Use of explicit inf-sup operators to solve indefinite problems

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Outline

- Well-posedness with the help of explicit inf-sup operators: $\mathtt{T}$-*coercivity*.
- Numerical approximation and convergence via $\mathtt{T}$-coercivity.
- Helmholtz equation in acoustics.
- Time-harmonic problems in electromagnetics.
- Transmission problems with sign changing coefficients.
- Conclusion.

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Abstract setting

- Let
  - V and W be two Hilbert spaces ;
  - $a(\cdot, \cdot)$ be a continuous sesquilinear form over $V \times W$;
  - f be an element of W', the dual space of W.

Aim: solve the Variational Formulation

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- [Hadamard] The Problem (VF) is *well-posed* if, and only if, for all f, it has one and only one solution u, with continuous dependence:

$$\exists C > 0,\ \forall f \in W',\ \|u\|_V \leq C\,\|f\|_{W'}.$$

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- [Lax-Milgram] OK provided that $a(\cdot,\cdot)$ is coercive!

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Abstract setting-2

- [Banach-Necas-Babuska] Introduce the two conditions

$$(BNB_1) \qquad \exists \alpha' > 0,\ \forall v \in V,\ \sup_{w \in W \setminus \{0\}} \frac{|a(v,w)|}{\|w\|_W} \geq \alpha' \|v\|_V.$$

$$(BNB_2) \qquad \forall w \in W:\ \{\forall v \in V,\ a(v,w) = 0\} \implies \{w = 0\}.$$

NB. Condition (BNB_1) is called an *inf-sup condition*, or a *stability condition*.

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- `Theorem (Well-posedness)` The two assertions below are equivalent:
  - (i) the Problem (VF) is well-posed ;
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- `Definition (T-coercivity)` The form $a(\cdot,\cdot)$ is `T`-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V,W),\ \text{bijective},\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v,\mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2 .$$

NB. In other words, the form $(v,v') \mapsto a(v,\mathtt{T}v')$ is *coercive on* $V \times V$.

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The operator `T` realizes conditions (BNB_1) and (BNB_2) explicitly.

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- `Theorem (Well-posedness)` The three assertions below are equivalent:
  - (i) the Problem (VF) with hermitian form is well-posed ;
  - (ii) the hermitian form $a(\cdot,\cdot)$ satisfies condition (BNB_1).
  - (iii) the hermitian form $a(\cdot,\cdot)$ is `T`-coercive.

Numerical approximation

- Conforming discretization:
  - let $(V_h)_h$ be finite dimensional vector subspaces of V ($\lim_{h\to 0}$ dim$(V_h) = +\infty$) ;
  - let $(W_h)_h$ be finite dimensional vector subspaces of W ($\lim_{h\to 0}$ dim$(W_h) = +\infty$).

Aim: solve the Discrete Variational Formulation

$$(DVF) \quad \textit{Find } u_h \in V_h \textit{ s.t. } \forall w_h \in W_h,\ a(u_h, w_h) = \langle f, w_h \rangle.$$

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NB. For simplicity, the discrete forms are assumed to be exact.

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- [Babuska-Brezzi] Introduce the uniform discrete inf-sup condition

$$(UDISC) \qquad \exists \alpha_\dagger > 0,\ \forall h > 0,\ \forall v_h \in V_h,\ \sup_{w_h \in W_h \setminus \{0\}} \frac{|a(v_h, w_h)|}{\|w_h\|_W} \geq \alpha_\dagger \|v_h\|_V .$$

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- `Definition (`$\mathtt{T}_h$`-coercivity)` The form $a(\cdot,\cdot)$ is *uniformly* $\mathtt{T}_h$*-coercive* if

$$\exists \alpha^\star, \beta^\star > 0,\ \forall h > 0,\ \exists \mathtt{T}_h \in \mathcal{L}(V_h, W_h),\ \forall v_h \in V_h,$$
$$|a(v_h, \mathtt{T}_h v_h)| \geq \alpha^\star \|v_h\|_V^2 \text{ and } |||\mathtt{T}_h||| \leq \beta^\star .$$

Numerical approximation-2

- `Theorem (approximation error)` The three assertions below are equivalent:
  (i) Problems (DVF) are well-posed with uniform continuous dependence ;
  (ii) the form $a(\cdot,\cdot)$ satisfies the uniform discrete inf-sup condition $(UDISC)$;
  (iii) the form $a(\cdot,\cdot)$ is uniformly $\mathtt{T}_h$-coercive.

  If one of these conditions is satisfied, the error $\|u-u_h\|_V$ is bounded by

$$(Strang) \qquad \|u-u_h\|_V \leq C \inf_{v_h\in V_h} \|u-v_h\|_V \; ,$$

  with C independent of f and h.

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- `Proposition (`$\mathtt{T}_h$`-coercivity)`
  Assume
  - $\exists \mathtt{T} \in \mathcal{L}(V,W)$, bijective, such that $(v,v') \mapsto a(v,\mathtt{T}v')$ is coercive on $V\times V$;
  - $\exists (\mathtt{T}_h)_h$, $\mathtt{T}_h \in \mathcal{L}(V_h,W_h)$ s.t. $\lim_{h\to 0}\left(\sup_{v_h\in V_h\setminus\{0\}} \frac{||(\mathtt{T}_h-\mathtt{T})(v_h)||_W}{||v_h||_V}\right) = 0$.

  Then, the form $a(\cdot,\cdot)$ is uniformly $\mathtt{T}_h$-coercive for h small enough.

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- Similar approach, see [Buffa-Costabel-Schwab'02] for BEM.
- *Non-conforming* discretization, see [Chung-Jr'1x] for DG.

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Helmholtz equation in acoustics

- Consider a bounded domain Ω of $\mathbb{R}^d$, with $d = 1, 2, 3$.
  We study the classical problem

$$\left\{ \begin{array}{l} \textit{Find } u \in H^1(\Omega) \textit{ such that} \\ \mathrm{div}\ (\sigma \nabla u) + \omega^2 \eta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{array} \right.$$

  - Above, f is a source, $\omega > 0$ is the given pulsation.
  - $\sigma, \eta \in L^\infty(\Omega)$, and $\exists \sigma_-, \eta_- > 0$ such that $\sigma > \sigma_-$ and $\eta > \eta_-$ a.e. in Ω.

  NB. Other boundary conditions are possible...

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  - Above, $f \in H^{-1}(\Omega)$.

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- Within our framework:
  - $V = W = H_0^1(\Omega)$.
  - $a^{ac}(v, w) = \displaystyle\int_\Omega (\sigma \nabla v \cdot \nabla w - \omega^2 \eta v w) \, d\Omega$.

How can one achieve $\mathtt{T}$-coercivity of the form $a^{ac}(\cdot, \cdot)$?

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- Choose the norms:
  - $v \mapsto \|v\|_0 := \left(\displaystyle\int_\Omega \eta v^2 \, d\Omega \right)^{1/2}$ in $L^2(\Omega)$.
  - $v \mapsto \|v\|_1 := \left(\displaystyle\int_\Omega \eta v^2 \, d\Omega + \int_\Omega \sigma |\nabla v|^2 \, d\Omega \right)^{1/2}$ in $H^1(\Omega)$.

Helmholtz equation in acoustics-2

- `Spectral Theorem`: $\exists (v_\ell)_{\ell \geq 0}$, a Hilbert basis of $H_0^1(\Omega)$ made up of eigenfunctions

$$\left\{ \begin{array}{l} \textit{Find } (v_\ell, \lambda_\ell) \in H_0^1(\Omega) \times \mathbb{R} \textit{ such that } v_\ell \neq 0 \textit{ and} \\ \displaystyle\int_\Omega \sigma \nabla v_\ell \cdot \nabla w \, d\Omega = \lambda_\ell \int_\Omega \eta v_\ell w \, d\Omega, \ \forall w \in H_0^1(\Omega). \end{array} \right.$$

In addition

- $(v_\ell)_{\ell \geq 0}$ is also an orthogonal basis of $L^2(\Omega)$;
- all eigenvalues are of *finite multiplicity* ;
- $\lambda_0 > 0$, and $\lim_{\ell \to \infty} \lambda_\ell = +\infty$.

NB. The eigenpairs are ordered by increasing values of the eigenvalues.

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- Choice of $\mathtt{T}^{ac}$:

  Let ℓ_{max} denote the largest index $\ell \geq 0$ such that $\lambda_\ell < \omega^2$. Introduce:

  - $V^- :=$ span$_{0 \leq \ell \leq \ell_{max}}(v_\ell)$, a *finite dimensional* vector subspace of $H_0^1(\Omega)$;
  - the orthogonal projection operator P^- from $H_0^1(\Omega)$ to V^-.

  NB. When ω^2 is smaller than λ_0, $\ell_{max} = -1$, $V^- = \{0\}$ and $\mathsf{P}^- = 0$...

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  Define $\mathtt{T}^{ac} := \mathtt{I}_{H^1_0(\Omega)} - 2\mathsf{P}^-$:

$$\mathtt{T}^{ac} v_\ell := \left\{ \begin{array}{l} -v_\ell \text{ if } 0 \leq \ell \leq \ell_{max} \\ +v_\ell \text{ if } \ell > \ell_{max}. \end{array} \right.$$

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- `Proposition` $a^{ac} : (v, w) \mapsto \displaystyle\int_\Omega (\sigma \nabla v \cdot \nabla w - \omega^2 \eta v w) \, d\Omega$ is $\mathtt{T}$-*coercive*:

$$\forall v \in H^1_0(\Omega), \ |a^{ac}(v, \mathtt{T}^{ac} v)| \geq \underline{\alpha} \, \|v\|^2_V, \quad \text{with } \underline{\alpha} := \min_{\ell \geq 0} \left| \frac{\lambda_\ell - \omega^2}{1 + \lambda_\ell} \right|.$$

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Helmholtz equation in acoustics-3

- Conforming discretization: Lagrange finite elements $\Longrightarrow (V_h)_h$...
  The Discrete Variational Formulation writes:

$$\textit{Find } u_h \in V_h \textit{ s.t. } a^{ac}(u_h, v_h) = -\langle f, v_h \rangle, \ \forall v_h \in V_h.$$

How can one achieve the uniform $\mathtt{T}_h$-coercivity of the form $a^{ac}(\cdot,\cdot)$?

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How can one achieve the uniform $\mathtt{T}_h$-coercivity of the form $a^{ac}(\cdot,\cdot)$?

- Idea (simple!): Build a suitable approximation of V^- in V_h.
  Choose approximations $(v_{\ell,h})_{0\leq\ell\leq\ell_{max}}$ of the basis vectors $(v_\ell)_{0\leq\ell\leq\ell_{max}}$, and set

$$V_h^- := \text{span}_{0\leq\ell\leq\ell_{max}}(v_{\ell,h}).$$

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- Idea (simple!): Build a suitable approximation of V^- in V_h.
  Because V^- is finite dimensional, one can find, *for h small enough*, a sequence of orthonormal families $(v_{\ell,h})_{0\le\ell\le\ell_{max},h}$ and a *uniform bound* δ ($\lim_{h\to 0}\delta(h) = 0$) s.t.

$$\|v_\ell - v_{\ell,h}\|_1 \le \delta(h),\ 0 \le \ell \le \ell_{max},\ \textit{for } h \textit{ small enough}.$$

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  Because V^- is finite dimensional, one can find, *for h small enough*, a sequence of orthonormal families $(v_{\ell,h})_{0\leq\ell\leq\ell_{max},h}$ and a *uniform bound* δ ($\lim_{h\to 0}\delta(h) = 0$) s.t.

$$\|v_\ell - v_{\ell,h}\|_1 \leq \delta(h), \; 0 \leq \ell \leq \ell_{max}, \textit{ for } h \textit{ small enough.}$$

- Introduce:
  - the orthogonal projection operator P_h^- from V_h to $V_h^- = \mathrm{span}_{0\leq\ell\leq\ell_{max}}(v_{\ell,h})$;
  - the operator $\mathtt{T}_h^{ac} := \mathtt{I}_{V_h} - 2\mathrm{P}_h^-$ of $\mathcal{L}(V_h)$.

Helmholtz equation in acoustics-3

- Conforming discretization: Lagrange finite elements $\Longrightarrow (V_h)_h$...
  The Discrete Variational Formulation writes:

$$\textit{Find } u_h \in V_h \textit{ s.t. } a^{ac}(u_h, v_h) = -\langle f, v_h \rangle, \ \forall v_h \in V_h.$$

How can one achieve the uniform $\mathtt{T}_h$-coercivity of the form $a^{ac}(\cdot,\cdot)$?

- Idea (simple!): Build a suitable approximation of V^- in V_h.
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- Introduce:
  - the orthogonal projection operator $\mathtt{P}_h^-$ from V_h to $V_h^- = \mathrm{span}_{0\le\ell\le\ell_{max}}(v_{\ell,h})$;
  - the operator $\mathtt{T}_h^{ac} := \mathtt{I}_{V_h} - 2\mathtt{P}_h^-$ of $\mathcal{L}(V_h)$.
- `Proposition` There holds $\lim_{h\to 0}\left(\sup_{v_h\in V_h\setminus\{0\}} \frac{||(\mathtt{T}_h^{ac} - \mathtt{T}^{ac})(v_h)||_1}{||v_h||_1}\right) = 0$.
  Hence, the discrete solution u_h converges to u, with a rate governed by $(Strang)$.

ENSTA

Time-harmonic problem in EM-ics

- Consider a bounded domain Ω of $\mathbb{R}^3$.
  We study the classical problem

$$\left\{ \begin{array}{l} \textit{Find } \boldsymbol{e} \in \boldsymbol{H}(\mathbf{curl};\Omega) \textit{ such that} \\ -\omega^2 \varepsilon \boldsymbol{e} + \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) = \boldsymbol{f} \text{ in } \Omega \\ \boldsymbol{e} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega. \end{array} \right.$$

  - Above, $\boldsymbol{f}$ is a source, $\omega > 0$ is the given pulsation.
  - $\varepsilon, \mu \in L^\infty(\Omega)$, and $\exists \varepsilon_-, \mu_- > 0$ such that $\varepsilon > \varepsilon_-$ and $\mu > \mu_-$ a.e. in Ω.

  NB. Other boundary conditions are possible...

ENSTA

Time-harmonic problem in EM-ics

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$$\left\{ \begin{array}{l} \textit{Find } \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \textit{ such that} \\ \displaystyle\int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\Omega - \omega^2\int_\Omega \varepsilon\boldsymbol{e}\cdot\boldsymbol{v}\,d\Omega = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\Omega,\ \forall\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega). \end{array}\right.$$

  - Above, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$.

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- Within our framework:
  - $V = W = \boldsymbol{H}_0(\mathbf{curl};\Omega)$.
  - $a^{EM}(\boldsymbol{v},\boldsymbol{w}) = \displaystyle\int_\Omega (\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w} - \omega^2\varepsilon\boldsymbol{v}\cdot\boldsymbol{w})\,d\Omega$.

How can one achieve $\mathtt{T}$-coercivity of the form $a^{EM}(\cdot,\cdot)$?

ENSTA

Time-harmonic problem in EM-ics

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  We study the classical problem

$$\begin{cases} \textit{Find } \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \textit{ such that} \\ \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\Omega - \omega^2\int_\Omega \varepsilon\boldsymbol{e}\cdot\boldsymbol{v}\,d\Omega = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\Omega,\ \forall\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega). \end{cases}$$

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How can one achieve $\mathtt{T}$-coercivity of the form $a^{EM}(\cdot,\cdot)$?

- Choose the norms:
  - $\boldsymbol{v}\mapsto \|\boldsymbol{v}\|_0 := \left(\int_\Omega \varepsilon|\boldsymbol{v}|^2\,d\Omega\right)^{1/2}$ in $\boldsymbol{L}^2(\Omega)$.
  - $\boldsymbol{v}\mapsto \|\boldsymbol{v}\|_{\mathbf{curl}} := \left(\int_\Omega \varepsilon|\boldsymbol{v}|^2\,d\Omega + \int_\Omega \mu^{-1}|\,\mathbf{curl}\,\boldsymbol{v}|^2\,d\Omega\right)^{1/2}$ in $\boldsymbol{H}(\mathbf{curl};\Omega)$.

ENSTA

Time-harmonic problem in EM-ics-2

- DIFFICULTY: the embedding of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ into $\boldsymbol{L}^2(\Omega)$ is *not compact*! Hence, the `Spectral Theorem` can not be applied “as is”...

Time-harmonic problem in EM-ics-2

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- `Proposition` There holds the decomposition

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) \quad = \quad \boldsymbol{G} \overset{\perp_{\mathbf{curl}}}{\oplus} \boldsymbol{W}_\varepsilon$$

where $\boldsymbol{G} := \nabla H^1_0(\Omega),\ \boldsymbol{W}_\varepsilon := \{\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\ :\ \mathrm{div}\,(\varepsilon\boldsymbol{w}) = 0\}.$

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- Idea: one can try and build two Hilbert bases:
  - one for $\boldsymbol{G}$ (cf. acoustics section): $(\boldsymbol{e}_\ell)_{\ell<0}$, with $\boldsymbol{e}_\ell := \nabla v_{-(1+\ell)}$ for $\ell < 0$;
  - one for $\boldsymbol{W}_\varepsilon$.

ENSTA

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$(\boldsymbol{e}_\ell)_{\ell<0}$ Hilbert basis of $\boldsymbol{G}$, with $\boldsymbol{e}_\ell := \nabla v_{-(1+\ell)}$ for $\ell < 0$.

- `Theorem` [Weber'80] $\boldsymbol{W}_\varepsilon$ is compactly embedded into $\boldsymbol{L}^2(\Omega)$.
- DIFFICULTY: $\boldsymbol{W}_\varepsilon$ is *not dense* in $\boldsymbol{L}^2(\Omega)$.

ENSTA

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- `Theorem` [Weber'80] $\boldsymbol{W}_\varepsilon$ is compactly embedded into $\boldsymbol{L}^2(\Omega)$.
- DIFFICULTY: $\boldsymbol{W}_\varepsilon$ is *not dense* in $\boldsymbol{L}^2(\Omega)$.
- New *pivot space*: $\boldsymbol{H}(\mathrm{div}\,\varepsilon 0;\Omega) := \{\boldsymbol{w} \in \boldsymbol{H}(\mathrm{div}\,\varepsilon;\Omega)\ :\ \mathrm{div}\,(\varepsilon \boldsymbol{w}) = 0\}$.
  (+) $\boldsymbol{W}_\varepsilon$ is compactly embedded into $\boldsymbol{H}(\mathrm{div}\,\varepsilon 0;\Omega)$;
  (+) one can prove that $\boldsymbol{W}_\varepsilon$ is dense in $\boldsymbol{H}(\mathrm{div}\,\varepsilon 0;\Omega)$.

ENSTA

Time-harmonic problem in EM-ics-2

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- `Proposition` There holds the decomposition

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$(\boldsymbol{e}_\ell)_{\ell<0}$ Hilbert basis of $\boldsymbol{G}$, with $\boldsymbol{e}_\ell := \nabla v_{-(1+\ell)}$ for $\ell < 0$.

- `Spectral Theorem`: $\exists (\boldsymbol{e}_\ell)_{\ell\geq 0}$ a Hilbert basis of $\boldsymbol{W}_\varepsilon$ made up of eigenfunctions

$$\left\{ \begin{array}{l} \textit{Find } (\boldsymbol{e}_\ell, \mu_\ell) \in \boldsymbol{W}_\varepsilon \times \mathbb{R} \textit{ such that } \boldsymbol{e}_\ell \neq 0 \textit{ and} \\ \displaystyle\int_\Omega (\varepsilon \boldsymbol{e}_\ell \cdot \boldsymbol{w} + \mu^{-1}\, \mathbf{curl}\, \boldsymbol{e}_\ell \cdot \mathbf{curl}\, \boldsymbol{w})\, d\Omega = (1+\mu_\ell) \int_\Omega \varepsilon \boldsymbol{e}_\ell \cdot \boldsymbol{w}\, d\Omega,\ \forall \boldsymbol{w} \in \boldsymbol{W}_\varepsilon. \end{array} \right.$$

  - all eigenvalues are of *finite multiplicity*;
  - $\mu_\ell = 0$ occurs K times, with $K+1$ number of c.c. of $\partial\Omega$, and $\lim_{\ell\to\infty} \mu_\ell = +\infty$.

  NB. The eigenpairs are ordered by increasing values of the eigenvalues.

ENSTA

Time-harmonic problem in EM-ics-3

- Conclusion: $(\boldsymbol{e}_\ell)_\ell$ is a Hilbert basis of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ such that

$$\forall \ell,\ \exists \mu_\ell \geq 0,\ (\boldsymbol{e}_\ell, \boldsymbol{w})_{\mathbf{curl}} = (1+\mu_\ell)\int_\Omega \varepsilon \boldsymbol{e}_\ell \cdot \boldsymbol{w}\, d\Omega,\ \forall \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega).$$

  - For $\ell < 0$: $\boldsymbol{e}_\ell \in \boldsymbol{G}$ and $\mu_\ell = 0$;
  - For $\ell \geq 0$: $\boldsymbol{e}_\ell \in \boldsymbol{W}_\varepsilon$ and μ_ℓ are eigenpairs, and
    - all eigenvalues are of *finite multiplicity*;
    - $\mu_\ell = 0$ occurs K times, and $\lim_{\ell\to\infty} \mu_\ell = +\infty$.

Time-harmonic problem in EM-ics-3

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NB. Given any $\omega > 0$, there is an infinite number of ℓ s.t. $\mu_\ell < \omega^2$.

ENSTA

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- Choice of $\mathtt{T}^{EM}$:

  Let ℓ_{max} denote the largest index ℓ such that $\mu_\ell < \omega^2$. Introduce:
  - $\boldsymbol{V}^- :=$ span$_{0\leq\ell\leq\ell_{max}}(\boldsymbol{e}_\ell)$, a *finite dimensional* vector subspace of $\boldsymbol{W}_\varepsilon$;
  - the orthogonal projection operator P^- from $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ to $\boldsymbol{V}^-$.

Time-harmonic problem in EM-ics-3

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Define $\mathtt{T}^{EM} := -\mathtt{i}_G + \mathtt{i}_{\boldsymbol{W}_\varepsilon} - 2\mathsf{P}^-$:

$$\mathtt{T}^{EM}\boldsymbol{e}_\ell := \begin{cases} -\boldsymbol{e}_\ell \text{ if } \ell \leq \ell_{max} \\ +\boldsymbol{e}_\ell \text{ if } \ell > \ell_{max}. \end{cases}$$

Time-harmonic problem in EM-ics-3

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Define $\mathtt{T}^{EM} := -\mathtt{i}_{\boldsymbol{G}} + \mathtt{i}_{\boldsymbol{W}_\varepsilon} - 2\mathsf{P}^-$.

- Proposition $a^{EM} : (\boldsymbol{v}, \boldsymbol{w}) \mapsto \int_\Omega (\mu^{-1}\, \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{w} - \omega^2 \varepsilon \boldsymbol{v} \cdot \boldsymbol{w})\, d\Omega$ is $\mathtt{T}$-*coercive*.

ENSTA

Time-harmonic problem in EM-ics-4

- Conforming discretization: Nédélec's first family finite elements $\Longrightarrow (\boldsymbol{V}_h)_h$...
  The Discrete Variational Formulation writes:

$$\textit{Find } \boldsymbol{e}_h \in \boldsymbol{V}_h \textit{ s.t. } a^{EM}(\boldsymbol{e}_h, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, d\Omega, \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

How can one achieve the uniform $\mathtt{T}_h$-coercivity of the form $a^{EM}(\cdot, \cdot)$?

Time-harmonic problem in EM-ics-4

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How can one achieve the uniform $\mathtt{T}_h$-coercivity of the form $a^{EM}(\cdot,\cdot)$?

DIFFICULTY: Given any $\omega > 0$, there is an *infinite number* of ℓ s.t. $\mu_\ell < \omega^2$.

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  DIFFICULTY: Given any $\omega > 0$, there is an *infinite number* of ℓ s.t. $\mu_\ell < \omega^2$.

- Idea:
  - split elements of $\boldsymbol{V}_h$ ($\approx$ *exact decomposition* $\boldsymbol{H}_0(\mathbf{curl};\Omega) = \boldsymbol{G} \oplus \boldsymbol{W}_\varepsilon$);
  - take the opposite of the gradient part ;
  - use the orthogonal projection on the other part (cf. acoustics section).

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  - take the opposite of the gradient part ;
  - use the orthogonal projection on the other part (cf. acoustics section).

DIFFICULTY: The discrete splitting needs to be *uniformly close* to the exact splitting.

Time-harmonic problem in EM-ics-5

- Given $\boldsymbol{v}_h \in \boldsymbol{V}_h$:
  - the exact splitting is $\exists!(\varphi, \boldsymbol{w}) \in H_0^1(\Omega) \times \boldsymbol{W}_\varepsilon,\ \boldsymbol{v}_h = \nabla\varphi + \boldsymbol{w}$.
  - a discrete splitting is $(\varphi_h, \boldsymbol{w}_h) \in V_h \times \boldsymbol{V}_h,\ \boldsymbol{v}_h = \nabla\varphi_h + \boldsymbol{w}_h$.

NB. Provided the orders of FE are appropriately chosen, there holds $\nabla V_h \subset \boldsymbol{V}_h$.

ENSTA

Time-harmonic problem in EM-ics-5

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  - a discrete splitting is $(\varphi_h, \boldsymbol{w}_h) \in V_h \times \boldsymbol{V}_h,\ \boldsymbol{v}_h = \nabla\varphi_h + \boldsymbol{w}_h$.

  NB. Provided the orders of FE are appropriately chosen, there holds $\nabla V_h \subset \boldsymbol{V}_h$.
- `Proposition (Uniform discrete splittings)`
  Assume that ε is *piecewise-constant*: there exists a discrete splitting such that

$$\|\nabla(\varphi - \varphi_h)\|_{\mathbf{curl}} = \|\boldsymbol{w} - \boldsymbol{w}_h\|_{\mathbf{curl}} \le C_r\, h^{\mathsf{s}}\, \|\boldsymbol{v}_h\|_{\mathbf{curl}},$$

  with $\mathsf{s} := \mathsf{s}(\Omega, \varepsilon) > 0$, $C_r > 0$ independent of $\boldsymbol{v}_h$.

ENSTA

Time-harmonic problem in EM-ics-5

- Given $\boldsymbol{v}_h \in \boldsymbol{V}_h$:
  - the exact splitting is $\exists!(\varphi, \boldsymbol{w}) \in H^1_0(\Omega) \times \boldsymbol{W}_\varepsilon,\ \boldsymbol{v}_h = \nabla\varphi + \boldsymbol{w}$.
  - a discrete splitting is $(\varphi_h, \boldsymbol{w}_h) \in V_h \times \boldsymbol{V}_h,\ \boldsymbol{v}_h = \nabla\varphi_h + \boldsymbol{w}_h$.

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  `Proof (main ingredients!)`
  - regular-singular splitting of elements of $\boldsymbol{W}_\varepsilon$, cf. [Costabel-Dauge-Nicaise'99];
  - edge element approximability of piecewise-smooth fields, cf. [Monk'03];
  - edge element interpolation of gradients, cf. [Nédélec'80].

ENSTA

Time-harmonic problem in EM-ics-5

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  with $\mathsf{s} := \mathsf{s}(\Omega, \varepsilon) > 0$, $C_r > 0$ independent of $\boldsymbol{v}_h$.
- Approximate $\boldsymbol{V}^-$ in $\boldsymbol{V}_h$, cf. acoustics section: $\boldsymbol{V}_h^- :=$ span$_{0\leq\ell\leq\ell_{max}}(\boldsymbol{e}_{\ell,h})$, with

$$\|\boldsymbol{e}_\ell - \boldsymbol{e}_{\ell,h}\|_{\mathbf{curl}} \leq \delta(h),\ 0 \leq \ell \leq \ell_{max},\ \textit{for } h \textit{ small enough } (\lim_{h\to 0} \delta(h) = 0).$$

Time-harmonic problem in EM-ics-5

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- Introduce:
  - the orthogonal projection operator P_h^- from $\boldsymbol{V}_h$ to $\boldsymbol{V}_h^-$;
  - the operator $\mathtt{T}_h^{EM}$ of $\mathcal{L}(\boldsymbol{V}_h)$ defined by $\mathtt{T}_h^{EM}(\boldsymbol{v}_h) := -\nabla\varphi_h + (\mathtt{I}_{\boldsymbol{V}_h} - 2\mathsf{P}_h^-)(\boldsymbol{w}_h)$.

Time-harmonic problem in EM-ics-5

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  - the exact splitting is $\exists!(\varphi, \boldsymbol{w}) \in H_0^1(\Omega) \times \boldsymbol{W}_\varepsilon,\ \boldsymbol{v}_h = \nabla\varphi + \boldsymbol{w}$.
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  NB. Provided the orders of FE are appropriately chosen, there holds $\nabla V_h \subset \boldsymbol{V}_h$.
- `Proposition (Uniform discrete splittings)`
  Assume that ε is *piecewise-constant*: there exists a discrete splitting such that

$$\|\nabla(\varphi - \varphi_h)\|_{\mathbf{curl}} = \|\boldsymbol{w} - \boldsymbol{w}_h\|_{\mathbf{curl}} \leq C_r\, h^{\mathsf{s}}\, \|\boldsymbol{v}_h\|_{\mathbf{curl}},$$

  with $\mathsf{s} := \mathsf{s}(\Omega, \varepsilon) > 0$, $C_r > 0$ independent of $\boldsymbol{v}_h$.
- Introduce:
  - the orthogonal projection operator P_h^- from $\boldsymbol{V}_h$ to $\boldsymbol{V}_h^-$;
  - the operator $\mathtt{T}_h^{EM}$ of $\mathcal{L}(\boldsymbol{V}_h)$ defined by $\mathtt{T}_h^{EM}(\boldsymbol{v}_h) := -\nabla\varphi_h + (\mathtt{I}_{\boldsymbol{V}_h} - 2\mathsf{P}_h^-)(\boldsymbol{w}_h)$.
- `Proposition` There holds $\lim_{h\to 0}\left(\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h\setminus\{0\}} \frac{||(\mathtt{T}_h^{EM} - \mathtt{T}^{EM})(\boldsymbol{v}_h)||_{\mathbf{curl}}}{||\boldsymbol{v}_h||_{\mathbf{curl}}}\right) = 0$.
  Hence, the discrete solution $\boldsymbol{e}_h$ converges to $\boldsymbol{e}$, with a rate governed by $(Strang)$.

ENSTA

Sign-changing coefficients

- Consider a scalar *transmission* problem, set in a bounded domain Ω of $\mathbb{R}^d$, $d = 1, 2, 3$.

$$\left\{ \begin{array}{l} \textit{Find } u \in H^1_0(\Omega) \textit{ such that} \\ \mathrm{div}\ (\sigma \nabla u) = f \text{ in } \Omega. \end{array} \right.$$

  - $\sigma \in L^\infty(\Omega)$ is a sign-changing coefficient: $\left\{ \begin{array}{l} \sigma > 0 \text{ in } \Omega_1, \text{ with } \mathrm{meas}(\Omega_1) > 0\,; \\ \sigma < 0 \text{ in } \Omega_2, \text{ with } \mathrm{meas}(\Omega_2) > 0. \end{array} \right.$
  - $\sigma^{-1} \in L^\infty(\Omega)$.

ENSTA

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  - $\sigma \in L^\infty(\Omega)$, is a sign-changing coefficient.
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NB. The “generalized” Helmholtz equation $\mathrm{div}\ (\sigma \nabla u) + \omega^2 \eta u = f$ with $\eta \in L^\infty(\Omega)$ can be analyzed similarly, cf. [BonnetBenDhia-Jr-Zwölf’10].

When $\sigma < 0$, this models a *metamaterial*.

One can also consider a Neumann b.c., cf. [BonnetBenDhia-Chesnel-Jr’12].

ENSTA

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- The main dificulty is that $(v, w) \mapsto \int_\Omega \sigma \nabla v \cdot \overline{\nabla w}\, d\Omega$ is *not coercive* in $H^1_0(\Omega)$.

ENSTA

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  Structure of spectrum? Use of the `Spectral Theorem`?

$\Longrightarrow$ New approach to achieve `T`-coercivity!

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  Structure of spectrum? Use of the `Spectral Theorem`?

  $\Longrightarrow$ New approach to achieve `T`-coercivity!
- We follow [BonnetBenDhia-Jr-Zwölf'10]:
  - Ω_1 and Ω_2 are domains of $\mathbb{R}^d$;
  - $\Sigma := \overline{\Omega_1} \cap \overline{\Omega_2}$ is the interface ;
  - $\Gamma_k := \partial\Omega \cap \partial\Omega_k$, $k = 1, 2$ are the boundaries.

ENSTA

Sign-changing coefficients-2

- For the transmission problem with sign-changing coefficient:
  - $V = H_0^1(\Omega)$;
  - the sesquilinear form is $a^{tr}(v,w) = \int_\Omega \sigma \nabla v \cdot \overline{\nabla w} \, d\Omega$.

NB. Complex-valued forms, to enable the introduction of *dissipation*...

ENSTA

Sign-changing coefficients-2

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  - $V = H^1_0(\Omega)$;
  - the sesquilinear form is $a^{tr}(v,w) = \int_\Omega \sigma \nabla v \cdot \overline{\nabla w} \, d\Omega$.
- Introduce $V_k := \{v_k \in H^1(\Omega_k) \,|\, v_{k\,|\Gamma_k} = 0\}$, $k = 1, 2$:

$$\boxed{V = \{v \,|\, v_{|\Omega_k} \in V_k,\ k = 1, 2,\ \text{Matching}_\Sigma(v_{|\Omega_1}, v_{|\Omega_2}) = 0\}},$$

with $\text{Matching}_\Sigma(v_1, v_2) := v_{1\,|\Sigma} - v_{2\,|\Sigma}$.

Sign-changing coefficients-2

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- Introduce $a_k^{tr}(v_k, w_k) := \displaystyle\int_{\Omega_k} \sigma \nabla v_k \cdot \overline{\nabla w_k}\, d\Omega$, $k = 1, 2$:

$$\boxed{\forall v, w \in V,\ a^{tr}(v,w) = a_1^{tr}(v_{|\Omega_1}, w_{|\Omega_1}) + a_2^{tr}(v_{|\Omega_2}, w_{|\Omega_2})}\ ;$$

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$$\forall v_1 \in V_1, \sigma_1^- \|\nabla v_1\|^2_{\boldsymbol{L}^2(\Omega_1)} \leq +a_1^{tr}(v_1, v_1) \leq \sigma_1^+ \|\nabla v_1\|^2_{\boldsymbol{L}^2(\Omega_1)};$$

$$\forall v_2 \in V_2, \sigma_2^- \|\nabla v_2\|^2_{\boldsymbol{L}^2(\Omega_2)} \leq -a_2^{tr}(v_2, v_2) \leq \sigma_2^+ \|\nabla v_2\|^2_{\boldsymbol{L}^2(\Omega_2)}.$$

NB. We have $0 < \sigma_k^- \leq \sigma_k^+ < \infty$, $k = 1, 2$.

ENSTA

Sign-changing coefficients-3

- First try:

$$\forall v \in H_0^1(\Omega), \quad \mathtt{T}_- \, v := \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{cases}.$$

NB. Given $v \in H_0^1(\Omega)$, we set $v_k := v_{|\Omega_k}$, $k = 1, 2$.

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$$\forall v \in H_0^1(\Omega), \quad \mathtt{T} \, v := \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 + 2R_1 \, v_1 & \text{in } \Omega_2 \end{cases}.$$

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Can one achieve $\mathtt{T}$-coercivity?

Sign-changing coefficients-4

- Some elementary computations:

$$
\begin{aligned}
|a^{tr}(v,\mathtt{T}v)| &= |a_1^{tr}(v_1,v_1) - a_2^{tr}(v_2,v_2) + 2a_2^{tr}(v_2,R_1\,v_1)| \\
&\geq |a_1^{tr}(v_1,v_1) - a_2^{tr}(v_2,v_2)| - 2|a_2^{tr}(v_2,R_1\,v_1)| \\
&\geq \sigma_1^{-}\,\|v_1\|_{V_1}^2 - a_2^{tr}(v_2,v_2) - 2|a_2^{tr}(v_2,R_1\,v_1)|
\end{aligned}
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ENSTA

Sign-changing coefficients-4

- Some elementary computations: let $\delta > 0$, apply Young's inequality

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&\geq |a_1^{tr}(v_1,v_1) - a_2^{tr}(v_2,v_2)| - 2|a_2^{tr}(v_2,R_1\,v_1)| \\
&\geq \sigma_1^-\,\|v_1\|_{V_1}^2 - a_2^{tr}(v_2,v_2) - 2|a_2^{tr}(v_2,R_1\,v_1)| \\
&\geq \sigma_1^-\,\|v_1\|_{V_1}^2 - a_2^{tr}(v_2,v_2) + \delta a_2^{tr}(v_2,v_2) + \delta^{-1}a_2^{tr}(R_1\,v_1,R_1\,v_1) \\
&\geq (\sigma_1^- - \delta^{-1}\sigma_2^+\,|||R_1|||^2)\|v_1\|_{V_1}^2 - (1-\delta)a_2^{tr}(v_2,v_2).
\end{aligned}
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\end{aligned}$$

- Hence, to obtain $|a^{tr}(v, \mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2$ with $\underline{\alpha} > 0$, it is *sufficient* that

$$\frac{\sigma_1^-}{\sigma_2^+} > |||R_1|||^2.$$

Sign-changing coefficients-5

- Third try: let $R_2 \in \mathcal{L}(V_2, V_1)$ s.t. for all $v_2 \in V_2$, Matching$_\Sigma(R_2 v_2, v_2) = 0$.

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- Conclusion: to achieve T-coercivity, one needs

$$\frac{\sigma_1^-}{\sigma_2^+} > \left(\inf_{R_1} |||R_1|||\right)^2 \quad \text{or} \quad \frac{\sigma_2^-}{\sigma_1^+} > \left(\inf_{R_2} |||R_2|||\right)^2.$$

ENSTA

Sign-changing coefficients-6

- How to choose the operators R_1 or R_2?
  - using traces on Σ, liftings, cf. [BonnetBenDhia-Jr-Zwölf'10], [Nicaise-Venel'11] ;
  - using geometrical transformations, cf. [BonnetBenDhia-Chesnel-Jr'12].

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- Numerical studies in $(V_h)_h$:
  - in general, one cannot build discrete operators $(\mathtt{T}_h)_h$ s.t.
    $\lim_{h\to 0}\left(\sup_{v_h\in V_h\setminus\{0\}}\frac{||(\mathtt{T}_h-\mathtt{T})(v_h)||_V}{||v_h||_V}\right)=0$;
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- *Safety net*: choose σ s.t. σ_1^-/σ_2^+ or σ_2^-/σ_1^+ are sufficiently large to ensure

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- Under this last assumption, convergence follows.

NB. One can also add dissipation, cf. [Chesnel-Jr'1x]:

(+) convergence follows without safety net ;

(–) convergence rate is reduced.

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Sign-changing coefficients-6

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- Under this last assumption, convergence follows.
- Numerical results:
  - conforming discretization, cf. [Chesnel-Jr'1x].
  - non-conforming discretization, cf. [Chung-Jr'1x] ;

ENSTA

Sign-changing coefficients-7

- In a symmetric domain. Here, $\Omega =]-1,1[\times]0,1[$, Ω_1 and Ω_2 are unit squares.
- $\sigma_k := \sigma_{|\Omega_k}$, $k=1,2$, are constant numbers, and $\sigma_2/\sigma_1 = -1.001$; $\omega = 0$.
- An *exact* piecewise smooth solution of the transmission problem is available.
- *Conforming discretization* using P_1 Lagrange FE.
- We study below the influence of the meshes (errors in L^2-norm ; $O(h^2)$ is expected).

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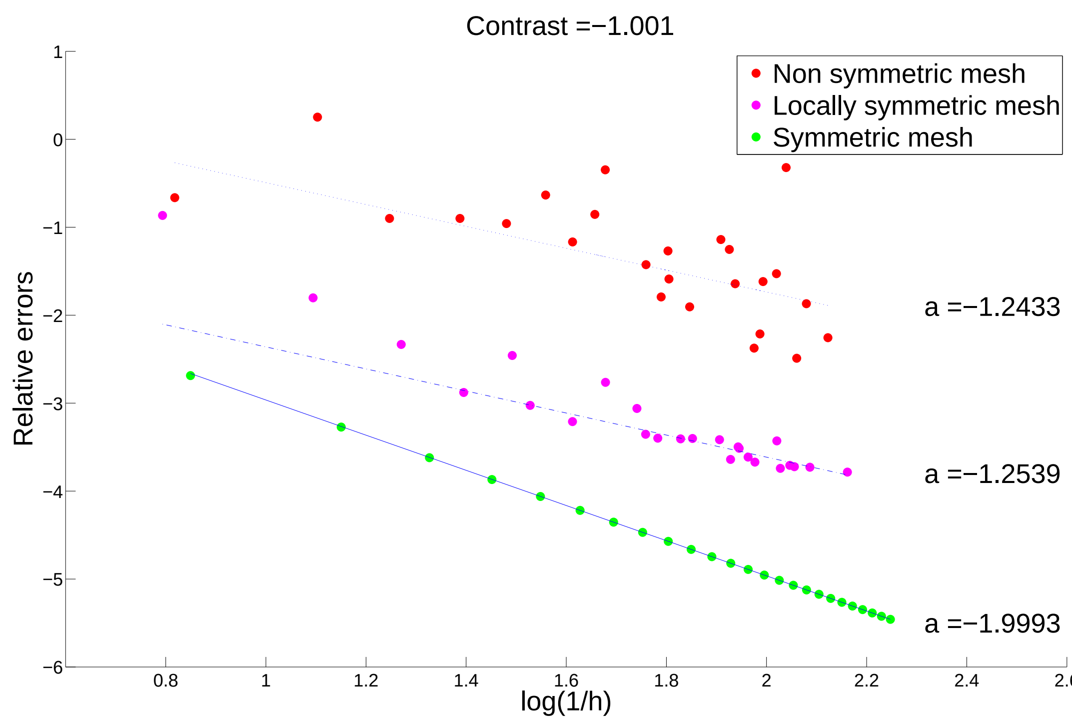
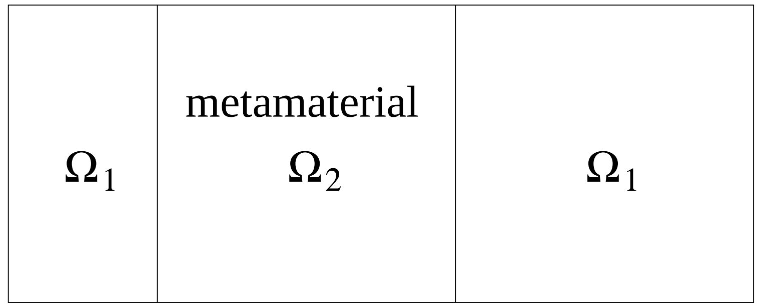


Sign-changing coefficients-8

- In a rectangle. Here, $\Omega =]0,5[\times]0,2[$, $\Omega_2 =]1,3[\times]0,2[$, and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$.



- $(\sigma_k)_{k=1,2}$ are constant numbers, and $\sigma_2/\sigma_1 = -1/3$; $\omega = 1.6$ and $\eta = \sigma^{-1}$.
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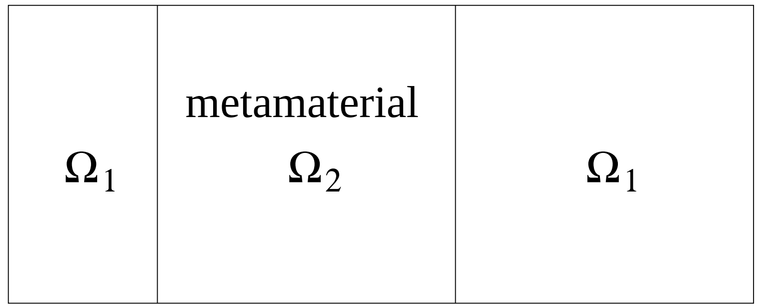
Sign-changing coefficients-8

- In a rectangle.



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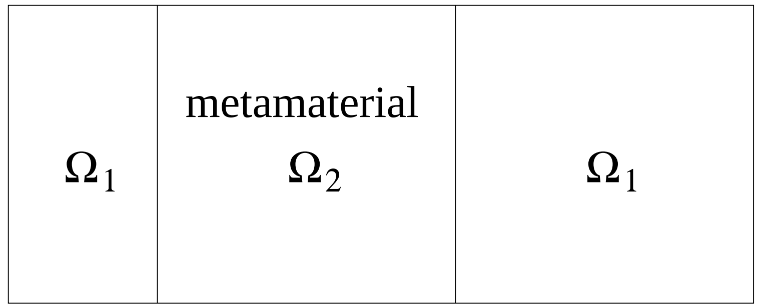

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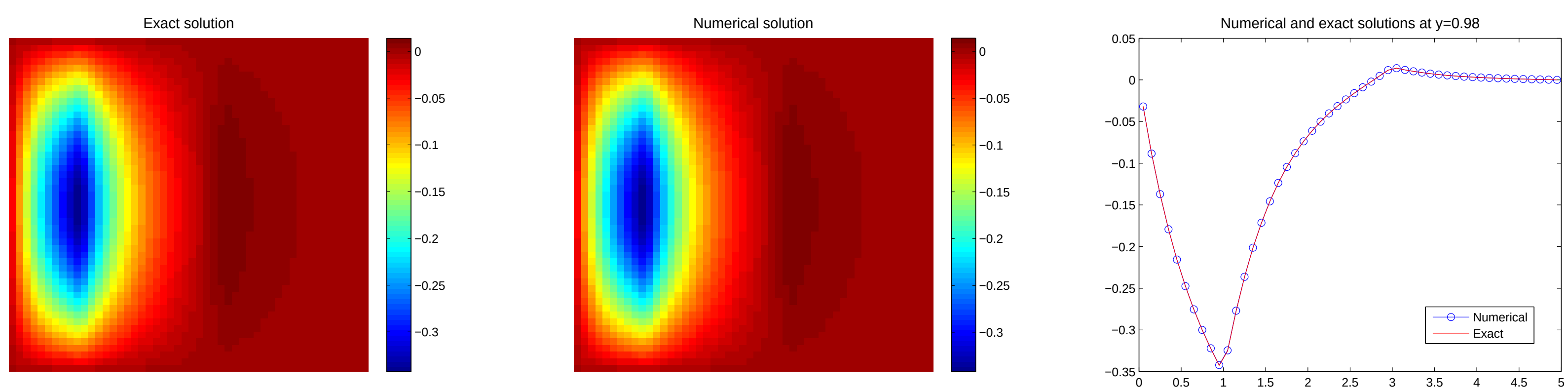

Conclusion/Perspectives

- $\mathtt{T}$-coercivity is versatile!
  - BEM for the classical Maxwell problem (cf. [Buffa-Costabel-Schwab'02]) ;
  - FEM for the classical scalar or Maxwell problems (cf. [Jr'12]) ;
  - Vol. Int. Eq. Methods for scattering from gratings (cf. [Lechleiter-Nguyen'1x]) ;
  - study of Interior Transmission Eigenvalue Problems:
    - scalar case (cf. [BonnetBenDhia-Chesnel-Haddar'11]) ;
    - Maxwell problem (cf. [Chesnel'1x]) ;
  - etc.

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Conclusion/Perspectives

- T-coercivity is versatile!

- Scalar problems *with sign-shifting coefficients*:
  - introduction of T-coercivity during WAVES'07 (cf. [BonnetBenDhia-Jr-Zwölf'10]) ;
  - numerical analysis when T-coercivity applies (cf. [BonnetBenDhia-Jr-Zwölf'10], [Nicaise-Venel'11], [Chesnel-Jr'1x], DG-approach [Chung-Jr'1x], etc.) ;
  - theoretical study of well-posedness (cf. [BonnetBenDhia-Chesnel-Jr'12]) ;
  - theoretical study of the *critical* cases (with [BonnetBenDhia-Chesnel-Claeys'1x]) ;
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- Maxwell problem(s) *with sign-shifting coefficients*:
  - $\mathbb{T}$-coercivity + side results during NELIA'11 (cf. [BonnetBenDhia-Chesnel-Jr'??]) ;
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- In the *critical* cases: are models derived from physics still relevant?
  - re-visit models (homogenization, multi-scale numerics, etc.).

    (A.N.R. METAMATH Project ; coordinator S. Fliss (POEMS)).

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