Electromagnetic wave propagation at classical material/meta-material interfaces

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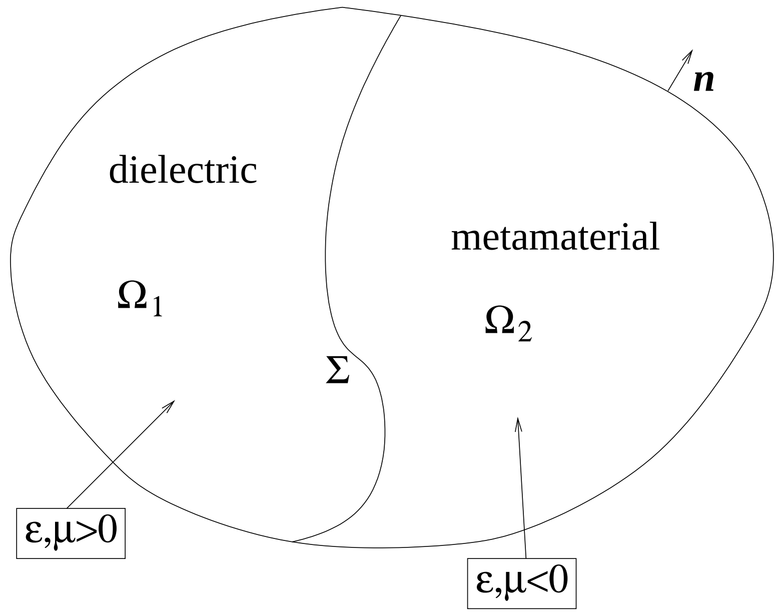
Motivation

- *Goal*: Solve numerically a time-harmonic problem in electromagnetism, set in a heterogeneous medium like below.

The domain Ω:



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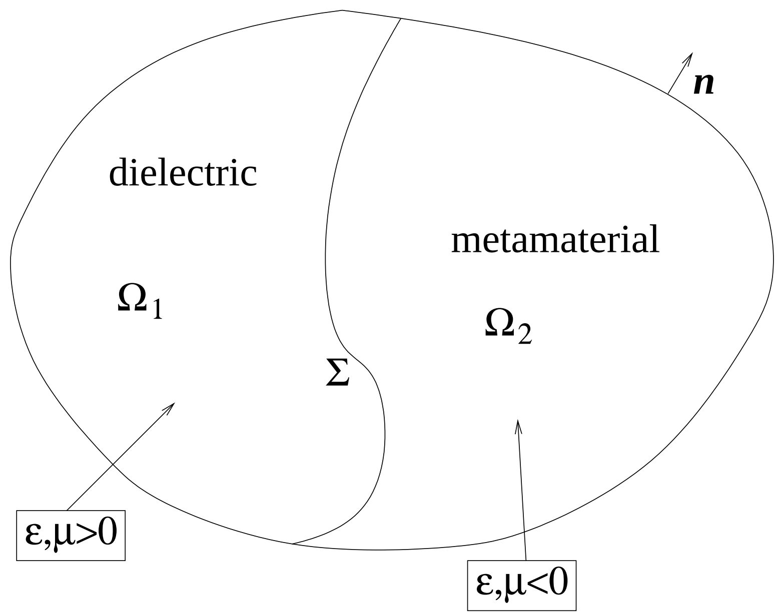

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- *Possible practical applications*: perfect lens, invisibility cloaking, etc.

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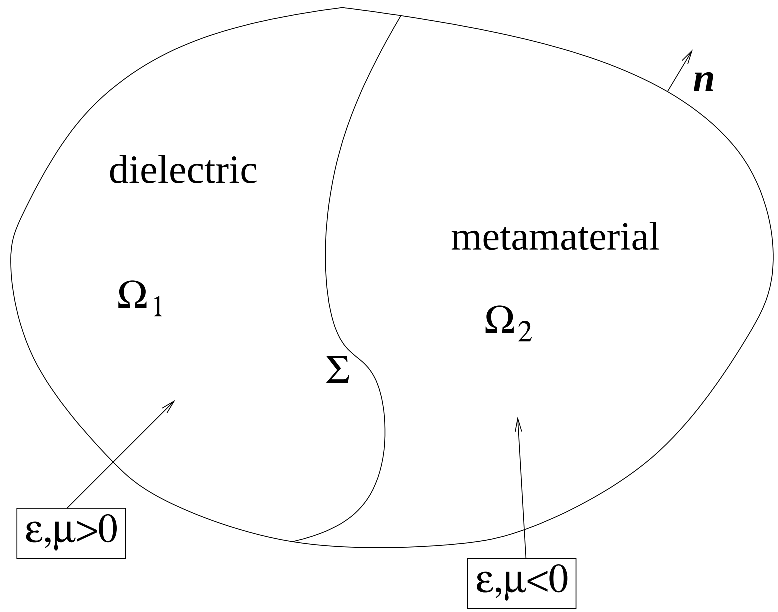

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- *Questions*:
  - Is the problem to be solved *well-posed*?
  - How to *compute* a numerical approximation of the solution?

Maxwell problem (electric field)

- Given $\omega > 0$ and source term $\mathcal{F} \in L^2(\Omega)^3$ ($\mathcal{F} := \imath\omega\mathcal{J}$, $\operatorname{div} \mathcal{F} = 0$).
  Find $\mathcal{E} \in L^2(\Omega)^3$ with $\mathbf{curl}\, \mathcal{E} \in L^2(\Omega)^3$ such that

$$\begin{cases} \mathbf{curl} \left(\dfrac{1}{\mu} \mathbf{curl}\, \mathcal{E} \right) - \omega^2 \varepsilon \mathcal{E} = \mathcal{F} & \text{in } \Omega \,; \\ \mathcal{E} \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

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- *When* $\varepsilon, \mu > 0$ ($\varepsilon, \mu, \varepsilon^{-1}, \mu^{-1} \in L^\infty(\Omega)$):
  which *functional space* to measure the electric field?
  which associated *discretization*?

◇ $\mathcal{H}_0(\mathbf{curl}\,;\Omega) := \{\mathcal{F} \in L^2(\Omega)^3 \,|\, \mathbf{curl}\,\mathcal{F} \in L^2(\Omega)^3,\ \mathcal{F} \times \mathbf{n}_{|\partial\Omega} = 0\}$ (Edge FE) .

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  ◇ $L^2(\Omega)^3$ (Discontinuous Galerkin FE) .
  Etc.

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  Our choice: $\boxed{\mathcal{X}_0(\varepsilon;\Omega) := \{\mathcal{F} \in \mathcal{H}_0(\mathbf{curl}\,;\Omega) \mid \operatorname{div} \varepsilon\mathcal{F} \in L^2(\Omega)\}}$ (CG FE) .
  (*Assumption*: no singular electric fields).

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- *Equivalent (Augmented) Variational Formulation*:
  Find $\mathcal{E} \in \mathcal{X}_0(\varepsilon;\Omega)$ such that

$$\int_\Omega \left(\frac{1}{\mu}\mathbf{curl}\,\mathcal{E}\cdot\mathbf{curl}\,\bar{\mathcal{E}}' + s\operatorname{div}\varepsilon\mathcal{E}\operatorname{div}\varepsilon\bar{\mathcal{E}}'\right)d\Omega - \omega^2\int_\Omega \varepsilon\mathcal{E}\cdot\bar{\mathcal{E}}'\,d\Omega = \int_\Omega \mathcal{F}\cdot\bar{\mathcal{E}}'\,d\Omega,\ \forall\mathcal{E}' \in \mathcal{X}_0(\varepsilon;\Omega)\,.$$

Well-posedness, when $\varepsilon, \mu > 0$...

Well-posedness stems from the two properties:

(1) coerciveness over $\mathcal{X}_0(\varepsilon; \Omega)$ of

$$a(\mathcal{E}, \mathcal{E}') := \int_\Omega \left(\frac{1}{\mu} \mathbf{curl}\, \mathcal{E} \cdot \mathbf{curl}\, \bar{\mathcal{E}}' + s \operatorname{div} \varepsilon \mathcal{E} \operatorname{div} \varepsilon \bar{\mathcal{E}}' \right) d\Omega \, .$$

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- ◇ These two ingredients fundamentally rely on: $\varepsilon > \varepsilon_\star > 0$ and $\mu > \mu_\star > 0$ a.e. in Ω.
- ◇ *Numerical convergence* then follows, for sufficiently small meshsize h...

Study of a scalar model problem

- *Assume* that the problem is independent of z.
  The third component $e := \mathcal{E}_z(x,y)$ is governed by
  find $e \in H^1(\Omega)$ such that

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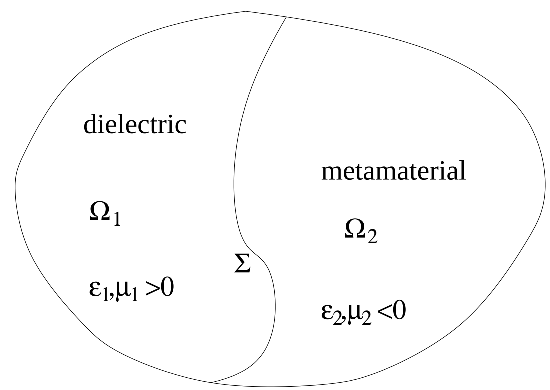
To fix ideas: ε and μ constant over Ω_i, $i = 1, 2$
($\varepsilon_i := \varepsilon_{|\Omega_i}$, $\mu_i := \mu_{|\Omega_i}$)



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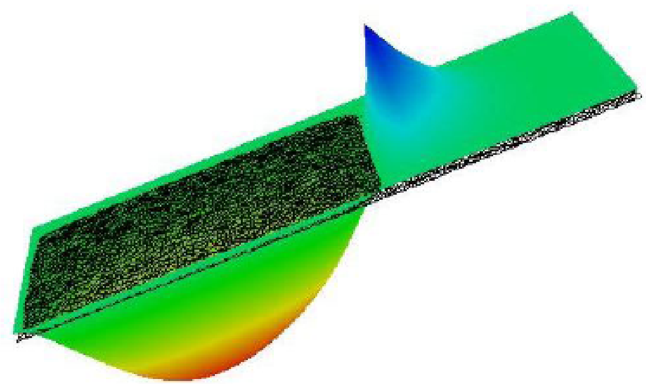
Jump of the trace of the normal derivative across the interface (with $\kappa_\mu < 0$)



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- Define the (negative) contrasts: $\kappa_\varepsilon := \dfrac{\varepsilon_1}{\varepsilon_2}$, $\kappa_\mu := \dfrac{\mu_1}{\mu_2}$.
- *State of the art*: [Costabel-Stephan'85], [Bonnet-Dauge-Ramdani'99], [Ramdani'99].
  - If $\kappa_\mu = -1$, the problem is always ill-posed.
  - If the interface Σ is smooth, then the problem is well-posed (except for resonance frequencies) as soon as $\kappa_\mu \neq -1$.
  - If Σ is piecewise smooth (ie. in the presence of corners), then the problem is well-posed (except for resonance frequencies) as soon as

$$\kappa_\mu \notin]\kappa_\mu^{\inf}, \kappa_\mu^{\sup}[\,, \text{ with } -1 \in]\kappa_\mu^{\inf}, \kappa_\mu^{\sup}[\,.$$

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Discretization (1): the two-field formulation

- *Introduce* the new – magnetic-like – "unknown" $\underline{\mathbf{h}}_2 := \left(\frac{1}{|\mu_2|} \mathbf{curl}\, e \right)_{|\Omega_2}$.

  Define a new formulation, with unknowns e over Ω and $\underline{\mathbf{h}}_2$ over Ω_2:

$$\left\{ \begin{array}{l} e \in H_0^1(\Omega) \\ \underline{\mathbf{h}}_2 \in \{\mathbf{p} \in H(\mathrm{curl}; \Omega_2) \,|\, \mathrm{div}\, |\mu_2| \mathbf{p} \in L^2(\Omega_2),\ |\mu_2| \mathbf{p} \cdot \mathbf{n}_{|\partial\Omega_2 \setminus \Sigma} = 0\} \end{array} \right. .$$

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- [Bonnet-Jr-Zwölf'07].

  Well-posedness can be recovered, provided $|\kappa_\mu|$ is "large enough".

  (The new formulation fits into the *coercive+compact framework*).

  Numerical convergence then follows.

  Added cost (related to $\underline{\mathbf{h}}_2^h$) reasonable if Ω_2 is "small" wrt Ω_1.

  Numerical experiments can be found in [Zwölf'07].

Discretization (2): the natural formulation

- *Discretize directly* the "standard" variational formulation.
  Find $e \in H^1_0(\Omega)$ such that

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  Everything goes well numerically, provided $|\kappa_\mu|$ is "large enough" (cf. [Zwölf'07]).

  Why?

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$$=: a_{scal}(e, e').$$

- [Bonnet-Jr-Zwölf'09].

  Replace the coercivity of the bilinear form $a_{scal}(\cdot,\cdot)$ by the more general $\mathbb{T}$-*coercivity*, where $\mathbb{T}$ is a bijective, continuous linear operator of $H_0^1(\Omega)$ ($\alpha > 0$):

$$a_{scal}(e, \mathbb{T}e) \geq \alpha \|e\|^2_{H_0^1(\Omega)},\ \forall e \in H_0^1(\Omega) \quad \Longleftarrow \quad \mathbb{T}v = \begin{cases} v_1 \text{ in } \Omega_1 \\ -v_2 + 2\mathcal{R}(v|_\Sigma) \text{ in } \Omega_2 \end{cases}.$$

  Then, the *coercive+compact framework* is recovered (for $|\kappa_\mu|$ "large enough").

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$=: a_{scal}(e, e')$.

- [Bonnet-Jr-Zwölf'09].

  For all h, let V^h be the discrete subspace of $H_0^1(\Omega)$.

  Define $\mathbb{T}^h \in \mathcal{L}(V^h)$ such that:
  - the form $a_{scal}(\cdot,\cdot)$ is $\mathbb{T}^h$-coercive, with a coercivity constant independent of h ;
  - the $(\mathbb{T}^h)_h$ are uniformly continuous.

  The *error estimate* is recovered (via a uniform stability estimate for a_{scal} over $(V^h)_h$):

$$\exists \mathcal{C} > 0\,, \exists h_0 > 0\,, \forall h \in]0, h_0] \quad \|u - u^h\|_{H_0^1(\Omega)} \le \mathcal{C} \inf_{v^h \in V^h} \|u - v^h\|_{H_0^1(\Omega)}\,.$$

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Back to the Maxwell problem

In addition to a compact embedding result, establish either

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(2) a well-posed two-field formulation.

- [Bonnet-Jr-Zwölf'08] on approach (2):
  - The embedding of $\mathcal{X}_0(\varepsilon;\Omega)$ into $L^2(\Omega)^3$ is compact.
  - The two-field formulation, with $\underline{\mathcal{H}}_2 := \left(\frac{1}{|\mu_2|}\mathbf{curl}\,\mathcal{E}\right)_{|\Omega_2}$:

$$\left\{\begin{array}{l} \mathcal{E} \in \{\mathcal{F} \in \mathcal{X}_0(\varepsilon;\Omega) \,|\, \operatorname{div} \varepsilon\mathcal{F} = 0\} \\ \underline{\mathcal{H}}_2 \in H(\mathbf{curl}\,;\Omega_2) \end{array}\right.$$

fits into the *coercive+compact framework*.

Back to the Maxwell problem

In addition to a compact embedding result, establish either

(1) $\mathbb{T}$- and uniform $\mathbb{T}^h$- coercivity, or

(2) a well-posed two-field formulation.

- [Bonnet-Jr-Zwölf'08] on approach (2):
  - The embedding of $\mathcal{X}_0(\varepsilon;\Omega)$ into $L^2(\Omega)^3$ is compact.
  - The two-field formulation, with $\underline{\mathcal{H}}_2 := \left(\frac{1}{|\mu_2|}\mathbf{curl}\,\mathcal{E}\right)_{|\Omega_2}$:

$$\begin{cases} \mathcal{E} \in \{\mathcal{F} \in \mathcal{X}_0(\varepsilon;\Omega) \,|\, \operatorname{div} \varepsilon\mathcal{F} = 0\} \\ \underline{\mathcal{H}}_2 \in H(\mathbf{curl}\,;\Omega_2) \end{cases}$$

  fits into the *coercive+compact framework*.

  Assumptions:
  - Compact embedding: smooth interface and $|\kappa_\varepsilon|$ "large enough".
  - Two-field formulation: $|\kappa_\mu|$ "large enough".

Numerical experiments

- In the unit cube, split in two halves (with $\Sigma := \{\frac{1}{2}\}\times]0,1[\times]0,1[$).
- An *exact* piecewise smooth solution is available.
- Discretization of the natural formulation (with $s_{|\Omega_i} = 1/(\mu_i\,\varepsilon_i^2)$):
  Find $\mathcal{E} \in \mathcal{X}_0(\varepsilon;\Omega)$ such that

$$\int_\Omega \frac{1}{\mu}\left(\mathbf{curl}\,\mathcal{E}\cdot\mathbf{curl}\,\bar{\mathcal{E}}' + \mathrm{div}\,\mathcal{E}\mathrm{div}\,\bar{\mathcal{E}}'\right)d\Omega - \omega^2\int_\Omega \varepsilon\mathcal{E}\cdot\bar{\mathcal{E}}'\,d\Omega = \int_\Omega \mathcal{F}\cdot\bar{\mathcal{E}}'\,d\Omega,\ \forall\mathcal{E}' \in \mathcal{X}_0(\varepsilon;\Omega)\,.$$

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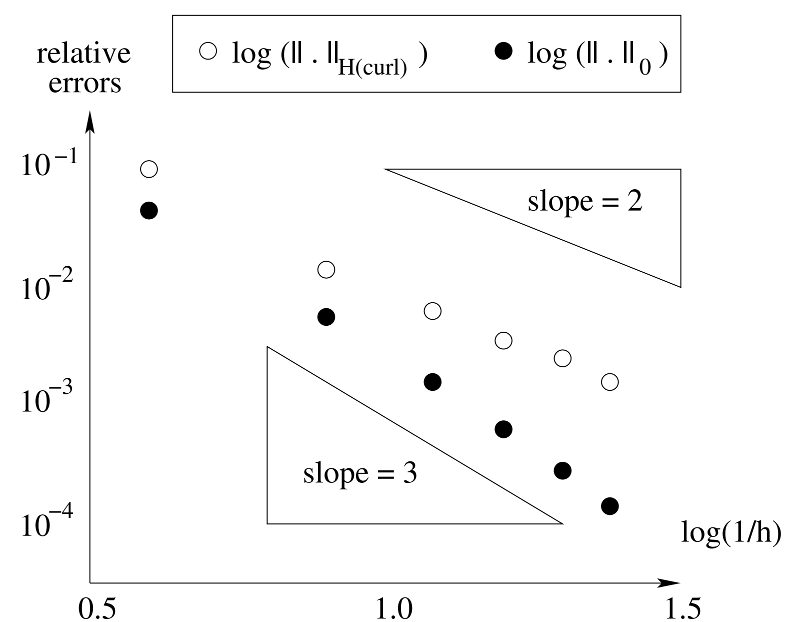
"Usual" case: $\omega = 4$,
$(\varepsilon_1,\mu_1) = (+1,+1)$,
$(\varepsilon_2,\mu_2) = (+1,+1)$,
with P_2 Lagrange FE.



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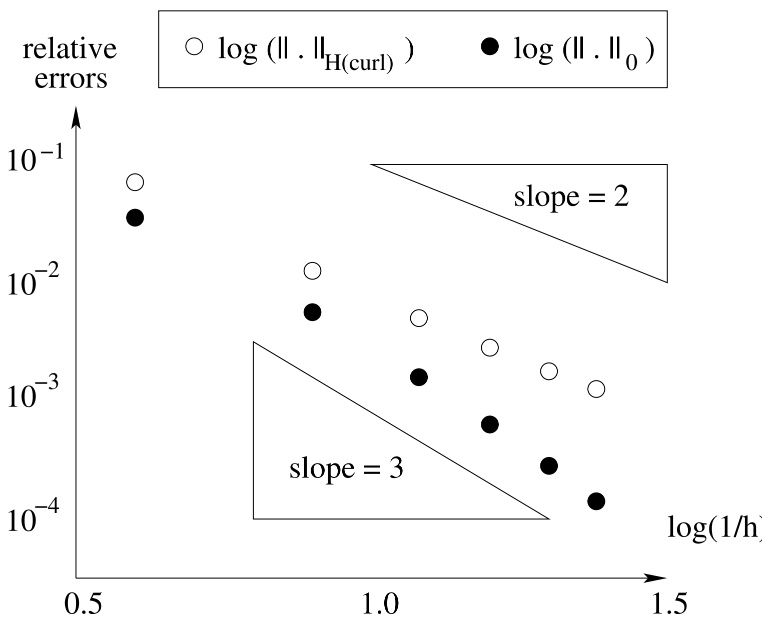
"Unusual" case: $\omega = 4$,
$(\varepsilon_1,\mu_1) = (+1,+1)$,
$(\varepsilon_2,\mu_2) = (-2,-\frac{1}{2})$,
with P_2 Lagrange FE.



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"Unusual" case: $\omega = 4$,
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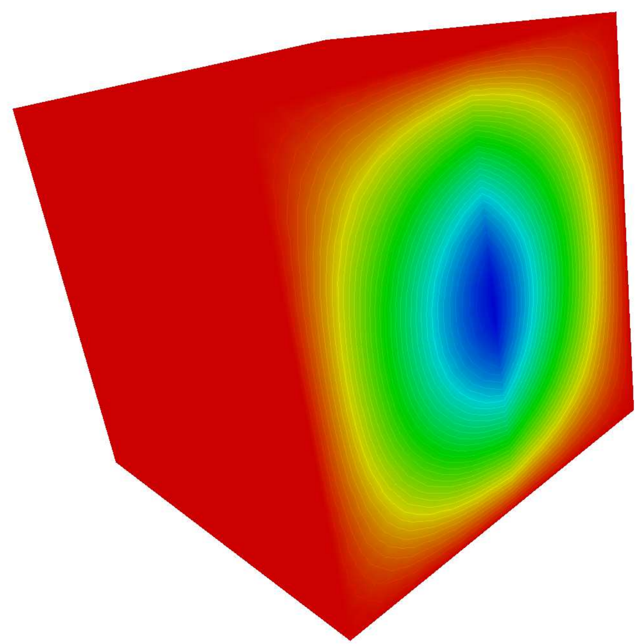
computed electric field ($\mathcal{E}_y^h$).



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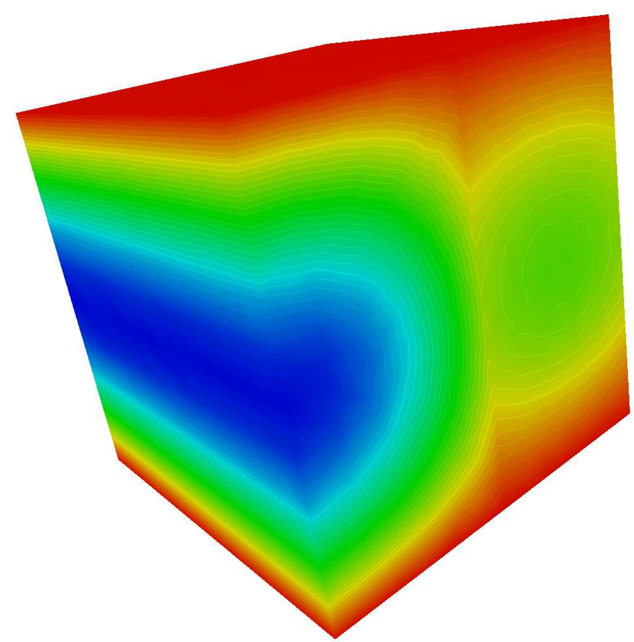
"Unusual" case: $\omega = 4$,
$(\varepsilon_1,\mu_1) = (+1,+1)$,
$(\varepsilon_2,\mu_2) = (-2,-\frac{1}{2})$,
computed magnetic field ($\mathcal{H}_z^h$).



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Perspectives

- For the Maxwell problem:
  - remove the regularity assumptions on the interface (allow corners and edges) ;
  - prove $\mathbb{T}$- and uniform $\mathbb{T}^h$- coercivity ;
  - enforce the divergence condition (with a Lagrange multiplier).

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Perspectives

- For the Maxwell problem:
  - remove the regularity assumptions on the interface (allow corners and edges) ;
  - prove $\mathbb{T}$- and uniform $\mathbb{T}^h$- coercivity ;
  - enforce the divergence condition (with a Lagrange multiplier).
- For both the scalar and the Maxwell problems, investigate the case when $\kappa_\mu \in]\kappa_\mu^{\inf}, \kappa_\mu^{\sup}[$, $\kappa_\varepsilon \in]\kappa_\varepsilon^{\inf}, \kappa_\varepsilon^{\sup}[$:
  - (re)define a mathematical framework ;
  - are the models derived from physics still relevant?

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