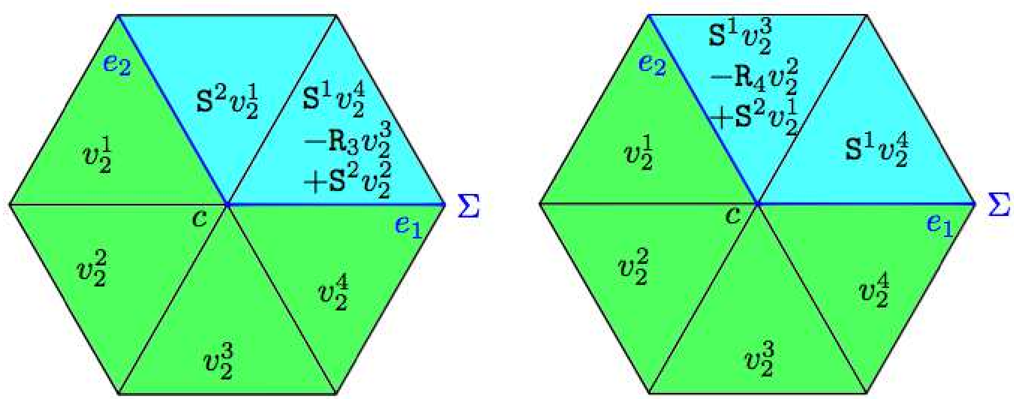
Optimality of $\mathtt{T}$-coercivity-3

- Study of elementary geometries:

1. Symmetric geometry (2D/3D)

2. Interface with an interior corner

Admissible operators $\mathtt{R}_{2\to 1}$:



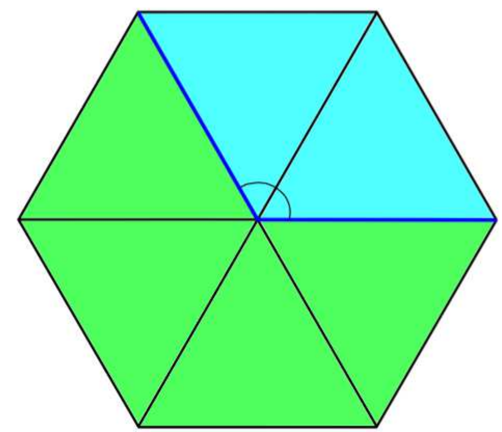
Optimality of $\mathtt{T}$-coercivity-3

- Study of elementary geometries:

1. Symmetric geometry (2D/3D)

2. Interface with an interior corner

   Operators $\mathtt{R}_{1\to 2}$, $\mathtt{R}_{2\to 1}$ combine rotation + symmetry geometrical transforms:
   (cf. [BonnetBenDhia-Carvalho-PC'18]).
   After "averaging" the operators:
   $|||\mathtt{R}^{opt}_{1\to 2}|||^2 = \max(\frac{2\pi-\alpha}{\alpha}, \frac{\alpha}{2\pi-\alpha})$;
   $|||\mathtt{R}^{opt}_{2\to 1}|||^2 = \max(\frac{2\pi-\alpha}{\alpha}, \frac{\alpha}{2\pi-\alpha})$.



Optimality of $\mathbb{T}$-coercivity-3

- Study of elementary geometries:

  1. Symmetric geometry (2D/3D)

  2. Interface with an interior corner

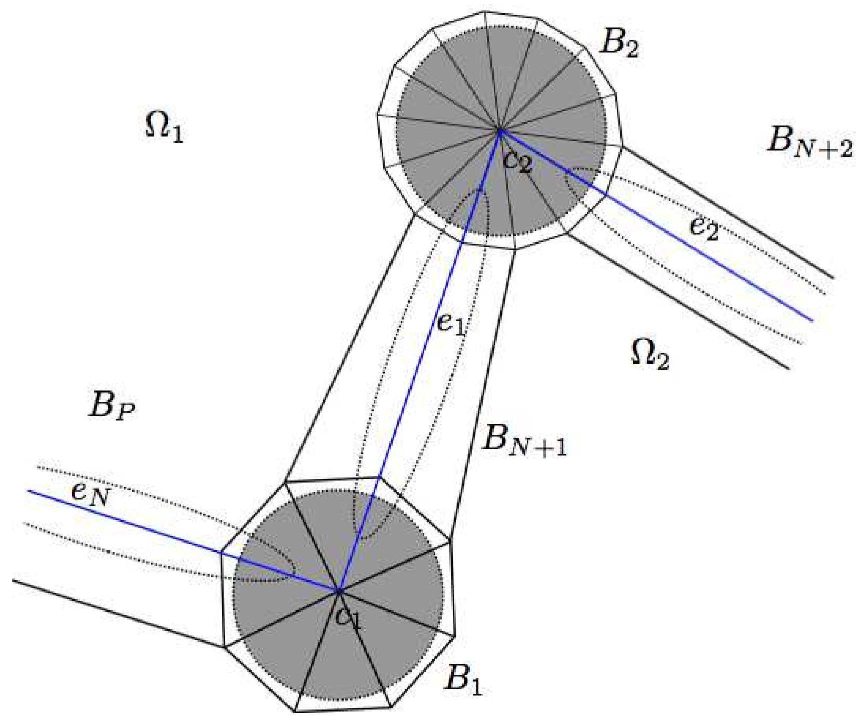
  3. Interface with a boundary corner

  4. Curved interface (2D/3D)

  Can handle all configurations in 2D geometries [BonnetBenDhia-Carvalho-PC'18].

Optimality of $\mathbb{T}$-coercivity-4

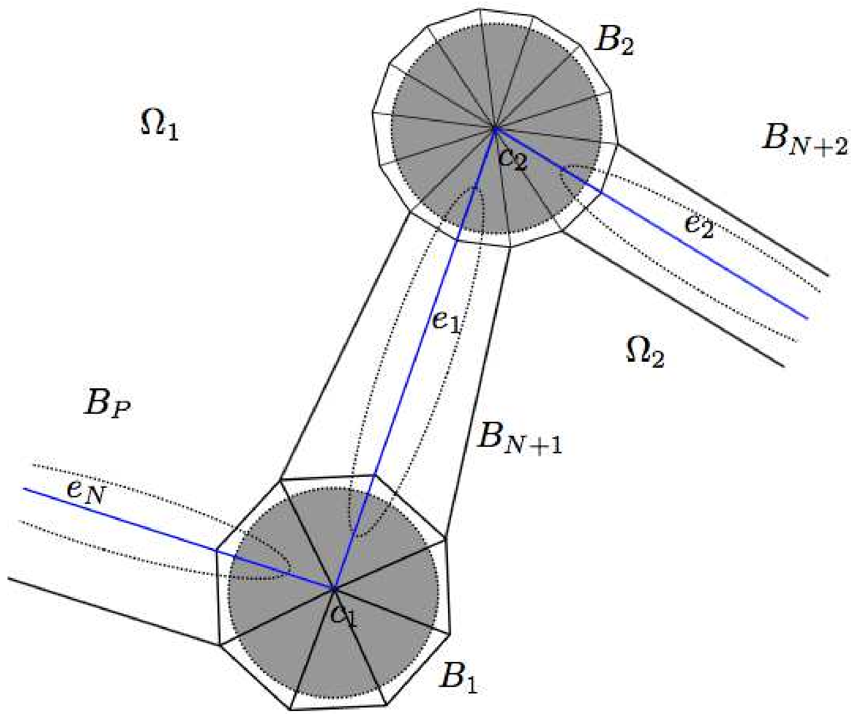
- Handle general geometries by *localization*.



Case of an inclusion: interface Σ with N corners and edges.

Optimality of $\mathtt{T}$-coercivity-4

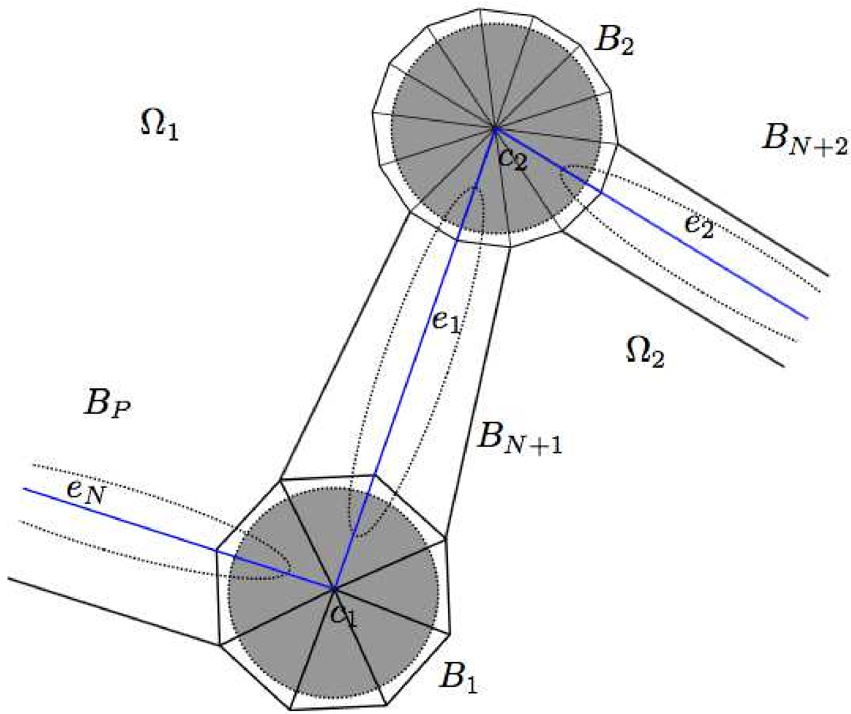
- Handle general geometries by *localization*.



Use geometrical transforms *near* each corner to define $\mathtt{R}^p_{1\to 2}$ *locally*, $p = 1, N$.

Optimality of $\mathtt{T}$-coercivity-4

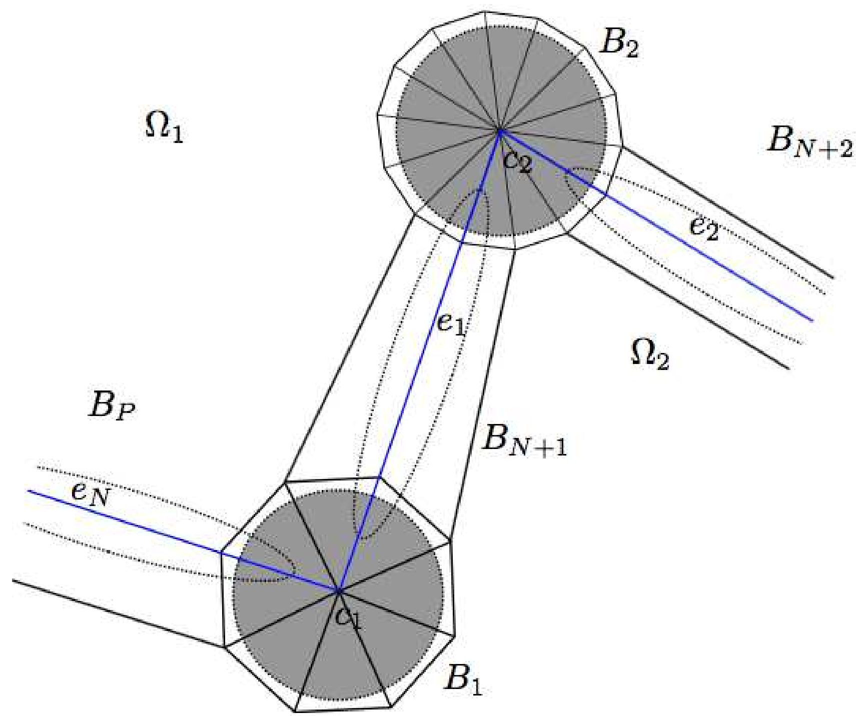
- Handle general geometries by *localization*.



Use geometrical transforms *near* each edge to define $\mathtt{R}^p_{1\to 2}$ *locally*, $p = N+1, 2N$.

Optimality of $\mathtt{T}$-coercivity-4

- Handle general geometries by *localization*.



Let $P = 2N$. Introduce $\mathtt{T}_1 \in \mathcal{L}(V)$ bijective, where $(\chi_p)_{1 \le p \le P}$ are cut-off functions:

$$\forall v \in V, \quad \mathtt{T}_1\, v := \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 + 2\sum_{p=1,P} \chi_p \mathtt{R}^p_{1\to 2}\, v_1 & \text{in } \Omega_2 \end{cases}.$$

Optimality of $\mathtt{T}$-coercivity-5

- Cf. [BonnetBenDhia-Chesnel-PC'12], [BonnetBenDhia-Carvalho-PC'18]: there exists a *critical interval* $I_\Sigma \subset]-\infty, 0[$ such that
  - if $\sigma_2/\sigma_1 \in I_\Sigma$: the scalar *transmission* problem is not well-posed ;
  - if $\sigma_2/\sigma_1 \notin I_\Sigma$: one has a Gärding inequality

$$\exists \underline{\alpha}_\sigma, \underline{\beta}_\sigma > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha}_\sigma\, |v|^2_{H^1(\Omega)} - \underline{\beta}_\sigma \|v\|^2_{L^2(\Omega)},$$

  i.e. the scalar *transmission* problem is well-posed (in the Fredholm sense).

- The *bounds* of I_Σ depend on the value of the angles at the corners of the interface ; e.g. a square-shaped metamaterial (within a dielectric): $I_\Sigma = [-3, -1/3]$.
- The critical interval I_Σ always contains -1 (for a piecewise smooth interface).

- If the interface is $\mathcal{C}^2$ without endpoints, $I_\Sigma = \{-1\}$ (cf. [Costabel-Stephan'85]).
- The model problem with $\omega \neq 0$ can be solved similarly.

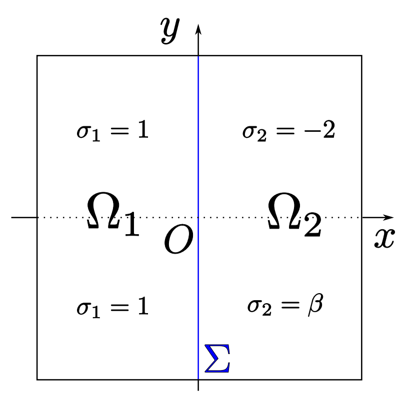
Comments

- General case, cf. [BonnetBenDhia-Chesnel-PC'12]:
  $\sigma, \sigma^{-1} \in L^\infty(\Omega)$, σ sign-changing coefficient.

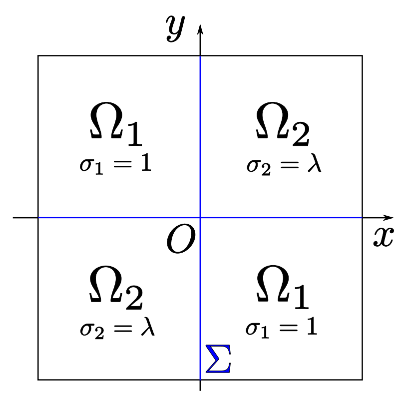
  Let $\sigma_1^+ = \sup_{\Omega_1} \sigma_1$, $\sigma_1^- = \inf_{\Omega_1} \sigma_1$; $|\sigma_2^+| = \sup_{\Omega_2} |\sigma_2|$, $|\sigma_2^-| = \inf_{\Omega_2} |\sigma_2|$:
  - one finds the sufficient condition:

$$\frac{\sigma_1^-}{|\sigma_2^+|} > \left(\inf_{\mathtt{R}_{1\to 2}} |||\mathtt{R}_{1\to 2}|||\right)^2 \quad \text{or} \quad \frac{|\sigma_2^-|}{\sigma_1^+} > \left(\inf_{\mathtt{R}_{2\to 1}} |||\mathtt{R}_{2\to 1}|||\right)^2 ;$$

  - moreover, only the knowledge of the coefficients at the interface is needed ;
  - there are (simple) cases not covered by the theory.



$\beta \in (-1, 0)$.



$\lambda < 0$.

Comments

- General case, cf. [BonnetBenDhia-Chesnel-PC'12]:
  $\sigma, \sigma^{-1} \in L^\infty(\Omega)$, σ sign-changing coefficient.

  Let $\sigma_1^+ = \sup_{\Omega_1} \sigma_1$, $\sigma_1^- = \inf_{\Omega_1} \sigma_1$; $|\sigma_2^+| = \sup_{\Omega_2} |\sigma_2|$, $|\sigma_2^-| = \inf_{\Omega_2} |\sigma_2|$:
  - one finds the sufficient condition:

$$\frac{\sigma_1^-}{|\sigma_2^+|} > \left(\inf_{\mathtt{R}_{1\to 2}} |||\mathtt{R}_{1\to 2}|||\right)^2 \quad \text{or} \quad \frac{|\sigma_2^-|}{\sigma_1^+} > \left(\inf_{\mathtt{R}_{2\to 1}} |||\mathtt{R}_{2\to 1}|||\right)^2 ;$$

  - moreover, only the knowledge of the coefficients at the interface is needed ;
  - there are (simple) cases not covered by the theory.

- T-coercivity using geometrical transforms is *sub-optimal* in some 3D domains.

  For instance, a cube-shaped metamaterial (within a dielectric):
  - this T-coercivity predicts $I_\Sigma \subseteq [-7, -1/7]$;
  - but using [Helsing-Perfekt'13], one may shrink I_Σ to $[-5.5359..., -1/3]$!?

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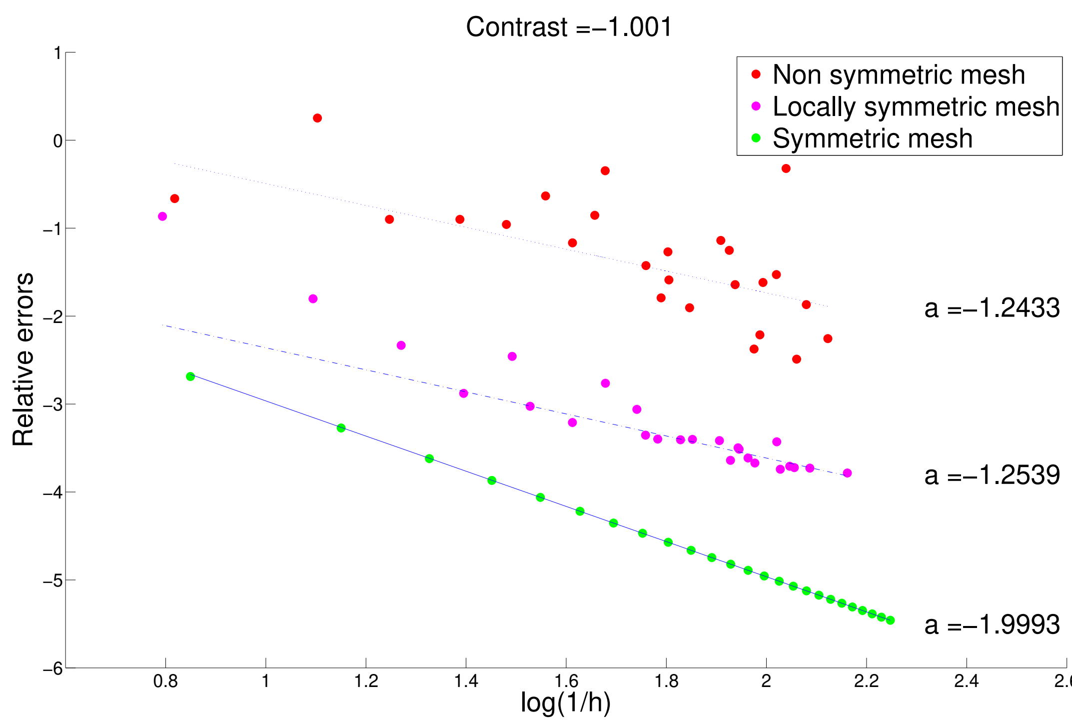
Numerics: no corners

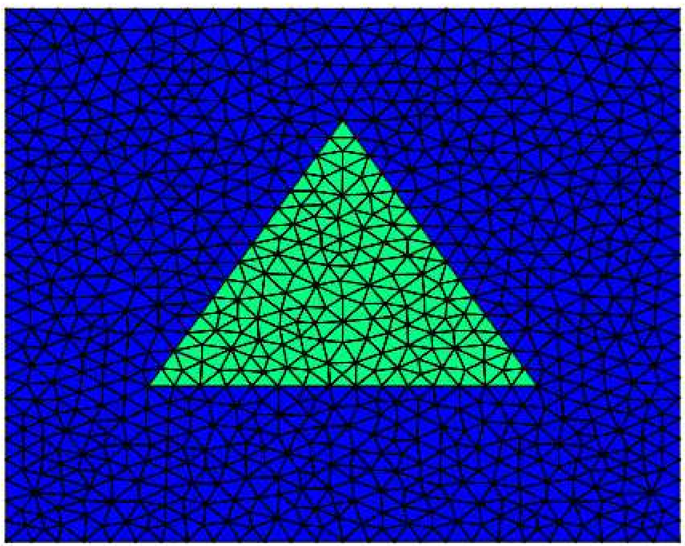
- Model scalar problem, *symmetric domain*: $\Omega_1 =]-1,0[\times]0,1[$, $\Omega_2 =]0,1[\times]0,1[$.
- An *exact* piecewise smooth solution is available, for a contrast $\sigma_2/\sigma_1 = -1.001$.
- Discretization using P_1 Lagrange finite elements.
- What is the influence of the meshes? (relative errors in L^2-norm ; $O(h^2)$ is expected).



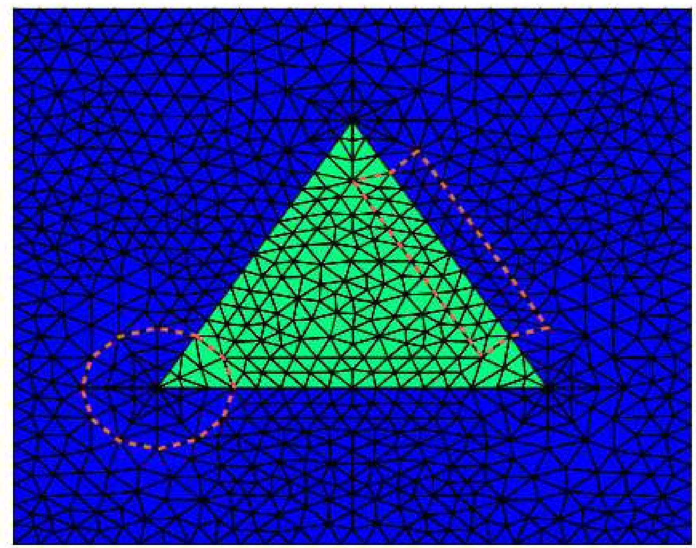
Meshes must/should be carefully designed: T-conform meshes!

Numerics: with corners

- Model scalar problem.
- No *exact* solution is available.
- Contrast: $\sigma_2/\sigma_1 = -5.2$ (critical interval $I_\Sigma = [-5, -1/5]$).
- Discretization using P_k Lagrange finite elements ($k = 1, 2, 3$).



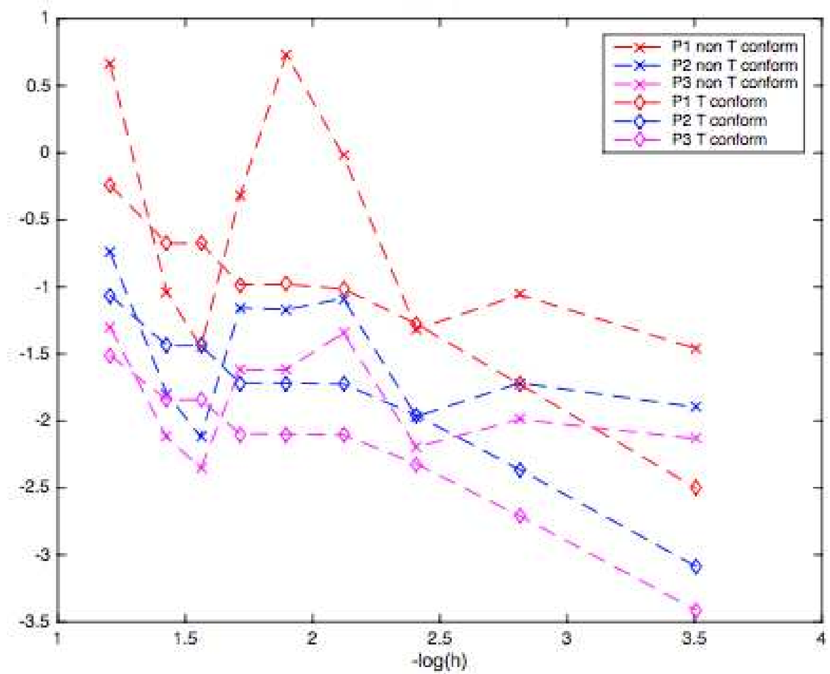
standard mesh



T-conform mesh

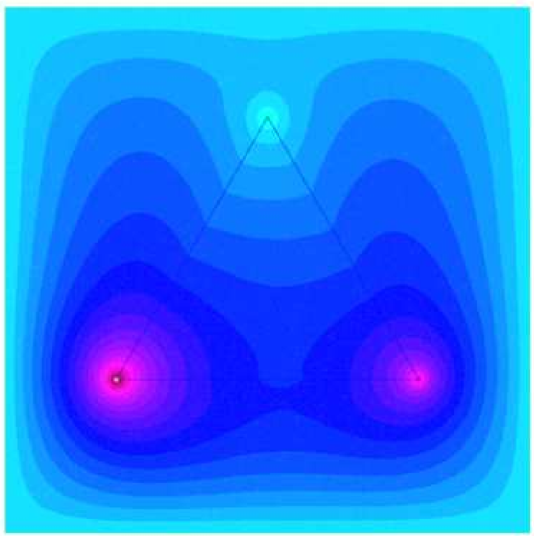
Numerics: with corners

- Model scalar problem.
- No *exact* solution is available.
- Discretization using P_k Lagrange finite elements, for $k = 1, 2, 3$.
- What is the influence of the meshes? (relative errors in L^2-norm).

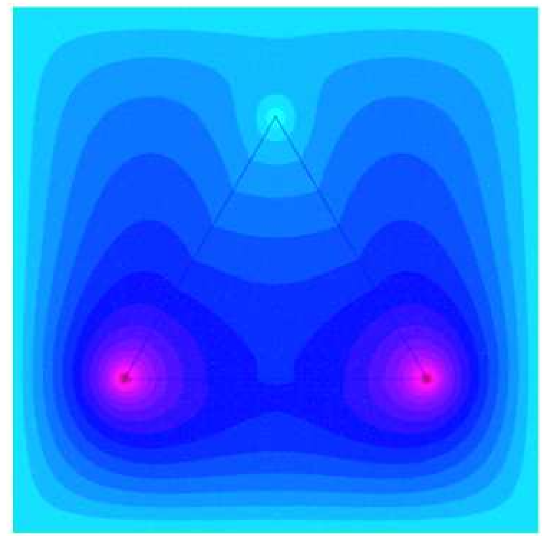


Numerics: with corners

- Model scalar problem.
- No *exact* solution is available.
- Discretization using P_3 Lagrange finite elements.
- Comparison of the computed solutions ($\approx 10^5$ dof).



standard mesh



T-conform mesh

Meshes must/should be carefully designed: T-conform meshes!

Numerical analysis

- Definition [Chesnel-PC'13], [BonnetBenDhia-Carvalho-PC'18]
  For $i = 1, 2$ and $p = 1, P$, let

$$\mathcal{T}^p_{h,i} := \{\tau \in \mathcal{T}_h \ : \ \tau \cap int(supp(\chi_p)) \cap \Omega_i \neq \emptyset\}.$$

  The meshes $(\mathcal{T}_h)_h$ are *locally* $\mathtt{T}$*-conform* if, for all $h \lesssim 1$, for all $p = 1, P$, for all $\tau \in \mathcal{T}^p_{h,1}$, the image of τ by the geometrical transforms underlying $\mathtt{R}_p$ belongs to $\mathcal{T}^p_{h,2}$.
  In other words, it is required that *the structure of the discrete spaces V_h is preserved locally with the help of the geometrical transforms*.

- Proposition [BonnetBenDhia-Carvalho-PC'18]
  Assume that $\sigma_2/\sigma_1 \notin I_\Sigma$, and that the exact problem is well-posed.
  If the meshes $(\mathcal{T}_h)_h$ are *locally* $\mathtt{T}$*-conform*, then, for h small enough, the discrete problem is well-posed in V_h. Moreover, the discrete solution u_h is such that

$$\|u - u_h\|_1 \leq C \inf_{v_h \in V_h} \|u - v_h\|_1$$

  with $C > 0$ independent of h.

Concluding remarks

- Scalar problems *with sign-shifting coefficients*:
  - introduction of $\mathtt{T}$-coercivity during WAVES'07 ;
  - theoretical study of well-posedness (cf. [BonnetBenDhia-Chesnel-PC'12]) ;
  - numerical analysis when (weak) $\mathtt{T}$-coercivity applies (cf. [BonnetBenDhia-PC-Zwölf'10], [Nicaise-Venel'11], [Chesnel-PC'13], [BonnetBenDhia-Carvalho-PC'18], etc.) ;
  - optimization-based numerical method (cf. [Abdulle-Huber-Lemaire'17]) ;
  - Boundary Integral Equations-based numerical method (cf. [Helsing-Karlsson'18]) ;
  - *a posteriori* error control (cf. [PC-Vohralik'18]).

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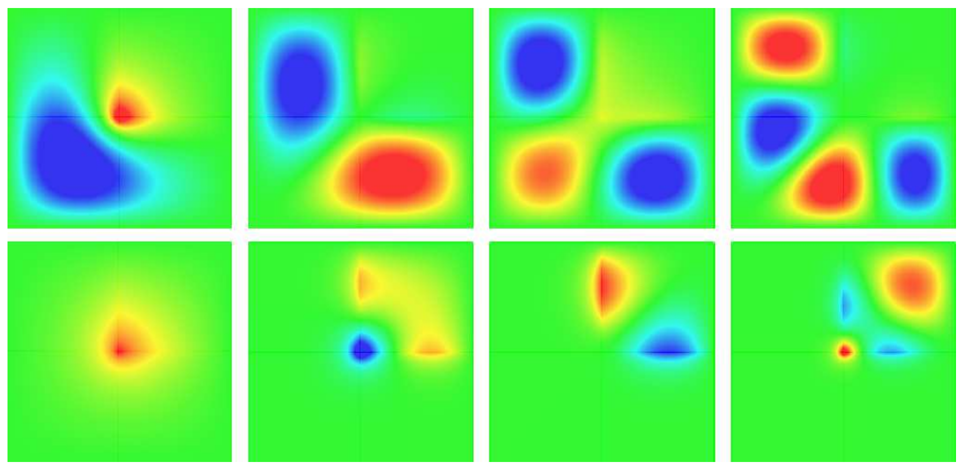
Concluding remarks

- Scalar problems *with sign-shifting coefficients*:

- Scalar eigenproblems *with sign-shifting coefficients*:
  - *localization* of eigenfunctions, *spectral correctness* if the meshes are *locally* T*-conform* [Carvalho-Chesnel-PC'17]. On a square minus square geometry:



Concluding remarks

- Scalar problems *with sign-shifting coefficients*:

- Scalar eigenproblems *with sign-shifting coefficients*:

- Maxwell problem(s) *with sign-shifting coefficients*:
  - (weak) $\mathtt{T}$-coercivity (cf. [BonnetBenDhia-Chesnel-PC'14a,b]) ;
  - Q numerical analysis? In progress...

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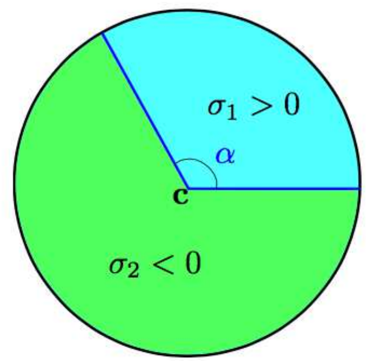
Concluding remarks

- Scalar problems *with sign-shifting coefficients*:
- Scalar eigenproblems *with sign-shifting coefficients*:
- Maxwell problem(s) *with sign-shifting coefficients*:
- Theoretical study of *critical* cases (cf. [BonnetBenDhia-Chesnel-Claeys'13]).

Extension: the SCM

- Inside the critical interval: $\sigma_2/\sigma_1 \in I_\Sigma \setminus \{-1\}$.
- In a simple geometry:

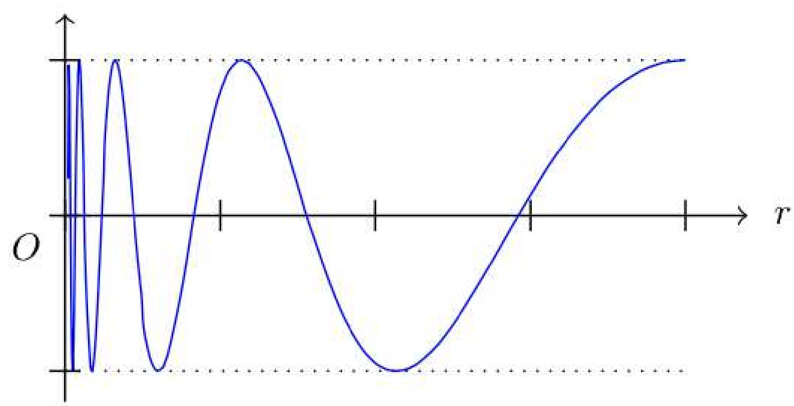


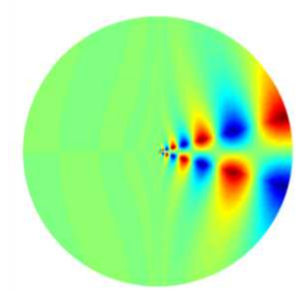
Extension: the SCM

- Inside the critical interval: $\sigma_2/\sigma_1 \in I_\Sigma \setminus \{-1\}$.

- Cf. [BonnetBenDhia-Chesnel-Claeys'13]:

  the scalar *transmission* problem is well-posed in $H_0^1(\Omega) \oplus \mathbb{C}(\zeta s)$, where s is an *hyper-oscillating* singularity of the form $s(r,\theta) = r^{\imath\lambda}\Phi(\theta)$, $\lambda \in \mathbb{R}$, and ζ is a smooth cutoff function.





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  Given $f \in L^2(\Omega)$, let $u = \tilde{u} + b(\zeta s)$ be the solution:
  - the regular part $\tilde{u}$ has some "extra" smoothness ;
  - $s \in \bigcap_{t\in[0,1[} H^t(\Omega)$ ($s \notin H^1(\Omega)$) ; $-\mathrm{div}\,(\sigma\,\mathbf{grad}\,s) = 0$ in Ω ;
  - the scalar *transmission* problem is also well-posed in $H^1_0(\Omega) \oplus \mathbb{C}(\zeta\overline{s})$.
- The Singular Complement Method, `work in progress with` C. Carvalho:
  - there exists a *dual singularity* z such that $b = \displaystyle\int_\Omega f\, z\, d\Omega$;
  - the regular part $\tilde{u}$ is governed by a Variational Formulation ;
  - b and $\tilde{u}$ can be recovered numerically.

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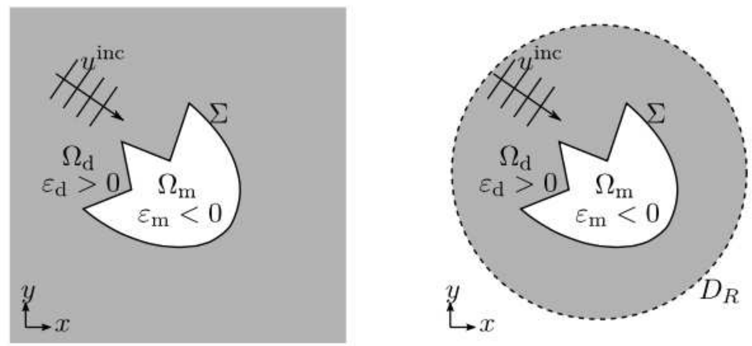
Extension: PML at corners

- Inside the critical interval: $\sigma_2/\sigma_1 \in I_\Sigma \setminus \{-1\}$.
- Scattering problem:



$$
\left\{
\begin{array}{l}
\textit{Find } u = u^{inc} + u^{sca} \textit{ such that} \\[2ex]
\mathrm{div}\, \left(\varepsilon^{-1}\nabla u\right) + k_0^2\, \mu u = 0 \quad \text{in } \mathbb{R}^2 \\[2ex]
\displaystyle\lim_{\xi \to +\infty} \int_{|\mathbf{x}|=\xi} \left| \frac{\partial u^{sca}}{\partial r} - iku^{sca} \right|^2 d\sigma = 0.
\end{array}
\right.
$$

The *radiation condition* is replaced by a boundary condition on ∂D_R, using a DtN map.

Extension: PML at corners

- Inside the critical interval: $\sigma_2/\sigma_1 \in I_\Sigma \setminus \{-1\}$.
- Scattering problem, using [BonnetBenDhia-Chesnel-Claeys'13] at the corners.
- Choice of the *hyper-oscillating* singularity $s_{\mathbf{c}}$ or $\overline{s_{\mathbf{c}}}$:
  - via *energy balance Eq.* (no energy should be brought into the system) ;
  - via the *limiting absorption principle* ;
  - both methods select the same singularity!

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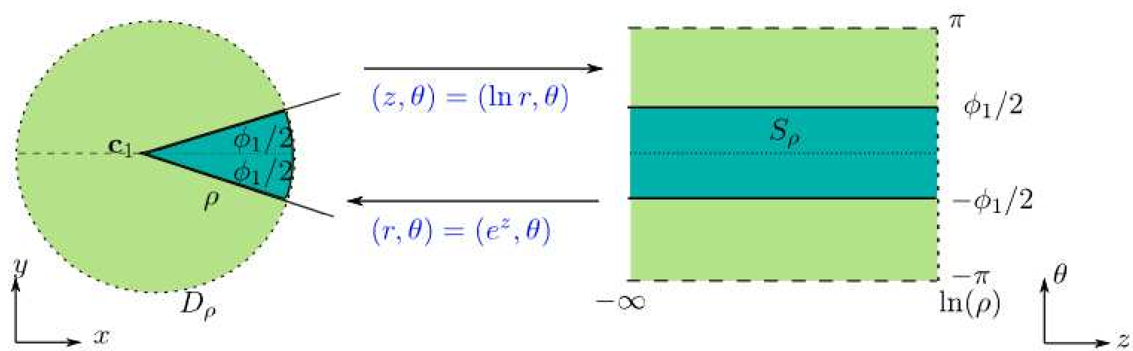
Extension: PML at corners

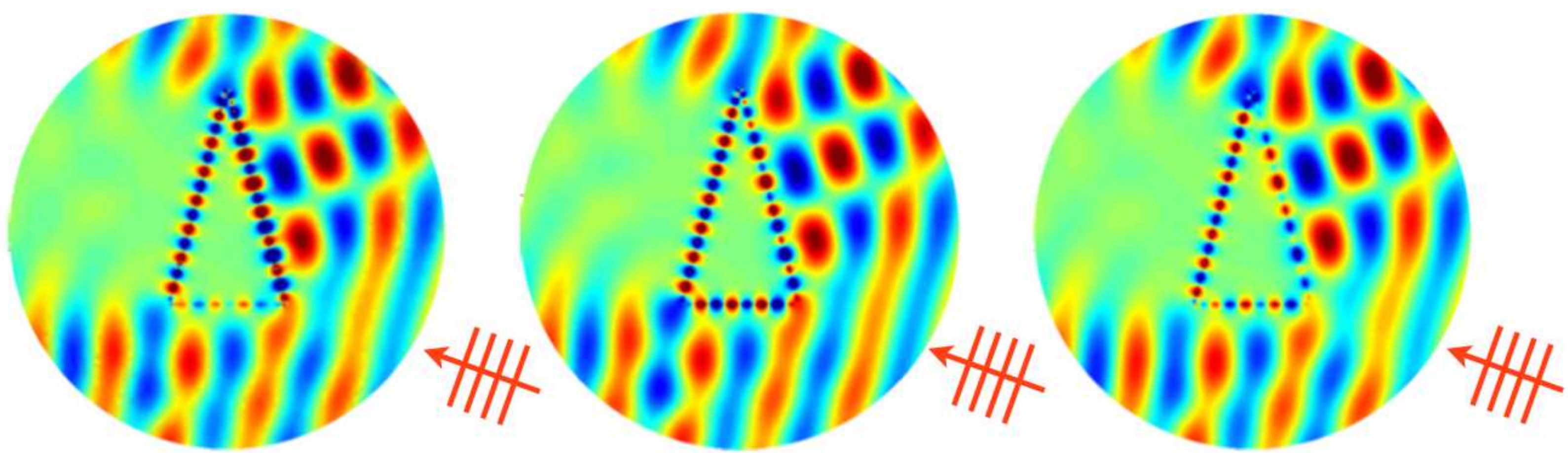
- Inside the critical interval: $\sigma_2/\sigma_1 \in I_\Sigma \setminus \{-1\}$.
- Scattering problem.
- Analogy with a semi-infinite waveguide:



  - In the waveguide, $\breve{s}_{\mathbf{c_1}}$ or $\overline{\breve{s}_{\mathbf{c_1}}}$ are *propagative modes*.
  - Use a Perfectly Matched Layer to bound artificially the waveguide.

Extension: PML at corners

- Inside the critical interval: $\sigma_2/\sigma_1 \in I_\Sigma \setminus \{-1\}$.
- Scattering problem, cf. [BonnetBenDhia-Carvalho-Chesnel-PC'16]:
  - Choice of the *hyper-oscillating* singularities $(s_{\mathbf{c}_k})_k$ or $(\overline{s_{\mathbf{c}_k}})_k$;
  - Use a Perfectly Matched Layer near each corner.
- Discretization using P_2 Lagrange FE (three meshsizes), *without PML at the corners*:



The numerical solution varies with the meshsize (especially near, or on, the interface).

Extension: PML at corners

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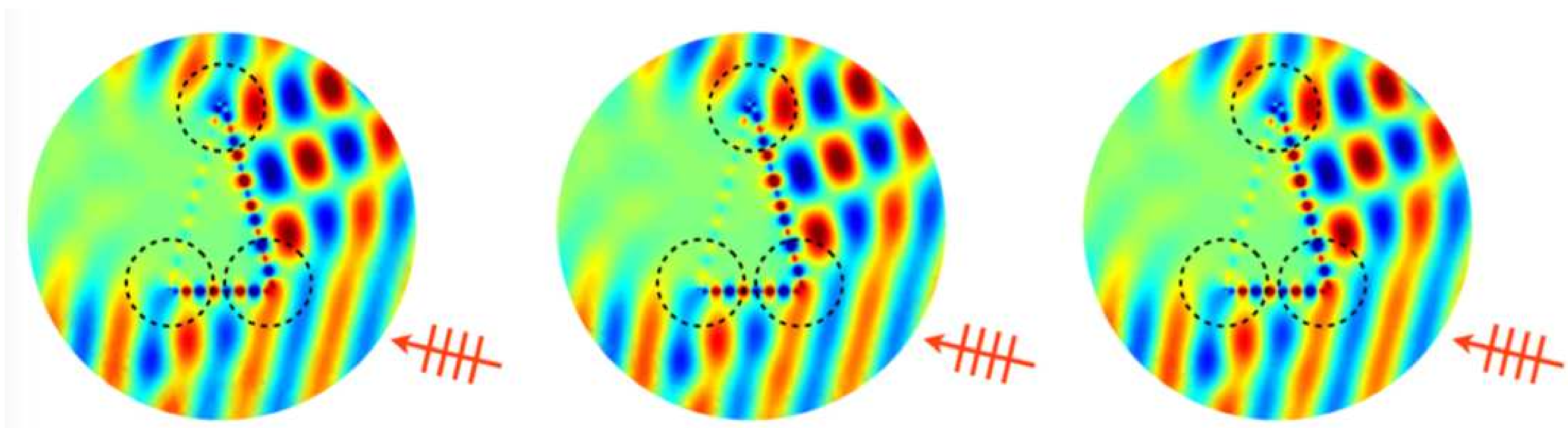

The numerical solution is independent of the meshsize.

Extension: PML at corners

- Inside the critical interval: $\sigma_2/\sigma_1 \in I_\Sigma \setminus \{-1\}$.
- Scattering problem, cf. [BonnetBenDhia-Carvalho-Chesnel-PC'16]:
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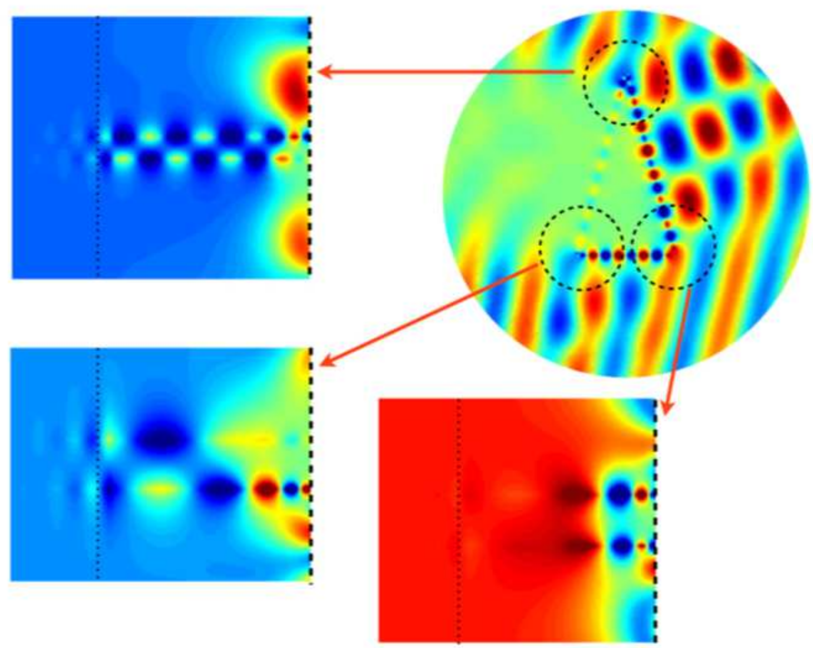
  - Use a Perfectly Matched Layer near each corner.
- Discretization using P_2 Lagrange FE, *with PML at the corners*. Close-up:



More concluding remarks

Inside the critical interval:

- Extensions with *hyper-oscillating* singularity work theoretically and numerically, cf. [BonnetBenDhia-Chesnel-Claeys'13], [BonnetBenDhia-Carvalho-Chesnel-PC'16]. `Work in progress with` C. Carvalho.

More concluding remarks

Inside the critical interval:

- Extensions with *hyper-oscillating* singularity work theoretically and numerically, cf. [BonnetBenDhia-Chesnel-Claeys'13], [BonnetBenDhia-Carvalho-Chesnel-PC'16]. `Work in progress with` C. Carvalho.
- Go back to the model *before homogenization*, with rapidly oscillating coefficients. Study:
  - the numerical homogenization, or the Heterogeneous Multiscale Method, for those models [Verfürth'17], [Ohlberger-Verfürth'18], [Verfürth'19].
  - how the *effective model* is influenced by interfaces, cf. PhD of C. Bénéteau on "enriched homogenization in presence of an interface" (supervision X. Claeys and S. Fliss).

More concluding remarks

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  - how the *effective model* is influenced by interfaces, cf. PhD of C. Bénéteau on "enriched homogenization in presence of an interface" (supervision X. Claeys and S. Fliss).

- The use of a nonlocal model is promising *numerically*, cf. [Borthagaray-PC'17]. "Localization" of the nonlocality (near the interface) seems possible.
  `Work in progress with` J.P. Borthagaray, cf. [Borthagaray-PC'19].

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