Numerical approximation of transmission problems with sign changing coefficients

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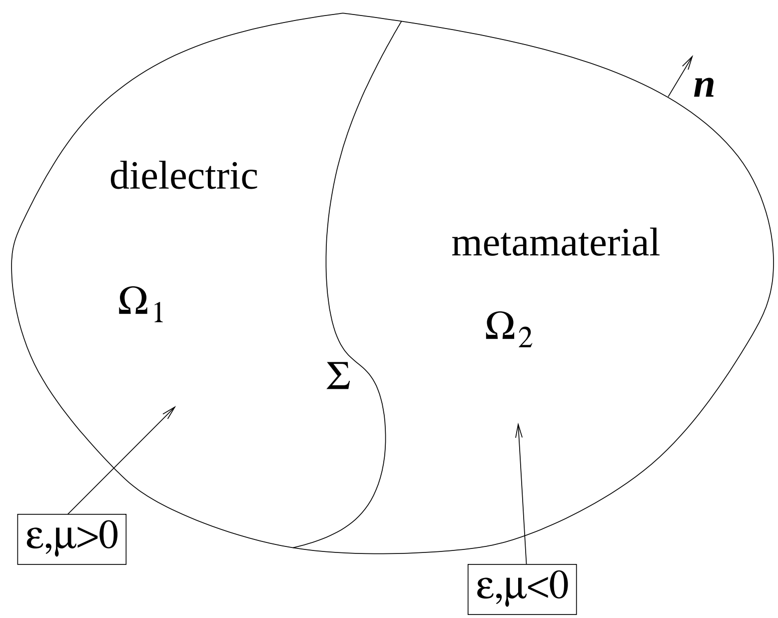
Sign-changing coefficients

- Consider a scalar *transmission* problem, set in a bounded domain Ω of $\mathbb{R}^d$, $d = 1, 2, 3$.

$$\begin{cases} \textit{Find } u \in H^1_0(\Omega) \textit{ such that} \\ -\mathrm{div}\,(\sigma \mathbf{grad}\, u) = f \text{ in } \Omega. \end{cases}$$

Motivation (EM-ics):
$\sigma := \varepsilon^{-1}$ or $\sigma := \mu^{-1}$.



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- Consider a scalar *transmission* problem, set in a bounded domain Ω of $\mathbb{R}^d$, $d = 1, 2, 3$.

$$\left\{ \begin{array}{l} \textit{Find } u \in H_0^1(\Omega) \textit{ such that} \\ -\mathrm{div}\,(\sigma \mathbf{grad}\, u) = f \text{ in } \Omega. \end{array} \right.$$

  - $\sigma \in L^\infty(\Omega)$ is a sign-changing coefficient: $\left\{ \begin{array}{l} \sigma > 0 \text{ in } \Omega_1, \text{ with meas}(\Omega_1) > 0\,; \\ \sigma < 0 \text{ in } \Omega_2, \text{ with meas}(\Omega_2) > 0. \end{array} \right.$
  - $\sigma^{-1} \in L^\infty(\Omega)$.

The parameter σ is discontinuous across Σ.

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  NB. The "generalized" Helmholtz equation $-\mathrm{div}\,(\sigma \mathbf{grad}\, u) - \omega^2 \eta u = f$ with $\eta \in L^\infty(\Omega)$ can be analyzed similarly, cf. [BonnetBenDhia-Jr-Zwölf'10].

  One can also consider a Neumann b.c., cf. [BonnetBenDhia-Chesnel-Jr'12].

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- The main difficulty is that $(v, w) \mapsto \int_\Omega \sigma\, \mathbf{grad}\, v \cdot \overline{\mathbf{grad}\, w}\, d\Omega$ is *not coercive* in $H_0^1(\Omega)$.

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$\Longrightarrow$ **Solve the problem with T-coercivity!**

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Abstract setting

- Let
  - V and W be two Hilbert spaces ;
  - $a(\cdot,\cdot)$ be a continuous sesquilinear form over $V \times W$;
  - f be an element of W', the dual space of W.

Aim: solve the Variational Formulation

$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = \langle f, w \rangle.$$

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- [Banach-Necas-Babuska] Introduce the two conditions

  $$(BNB_1) \qquad \exists \alpha' > 0,\ \forall v \in V,\ \sup_{w \in W \setminus \{0\}} \frac{|a(v,w)|}{\|w\|_W} \geq \alpha' \, \|v\|_V.$$

  $$(BNB_2) \qquad \forall w \in W :\ \{\forall v \in V,\ a(v,w) = 0\} \implies \{w = 0\}.$$

  NB. Condition (BNB_1) is called an *inf-sup condition*, or a *stability condition*.

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- The form $a(\cdot,\cdot)$ is $\mathtt{T}$-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V,W),\ \text{bijective},\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2.$$

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- `Theorem (Well-posedness)` The three assertions below are equivalent:
  - (i) the Problem (VF) is well-posed ;
  - (ii) the form $a(\cdot,\cdot)$ satisfies conditions (BNB_1) and (BNB_2).
  - (iii) the form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive.

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The operator $\mathtt{T}$ realizes conditions (BNB_1) and (BNB_2) explicitly.

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Abstract setting-2

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- `Theorem (Well-posedness)` The three assertions below are equivalent:
  (i) the Problem (VF) with hermitian form is well-posed ;
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  (iii) the hermitian form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive.

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Practical T-coercivity

- In the case of the scalar *transmission* problem:
  - Ω, Ω_1 and Ω_2 are domains of $\mathbb{R}^d$, $d \geq 1$: $\Omega_1 \cap \Omega_2 = \emptyset$, $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$;
  - the *interface* is $\Sigma := \overline{\Omega_1} \cap \overline{\Omega_2}$; the boundaries are $\Gamma_k := \partial\Omega \cap \partial\Omega_k$, $k = 1, 2$;

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  - $V := H_0^1(\Omega)$; the form is $a(v, w) := \int_\Omega \sigma \, \mathbf{grad}\, v \cdot \overline{\mathbf{grad}\, w} \, d\Omega$.
  - $0 < \sigma_1^- \leq \sigma \leq \sigma_1^+ < \infty$ in Ω_1 ; $0 < \sigma_2^- \leq -\sigma \leq \sigma_2^+ < \infty$ in Ω_2.

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- Introduce $V_k := \{v_k \in H^1(\Omega_k) \,|\, v_{k|\Gamma_k} = 0\}$, $k = 1, 2$:

$$\boxed{V = \{v \,|\, v_{|\Omega_k} \in V_k,\ k = 1, 2,\ \text{Matching}_\Sigma(v_{|\Omega_1}, v_{|\Omega_2}) = 0\}},$$

with $\text{Matching}_\Sigma(v_1, v_2) := v_{1|\Sigma} - v_{2|\Sigma}$.

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Practical T-coercivity-2

- First try:

$$\forall v \in H_0^1(\Omega), \quad \mathtt{T}_- \, v := \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{cases} .$$

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(+) Obviously, $(\mathtt{T}_-)^2 = \mathtt{I}$.

(–) But $\mathtt{T}_- \not\in \mathcal{L}(H_0^1(\Omega))$, because the matching condition is not enforced.

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(+) $\mathtt{T}_1 \in \mathcal{L}(H_0^1(\Omega))$.

(+) One checks easily that $(\mathtt{T}_1)^2 = \mathtt{I}$!

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Can one achieve $\mathtt{T}$-coercivity with $\mathtt{T}_1$?

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To obtain T-coercivity with $\mathtt{T}_1$, one needs $\dfrac{\sigma_1^-}{\sigma_2^+} > |||R_1|||^2$.

Practical T-coercivity-3

- Third try: let $R_2 \in \mathcal{L}(V_2, V_1)$ s.t. for all $v_2 \in V_2$, Matching$_\Sigma(R_2 v_2, v_2) = 0$.

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- Conclusion: to achieve $\mathtt{T}$-coercivity with $\mathtt{T}_1$ or $\mathtt{T}_2$, one needs

$$\frac{\sigma_1^-}{\sigma_2^+} > \left(\inf_{R_1} |||R_1|||\right)^2 \quad \text{or} \quad \frac{\sigma_2^-}{\sigma_1^+} > \left(\inf_{R_2} |||R_2|||\right)^2.$$

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- How to choose the operators R_1, R_2?
  - using traces on Σ, liftings, cf. [BonnetBenDhia-Jr-Zwölf'10], [Nicaise-Venel'11] ;
  - using geometrical transformations, cf. [BonnetBenDhia-Chesnel-Jr'12], [BonnetBenDhia-Carvalho-Jr].

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Optimality of T-coercivity

- Study of an elementary setting:
  - piecewise constant coefficient σ ;

    in this case, $\sigma_1^- = \sigma_1^+ = \sigma_1$, and $\sigma_2^- = \sigma_2^+ = |\sigma_2|$;

    define the *contrast* $\kappa_\sigma := \dfrac{\sigma_2}{\sigma_1} \in]-\infty, 0[$.

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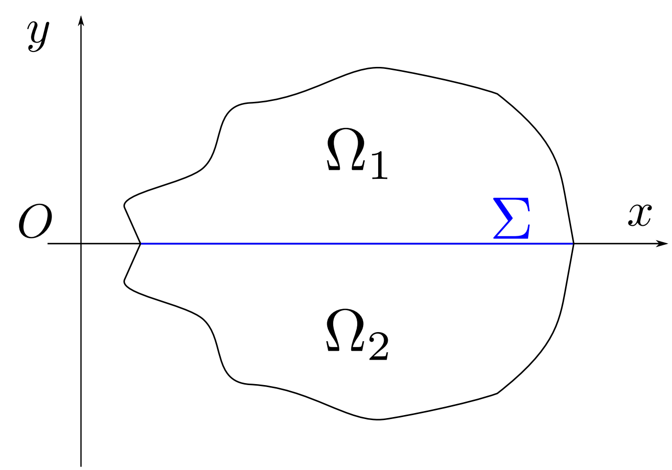
- $\sigma_1 \neq -\sigma_2$, in a symmetric geometry.

Sample symmetric geometry:



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  Let $R_1 \in \mathcal{L}(V_1, V_2)$ s.t. for all $v_1 \in V_1$, $R_1 v_1(x,y) = v_1(x,-y)$, a.e. in Ω_2.

  One finds $|||R_1||| = 1$.

  To achieve $\mathtt{T}$-coercivity, one needs $\dfrac{\sigma_1}{|\sigma_2|} > 1$.

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  The scalar *transmission* problem is well-posed when $\kappa_\sigma \neq -1$.

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  The scalar *transmission* problem is ill-posed when $\kappa_\sigma = -1$ (*Critical case*.)

- Conclusion: The scalar *transmission* problem is well-posed iff $\kappa_\sigma \neq -1$.

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Optimality of T-coercivity-2

- Study of simple geometries (on a *piecewise straight* interface Σ):

  1. Symmetric geometry

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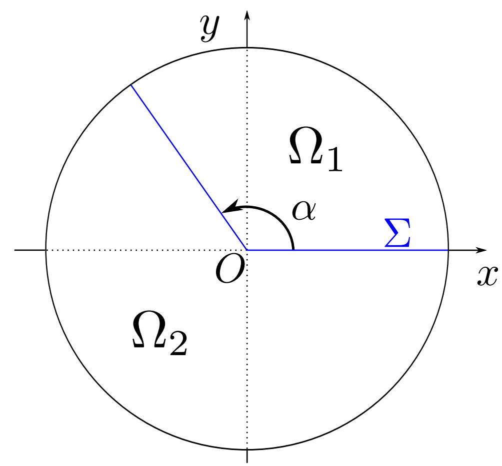
  Operators R_1, R_2 combine rotation + angle dilation:
  $(R_1\, v_1)(\rho,\theta) = v_1(\rho, \frac{\alpha}{2\pi-\alpha}\,(2\pi-\theta))$;
  $(R_2\, v_2)(\rho,\theta) = v_2(\rho, 2\pi - \frac{2\pi-\alpha}{\alpha}\theta)$.



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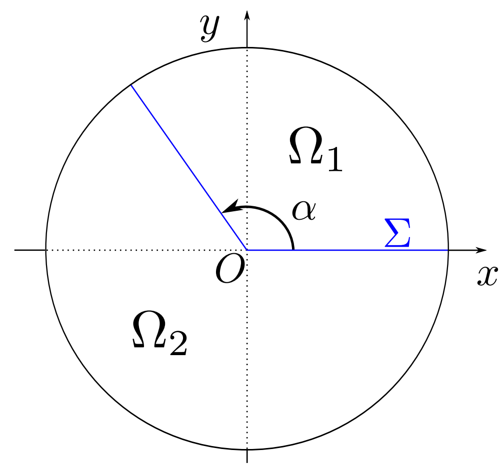

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$\ell = 1, 2$: $|||R_\ell|||^2 \leq \max(\frac{2\pi-\alpha}{\alpha}, \frac{\alpha}{2\pi-\alpha})$

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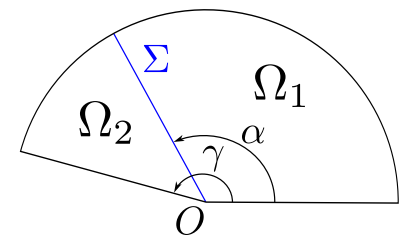
Operators R_1, R_2: similar to 2. (+ continuation by 0)



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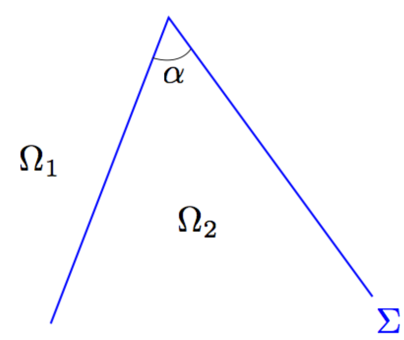

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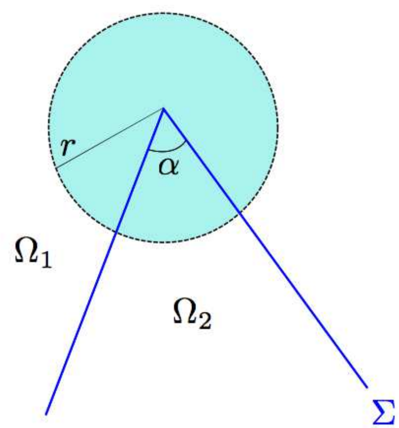

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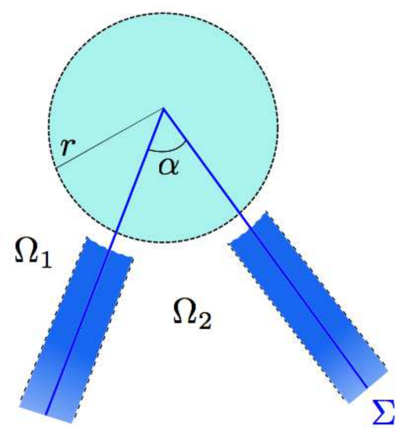

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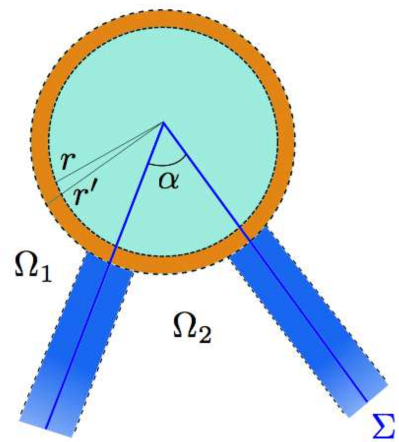

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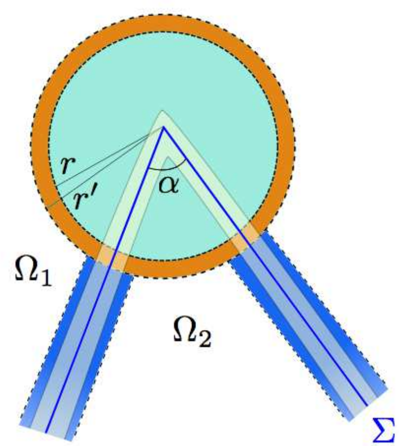

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  There exists an interval $I_\Sigma \subset]-\infty, 0[$ s.t. if $\kappa_\sigma \notin I_\Sigma$, one has a Garding inequality

$$\exists C_\sigma, C'_\sigma > 0,\ \forall v \in H^1_0(\Omega),\ |a(v, \mathtt{T}v)| \geq C_\sigma\, |v|_1^2 - C'_\sigma \|v\|_0^2.$$

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If $\kappa_\sigma \notin I_\Sigma$, then the scalar *transmission* problem is well-posed in the Fredholm sense.

- In this case, the associated operator is Fredholm of index 0.
- The interval I_Σ is *optimal* in the sense that if $\kappa_\sigma \in I_\Sigma$, then the scalar *transmission* problem is not well-posed in the Fredholm sense.
- The *bounds* of I_Σ depend on the value of the angles at the corners.

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- The *bounds* of I_Σ depend on the value of the angles at the corners.
- The interval I_Σ always contains -1.
- If the interface is $\mathcal{C}^1$ without endpoints, $I_\Sigma = \{-1\}$ (cf. [Costabel-Stephan'85]).
- The "generalized" Helmholtz equation can be solved similarly.

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Numerical experiments: no corners

- In a symmetric domain, made up of $\Omega_1 =]-1,0[\times]0,1[$, $\Omega_2 =]0,1[\times]0,1[$.
- An *exact* piecewise smooth solution is available.
- Contrast: $\kappa_\sigma = -1.001$.
- Conforming discretization using P_1 Lagrange finite elements:
  - $(\mathcal{T}_h)_h$ a regular family of meshes;
  - $(V_h)_h$ (discrete) subspaces of $H^1_0(\Omega)$;
  - `Freefem++` software.

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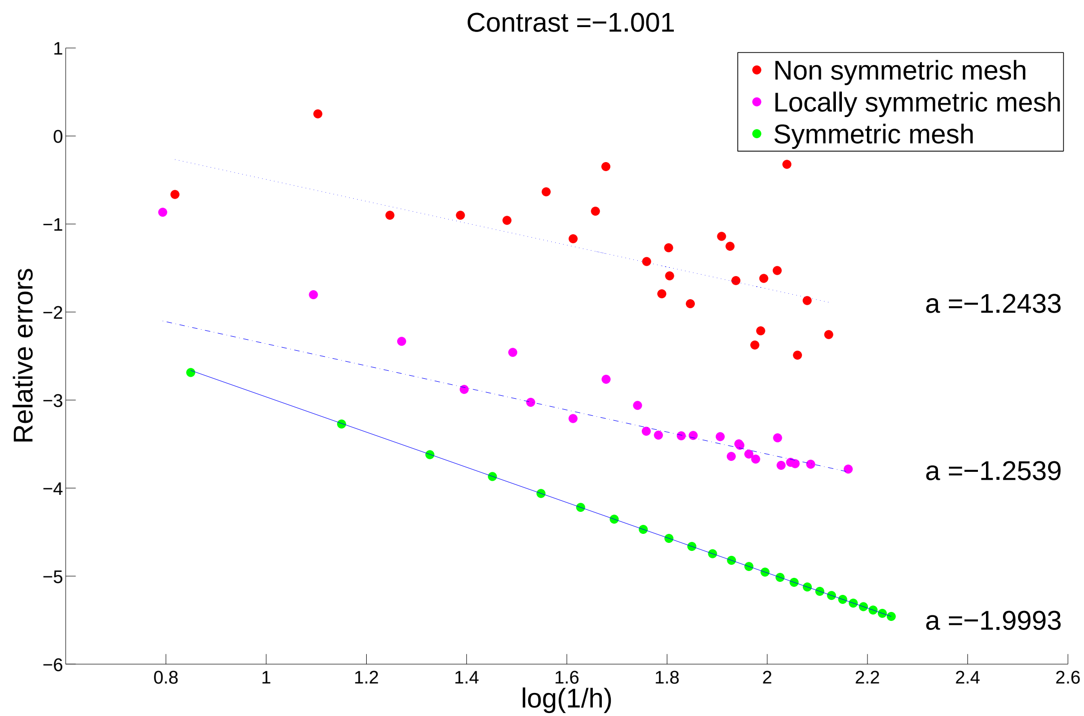

Numerical analysis

- Let $(\mathtt{T}_h)_h$ denote approximations of $\mathtt{T}$.
- The meshes $(\mathcal{T}_h)_h$ are *locally* $\mathtt{T}_h$*-conform* if
  there exists $h_0 > 0$ s.t. for all $h < h_0$, $\mathcal{T}_h$ is *locally invariant* by the geometrical transformations defining $\mathtt{T}_h$, in a *fixed* neighborhood of the interface Σ.

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$$\|u - u_h\|_1 \leq C \inf_{v_h \in V_h} \|u - v_h\|_1$$

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- Hence, it is required that the discrete spaces V_h are *locally invariant* at the interface.

Numerical experiments: no corners-2

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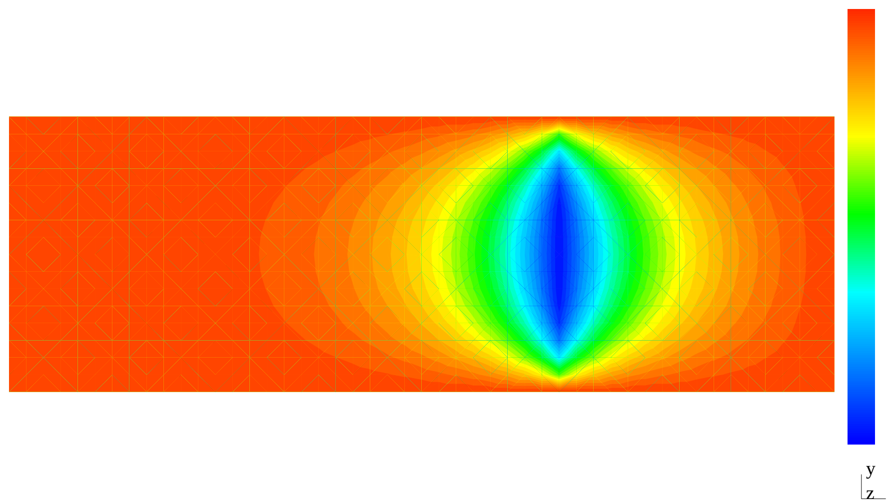
- Computed solution after 10 iterations:



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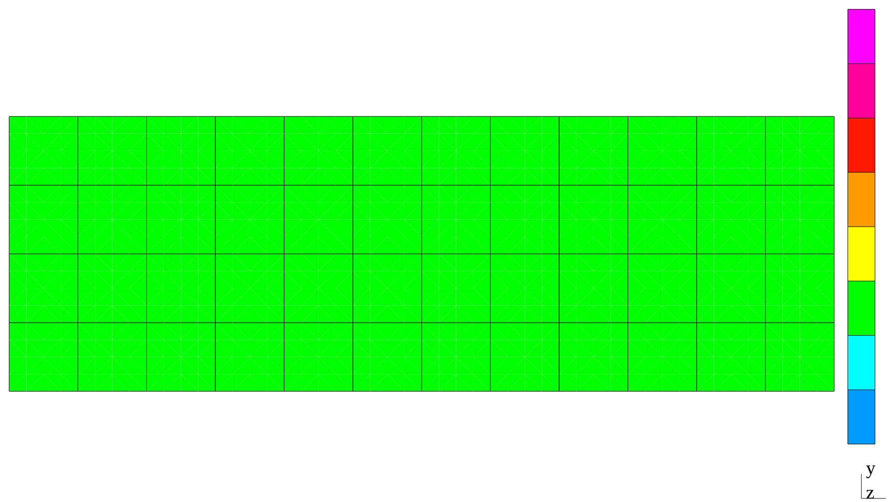
- Initial mesh (with degrees of approximation):



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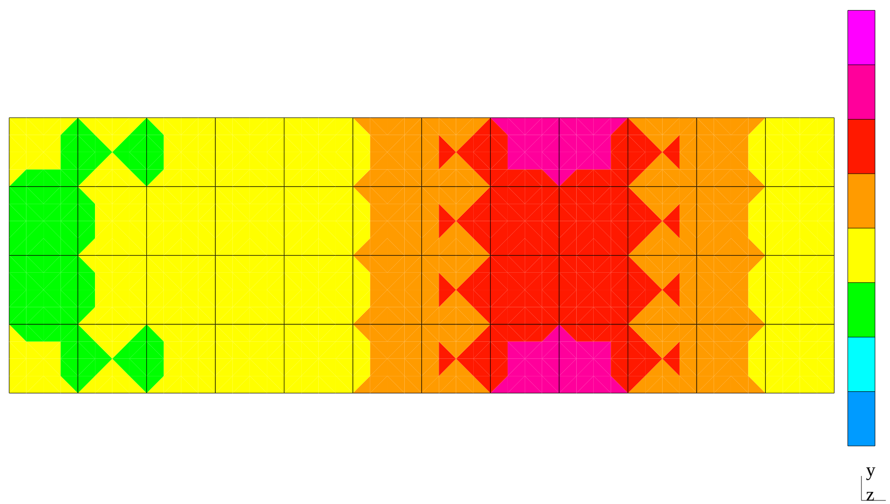
- Final mesh (with degrees of approximation):



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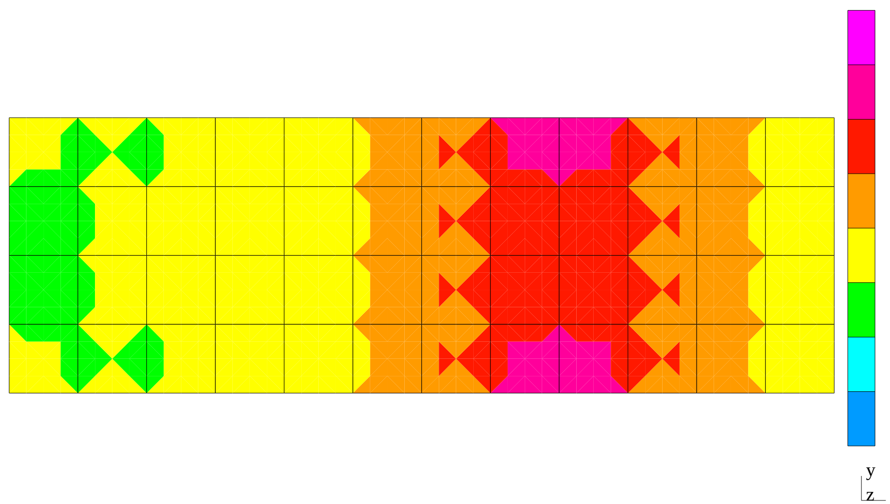

Using adaptivity yields locally symmetric meshes, with locally symmetric degree of the approximation: the final discrete spaces are *locally invariant* at the interface.

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Numerical experiments: with corners

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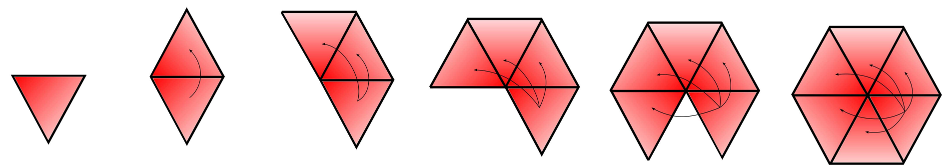
  Example with $\alpha = \pi/3$: going from Ω_2 to Ω_1.



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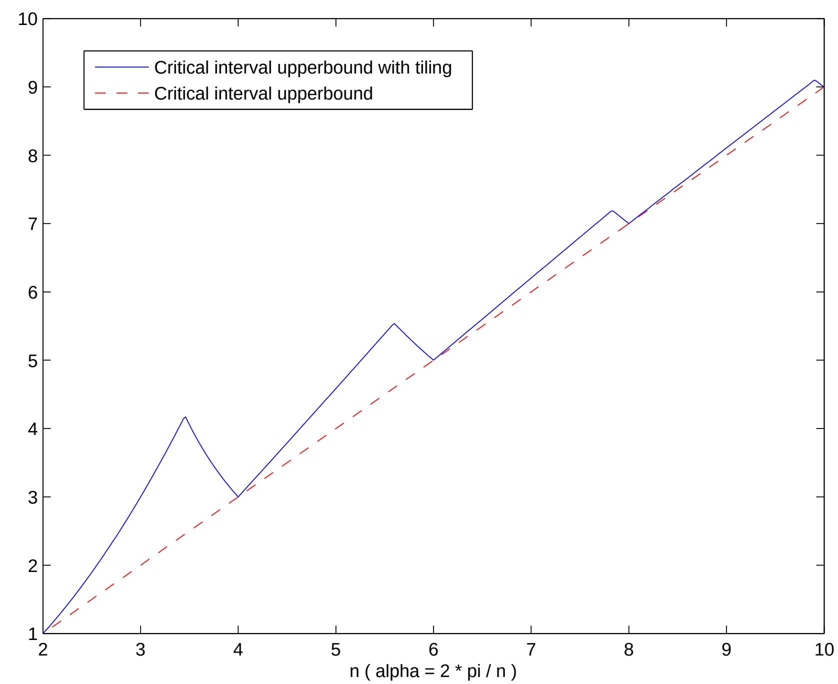

Numerical experiments: with corners-2

- Consider finally an *eigenproblem*.

$$\begin{cases} \textit{Find } u \in H_0^1(\Omega) \setminus \{0\},\ \lambda \in \mathbb{C} \textit{ such that} \\ -\mathrm{div}\,(\sigma\,\mathbf{grad}\,u) = \lambda\eta u \text{ in } \Omega. \end{cases}$$

($\eta \in L^\infty(\Omega)$, $0 < \eta_- \leq \eta$ in Ω).

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($\eta \in L^\infty(\Omega)$, $0 < \eta_- \le \eta$ in Ω).

- One can use the classical theory (cf. [Osborn'75]) to carry out the numerical analysis:
  - all eigenvalues are real;
  - there are two sequences of eigenvalues with limits $-\infty$, $+\infty$;
  - convergence theory follows from the error estimate for the direct problem.

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- One can use the classical theory (cf. [Osborn'75]) to carry out the numerical analysis.
- Droplet-shape domain Ω ($\alpha = \pi/6$); contrast $\kappa_\sigma = -13$, $\eta = 1$.
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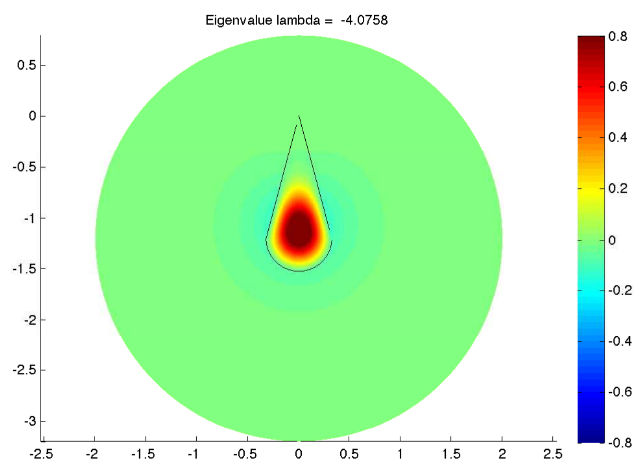

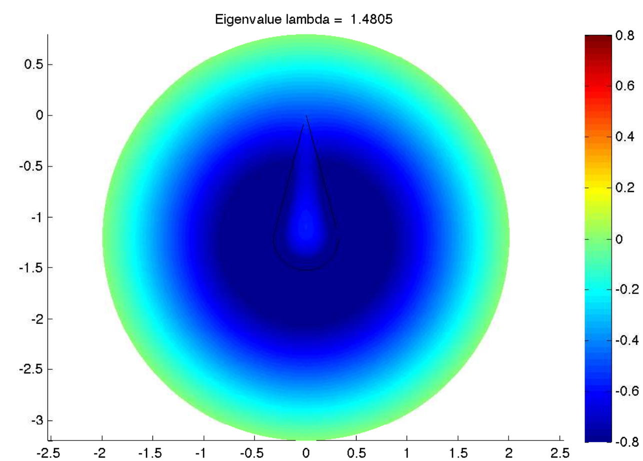

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Conclusion/Perspectives

- T-coercivity is versatile!
  - BEM for the classical Maxwell problem (cf. [Buffa-Costabel-Schwab'02]) ;
  - FEM for the classical scalar or Maxwell problems (cf. [Jr'12]) ;
  - Vol. Int. Eq. Methods for scattering from gratings (cf. [Lechleiter-Nguyen'13]) ;
  - study of Interior Transmission Eigenvalue Problems:
    - scalar case (cf. [BonnetBenDhia-Chesnel-Haddar'11]) ;
    - Maxwell problem (cf. [Chesnel'12]) ;
  - etc.

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Conclusion/Perspectives

- T-coercivity is versatile!
- Scalar problems *with sign-shifting coefficients*:
  - introduction of T-coercivity during WAVES'07 ;
  - numerical analysis when T-coercivity applies (cf. [BonnetBenDhia-Jr-Zwölf'10], [Nicaise-Venel'11], [Chesnel-Jr'13], DG-approach [Chung-Jr'13], etc.) ;
  - theoretical study of well-posedness (cf. [BonnetBenDhia-Chesnel-Jr'12]) ;
  - theoretical study of the *critical* cases (cf. [BonnetBenDhia-Chesnel-Claeys'13]) ;
  - † discretization and numerical analysis of the *critical* cases.
- Maxwell problem(s) *with sign-shifting coefficients*:
  - T-coercivity + side results during NELIA'11 (cf. [BonnetBenDhia-Chesnel-Jr'1x]) ;
  - † numerical analysis when T-coercivity applies.
- In the *critical* cases: are models derived from physics still relevant?
  - † re-visit models (homogenization, multi-scale numerics, etc.).
  - † define *ad hoc* numerical methods.

  (A.N.R. METAMATH Project ; coordinator S. Fliss (POEMS)).

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